The 1925 Born and Jordan paper "On quantum mechanics"

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The 1925 paper "On quantum mechanics" by M. Born and P. Jordan, and the sequel "On quantum mechanics II" by M. Born, W. Heisenberg, and P. Jordan, developed Heisenberg's pioneering theory into the first complete formulation of quantum mechanics. The Born and Jordan paper is the subject of the present article. This paper introduced matrices to physicists. We discuss the original postulates of quantum mechanics, present the two-part discovery of the law of commutation, and clarify the origin of Heisenberg's equation. We show how the 1925 proof of energy conservation and Bohr's frequency condition served as the gold standard with which to measure the validity of the new quantum mechanics.
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I. INTRODUCTION

The name "quantum mechanics" was coined by Max Born.[1] For Born and others, quantum mechanics denoted a canonical theory of atomic and electronic motion of the same level of generality and consistency as classical mechanics. The transition from classical mechanics to a true quantum mechanics remained an elusive goal prior to 1925.

Heisenberg made the breakthrough in his historic 1925 paper, "Quantum-theoretical reinterpretation of kinematic and mechanical relations."[2] Heisenberg's bold idea was to retain the classical equations of Newton but to replace the classical position coordinate with a "quantum-theoretical quantity." The new position quantity contains information about the measurable line spectrum of an atom rather than the unobservable orbit of the electron. Born realized that Heisenberg's kinematical rule for multiplying position quantities was equivalent to the mathematical rule for multiplying matrices. The next step was to formalize Heisenberg's theory using the language of matrices.

The first comprehensive exposition on quantum mechanics in matrix form was written by Born and Jordan,[4] and the sequel was written by Born, Heisenberg, and Jordan.[5] Dirac independently discovered the general equations of quantum mechanics without using matrix theory.[6] These papers developed a Hamiltonian mechanics of the atom in a completely new quantum (noncommutative) format. These papers ushered in a new era in theoretical physics where Hermitian matrices, commutators, and eigenvalue problems became the mathematical trademark of the atomic world. We discuss the first paper "On quantum mechanics."[4]

This formulation of quantum mechanics, now referred to as matrix mechanics,[7] marked one of the most intense periods of discovery in physics. The ideas and formalism behind the original matrix mechanics are absent in most textbooks. Recent articles discuss the correspondence between classical harmonics and quantum jumps,[8] the calculational details of Heisenberg's paper,[9] and the role of Born in the creation of quantum theory.[10] References 11–19 represent a sampling of the many sources on the development of quantum mechanics.

Given Born and Jordan's pivotal role in the discovery of quantum mechanics, it is natural to wonder why there are no equations named after them,[20] and why they did not share the Nobel Prize with others.[21] In 1933 Heisenberg wrote Born saying "The fact that I am to receive the Nobel Prize alone, for work done in Göttingen in collaboration—you, Jordan, and I, this fact depresses me and I hardly know what to write to you. I am, of course, glad that our common efforts are now appreciated, and I enjoy the recollection of the beautiful time of collaboration. I also believe that all good physicists know how great was your and Jordan's contribution to the structure of quantum mechanics—and this remains unchanged by a wrong decision from outside. Yet I myself can do nothing but thank you again for all the fine collaboration and feel a little ashamed."[23]

Engraved on Max Born's tombstone is a one-line epitaph: $pq-qp=h/2\pi i$. Born composed this elegant equation in early July 1925 and called it "die verschärfte Quantenbedingung"[4]—the sharpened quantum condition. This equation is now known as the law of commutation and is the hallmark of quantum algebra.

In the contemporary approach to teaching quantum mechanics, matrix mechanics is usually introduced after a thorough discussion of wave mechanics. The Heisenberg picture is viewed as a unitary transformation of the Schrödinger picture.[24] How was matrix mechanics formulated in 1925 when the Schrödinger picture was nowhere in sight? The Born and Jordan paper[4] represents matrix mechanics in its purest form.

II. BACKGROUND TO "ON QUANTUM MECHANICS"

Heisenberg's program, as indicated by the title of his paper,[2] consisted of constructing quantum-theoretical relations by reinterpreting the classical relations. To appreciate what Born and Jordan did with Heisenberg's reinterpretations, we discuss in the Appendix four key relations from Heisenberg's paper.[2] Heisenberg wrote the classical and quantum versions of each relation in parallel—as formula couplets. Heisenberg has been likened to an "expert decoder who reads a cryptogram."[25] The correspondence principle[8,26] acted as a "code book" for translating a classical relation into its quantum counterpart. Unlike his predecessors who used the correspondence principle to produce specific relations, Heisenberg produced an entirely new theory—complete with a new representation of position and a new rule of multiplication, together with an equation of motion and a quantum condition whose solution determined the atomic observables (energies, frequencies, and transition amplitudes).

Matrices are not explicitly mentioned in Heisenberg's paper. He did not arrange his quantum-theoretical quantities into a table or array. In looking back on his discovery, Heisenberg wrote, "At that time I must confess I did not know what a matrix was and did not know the rules of matrix multiplication."[18] In the last sentence of his paper he wrote "whether this method after all represents far too rough an approach to the physical program of constructing a theoretical quantum mechanics, an obviously very involved problem at the moment, can be decided only by a more intensive mathematical investigation of the method which has been very superficially employed here."[27]

Born took up Heisenberg's challenge to pursue "a more intensive mathematical investigation." At the time Heisenberg wrote his paper, he was Born's assistant at the University of Göttingen. Born recalls the moment of inspiration when he realized that position and momentum were matrices:[28]

> After having sent Heisenberg's paper to the Zeitschrift für Physik for publication, I began to ponder about his symbolic multiplication, and was soon so involved in it…For I felt there was something fundamental behind it…And one morning, about 10 July 1925, I suddenly saw the light: Heisenberg's symbolic multiplication was nothing but the matrix calculus, well known to me since my student days from the lectures of Rosanes in Breslau.
>
> I found this by just simplifying the notation a little: instead of $q(n,n+\tau)$, where n is the quantum number of one state and τ the integer indicating the transition, I wrote $q(n,m)$, and rewriting Heisenberg's form of Bohr's quantum condition, I recognized at once its formal significance. It meant that the two matrix products $\mathbf{pq}$ and $\mathbf{qp}$ are not identical. I was familiar with the fact that matrix multiplication is not commutative; therefore I was not too much puzzled by this result. Closer inspection showed that Heisenberg's formula gave only the value of the diagonal elements $(m=n)$ of the matrix $\mathbf{pq}-\mathbf{qp}$; it said they were all equal and had the value $h/2\pi i$ where h is Planck's constant and $i=\sqrt{-1}$. But what were the other elements $(m\neq n)$?
>
> Here my own constructive work began. Repeating Heisenberg's calculation in matrix notation, I soon convinced myself that the only reasonable value of the nondiagonal elements should be zero, and I wrote the strange equation
>
> $$\mathbf{pq}-\mathbf{qp}=\frac{h}{2\pi i}\mathbf{1}, \tag{1}$$
>
> where $\mathbf{1}$ is the unit matrix. But this was only a guess, and all my attempts to prove it failed.

On 19 July 1925, Born invited his former assistant Wolfgang Pauli to collaborate on the matrix program. Pauli declined the invitation.[29] The next day, Born asked his student Pascual Jordan to assist him. Jordan accepted the invitation and in a few days proved Born's conjecture that all nondiagonal elements of $\mathbf{pq}-\mathbf{qp}$ must vanish. The rest of the new quantum mechanics rapidly solidified. The Born and Jordan paper was received by the Zeitschrift für Physik on 27 September 1925, two months after Heisenberg's paper was received by the same journal. All the essentials of matrix mechanics as we know the subject today fill the pages of this paper.

In the abstract Born and Jordan wrote "The recently published theoretical approach of Heisenberg is here developed into a systematic theory of quantum mechanics (in the first place for systems having one degree of freedom) with the aid of mathematical matrix methods."[30] In the introduction they go on to write "The physical reasoning which led Heisenberg to this development has been so clearly described by him that any supplementary remarks appear superfluous. But, as he himself indicates, in its formal, mathematical aspects his approach is but in its initial stages. His hypotheses have been applied only to simple examples without being fully carried through to a generalized theory. Having been in an advantageous position to familiarize ourselves with his ideas throughout their formative stages, we now strive (since his investigations have been concluded) to clarify the mathematically formal content of his approach and present some of our results here. These indicate that it is in fact possible, starting with the basic premises given by Heisenberg, to build up a closed mathematical theory of quantum mechanics which displays strikingly close analogies with classical mechanics, but at the same time preserves the characteristic features of quantum phenomena."[31]

The reader is introduced to the notion of a matrix in the third paragraph of the introduction: "The mathematical basis of Heisenberg's treatment is the *law of multiplication* of quantum-theoretical quantities, which he derived from an ingenious consideration of correspondence arguments. The development of his formalism, which we give here, is based upon the fact that this rule of multiplication is none other than the well-known mathematical rule of *matrix multiplication*. The infinite square array which appears at the start of the next section, termed a *matrix*, is a representation of a physical quantity which is given in classical theory as a function of time. The mathematical method of treatment inherent in the new quantum mechanics is thereby characterized by the employment of *matrix analysis* in place of the usual number analysis."

The Born-Jordan paper[4] is divided into four chapters. Chapter 1 on "Matrix calculation" introduces the mathematics (algebra and calculus) of matrices to physicists. Chapter 2 on "Dynamics" establishes the fundamental postulates of quantum mechanics, such as the law of commutation, and derives the important theorems, such as the conservation of energy. Chapter 3 on "Investigation of the anharmonic oscillator" contains the first rigorous (correspondence free) calculation of the energy spectrum of a quantum-mechanical harmonic oscillator. Chapter 4 on "Remarks on electrodynamics" contains a procedure—the first of its kind—to quantize the electromagnetic field. We focus on the material in Chap. 2 because it contains the essential physics of matrix mechanics.

III. THE ORIGINAL POSTULATES OF QUANTUM MECHANICS

Current presentations of quantum mechanics frequently are based on a set of postulates.[32] The Born–Jordan postulates of quantum mechanics were crafted before wave mechanics was formulated and thus are quite different than the Schrödinger-based postulates in current textbooks. The original postulates come as close as possible to the classical-mechanical laws while maintaining complete quantum-mechanical integrity.

Section III, "The basic laws," in Chap. 2 of the Born–Jordan paper is five pages long and contains approximately thirty equations. We have imposed a contemporary postulatory approach on this section by identifying five fundamental passages from the text. We call these five fundamental ideas "the postulates." We have preserved the original phrasing, notation, and logic of Born and Jordan. The labeling and the naming of the postulates is ours.

Postulate 1. Position and Momentum. Born and Jordan introduce the position and momentum matrices by writing that[33]

> The dynamical system is to be described by the spatial coordinate $\mathbf{q}$ and the momentum $\mathbf{p}$, these being represented by the matrices
>
> $$(\mathbf{q}=q(nm)e^{2\pi i\nu(nm)t}),$$
> $$(\mathbf{p}=p(nm)e^{2\pi i\nu(nm)t}). \tag{2}$$
>
> Here the $\nu(nm)$ denote the quantum-theoretical frequencies associated with the transitions between states described by the *quantum numbers* n and m. The matrices (2) are to be Hermitian, e.g., on transposition of the matrices, each element is to go over into its complex conjugate value, a condition which should apply for all real t. We thus have
>
> $$q(nm)q(mn)=|q(nm)|^2 \tag{3}$$
>
> and
>
> $$\nu(nm)=-\nu(mn). \tag{4}$$
>
> If q is a *Cartesian* coordinate, then the expression (3) is a measure of the *probabilities* of the transitions $n\rightleftarrows m$.

The preceding passage placed Hermitian matrices into the physics limelight. Prior to the Born–Jordan paper, matrices were rarely seen in physics.[34] Hermitian matrices were even stranger. Physicists were reluctant to accept such an abstract mathematical entity as a description of physical reality.

For Born and Jordan, $\mathbf{q}$ and $\mathbf{p}$ do not specify the position and momentum of an electron in an atom. Heisenberg stressed that quantum theory should focus only on the observable properties, namely the frequency and intensity of the atomic radiation and not the position and period of the electron. The quantities $\mathbf{q}$ and $\mathbf{p}$ represent position and momentum in the sense that $\mathbf{q}$ and $\mathbf{p}$ satisfy matrix equations of motion that are identical in form to those satisfied by the position and momentum of classical mechanics. In the Bohr atom the electron undergoes periodic motion in a well defined orbit around the nucleus with a certain classical frequency. In the Heisenberg–Born–Jordan atom there is no longer an orbit, but there is some sort of periodic "quantum motion" of the electron characterized by the set of frequencies $\nu(nm)$ and amplitudes $q(nm)$. Physicists believed that something inside the atom must vibrate with the right frequencies even though they could not visualize what the quantum oscillations looked like. The mechanical properties $(\mathbf{q},\mathbf{p})$ of the quantum motion contain complete information on the spectral properties (frequency, intensity) of the emitted radiation.

The diagonal elements of a matrix correspond to the states, and the off-diagonal elements correspond to the transitions. An important property of all dynamical matrices is that the diagonal elements are independent of time. The Hermitian rule in Eq. (4) implies the relation $\nu(nn)=0$. Thus the time factor of the nth diagonal term in any matrix is $e^{2\pi i\nu(nn)t}=1$. As we shall see, the time-independent entries in a diagonal matrix are related to the constant values of a conserved quantity.

In their purely mathematical introduction to matrices (Chap. 1), Born and Jordan use the following symbols to denote a matrix

$$\mathbf{a}=(a(nm))=\begin{pmatrix} a(00) & a(01) & a(02) & \dots \\ a(10) & a(11) & a(12) & \\ a(20) & a(21) & a(22) & \\ \vdots & & & \ddots \end{pmatrix}. \tag{5}$$

The bracketed symbol $(a(nm))$, which displays inner elements $a(nm)$ contained within outer brackets ( ), is the shorthand notation for the array in Eq. (5). By writing the matrix elements as $a(nm)$, rather than a_{nm}, Born and Jordan made direct contact with Heisenberg's quantum-theoretical quantities $a(n,n-\alpha)$ (see the Appendix). They wrote[35] "Matrix multiplication is defined by the rule 'rows times columns,' familiar from the ordinary theory of determinants:

$$\mathbf{a}=\mathbf{bc} \text{ means } a(nm)=\sum_{k=0}^{\infty} b(nk)c(km)\,.\text{"} \tag{6}$$

This multiplication rule was first given (for finite square matrices) by Arthur Cayley.[36] Little did Cayley know in 1855 that his mathematical "row times column" expression $b(nk)c(km)$ would describe the physical process of an electron making the transition $n\to k\to m$ in an atom.

Born and Jordan wrote in Postulate 1 that the quantity $|q(nm)|^2$ provides "a measure of the *probabilities* of the transitions $n\rightleftarrows m$." They justify this profound claim in the last chapter.[37] Born and Jordan's one-line claim about transition probabilities is the only statistical statement in their postulates. Physics would have to wait several months before Schrödinger's wave function $\Psi(x)$ and Born's probability function $|\Psi(x)|^2$ entered the scene. Born discovered the connection between $|\Psi(x)|^2$ and position probability, and was also the first physicist (with Jordan) to formalize the connection between $|q(nm)|^2$ and the transition probability via a "quantum electrodynamic" argument.[38] As a pioneer statistical interpreter of quantum mechanics, it is interesting to speculate that Born might have discovered how to form a linear superposition of the periodic matrix elements $q(nm)e^{2\pi i\nu(nm)t}$ in order to obtain another statistical object, namely the expectation value $\langle\mathbf{q}\rangle$. Early on, Born, Heisenberg, and Jordan did superimpose matrix elements,[47] but did not supply the statistical interpretation.

Postulate 2. Frequency Combination Principle. After defining $\mathbf{q}$ and $\mathbf{p}$, Born and Jordan wrote[39] "Further, we shall require that

$$\nu(jk) + \nu(kl) + \nu(lj) = 0\text{ ."} \tag{7}$$

The frequency sum rule in Eq. (7) is the fundamental constraint on the quantum-theoretical frequencies. This rule is based on the Ritz combination principle, which explains the relations of the spectral lines of atomic spectroscopy.[40] Equation (7) is the quantum analogue of the "Fourier combination principle", $\nu(k-j)+\nu(l-k)+\nu(j-l)=0$, where $\nu(\alpha)=\alpha\nu(1)$ is the frequency of the αth harmonic component of a Fourier series. The frequency spectrum of classical periodic motion obeys this Fourier sum rule. The equal Fourier spacing of classical lines is replaced by the irregular Ritzian spacing of quantal lines. In the correspondence limit of large quantum numbers and small quantum jumps the atomic spectrum of Ritz reduces to the harmonic spectrum of Fourier.[8,26] Because the Ritz rule was considered an exact law of atomic spectroscopy, and because Fourier series played a vital role in Heisenberg's analysis, it made sense for Born and Jordan to posit the frequency rule in Eq. (7) as a basic law.

One might be tempted to regard Eq. (7) as equivalent to the Bohr frequency condition, $E(n)-E(m)=h\nu(nm)$, where $E(n)$ is the energy of the stationary state n. For Born and Jordan, Eq. (7) says nothing about energy. They note that Eqs. (4) and (7) imply that there exists spectral terms W_n such that

$$h\nu(nm) = W_n - W_m. \tag{8}$$

At this postulatory stage, the term W_n of the spectrum is unrelated to the energy $E(n)$ of the state. Heisenberg emphasized this distinction between "term" and "energy" in a letter to Pauli summarizing the Born–Jordan theory.[41] Born and Jordan adopt Eq. (7) as a postulate–one based solely on the observable spectral quantities $\nu(nm)$ without reference to any mechanical quantities $E(n)$. The Bohr frequency condition is not something they assume *a priori*, it is something that must be rigorously proved.

The Ritz rule insures that the nm element of any dynamical matrix (any function of $\mathbf{p}$ and $\mathbf{q}$) oscillates with the same frequency $\nu(nm)$ as the nm element of $\mathbf{p}$ and $\mathbf{q}$. For example, if the $3\rightarrow2$ elements of $\mathbf{p}$ and $\mathbf{q}$ oscillate at 500 MHz, then the $3\rightarrow2$ elements of $\mathbf{p}^2$, $\mathbf{q}^2$, $\mathbf{pq}$, $\mathbf{q}^3$, $\mathbf{p}^2+\mathbf{q}^2$, etc. each oscillate at 500 MHz. In all calculations involving the canonical matrices $\mathbf{p}$ and $\mathbf{q}$, no new frequencies are generated. A consistent quantum theory must preserve the frequency spectrum of a particular atom because the spectrum is the spectroscopic signature of the atom. The calculations must not change the identity of the atom. Based on the rules for manipulating matrices and combining frequencies, Born and Jordan wrote that "it follows that a function $\mathbf{g}(\mathbf{pq})$ invariably takes on the form

$$\mathbf{g} = (g(nm)e^{2\pi i\nu(nm)t}) \tag{9}$$

and the matrix $(g(nm))$ therein results from identically the same process applied to the matrices $(q(nm))$, $(p(nm))$ as was employed to find $\mathbf{g}$ from $\mathbf{q}$, $\mathbf{p}$."[42] Because $e^{2\pi i\nu(nm)t}$ is the universal time factor common to all dynamical matrices, they note that it can be dropped from Eq. (2) in favor of the shorter notation $\mathbf{q}=(q(nm))$ and $\mathbf{p}=(p(nm))$.

Why does the Ritz rule insure that the time factors of $\mathbf{g}(\mathbf{pq})$ are identical to the time factors of $\mathbf{p}$ and $\mathbf{q}$? Consider the potential energy function $\mathbf{q}^2$. The nm element of $\mathbf{q}^2$, which we denote by $\mathbf{q}^2(nm)$, is obtained from the elements of $\mathbf{q}$ via the multiplication rule

$$\mathbf{q}^2(nm) = \sum_k q(nk)e^{2\pi i\nu(nk)t}q(km)e^{2\pi i\nu(km)t}. \tag{10}$$

Given the Ritz relation $\nu(nm)=\nu(nk)+\nu(km)$, which follows from Eqs. (4) and (7), Eq. (10) reduces to

$$\mathbf{q}^2(nm) = \left[\sum_k q(nk)q(km)\right]e^{2\pi i\nu(nm)t}. \tag{11}$$

It follows that the nm time factor of $\mathbf{q}^2$ is the same as the nm time factor of $\mathbf{q}$.

We see that the theoretical rule for multiplying mechanical amplitudes, $a(nm)=\Sigma_k b(nk)c(km)$, is intimately related to the experimental rule for adding spectral frequencies, $\nu(nm)=\nu(nk)+\nu(km)$. The Ritz rule occupied a prominent place in Heisenberg's discovery of the multiplication rule (see the Appendix). Whenever a contemporary physicist calculates the total amplitude of the quantum jump $n\rightarrow k\rightarrow m$, the steps involved can be traced back to the frequency combination principle of Ritz.

Postulate 3. The Equation of Motion. Born and Jordan introduce the law of quantum dynamics by writing[43]

> In the case of a Hamilton function having the form
>
> $$\mathbf{H} = \frac{1}{2m}\mathbf{p}^2 + \mathbf{U}(\mathbf{q}), \tag{12}$$
>
> we shall assume, as did Heisenberg, that the equations of motion have just the same form as in the classical theory, so that we can write:
>
> $$\dot{\mathbf{q}} = \frac{\partial\mathbf{H}}{\partial\mathbf{p}} = \frac{1}{m}\mathbf{p}, \tag{13a}$$
>
> $$\dot{\mathbf{p}} = -\frac{\partial\mathbf{H}}{\partial\mathbf{q}} = -\frac{\partial\mathbf{U}}{\partial\mathbf{q}}. \tag{13b}$$

This Hamiltonian formulation of quantum dynamics generalized Heisenberg's Newtonian approach.[44] The assumption by Heisenberg and Born and Jordan that quantum dynamics looks the same as classical dynamics was a bold and deep assumption. For them, the problem with classical mechanics was not the dynamics (the form of the equations of motion), but rather the kinematics (the meaning of position and momentum).

Postulate 4. Energy Spectrum. Born and Jordan reveal the connection between the allowed energies of a conservative system and the numbers in the Hamiltonian matrix:

> "The diagonal elements $H(nn)$ of $\mathbf{H}$ are interpreted, according to Heisenberg, as the *energies of the various states of the system*."[45]

This statement introduced a radical new idea into mainstream physics: calculating an energy spectrum reduces to finding the components of a diagonal matrix.[46] Although Born and Jordan did not mention the word eigenvalue in Ref. 4, Born, Heisenberg, and Jordan would soon formalize the idea of calculating an energy spectrum by solving an eigenvalue problem.[5] The ad hoc rules for calculating a quantized energy in the old quantum theory were replaced by a systematic mathematical program.

Born and Jordan considered exclusively conservative systems for which $\mathbf{H}$ does not depend explicitly on time. The connection between conserved quantities and diagonal matrices will be discussed later. For now, recall that the diagonal elements of any matrix are independent of time. For the special case where all the non-diagonal elements of a dynamical matrix $\mathbf{g}(\mathbf{pq})$ vanish, the quantity $\mathbf{g}$ is a constant of the motion. A postulate must be introduced to specify the physical meaning of the constant elements in $\mathbf{g}$.

In the old quantum theory it was difficult to explain why the energy was quantized. The discontinuity in energy had to be postulated or artificially imposed. Matrices are naturally quantized. The quantization of energy is built into the discrete row-column structure of the matrix array. In the old theory Bohr's concept of a stationary state of energy E_n was a central concept. Physicists grappled with the questions: Where does E_n fit into the theory? How is E_n calculated? Bohr's concept of the energy of the stationary state finally found a rigorous place in the new matrix scheme.[47]

Postulate 5. The Quantum Condition. Born and Jordan state that the elements of $\mathbf{p}$ and $\mathbf{q}$ for any quantum mechanical system must satisfy the "quantum condition":

$$\sum_k (p(nk)q(kn) - q(nk)p(kn)) = \frac{h}{2\pi i}. \tag{14}$$

Given the significance of Eq. (14) in the development of quantum mechanics, we quote Born and Jordan's "derivation" of this equation:

> The equation
>
> $$J = \oint p dq = \int_0^{1/\nu} p\dot{q} dt \tag{15}$$
>
> of "classical" quantum theory can, on introducing the Fourier expansions of p and q,
>
> $$p = \sum_{\tau=-\infty}^{\infty} p_\tau e^{2\pi i \nu \tau t}, \qquad q = \sum_{\tau=-\infty}^{\infty} q_\tau e^{2\pi i \nu \tau t}, \tag{16}$$
>
> be transformed into
>
> $$1 = 2\pi i \sum_{\tau=-\infty}^{\infty} \tau \frac{\partial}{\partial J}(q_\tau p_{-\tau}). \tag{17}$$
>
> The following expressions should correspond:
>
> $$\sum_{\tau=-\infty}^{\infty} \tau \frac{\partial}{\partial J}(q_\tau p_{-\tau}) \tag{18}$$
>
> with
>
> $$\frac{1}{h}\sum_{\tau=-\infty}^{\infty} (q(n+\tau,n)p(n,n+\tau) - q(n,n-\tau)p(n-\tau,n)), \tag{19}$$
>
> where in the right-hand expression those $q(nm)$, $p(nm)$ which take on a negative index are to be set equal to zero. In this way we obtain the quantization condition corresponding to Eq. (17) as
>
> $$\sum_k (p(nk)q(kn) - q(nk)p(kn)) = \frac{h}{2\pi i}. \tag{20}$$
>
> This is a system of infinitely many equations, namely one for each value of n.[48]

Why did Born and Jordan take the derivative of the action integral in Eq. (15) to arrive at Eq. (17)? Heisenberg performed a similar maneuver (see the Appendix). One reason is to eliminate any explicit dependence on the integer variable n from the basic laws. Another reason is to generate a differential expression that can readily be translated via the correspondence principle into a difference expression containing only transition quantities. In effect, a state relation is converted into a change-in-state relation. In the old quantum theory the Bohr–Sommerfeld quantum condition, $\oint pdq = nh$, determined how all state quantities depend on n. Such an ad hoc quantization algorithm has no proper place in a rigorous quantum theory, where n should not appear explicitly in any of the fundamental laws. The way in which $q(nm)$, $p(nm)$, $\nu(nm)$ depend on (nm) should not be artificially imposed, but should be naturally determined by fundamental relations involving only the canonical variables $\mathbf{q}$ and $\mathbf{p}$, without any explicit dependence on the state labels n and m. Equation (20) is one such fundamental relation.

In 1924 Born introduced the technique of replacing differentials by differences to make the "formal passage from classical mechanics to a 'quantum mechanics'."[49] This correspondence rule played an important role in allowing Born and others to develop the equations of quantum mechanics.[50] To motivate Born's rule note that the fundamental orbital frequency of a classical periodic system is equal to dE/dJ (E is energy and $J = \oint pdq$ is an action),[51] whereas the spectral frequency of an atomic system is equal to $\Delta E/h$. Hence, the passage from a classical to a quantum frequency is made by replacing the derivative dE/dJ by the difference $\Delta E/h$.[52] Born conjectured that this correspondence is valid for any quantity Φ. He wrote "We are therefore as good as forced to adopt the rule that we have to replace a classically calculated quantity, whenever it is of the form $\tau\partial\Phi/\partial J$ by the linear average or difference quotient $[\Phi(n+\tau) - \Phi(n)]/h$."[53] The correspondence between Eqs. (18) and (19) follows from

Born's rule by letting Φ be $\Phi(n)=q(n,n-\tau)p(n-\tau,n)$, where $q(n,n-\tau)$ corresponds to q_τ and $p(n-\tau,n)$ corresponds to $p_{-\tau}$ or p_τ^*.

Born and Jordan remarked that Eq. (20) implies that **p** and **q** can never be finite matrices.[54] For the special case $\mathbf{p}=m\dot{\mathbf{q}}$ they also noted that the general condition in Eq. (20) reduces to Heisenberg's form of the quantum condition (see the Appendix). Heisenberg did not realize that his quantization rule was a relation between **pq** and **qp**.[55]

Planck's constant h enters into the theory via the quantum condition in Eq. (20). The quantum condition expresses the following deep law of nature: *All the diagonal components of* $\mathbf{pq}-\mathbf{qp}$ *must equal the universal constant* $h/2\pi i$.

What about the nondiagonal components of $\mathbf{pq}-\mathbf{qp}$? Born claimed that they were all equal to zero. Jordan proved Born's claim. It is important to emphasize that Postulate 5 says nothing about the nondiagonal elements. Born and Jordan were careful to distinguish the postulated statements (laws of nature) from the derivable results (consequences of the postulates). Born's development of the diagonal part of $\mathbf{pq}-\mathbf{qp}$ and Jordan's derivation of the nondiagonal part constitute the two-part discovery of the law of commutation.

IV. THE LAW OF COMMUTATION

Born and Jordan write the following equation in Sec. IV of "On quantum mechanics":

$$\mathbf{pq}-\mathbf{qp}=\frac{h}{2\pi i}\mathbf{1}. \tag{21}$$

They call Eq. (21) the "sharpened quantum condition" because it sharpened the condition in Eq. (20), which only fixes the diagonal elements, to one which fixes all the elements. In a letter to Pauli, Heisenberg referred to Eq. (21) as a "fundamental law of this mechanics" and as "Born's very clever idea."[56] Indeed, the commutation law in Eq. (21) is one of the most fundamental relations in quantum mechanics. This equation introduces Planck's constant and the imaginary number i into the theory in the most basic way possible. It is the golden rule of quantum algebra and makes quantum calculations unique. The way in which all dynamical properties of a system depend on h can be traced back to the simple way in which $\mathbf{pq}-\mathbf{qp}$ depend on h. In short, the commutation law in Eq. (21) stores information on the discontinuity, the non-commutativity, the uncertainty, and the complexity of the quantum world.

In their paper Born and Jordan proved that the off-diagonal elements of $\mathbf{pq}-\mathbf{qp}$ are equal to zero by first establishing a "diagonality theorem," which they state as follows: "If $\nu(nm)\neq 0$ when $n\neq m$, a condition which we wish to assume, then the formula $\dot{\mathbf{g}}=0$ denotes that **g** is a diagonal matrix with $g(nm)=\delta_{nm}g(nn)$."[57] This theorem establishes the connection between the structural (diagonality) and the temporal (constancy) properties of a dynamical matrix. It provided physicists with a whole new way to look at conservation principles: *In quantum mechanics, conserved quantities are represented by diagonal matrices.*[58]

Born and Jordan proved the diagonality theorem as follows. Because all dynamical matrices $\mathbf{g}(\mathbf{pq})$ have the form in Eq. (9), the time derivative of **g** is

$$\dot{\mathbf{g}}=2\pi i(\nu(nm)g(nm)e^{2\pi i\nu(nm)t}). \tag{22}$$

If $\dot{\mathbf{g}}=0$, then Eq. (22) implies the relation $\nu(nm)g(nm)=0$ for all (nm). This relation is always true for the diagonal elements because $\nu(nn)$ is always equal to zero. For the off-diagonal elements, the relation $\nu(nm)g(nm)=0$ implies that $g(nm)$ must equal zero, because it is assumed that $\nu(nm)\neq 0$ for $n\neq m$. Thus, **g** is a diagonal matrix.

Hence, to show that $\mathbf{pq}-\mathbf{qp}$ is a diagonal matrix, Born and Jordan showed that the time derivative of $\mathbf{pq}-\mathbf{qp}$ is equal to zero. They introduced the matrix $\mathbf{d}\equiv\mathbf{pq}-\mathbf{qp}$ and expressed the time derivative of **d** as

$$\dot{\mathbf{d}}=\dot{\mathbf{p}}\mathbf{q}+\mathbf{p}\dot{\mathbf{q}}-\dot{\mathbf{q}}\mathbf{p}-\mathbf{q}\dot{\mathbf{p}}. \tag{23}$$

They used the canonical equations of motion in Eq. (13) to write Eq. (23) as

$$\dot{\mathbf{d}}=\mathbf{q}\frac{\partial\mathbf{H}}{\partial\mathbf{q}}-\frac{\partial\mathbf{H}}{\partial\mathbf{q}}\mathbf{q}+\mathbf{p}\frac{\partial\mathbf{H}}{\partial\mathbf{p}}-\frac{\partial\mathbf{H}}{\partial\mathbf{p}}\mathbf{p}. \tag{24}$$

They next demonstrated that the combination of derivatives in Eq. (24) leads to a vanishing result[59] and say that "it follows that $\dot{\mathbf{d}}=0$ and **d** is a diagonal matrix. The diagonal elements of **d** are, however, specified by the quantum condition (20). Summarizing, we obtain the equation

$$\mathbf{pq}-\mathbf{qp}=\frac{h}{2\pi i}\mathbf{1}, \tag{25}$$

on introducing the unit matrix **1**. We call Eq. (25) the 'sharpened quantum condition' and base all further conclusions on it."[60] Fundamental results that propagate from Eq. (25) include the equation of motion, $\dot{\mathbf{g}}=(2\pi i/h)(\mathbf{Hg}-\mathbf{gH})$ (see Sec. V), the Heisenberg uncertainty principle, $\Delta p\Delta q\geqslant h/4\pi$, and the Schrödinger operator, $p=(h/2\pi i)d/dq$.

It is important to emphasize the two distinct origins of $\mathbf{pq}-\mathbf{qp}=(h/2\pi i)\mathbf{1}$. The diagonal part, $(\mathbf{pq}-\mathbf{qp})_{\text{diagonal}}=h/2\pi i$ is a *law*—an exact decoding of the approximate law $\oint pdq=nh$. The nondiagonal part, $(\mathbf{pq}-\mathbf{qp})_{\text{nondiagonal}}=0$, is a *theorem*—a logical consequence of the equations of motion. From a practical point of view Eq. (25) represents vital information on the line spectrum of an atom by defining a system of algebraic equations that place strong constraints on the magnitudes of $q(nm)$, $p(nm)$, and $\nu(nm)$.

V. THE EQUATION OF MOTION

Born and Jordan proved that the equation of motion describing the time evolution of any dynamical quantity $\mathbf{g}(\mathbf{pq})$ is

$$\dot{\mathbf{g}}=\frac{2\pi i}{h}(\mathbf{Hg}-\mathbf{gH}). \tag{26}$$

Equation (26) is now often referred to as the Heisenberg equation.[61] In Ref. 2 the only equation of motion is Newton's second law, which Heisenberg wrote as $\ddot{x}+f(x)=0$ (see the Appendix).

The "commutator" of mechanical quantities is a recurring theme in the Born–Jordan theory. The quantity $\mathbf{pq}-\mathbf{qp}$ lies at the core of their theory. Equation (26) reveals how the quantity $\mathbf{Hg}-\mathbf{gH}$ is synonymous with the time evolution of **g**. Thanks to Born and Jordan, as well as Dirac who established the connection between commutators and classical Poisson brackets,[6] the commutator is now an integral part of modern quantum theory. The change in focus from commuting variables to noncommuting variables represents a paradigm shift in quantum theory.

The original derivation of Eq. (26) is different from present-day derivations. In the usual textbook presentation Eq. (26) is derived from a unitary transformation of the states and operators in the Schrödinger picture.[24] In 1925, the Schrödinger picture did not exist. To derive Eq. (26) from their postulates Born and Jordan developed a new quantum-theoretical technology that is now referred to as "commutator algebra." They began the proof by stating the following generalizations of Eq. (25):

$$\mathbf{p}^n\mathbf{q} = \mathbf{q}\mathbf{p}^n + n\frac{h}{2\pi i}\mathbf{p}^{n-1}, \tag{27}$$

$$\mathbf{q}^n\mathbf{p} = \mathbf{p}\mathbf{q}^n - n\frac{h}{2\pi i}\mathbf{q}^{n-1}, \tag{28}$$

which can readily be derived by induction. They considered Hamiltonians of the form

$$\mathbf{H} = \mathbf{H}_1(\mathbf{p}) + \mathbf{H}_2(\mathbf{q}), \tag{29}$$

where $\mathbf{H}_1(\mathbf{p})$ and $\mathbf{H}_2(\mathbf{q})$ are represented by power series

$$\mathbf{H}_1 = \sum_s a_s\mathbf{p}^s,$$

$$\mathbf{H}_2 = \sum_s b_s\mathbf{q}^s. \tag{30}$$

After writing these expressions, they wrote[62] "Formulae (27) and (28) indicate that

$$\mathbf{H}\mathbf{q} - \mathbf{q}\mathbf{H} = \frac{h}{2\pi i}\frac{\partial\mathbf{H}}{\partial\mathbf{p}}, \tag{31}$$

$$\mathbf{H}\mathbf{p} - \mathbf{p}\mathbf{H} = -\frac{h}{2\pi i}\frac{\partial\mathbf{H}}{\partial\mathbf{q}}. \tag{32}$$

Comparison with the equations of motion (13) yields

$$\dot{\mathbf{q}} = \frac{2\pi i}{h}(\mathbf{H}\mathbf{q} - \mathbf{q}\mathbf{H}), \tag{33}$$

$$\dot{\mathbf{p}} = \frac{2\pi i}{h}(\mathbf{H}\mathbf{p} - \mathbf{p}\mathbf{H}). \tag{34}$$

Denoting the matrix $\mathbf{Hg}-\mathbf{gH}$ by $\left|\begin{smallmatrix}\mathbf{H}\\ \mathbf{g}\end{smallmatrix}\right|$ for brevity, one has

$$\left|\begin{matrix}\mathbf{H}\\ \mathbf{ab}\end{matrix}\right| = \left|\begin{matrix}\mathbf{H}\\ \mathbf{a}\end{matrix}\right|\mathbf{b} + \mathbf{a}\left|\begin{matrix}\mathbf{H}\\ \mathbf{b}\end{matrix}\right|, \tag{35}$$

from which generally for $\mathbf{g}=\mathbf{g}(\mathbf{pq})$ one may conclude that

$$\dot{\mathbf{g}} = \frac{2\pi i}{h}\left|\begin{matrix}\mathbf{H}\\ \mathbf{g}\end{matrix}\right| = \frac{2\pi i}{h}(\mathbf{Hg} - \mathbf{gH})\text{ ."} \tag{36}$$

The derivation of Eq. (36) clearly displays Born and Jordan's expertise in commutator algebra. The essential step to go from Eq. (27) to Eq. (31) is to note that Eq. (27) can be rewritten as a commutator-derivative relation, $\mathbf{p}^n\mathbf{q}-\mathbf{q}\mathbf{p}^n = (h/2\pi i)d\mathbf{p}^n/d\mathbf{p}$, which is equivalent to the nth term of the series representation of Eq. (31). The generalized commutation rules in Eqs. (27) and (28), and the relation between commutators and derivatives in Eqs. (31) and (32) are now standard operator equations of contemporary quantum theory.

With the words, "Denoting the matrix $\mathbf{Hg}-\mathbf{gH}$ by $\left|\begin{smallmatrix}\mathbf{H}\\ \mathbf{g}\end{smallmatrix}\right|$," Born and Jordan formalized the notion of a commutator and introduced physicists to this important quantum-theoretical object. The appearance of Eq. (36) in Ref. 4 marks the first printed statement of the general equation of motion for a dynamical quantity in quantum mechanics.

VI. THE ENERGY THEOREMS

Heisenberg, Born, and Jordan considered the conservation of energy and the Bohr frequency condition as universal laws that should emerge as logical consequences of the fundamental postulates. Proving energy conservation and the frequency condition was the ultimate measure of the power of the postulates and the validity of the theory.[63] Born and Jordan began Sec. IV of Ref. 4 by writing "The content of the preceding paragraphs furnishes the basic rules of the new quantum mechanics in their entirety. All the other laws of quantum mechanics, whose general validity is to be verified, must be *derivable* from these basic tenets. As instances of such laws to be proved, the law of energy conservation and the Bohr frequency condition primarily enter into consideration."[64]

The energy theorems are stated as follows:[65]

$$\dot{\mathbf{H}} = 0 \quad \text{(energy conservation)}, \tag{37}$$

$$h\nu(nm) = H(nn) - H(mm) \quad \text{(frequency condition)}. \tag{38}$$

Equations (37) and (38) are remarkable statements on the temporal behavior of the system and the logical structure of the theory.[66] Equation (37) says that $\mathbf{H}$, which depends on the matrices $\mathbf{p}$ and $\mathbf{q}$ is always a constant of the motion even though $\mathbf{p}=\mathbf{p}(t)$ and $\mathbf{q}=\mathbf{q}(t)$ depend on time. In short, the t in $\mathbf{H}(\mathbf{p}(t),\mathbf{q}(t))$ must completely disappear. Equation (37) reveals the time independence of $\mathbf{H}$, and Eq. (38) specifies how $\mathbf{H}$ itself determines the time dependence of all other dynamical quantities.

Why should $\nu(nm)$, $H(nn)$, and $H(mm)$ be related? These quantities are completely different structural elements of different matrices. The parameter $\nu(nm)$ is a transition quantity that characterizes the off-diagonal, time-dependent part of $\mathbf{q}$ and $\mathbf{p}$. In contrast, $H(nn)$ is a state quantity that characterizes the diagonal, time-independent part of $\mathbf{H}(\mathbf{pq})$. It is a nontrivial claim to say that these mechanical elements are related.

It is important to distinguish between the Bohr meaning of $E_n-E_m=h\nu$ and the Born–Jordan meaning of $H(nn)-H(mm)=h\nu(nm)$. For Bohr, E_n denotes the mechanical energy of the electron and ν denotes the spectral frequency of the radiation. In the old quantum theory there exists ad hoc, semiclassical rules to calculate E_n. There did not exist any mechanical rules to calculate ν, independent of E_n and E_m. The relation between E_n-E_m and ν was postulated. Born and Jordan did not postulate any connection between $H(nn)$, $H(mm)$, and $\nu(nm)$. The basic mechanical laws (law of motion and law of commutation) allow them to calculate the frequencies $\nu(nm)$ which paramaterize $\mathbf{q}$ and the energies $H(nn)$ stored in $\mathbf{H}$. The theorem in Eq. (38) states that the calculated values of the mechanical parameters $H(nn)$, $H(mm)$, and $\nu(nm)$ will always satisfy the relation $H(nn) - H(mm) = h\nu(nm)$.

The equation of motion (36) is the key to proving the energy theorems. Born and Jordan wrote "In particular, if in Eq. (36) we set $\mathbf{g} = \mathbf{H}$, we obtain

$$\dot{\mathbf{H}} = 0. \tag{39}$$

Now that we have verified the energy-conservation law and recognized the matrix $\mathbf{H}$ to be diagonal [by the diagonality theorem, $\dot{\mathbf{H}} = 0 \Rightarrow \mathbf{H}$ is diagonal], Eqs. (33) and (34) can be put into the form

$$h\nu(nm)q(nm) = (H(nn) - H(mm))q(nm), \tag{40}$$

$$h\nu(nm)p(nm) = (H(nn) - H(mm))p(nm), \tag{41}$$

from which the frequency condition follows."[67] Given the importance of this result, it is worthwhile to elaborate on the proof. Because the nm component of any matrix $\mathbf{g}$ is $g(nm)e^{2\pi i\nu(nm)t}$, the nm component of the matrix relation in Eq. (33) is

$$\begin{aligned} 2\pi i\nu(nm)q(nm)e^{2\pi i\nu(nm)t} &= \frac{2\pi i}{h}\sum_k (H(nk)q(km) \\ &\quad - q(nk)H(km))e^{2\pi i[\nu(nk)+\nu(km)]t}. \end{aligned} \tag{42}$$

Given the diagonality of $\mathbf{H}$, $H(nk) = H(nn)\delta_{nk}$ and $H(km) = H(mm)\delta_{km}$, and the Ritz rule, $\nu(nk) + \nu(km) = \nu(nm)$, Eq. (42) reduces to

$$\nu(nm) = \frac{1}{h}(H(nn) - H(mm)). \tag{43}$$

In this way Born and Jordan demonstrated how Bohr's frequency condition, $h\nu(nm) = H(nn) - H(mm)$, is simply a scalar component of the matrix equation, $h\dot{\mathbf{q}} = 2\pi i(\mathbf{Hq} - \mathbf{qH})$. In any presentation of quantum mechanics it is important to explain how and where Bohr's frequency condition logically fits into the formal structure.[68]

According to Postulate 4, the nth diagonal element $H(nn)$ of $\mathbf{H}$ is equal to the energy of the nth stationary state. Logically, this postulate is needed to interpret Eq. (38) as the original frequency condition conjectured by Bohr. Born and Jordan note that Eqs. (8) and (38) imply that the mechanical energy $H(nn)$ is related to the spectral term W_n as follows: $W_n = H(nn) + \text{constant}$.[69]

This mechanical proof of the Bohr frequency condition established an explicit connection between time evolution and energy. In the matrix scheme all mechanical quantities ($\mathbf{p}$, $\mathbf{q}$, and $\mathbf{g}(\mathbf{pq})$) evolve in time via the set of factors $e^{2\pi i\nu(nm)t}$, where $\nu(nm) = (H(nn) - H(mm))/h$. Thus, all $\mathbf{g}$-functions have the form[70]

$$\mathbf{g} = (g(nm)e^{2\pi i(H(nn)-H(mm))t/h}). \tag{44}$$

Equation (44) exhibits how the difference in energy between state n and state m is the "driving force" behind the time evolution (quantum oscillations) associated with the change of state $n \to m$.

In the introduction of their paper, Born and Jordan write "With the aid of [the equations of motion and the quantum condition], one can prove the general validity of the *law of conservation of energy* and the *Bohr frequency relation* in the sense conjectured by Heisenberg: this proof could not be carried through in its entirety by him even for the simple examples which he considered."[71] Because $\mathbf{p}$ and $\mathbf{q}$ do not commute, the mechanism responsible for energy conservation in quantum mechanics is significantly different than the classical mechanism. Born and Jordan emphasize this difference by writing "Whereas in classical mechanics energy conservation ($\dot{H} = 0$) is directly apparent from the canonical equations, the same law of energy conservation in quantum mechanics, $\dot{\mathbf{H}} = 0$ lies, as one can see, more deeply hidden beneath the surface. That its demonstrability from the assumed postulates is far from being trivial will be appreciated if, following more closely the classical method of proof, one sets out to prove $\mathbf{H}$ to be constant simply by evaluating $\dot{\mathbf{H}}$."[72]

We carry out Born and Jordan's suggestion "to prove $\mathbf{H}$ to be constant simply by evaluating $\dot{\mathbf{H}}$" for the special Hamiltonian

$$\mathbf{H} = \mathbf{p}^2 + \mathbf{q}^3. \tag{45}$$

In order to focus on the energy calculus of the $\mathbf{p}$ and $\mathbf{q}$ matrices, we have omitted the scalar coefficients in Eq. (45). If we write Eq. (45) as $\mathbf{H} = \mathbf{pp} + \mathbf{qqq}$, calculate $\dot{\mathbf{H}}$, and use the equations of motion $\dot{\mathbf{q}} = 2\mathbf{p}$, $\dot{\mathbf{p}} = -3\mathbf{q}^2$, we find[73]

$$\dot{\mathbf{H}} = \mathbf{q}(\mathbf{pq} - \mathbf{qp}) + (\mathbf{qp} - \mathbf{pq})\mathbf{q}. \tag{46}$$

Equation (46) reveals how the value of $\mathbf{pq} - \mathbf{qp}$ uniquely determines the value of $\dot{\mathbf{H}}$. The quantum condition, $\mathbf{pq} - \mathbf{qp} = (h/2\pi i)\mathbf{1}$, reduces Eq. (46) to $\dot{\mathbf{H}} = 0$. In classical mechanics the classical condition, $pq - qp = 0$, is taken for granted in proving energy conservation. In quantum mechanics the condition that specifies the nonzero value of $\mathbf{pq} - \mathbf{qp}$ plays a nontrivial role in establishing energy conservation. This nontriviality is what Born and Jordan meant when they wrote that energy conservation in quantum mechanics "lies more deeply hidden beneath the surface."

Proving the law of energy conservation and the Bohr frequency condition was the decisive test of the theory—the final validation of the new quantum mechanics. All of the pieces of the "quantum puzzle" now fit together. After proving the energy theorems, Born and Jordan wrote that "The fact that energy-conservation and frequency laws could be proved in so general a context would seem to us to furnish strong grounds to hope that this theory embraces truly deep-seated physical laws."[74]

VII. CONCLUSION

To put the discovery of quantum mechanics in matrix form into perspective, we summarize the contributions of Heisenberg and Born–Jordan. Heisenberg's breakthrough consists of four quantum-theoretical reinterpretations (see the Appendix):

1. Replace the position coordinate $x(t)$ by the set of transition components $a(n,n-\alpha)e^{i\omega(n,n-\alpha)t}$.
2. Replace $x^2(t)$ with the set $\Sigma_\alpha a(n,n-\alpha)e^{i\omega(n,n-\alpha)t}a(n-\alpha,n-\beta)e^{i\omega(n-\alpha,n-\beta)t}$.
3. Keep Newton's second law, $\ddot{x} + f(x) = 0$, but replace x as before.

4. Replace the old quantum condition, $nh=\oint m\dot{x}^2 dt$, with $h=4\pi m\Sigma_\alpha\{|a(n+\alpha,n)|^2\omega(n+\alpha,n)-|a(n,n-\alpha)|^2\omega(n,n-\alpha)\}$.

The quantum mechanics of Born and Jordan consists of five postulates:

1. $\mathbf{q}=(q(nm)e^{2\pi i\nu(nm)t})$, $\mathbf{p}=(p(nm)e^{2\pi i\nu(nm)t})$,
2. $\nu(jk)+\nu(kl)+\nu(lj)=0$,
3. $\dot{\mathbf{q}}=\partial\mathbf{H}/\partial\mathbf{p}$, $\dot{\mathbf{p}}=-\partial\mathbf{H}/\partial\mathbf{q}$,
4. $E_n=H(nn)$, and
5. $(\mathbf{pq}-\mathbf{qp})_{\text{diagonal}}=h/2\pi i$,

and four theorems

1. $(\mathbf{pq}-\mathbf{qp})_{\text{nondiagonal}}=0$,
2. $\dot{\mathbf{g}}=(2\pi i/h)(\mathbf{Hg}-\mathbf{gH})$,
3. $\dot{\mathbf{H}}=0$, and
4. $h\nu(nm)=H(nn)-H(mm)$.

Quantum mechanics evolved at a rapid pace after the papers of Heisenberg and Born–Jordan. Dirac's paper was received on 7 November 1925.[6] Born, Heisenberg, and Jordan's paper was received on 16 November 1925.[5] The first "textbook" on quantum mechanics appeared in 1926.[75] In a series of papers during the spring of 1926, Schrödinger set forth the theory of wave mechanics.[76] In a paper received June 25, 1926 Born introduced the statistical interpretation of the wave function.[77] The Nobel Prize was awarded to Heisenberg in 1932 (delayed until 1933) to Schrödinger and Dirac in 1933, and to Born in 1954.

APPENDIX: HEISENBERG'S FOUR BREAKTHROUGH IDEAS

We divide Heisenberg's paper[2] into four major reinterpretations. For the most part we will preserve Heisenberg's original notation and arguments.

Reinterpretation 1: Position. Heisenberg considered one-dimensional periodic systems. The classical motion of the system (in a stationary state labeled n) is described by the time-dependent position $x(n,t)$.[78] Heisenberg represents this periodic function by the Fourier series

$$x(n,t)=\sum_\alpha a_\alpha(n)e^{i\alpha\omega(n)t}. \tag{A1}$$

Unless otherwise noted, sums over integers go from $-\infty$ to ∞. The αth Fourier component related to the nth stationary state has amplitude $a_\alpha(n)$ and frequency $\alpha\omega(n)$. According to the correspondence principle, the αth Fourier component of the classical motion in the state n corresponds to the quantum jump from state n to state $n-\alpha$.[8,26] Motivated by this principle, Heisenberg replaced the classical component $a_\alpha(n)e^{i\alpha\omega(n)t}$ by the transition component $a(n,n-\alpha)e^{i\omega(n,n-\alpha)t}$.[79] We could say that the Fourier harmonic is replaced by a "Heisenberg harmonic." Unlike the sum over the classical components in Eq. (A1), Heisenberg realized that a similar sum over the transition components is meaningless. Such a quantum Fourier series could not describe the electron motion in one stationary state (n) because each term in the sum describes a transition process associated with two states (n and $n-\alpha$).

Heisenberg's next step was bold and ingenious. Instead of reinterpreting $x(t)$ as a *sum* over transition components, he represented the position by the *set* of transition components. We symbolically denote Heisenberg's reinterpretation as

$$x\rightarrow\{a(n,n-\alpha)e^{i\omega(n,n-\alpha)t}\}. \tag{A2}$$

Equation (A2) is the first breakthrough relation.

Reinterpretation 2: Multiplication. To calculate the energy of a harmonic oscillator, Heisenberg needed to know the quantity x^2. How do you square a set of transition components? Heisenberg posed this fundamental question twice in his paper.[80] His answer gave birth to the algebraic structure of quantum mechanics. We restate Heisenberg's question as "If x is represented by $\{a(n,n-\alpha)e^{i\omega(n,n-\alpha)t}\}$ and x^2 is represented by $\{b(n,n-\beta)e^{i\omega(n,n-\beta)t}\}$, how is $b(n,n-\beta)$ related to $a(n,n-\alpha)$?"

Heisenberg answered this question by reinterpreting the square of a Fourier series with the help of the Ritz principle. He evidently was convinced that quantum multiplication, whatever it looked like, must reduce to Fourier-series multiplication in the classical limit. The square of Eq. (A1) gives

$$x^2(n,t)=\sum_\beta b_\beta(n)e^{i\beta\omega(n)t}, \tag{A3}$$

where the βth Fourier amplitude is

$$b_\beta(n)=\sum_\alpha a_\alpha(n)a_{\beta-\alpha}(n). \tag{A4}$$

In the new quantum theory Heisenberg replaceed Eqs. (A3) and (A4) with

$$x^2\rightarrow\{b(n,n-\beta)e^{i\omega(n,n-\beta)t}\}, \tag{A5}$$

where the $n\rightarrow n-\beta$ transition amplitude is

$$b(n,n-\beta)=\sum_\alpha a(n,n-\alpha)a(n-\alpha,n-\beta). \tag{A6}$$

In constructing Eq. (A6) Heisenberg uncovered the symbolic algebra of atomic processes.

The logic behind the quantum rule of multiplication can be summarized as follows. Ritz's sum rule for atomic frequencies, $\omega(n,n-\beta)=\omega(n,n-\alpha)+\omega(n-\alpha,n-\beta)$, implies the product rule for Heisenberg's kinematic elements, $e^{i\omega(n,n-\beta)t}=e^{i\omega(n,n-\alpha)t}e^{i\omega(n-\alpha,n-\beta)t}$, which is the backbone of the multiplication rule in Eq. (A6). Equation (A6) allowed Heisenberg to algebraically manipulate the transition components.

Reinterpretation 3: Motion. Equations (A2), (A5), and (A6) represent the new "kinematics" of quantum theory—the new meaning of the position x. Heisenberg next turned his attention to the new "mechanics." The goal of Heisenberg's mechanics is to determine the amplitudes, frequencies, and energies from the given forces. Heisenberg noted that in the old quantum theory $a_\alpha(n)$ and $\omega(n)$ are determined by solving the classical equation of motion

$$\ddot{x}+f(x)=0, \tag{A7}$$

and quantizing the classical solution—making it depend on n—via the quantum condition

$$\oint m\dot{x}dx=nh. \tag{A8}$$

In Eqs. (A7) and (A8) $f(x)$ is the force (per mass) function and m is the mass.

Heisenberg assumed that Newton's second law in Eq. (A7) is valid in the new quantum theory provided that the classical quantity x is replaced by the set of quantities in Eq. (A2), and $f(x)$ is calculated according to the new rules of amplitude algebra. Keeping the same form of Newton's law of dynamics, but adopting the new kinematic meaning of x is the third Heisenberg breakthrough.

Reinterpretation 4: Quantization. How did Heisenberg reinterpret the old quantization condition in Eq. (A8)? Given the Fourier series in Eq. (A1), the quantization condition, $nh=\oint m\dot{x}^2dt$, can be expressed in terms of the Fourier parameters $a_\alpha(n)$ and $\omega(n)$ as

$$nh = 2\pi m \sum_{\alpha} |a_\alpha(n)|^2 \alpha^2 \omega(n). \tag{A9}$$

For Heisenberg, setting $\oint p dx$ equal to an integer multiple of h was an arbitrary rule that did not fit naturally into the dynamical scheme. Because his theory focuses exclusively on transition quantities, Heisenberg needed to translate the old quantum condition that fixes the properties of the states to a new condition that fixes the properties of the transitions between states. Heisenberg believed[14] that what matters is the difference between $\oint p dx$ evaluated for neighboring states: $[\oint p dx]_n - [\oint p dx]_{n-1}$. He therefore took the derivative of Eq. (A9) with respect to n to eliminate the forced n dependence and to produce a differential relation that can be reinterpreted as a difference relation between transition quantities. In short, Heisenberg converted

$$h = 2\pi m \sum_{\alpha} \alpha \frac{d}{dn}(|a_\alpha(n)|^2 \alpha \omega(n)) \tag{A10}$$

to

$$\begin{aligned} h = 4\pi m \sum_{\alpha=0}^{\infty} \{ &|a(n+\alpha,n)|^2 \omega(n+\alpha,n) \\ &- |a(n,n-\alpha)|^2 \omega(n,n-\alpha)\}. \end{aligned} \tag{A11}$$

In a sense Heisenberg's "amplitude condition" in Eq. (A11) is the counterpart to Bohr's frequency condition (Ritz's frequency combination rule). Heisenberg's condition relates the amplitudes of different lines within an atomic spectrum and Bohr's condition relates the frequencies. Equation (A11) is the fourth Heisenberg breakthrough.[81]

Equations (A7) and (A11) constitute Heisenberg's new mechanics. In principle, these two equations can be solved to find $a(n,n-\alpha)$ and $\omega(n,n-\alpha)$. No one before Heisenberg knew how to calculate the amplitude of a quantum jump. Equations (A2), (A6), (A7), and (A11) define Heisenberg's program for constructing the line spectrum of an atom from the given force on the electron.

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[33]Reference 3, p. 287, paper 13.
[34]Reference 12, pp. 217–218.
[35]Reference 3, p. 280, paper 13.
[36]A. Cayley, "Sept différents mémoires d'analyse," Mathematika **50**, 272–

317 (1855); A. Cayley, "A memoir on the theory of matrices," Philos. Trans. R. Soc. London, Ser. A **148**, 17–37 (1858).

[37]The last chapter (Bemerkungen zur Elektrodynamik) is not translated in Ref. 3. See Ref. 13, pp. 87–90 for a discussion of the contents of this section.

[38]In Heisenberg's paper (see Ref. 2) the connection between $|q(nm)|^2$ and transition probability is implied but not discussed. See Ref. 3, pp. 30–32, for a discussion of Heisenberg's assertion that the transition amplitudes determine the transition probabilities. The relation between a squared amplitude and a transition probability originated with Bohr who conjectured that the squared Fourier amplitude of the classical electron motion provides a measure of the transition probability (see Refs. 8 and 26). The correspondence between classical intensities and quantum probabilities was studied by several physicists including H. Kramers, *Intensities of Spectral Lines* (A. F. Host and Sons, Kobenhaven, 1919); R. Ladenburg in Ref. 3, paper 4, and J. H. Van Vleck, "Quantum principles and line spectra," Bulletin of the National Research Council, Washington, DC, 1926, pp. 118–153.

[39]Reference 3, p. 287, paper 13.

[40]W. Ritz, "Über ein neues Gesetz der Serienspektren," Phys. Z. **9**, 521–529 (1908); W. Ritz, "On a new law of series spectra," Astrophys. J. **28**, 237–243 (1908). The Ritz combination principle was crucial in making sense of the regularities in the line spectra of atoms. It was a key principle that guided Bohr in constructing a quantum theory of line spectra. Observations of spectral lines revealed that pairs of line frequencies combine (add) to give the frequency of another line in the spectrum. The Ritz combination rule is $\nu(nk)+\nu(km)=\nu(nm)$, which follows from Eqs. (4) and (7). As a universal, exact law of spectroscopy, the Ritz rule provided a powerful tool to analyze spectra and to discover new lines. Given the measured frequencies ν_1 and ν_2 of two known lines in a spectrum, the Ritz rule told spectroscopists to look for new lines at the frequencies $\nu_1+\nu_2$ or $\nu_1-\nu_2$.

[41]In the letter dated 18 September 1925 Heisenberg explained to Pauli that the frequencies ν_{ik} in the Born–Jordan theory obey the "combination relation $\nu_{ik}+\nu_{kl}=\nu_{il}$ or $\nu_{ik}=(W_i-W_k)/h$ but naturally it is *not* to be assumed that W is the energy." See Ref. 3, p. 45.

[42]Reference 3, p. 287, paper 13.

[43]Reference 3, p. 289, paper 13.

[44]Born and Jordan devote a large portion of Chap. 1 to developing a matrix calculus to give meaning to matrix derivatives such as $d\mathbf{q}/dt$ and $\partial\mathbf{H}/\partial\mathbf{p}$. They introduce the process of "symbolic differentiation" for constructing the derivative of a matrix with respect to another matrix. For a discussion of Born and Jordan's matrix calculus, see Ref. 13, pp. 68–71. To deal with arbitrary Hamiltonian functions, Born and Jordan formulated a more general dynamical law by converting the classical action principal, $\int L dt=$extremum, into a quantal action principal, $D(\mathbf{p}\dot{\mathbf{q}}-\mathbf{H}(\mathbf{pq}))=$extremum, where D denotes the trace (diagonal sum) of the Lagrangian matrix, $\mathbf{p}\dot{\mathbf{q}}-\mathbf{H}$. See Ref. 3, pp. 289–290.

[45]Reference 3, p. 292. This statement by Born and Jordan appears in Sec. IV of their paper following the section on the basic laws. We have included it with the postulates because it is a deep assumption with far-reaching consequences.

[46]In contemporary language the states labeled $n=0,1,2,3,\ldots$ in Heisenberg's paper and the Born–Jordan paper are exact stationary states (eigenstates of $\mathbf{H}$). The Hamiltonian matrix is automatically a diagonal matrix with respect to this basis.

[47]Although Heisenberg, Born, and Jordan made the "energy of the state" and the "transition between states" rigorous concepts, it was Schrödinger who formalized the concept of the "state" itself. It is interesting to note that "On quantum mechanics II" by Born, Heisenberg, and Jordan was published before Schrödinger and implicitly contains the first mathematical notion of a quantum state. In this paper (Ref. 3, pp. 348–353), each Hermitian matrix $\mathbf{a}$ is associated with a "bilinear form" $\Sigma_{nm}a(nm)x_n x_m^*$. Furthermore, they identified the "energy spectrum" of a system with the set of "eigenvalues" W in the equation $Wx_k-\Sigma_l H(kl)x_l=0$. In present-day symbolic language the bilinear form and eigenvalue problem are $\langle\Psi|\mathbf{a}|\Psi\rangle$ and $\mathbf{H}|\Psi\rangle=\mathbf{W}|\Psi\rangle$, respectively, where the variables x_n are the expansion coefficients of the quantum state $|\Psi\rangle$. At the time, they did not realize the physical significance of their eigenvector $(x_1,x_2,\ldots)$ as representing a stationary state.

[48]Reference 3, pp. 290–291, paper 13.

[49]This is the sentence from Born's 1924 paper (See Ref. 1) where the name "quantum mechanics" appears for the first time in the physics literature [Ref. 3, p. 182].

[50]See the chapter "The transition to quantum mechanics" in Ref. 12, pp. 181–198 for applications of "Born's correspondence rule." The most important application was deriving Kramer's dispersion formula. See Ref. 3, papers 6–10 and Ref. 14.

[51]Reference 11, pp. 144–145.

[52]The exact relation between the orbital frequency and the optical frequency is derived as follows. Consider the transition from state n of energy $E(n)$ to state $n-\tau$ of energy $E(n-\tau)$. In the limit $n\gg\tau$, that is, large "orbit" and small "jump," the difference $E(n)-E(n-\tau)$ is equal to the derivative $\tau dE/dn$. Given the old quantum condition $J=nh$, it follows that $dE/dn=hdE/dJ$. Thus for $n\gg\tau$ and $J=nh$, we have the relation $[E(n)-E(n-\tau)]/h=\tau dE/dJ$, or equivalently, $\nu(n,n-\tau)=\tau\nu(n)$. This relation proves an important correspondence theorem: In the limit $n\gg\tau$, the frequency $\nu(n,n-\tau)$ associated with the quantum jump $n\to n-\tau$ is equal to the frequency $\tau\nu(n)$ associated with the τth harmonic of the classical motion in the state n. See Refs. 8 and 26.

[53]Reference 3, p. 191, paper 7.

[54]Suppose that the number of states is finite and equal to the integer N. Then, according to Eq. (20), the diagonal sum (trace) of $\mathbf{pq}-\mathbf{qp}$ would be $D(\mathbf{pq}-\mathbf{qp})=Nh/2\pi i$. This nonzero value of the trace contradicts the purely mathematical relation $D(\mathbf{pq}-\mathbf{qp})=0$, which must be obeyed by all finite matrices.

[55]Heisenberg interview quoted in Ref. 12, p. 281, footnote 45.

[56]Reference 17, p. 361.

[57]Reference 3, p. 288, paper 13. The name "Diagonality theorem" is ours. The condition $\nu(nm)\neq 0$ when $n\neq m$ implies that the system is nondegenerate.

[58]In contemporary language a conserved quantity is an operator that commutes with the Hamiltonian operator $\mathbf{H}$. For such commuting operators there exists a common set of eigenvectors. In the energy eigenbasis that underlies the Born–Jordan formulation, the matrices representing $\mathbf{H}$ and all conserved quantities are automatically diagonal.

[59]Born and Jordan's proof that Eq. (24) vanishes is based on a purely mathematical property of "symbolic differentiation" discussed in Sec. II of their paper (See Ref. 4). For a separable Hamiltonian of the form $\mathbf{H}=\mathbf{p}^2/2m+\mathbf{U}(\mathbf{q})$, the proof is simpler. For this case Eq. (24) becomes $\dot{\mathbf{d}}=\mathbf{q}(\partial\mathbf{U}/\partial\mathbf{q})-(\partial\mathbf{U}/\partial\mathbf{q})\mathbf{q}+\mathbf{p}(\mathbf{p}/m)-(\mathbf{p}/m)\mathbf{p}$. Because $\mathbf{p}$ and $\mathbf{q}$ are separated in this expression, we do not have to consider the inequality $\mathbf{pq}\neq\mathbf{qp}$. The expression reduces to $\dot{\mathbf{d}}=0$.

[60]Reference 3, p. 292, paper 13. In Ref. 4, Born and Jordan refer to $\mathbf{pq}-\mathbf{qp}=(h/2\pi i)\mathbf{1}$ as the "vershärfte Quantenbedingung," which has been translated as "sharpened quantum condition" (Ref. 13, p. 77), "stronger quantum condition" (Ref. 3, p. 292), and "exact quantum condition" (Ref. 12, p. 220).

[61]J. J. Sakurai, *Modern Quantum Mechanics* (Addison-Wesley, San Francisco, 1994), pp. 83–84; A. Messiah, *Quantum Mechanics* (J Wiley, New York, 1958), Vol. I, p. 316.

[62]Reference 3, p. 293, paper 13.

[63]Proving the frequency condition—the second general principle of Bohr—was especially important because this purely quantal condition was generally regarded as a safely established part of physics. Prior to Born and Jordan's mechanical proof of the frequency condition, there existed a "thermal proof" given by Einstein in his historic paper, "On the quantum theory of radiation," Phys. Z. **18**, 121 (1917), translated in Ref. 3, pp, 63–77. In this paper Einstein provides a completely new derivation of Planck's thermal radiation law by introducing the notion of transition probabilities (A and B coefficients). Bohr's frequency condition emerges as the condition necessary to reduce the Boltzmann factor $\exp[(E_n-E_m)/kT]$ in Einstein's formula to the "Wien factor" $\exp(h\nu/kT)$ in Planck's formula.

[64]Reference 3, p. 291, paper 13.

[65]Reference 3, pp. 291–292, paper 13. Born and Jordan do not refer to the consequences in Eqs. (37) and (38) as theorems. The label "Energy theorems" is ours.

[66]Instead of postulating the equations of motion and deriving the energy theorems, we could invert the proof and postulate the energy theorems and derive the equations of motion. This alternate logic is mentioned in Ref. 3, p. 296 and formalized in Ref. 5 (Ref. 3, p. 329). Also see J. H. Van Vleck, "Note on the postulates of the matrix quantum dynamics," Proc. Natl. Acad. Sci. U.S.A. **12**, 385–388 (1926).

[67]Reference 3, pp. 293–294, paper 13. The proof of the energy theorems was based on separable Hamiltonians defined in Eq. (29). To generalize the proof Born and Jordan consider more general Hamiltonian functions

$\mathbf{H}(\mathbf{pq})$ and discover the need to symmetrize the functions. For example, for $\mathbf{H}^*=\mathbf{p}^2\mathbf{q}$, it does not follow that $\dot{\mathbf{H}}^*=0$. However, they note that $\mathbf{H}=(\mathbf{p}^2\mathbf{q}+\mathbf{q}\mathbf{p}^2)/2$ yields the same equations of motion as $\mathbf{H}^*$ and also conserves energy, $\dot{\mathbf{H}}=0$. The symmetrization rule reflects the noncommutativity of $\mathbf{p}$ and $\mathbf{q}$.

[68] In the Heisenberg, Born–Jordan approach the transition components of the "matter variables" $\mathbf{q}$ and $\mathbf{p}$ are simply assumed to oscillate in time with the radiation frequencies. In contemporary texts a rigorous proof of Bohr's frequency condition involves an analysis of the interaction between matter and radiation (radiative transitions) using time-dependent perturbation theory. See Ref. 24, Chap. 19.

[69] Reference 3, p. 292, paper 13.

[70] Using the language of state vectors and bra-kets, the matrix element of an operator $\mathbf{g}$ is $g_{nm}(t)=\langle\Psi_n(t)|\mathbf{g}|\Psi_m(t)\rangle$, where the energy eigenstate is $|\Psi_n(t)\rangle=\exp(-2\pi i E_n t/h)|\Psi_n(0)\rangle$. This Schrödinger element is equivalent to the Born–Jordan element in Eq. (44).

[71] Reference 3, p. 279, paper 13. Heisenberg was able to demonstrate energy conservation and Bohr's frequency condition for two systems (anharmonic oscillator and rotator). The anharmonic oscillator analysis was limited to second-order perturbation theory.

[72] Reference 3. Born and Jordan do not pursue this direct method of proof noting that for the most general Hamiltonians the calculation "becomes so exceedingly involved that it seems hardly feasible." (Ref. 3, p. 296). In a footnote on p. 296, they note that for the special case $\mathbf{H}=\mathbf{p}^2/2m+U(\mathbf{q})$, the proof can be carried out immediately. The details of this proof can be found in Ref. 73.

[73] J. J. Prentis and W. A. Fedak, "Energy conservation in quantum mechanics," Am. J. Phys. **72**, 580–590 (2004).

[74] Reference 3, p. 296, paper 13.

[75] M. Born, *Problems of Atomic Dynamics* (MIT Press, Cambridge, 1970).

[76] E. Schrödinger, *Collected Papers on Wave Mechanics* (Chelsea, New York, 1978).

[77] M. Born, "Zur Quantenmechanik der Stoßvorgänge," Z. Phys. **37**, 863–867 (1926).

[78] Heisenberg's "classical" quantity $x(n,t)$ is the classical solution $x(t)$ of Newton's equation of motion subject to the old quantum condition $\oint m\dot{x}dx=nh$. For example, given the purely classical position function $x(t)=a\cos\omega t$ of a harmonic oscillator, the condition $\oint m\dot{x}^2dt=nh$ quantizes the amplitude, making a depend on n as follows: $a(n)=\sqrt{nh/\pi m\omega}$. Thus, the motion of the harmonic oscillator in the stationary state n is described by $x(n,t)=\sqrt{nh/\pi m\omega}\cos\omega t$.

[79] The introduction of transition components $a(n,n-\tau)e^{i\omega(n,n-\alpha)t}$ into the formalism was a milestone in the development of quantum theory. The one-line abstract of Heisenberg's paper reads "The present paper seeks to establish a basis for theoretical quantum mechanics founded exclusively upon relationships between quantities which in principle are observable" (Ref. 3, p. 261). For Heisenberg, the observable quantities were $a(n,n-\tau)$ and $\omega(n,n-\tau)$, that is, the amplitudes and the frequencies of the spectral lines. Prior to 1925, little was known about transition amplitudes. There was a sense that Einstein's transition probabilities were related to the squares of the transition amplitudes. Heisenberg made the transition amplitudes (and frequencies) the central quantities of his theory. He discovered how to manipulate them, relate them, and calculate their values.

[80] Reference 3, pp. 263–264, paper 12.

[81] Heisenberg notes (Ref. 3, p. 268, paper 12) that Eq. (A11) is equivalent to the sum rule of Kuhn and Thomas (Ref. 3, paper 11). For a discussion of Heisenberg's development of the quantum condition, see Mehra and H. Rechenberg, *The Historical Development of Quantum Theory* (Springer, New York, 1982), Vol. 2, pp. 243–245, and Ref. 14.

SCIENTIFIC APTITUDE AND AUTISM

There's even some evidence that scientific abilities are associated with traits characteristic of autism, the psychological disorder whose symptoms include difficulties in social relationships and communication, or its milder version, Asperger syndrome. One recent study, for instance, examined different groups according to the Autism-Spectrum Quotient test, which measures autistic traits. Scientists scored higher than nonscientists on this test, and within the sciences, mathematicians, physical scientists, and engineers scored higher than biomedical scientists.

Sidney Perkowitz, *Hollywood Science: Movies, Science, and the End of the World* (Columbia University Press, 2007), p. 170.

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ON THE EINSTEIN PODOLSKY ROSEN PARADOX*

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I. Introduction

THE paradox of Einstein, Podolsky and Rosen [1] was advanced as an argument that quantum mechanics could not be a complete theory but should be supplemented by additional variables. These additional variables were to restore to the theory causality and locality [2]. In this note that idea will be formulated mathematically and shown to be incompatible with the statistical predictions of quantum mechanics. It is the requirement of locality, or more precisely that the result of a measurement on one system be unaffected by operations on a distant system with which it has interacted in the past, that creates the essential difficulty. There have been attempts [3] to show that even without such a separability or locality requirement no "hidden variable" interpretation of quantum mechanics is possible. These attempts have been examined elsewhere [4] and found wanting. Moreover, a hidden variable interpretation of elementary quantum theory [5] has been explicitly constructed. That particular interpretation has indeed a grossly non-local structure. This is characteristic, according to the result to be proved here, of any such theory which reproduces exactly the quantum mechanical predictions.

II. Formulation

With the example advocated by Bohm and Aharonov [6], the EPR argument is the following. Consider a pair of spin one-half particles formed somehow in the singlet spin state and moving freely in opposite directions. Measurements can be made, say by Stern-Gerlach magnets, on selected components of the spins $\vec{\sigma}_1$ and $\vec{\sigma}_2$. If measurement of the component $\vec{\sigma}_1 \cdot \vec{a}$, where $\vec{a}$ is some unit vector, yields the value $+1$ then, according to quantum mechanics, measurement of $\vec{\sigma}_2 \cdot \vec{a}$ must yield the value -1 and vice versa. Now we make the hypothesis [2], and it seems one at least worth considering, that if the two measurements are made at places remote from one another the orientation of one magnet does not influence the result obtained with the other. Since we can predict in advance the result of measuring any chosen component of $\vec{\sigma}_2$, by previously measuring the same component of $\vec{\sigma}_1$, it follows that the result of any such measurement must actually be predetermined. Since the initial quantum mechanical wave function does *not* determine the result of an individual measurement, this predetermination implies the possibility of a more complete specification of the state.

Let this more complete specification be effected by means of parameters λ. It is a matter of indifference in the following whether λ denotes a single variable or a set, or even a set of functions, and whether the variables are discrete or continuous. However, we write as if λ were a single continuous parameter. The result A of measuring $\vec{\sigma}_1 \cdot \vec{a}$ is then determined by $\vec{a}$ and λ, and the result B of measuring $\vec{\sigma}_2 \cdot \vec{b}$ in the same instance is determined by $\vec{b}$ and λ, and

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$$A(\vec{a}, \lambda) = \pm 1, \; B(\vec{b}, \lambda) = \pm 1. \tag{1}$$

The vital assumption [2] is that the result B for particle 2 does not depend on the setting $\vec{a}$, of the magnet for particle 1, nor A on $\vec{b}$.

If $\rho(\lambda)$ is the probability distribution of λ then the expectation value of the product of the two components $\vec{\sigma}_1 \cdot \vec{a}$ and $\vec{\sigma}_2 \cdot \vec{b}$ is

$$P(\vec{a}, \vec{b}) = \int d\lambda \, \rho(\lambda) A(\vec{a}, \lambda) B(\vec{b}, \lambda) \tag{2}$$

This should equal the quantum mechanical expectation value, which for the singlet state is

$$\langle \vec{\sigma}_1 \cdot \vec{a} \; \vec{\sigma}_2 \cdot \vec{b} \rangle = -\vec{a} \cdot \vec{b} \, . \tag{3}$$

But it will be shown that this is not possible.

Some might prefer a formulation in which the hidden variables fall into two sets, with A dependent on one and B on the other; this possibility is contained in the above, since λ stands for any number of variables and the dependences thereon of A and B are unrestricted. In a complete physical theory of the type envisaged by Einstein, the hidden variables would have dynamical significance and laws of motion; our λ can then be thought of as initial values of these variables at some suitable instant.

III. Illustration

The proof of the main result is quite simple. Before giving it, however, a number of illustrations may serve to put it in perspective.

Firstly, there is no difficulty in giving a hidden variable account of spin measurements on a single particle. Suppose we have a spin half particle in a pure spin state with polarization denoted by a unit vector $\vec{p}$. Let the hidden variable be (for example) a unit vector $\vec{\lambda}$ with uniform probability distribution over the hemisphere $\vec{\lambda} \cdot \vec{p} > 0$. Specify that the result of measurement of a component $\vec{\sigma} \cdot \vec{a}$ is

$$\operatorname{sign} \; \vec{\lambda} \cdot \vec{a}' \; , \tag{4}$$

where $\vec{a}'$ is a unit vector depending on $\vec{a}$ and $\vec{p}$ in a way to be specified, and the sign function is $+1$ or -1 according to the sign of its argument. Actually this leaves the result undetermined when $\lambda \cdot a' = 0$, but as the probability of this is zero we will not make special prescriptions for it. Averaging over $\vec{\lambda}$ the expectation value is

$$\langle \vec{\sigma} \cdot \vec{a} \rangle = 1 - 2\theta'/\pi \, , \tag{5}$$

where θ' is the angle between $\vec{a}'$ and $\vec{p}$. Suppose then that $\vec{a}'$ is obtained from $\vec{a}$ by rotation towards $\vec{p}$ until

$$1 - \frac{2\theta'}{\pi} = \cos\theta \tag{6}$$

where θ is the angle between $\vec{a}$ and $\vec{p}$. Then we have the desired result

$$\langle \vec{\sigma} \cdot \vec{a} \rangle = \cos\theta \tag{7}$$

So in this simple case there is no difficulty in the view that the result of every measurement is determined by the value of an extra variable, and that the statistical features of quantum mechanics arise because the value of this variable is unknown in individual instances.

Secondly, there is no difficulty in reproducing, in the form (2), the only features of (3) commonly used in verbal discussions of this problem:

$$\left.\begin{aligned} P(\vec{a}, \vec{a}) &= -P(\vec{a}, -\vec{a}) = -1 \\ P(\vec{a}, \vec{b}) &= 0 \ \text{ if } \ \vec{a} \cdot \vec{b} = 0 \end{aligned}\right\} \tag{8}$$

For example, let λ now be unit vector $\vec{\lambda}$, with uniform probability distribution over all directions, and take

$$\left.\begin{aligned} A(\vec{a}, \vec{\lambda}) &= \operatorname{sign} \ \vec{a} \cdot \vec{\lambda} \\ B(a, b) &= -\operatorname{sign} \ \vec{b} \cdot \vec{\lambda} \end{aligned}\right\} \tag{9}$$

This gives

$$P(\vec{a}, \vec{b}) = -1 + \frac{2}{\pi}\theta , \tag{10}$$

where θ is the angle between a and b, and (10) has the properties (8). For comparison, consider the result of a modified theory [6] in which the pure singlet state is replaced in the course of time by an isotropic mixture of product states; this gives the correlation function

$$-\frac{1}{3}\vec{a} \cdot \vec{b} \tag{11}$$

It is probably less easy, experimentally, to distinguish (10) from (3), than (11) from (3).

Unlike (3), the function (10) is not stationary at the minimum value -1 (at $\theta = 0$). It will be seen that this is characteristic of functions of type (2).

Thirdly, and finally, there is no difficulty in reproducing the quantum mechanical correlation (3) if the results A and B in (2) are allowed to depend on $\vec{b}$ and $\vec{a}$ respectively as well as on $\vec{a}$ and $\vec{b}$. For example, replace $\vec{a}$ in (9) by $\vec{a}'$, obtained from $\vec{a}$ by rotation towards $\vec{b}$ until

$$1 - \frac{2}{\pi}\theta' = \cos\theta ,$$

where θ' is the angle between $\vec{a}'$ and $\vec{b}$. However, for given values of the hidden variables, the results of measurements with one magnet now depend on the setting of the distant magnet, which is just what we would wish to avoid.

IV. Contradiction

The main result will now be proved. Because ρ is a normalized probability distribution,

$$\int d\lambda\, \rho(\lambda) = 1 , \tag{12}$$

and because of the properties (1), P in (2) cannot be less than -1. It can reach -1 at $\vec{a} = \vec{b}$ only if

$$A(\vec{a}, \lambda) = -B(\vec{a}, \lambda) \tag{13}$$

except at a set of points λ of zero probability. Assuming this, (2) can be rewritten

$$P(\vec{a}, \vec{b}) = -\int d\lambda\, \rho(\lambda)\, A(\vec{a}, \lambda)\, A(\vec{b}, \lambda) . \tag{14}$$

It follows that $\vec{c}$ is another unit vector

$$P(\vec{a}, \vec{b}) - P(\vec{a}, \vec{c}) = -\int d\lambda\, \rho(\lambda)\, [A(\vec{a}, \lambda)\, A(\vec{b}, \lambda) - A(\vec{a}, \lambda)\, A(\vec{c}, \lambda)]$$
$$= \int d\lambda\, \rho(\lambda)\, A(\vec{a}, \lambda)\, A(\vec{b}, \lambda)\, [A(\vec{b}, \lambda)\, A(\vec{c}, \lambda) - 1]$$

using (1), whence

$$|P(\vec{a}, \vec{b}) - P(\vec{a}, \vec{c})| \le \int d\lambda\, \rho(\lambda)\, [1 - A(\vec{b}, \lambda)\, A(\vec{c}, \lambda)]$$

The second term on the right is $P(\vec{b}, \vec{c})$, whence

$$1 + P(\vec{b}, \vec{c}) \ge |P(\vec{a}, \vec{b}) - P(\vec{a}, \vec{c})| \tag{15}$$

Unless P is constant, the right hand side is in general of order $|\vec{b} - \vec{c}|$ for small $|\vec{b} - \vec{c}|$. Thus $P(\vec{b}, \vec{c})$ cannot be stationary at the minimum value (-1 at $\vec{b} = \vec{c}$) and cannot equal the quantum mechanical value (3).

Nor can the quantum mechanical correlation (3) be arbitrarily closely approximated by the form (2). The formal proof of this may be set out as follows. We would not worry about failure of the approximation at isolated points, so let us consider instead of (2) and (3) the functions

$$\bar{P}(\vec{a}, \vec{b}) \quad \text{and} \quad \overline{-\vec{a} \cdot \vec{b}}$$

where the bar denotes independent averaging of $P(\vec{a}', \vec{b}')$ and $-\vec{a}' \cdot \vec{b}'$ over vectors $\vec{a}'$ and $\vec{b}'$ within specified small angles of $\vec{a}$ and $\vec{b}$. Suppose that for all $\vec{a}$ and $\vec{b}$ the difference is bounded by ϵ:

$$|\bar{P}(\vec{a}, \vec{b}) + \vec{a} \cdot \vec{b}| \le \epsilon \tag{16}$$

Then it will be shown that ϵ cannot be made arbitrarily small.

Suppose that for all a and b

$$|\overline{\vec{a} \cdot \vec{b}} - \vec{a} \cdot \vec{b}| \le \delta \tag{17}$$

Then from (16)

$$|\bar{P}(\vec{a}, \vec{b}) + \vec{a} \cdot \vec{b}| \le \epsilon + \delta \tag{18}$$

From (2)

$$\bar{P}(\vec{a}, \vec{b}) = \int d\lambda\, \rho(\lambda)\, \bar{A}(\vec{a}, \lambda)\, \bar{B}(\vec{b}, \lambda) \tag{19}$$

where

$$|\bar{A}(\vec{a}, \lambda)| \le 1 \quad \text{and} \quad |\bar{B}(\vec{b}, \lambda)| \le 1 \tag{20}$$

From (18) and (19), with $\vec{a} = \vec{b}$,

$$d\lambda\, \rho(\lambda)\, [\bar{A}(\vec{b}, \lambda)\, \bar{B}(\vec{b}, \lambda) + 1] \le \epsilon + \delta \tag{21}$$

From (19)

$$\bar{P}(\vec{a}, \vec{b}) - \bar{P}(\vec{a}, \vec{c}) = \int d\lambda\, \rho(\lambda)\, [\bar{A}(\vec{a}, \lambda)\, \bar{B}(\vec{b}, \lambda) - \bar{A}(\vec{a}, \lambda)\, \bar{B}(\vec{c}, \lambda)]$$
$$= \int d\lambda\, \rho(\lambda)\, \bar{A}(\vec{a}, \lambda)\, \bar{B}(\vec{b}, \lambda)\, [1 + \bar{A}(\vec{b}, \lambda)\, \bar{B}(\vec{c}, \lambda)]$$
$$- \int d\lambda\, \rho(\lambda)\, \bar{A}(\vec{a}, \lambda)\, \bar{B}(\vec{c}, \lambda)\, [1 + \bar{A}(\vec{b}, \lambda)\, \bar{B}(\vec{b}, \lambda)]$$

Using (20) then

$$|\bar{P}(\vec{a}, \vec{b}) - \bar{P}(\vec{a}, \vec{c})| \leq \int d\lambda \propto(\lambda) [1 + \bar{A}(\vec{b}, \lambda) \bar{B}(\vec{c}, \lambda)] + \int d\lambda \rho(\lambda) [1 + \bar{A}(\vec{b}, \lambda) \bar{B}(\vec{b}, \lambda)]$$

Then using (19) and 21)

$$|\bar{P}(\vec{a}, \vec{b}) - \bar{P}(\vec{a}, \vec{c})| \leq 1 + \bar{P}(\vec{b}, \vec{c}) + \epsilon + \delta$$

Finally, using (18),

$$|\vec{a} \cdot \vec{c} - \vec{a} \cdot \vec{b}| - 2(\epsilon + \delta) \leq 1 - \vec{b} \cdot \vec{c} + 2(\epsilon + \delta)$$

or

$$4(\epsilon + \delta) \geq |\vec{a} \cdot \vec{c} - \vec{a} \cdot \vec{b}| + \vec{b} \cdot \vec{c} - 1 \tag{22}$$

Take for example $\vec{a} \cdot \vec{c} = 0, \vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{c} = 1/\sqrt{2}$ Then

$$4(\epsilon + \delta) \geq \sqrt{2} - 1$$

Therefore, for small finite δ, ϵ cannot be arbitrarily small.

Thus, the quantum mechanical expectation value cannot be represented, either accurately or arbitrarily closely, in the form (2).

V. Generalization

The example considered above has the advantage that it requires little imagination to envisage the measurements involved actually being made. In a more formal way, assuming [7] that any Hermitian operator with a complete set of eigenstates is an "observable", the result is easily extended to other systems. If the two systems have state spaces of dimensionality greater than 2 we can always consider two dimensional subspaces and define, in their direct product, operators $\vec{\sigma}_1$ and $\vec{\sigma}_2$ formally analogous to those used above and which are zero for states outside the product subspace. Then for at least one quantum mechanical state, the "singlet" state in the combined subspaces, the statistical predictions of quantum mechanics are incompatible with separable predetermination.

VI. Conclusion

In a theory in which parameters are added to quantum mechanics to determine the results of individual measurements, without changing the statistical predictions, there must be a mechanism whereby the setting of one measuring device can influence the reading of another instrument, however remote. Moreover, the signal involved must propagate instantaneously, so that such a theory could not be Lorentz invariant.

Of course, the situation is different if the quantum mechanical predictions are of limited validity. Conceivably they might apply only to experiments in which the settings of the instruments are made sufficiently in advance to allow them to reach some mutual rapport by exchange of signals with velocity less than or equal to that of light. In that connection, experiments of the type proposed by Bohm and Aharonov [6], in which the settings are changed during the flight of the particles, are crucial.

I am indebted to Drs. M. Bander and J. K. Perring for very useful discussions of this problem. The first draft of the paper was written during a stay at Brandeis University; I am indebted to colleagues there and at the University of Wisconsin for their interest and hospitality.

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Is the moon there when nobody looks? Reality and the quantum theory

Einstein maintained that quantum metaphysics entails spooky actions at a distance; experiments have now shown that what bothered Einstein is not a debatable point but the observed behaviour of the real world.

N. David Mermin

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He says he *is* bothered by Bell's theorem, but may have rocks in his head anyway.]

Quantum mechanics is magic[1]

In May 1935, Albert Einstein, Boris Podolsky and Nathan Rosen published[2] an argument that quantum mechanics fails to provide a complete description of physical reality. Today, 50 years later, the EPR paper and the theoretical and experimental work it inspired remain remarkable for the vivid illustration they provide of one of the most bizarre aspects of the world revealed to us by the quantum theory.

Einstein's talent for saying memorable things did him a disservice when he declared "God does not play dice." for it has been held ever since the basis for his opposition to quantum mechanics was the claim that a fundamental understanding of the world can only be statistical.

But the EPR paper, his most powerful attack on the quantum theory, focuses on quite a different aspect: the doctrine that physical properties have in general no objective reality independent of the act of observation. As Pascual Jordan put it[3]:

"Observations not only disturb what has to be measured, they produce it….We compel [the electron] to assume a definite position…. We ourselves produce the results of measurements."

Jordan's statement is something of a truism for contemporary physicists. Underlying it, we have all been taught, is the disruption of what is being measured by the act of measurement, made unavoidable by the existence of the quantum of action, which generally makes it impossible even in principle to construct probes that can yield the information classical intuition expects to be there.

Einstein didn't like this. He wanted things out there to have properties, whether or not they were measured[4]:

"We often discussed his notions on objective reality. I recall that during one walk Einstein suddenly stopped, turned to me and asked whether I really believed that the moon exists only when I look at it."

The EPR paper describes a situation ingeniously contrived to force the quantum theory into asserting that properties in a space-time region **B** are the result of an act of measurement in another space-time region **A**, so far from **B** that there is no possibility of the measurement in **A** exerting an influence on region **B** by any known dynamical mechanism. Under these conditions, Einstein maintained that the properties in **A** must have existed all along.

Spooky actions at a distance

Many of his simplest and most explicit statements of this position can be found in Einstein's correspondence with Max Born.[5] Throughout the book (which sometimes reads like a Nabokov novel), Born, pained by Einstein's distaste for the statistical character of the quantum theory, repeatedly fails, both in his letters and in his later commentary on the correspondence, to understand what is really bothering Einstein. Einstein tries over and over again, without success, to make himself clear. In March 1948, for example, he writes:

"That which really exists in B should …not depend on what kind of measurement is carried out in part of space A; it should also be independent of whether or not any measurement at all is carried out in space A. If one adheres to this program, one can hardly consider the quantum-theoretical description as a complete representation of the physically real. If one tries to do so in spite of this, one has to assume that the physically real in B suffers a sudden change as a result of a measurement in A. My instinct for physics bristles at this."

Or, in March 1947:

"I cannot seriously believe in [the quantum theory] because it cannot be reconciled with the idea that physics should represent a reality in time and space, free from spooky actions at a distance."

The "spooky actions at a distance" (spukhafte Fernwirkungen) are the acquisition of a definite value of a property by the system in region **B** by virtue of the measurement carried out in region **A**. The EPR paper presents a wavefunction that describes two correlated particles, localized in regions **A** and **B**, far apart.

In this particular two-particle state one can learn (in the sense of being able to predict with certainty the result of a subsequent measurement) either the position or the momentum of the particle in region **B** as a result of measuring the corresponding property of the particle in region **A**. If "that which really exists" in region **B** does not depend on what kind of measurement is carried out in region **A**, then the particle in region **B** must have had both a definite position and a definite momentum all along.

Because the quantum theory is intrinsically incapable of assigning values to both quantities at once, it must provide an incomplete description of the physically real. Unless, of couse, one asserts that it is only by virtue of the position (or momentum) measurement in **A** that the particle in **B** acquires its position (or momentum): spooky actions at a distance.

At a dramatic moment Pauli appears in the *Born-Einstein Letters*, writing Born from Princeton in 1954 with his famous tact on display:

"Einstein gave me your manuscript to read; he was *not at all* annoyed with you, but only said you were a person who will not listen. This agrees with the impression I have formed myself insofar as I was unable to recognize Einstein whenever you talked about him in either your letter or your manuscript. It seemed to me as if you had erected some dummy Einstein for yourself, which you then knocked down with great pomp. In particular, Einstein does not consider the concept of 'determinism' to be as fundamental as it is frequently held to be (as he told me emphatically many times)… In the same way, he *disputes* that he uses as criterion for the admissibility of a theory the question: *Is it rigorously deterministic?* "

Pauli goes on to state the real nature of Einstein's "philosophical prejudice" to Born, emphasizing that "Einstein's point of departure is 'realistic' rather than 'deterministic'." According to Pauli the proper grounds for challenging Einstein's view are simply that:

"One should no more rack one's brain about the problem of whether something one cannot know anything about exists all the same, than about the ancient question of how many angels are able to sit on the point of a needle. But it seems to me that Einstein's questions are ultimately always of this kind."

Faced with spooky actions at a distance, Einstein preferred to believe that things one cannot know anything about (such as the momentum of a particle with a definite position) do exist all the same.

In April 1948 he wrote to Born:

"Those physicists who regard the descriptive methods of quantum mechanics as definitive in principle would…drop the requirement for the independent existence of the physical reality present in different parts of space; they would be justified in pointing out that the quantum theory nowhere makes explicit use of this requirement. I admit this, but would point out: when I consider the the physical phenomena known to me, and especially those which are being so successfully encompassed by quantum mechanics, I still cannot find any fact anywhere which would make it appear likey that [the] requirement will have to be abandoned. I am therefore inclined to believe that the description of quantum mechanics…has to be regarded as an incomplete and indirect description of reality…"

A fact is found

The theoretical answer to this challenge to provide "any fact anywhere" was given in 1964 by John S.Bell, in a famous paper[6] in the short-lived journal *Physics.* Using a gedanken experiment invented[7] by David Bohm, in which "properties one cannot know anything about" (the simultaneous values of the spin of a particle along several distinct directions) are required to exist by EPR line of reasoning, Bell showed ("Bell's theorem") that the nonexistence of these properties is a direct consequence of the quantitative numerical predictions of the quantum theory. The conclusion is quite independent of whether or not one believes that the quantum theory offers a complete description of physical reality.

If the data in such an experiment are in agreement with the numerical predictions of the quantum theory, then Einstein's philosophical position has to be wrong.

In the last few years, in a beautiful series of experiments, Alain Aspect and his collaborators at the University of Paris's Institute of Theoretical and Applied Optics in Orsay provided[8] the experimental answer to Einstein's challenge by performing a version of the EPR experiment under conditions in which Bell's type of analysis applied.

They showed that the quantum-theoretic predictions were indeed obeyed. Thirty years after Einstein's challenge, a fact -not a metaphysical doctrine- was provided to refute him.

Attitudes toward this particular 50-year sequence of intellectual history and scientific discovery vary widely.[9] From the very start Bohr certainly took it seriously. Leon Rosenfeld describes[10] the impact of the EPR argument:

"This onslaught came down upon us as a bolt from the blue. Its effect on Bohr was remarkable….A new worry could not have come at a less propitious time. Yet, as soon as Bohr had heard my report of Einstein's argument, everything else was abandoned."

Bell's contribution has become celebrated in what might be called semi-popular culture. We read, for example, in *The Dancing Wu Li Masters* that[11]:

"Some physicists are convinced that [Bell's theorem] is the most important single work, perhaps, in the history of physics."

And indeed, Henry Stapp, a particle theorist at Berkeley, writes that[12]:

"Bell's theorem is the most profound discovery of science."

At the other end of the spectrum, Abraham Pais, in his recent biography of Einstein, writes[13] of the EPR article that "bolt from the blue" the basis for "the most profound discovery of science" :

"The only part of this article which will ultimately survive, I believe, is…a phrase ['No reasonable definition of reality could be expected to permit this'] which so poignantly summarizes Einstein's views on quantum mechanics in his later years."

I think it is fair to say that more physicists would side with Pais than with Stapp, but between the majority position of near indifference and the minority position of wild extravagance is an attitude I would characterize as balanced. This was expressed to me most succintly by a distinguished Princeton physicist on the occasion of my asking how he thought Einstein would have reacted to Bell's theorem.

He said that Einstein would have gone home and thought about it hard for several weeks that he couldn't guess what he would then have said, except that it would have been extremely interesting. He was sure that Einstein would have been very bothered by Bell's theorem.

Then he added:

"Anybody who's not bothered by Bell's theorem has to have rocks in his head."

To this moderate point of view I would only add the observation that contemporary physicists come in two varieties.

Type 1 physicists are bothered by EPR and Bell's theorem.

Type 2 (the majority) are not, but one has to distinguish two subvarieties.

Type 2a physicists explain why they are not bothered. Their explanations tend either to miss the point entirely (like Born's to Einstein) or to contain physical assertions that can be shown to be false.

Type 2b are not bothered and refuse to explain why. Their position is unassailable. (There is a variant of type 2b who say that Bohr straightened out[14] the whole business, but refuse to explain how.)

A gedanken demonstration

To enable you to test which category you belong to, I shall describe, in black-box terms, a very simple version of Bell's gedanken experiment, deferring to the very end any reference whatever either to the underlying mechanism that makes the gadget work or to the quantum-theoretic analysis that accounts for the data. Perhaps this backwards way of proceeding will make it easier for you to lay aside your quantum theoretic prejudices and decide afresh whether what I describe is or is not strange.[15]

What I have in mind is a simple gedanken demonstration. The apparatus comes in three pieces. Two of them (**A** and **B**) function as detectors.

They are far apart from each other (in the analogous Aspect experiments over 10 meters apart). Each detector has a switch that can be set to one of three positions; each detector responds to an event by flashing either a red light or a green one. The third piece (**C**), midway between **A** and **B**, functions as a source. (See **figure 1**.)

There are no connections between the pieces, no mechanical connections, no electromagnetic connections, nor any other known kinds of relevant connections. (I promise that when you learn what is inside the black boxes you will agree that there are no connections.)

The detectors are thus incapable of signaling to each other or to the source via any known mechanism, and with the exception of the "particles" described below, the source has no way of signaling to the detectors. The demonstration proceeds as follows:

The switch of each detector is independently and randomly set to one of its three positions, and a button is pushed on the source; a little after that, each detector flashes either red or green. The settings of the switches and the colors that flash are recorded, and then the whole thing is repeated over and over again.

The data consist of a pair of numbers and a pair of colors for each run. **A** run, for example, in which **A** was set to 3, **B** was set to 2, **A** flashed red, and **B** flashed green, would be recorded as "**32RG**", as shown in **figure 2**.

Because there are no built-in connections between the source **C** and the detectors **A** and **B**, the link between the pressing of the button and the flashing of the light on a detector can only be provided by the passage of something (which we shall call a "particle", though you can call it anything you like) between the source and that detector. This can easily be tested; for example, by putting a brick between the source and a detector. In subsequent runs, that detector will not flash. When the brick is removed, everything works as before.

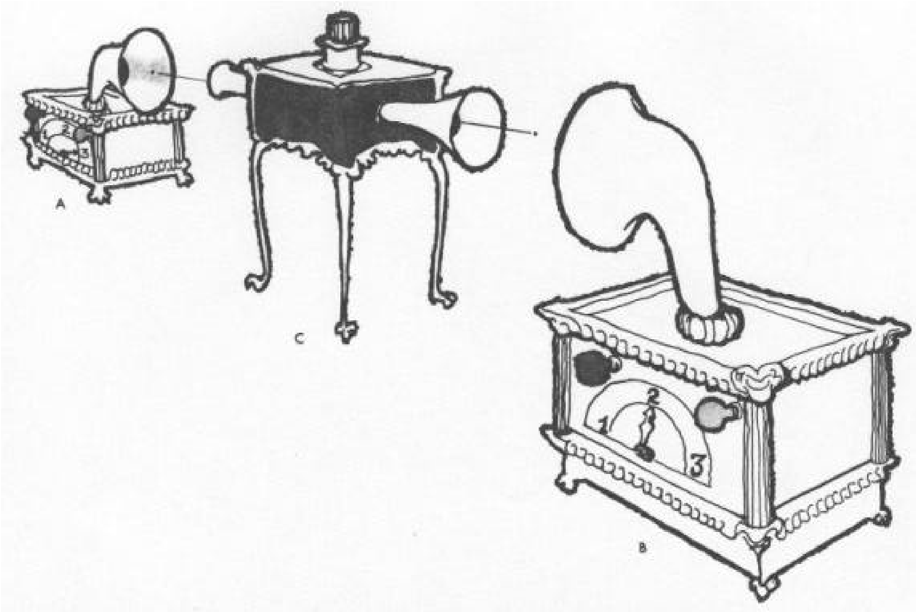
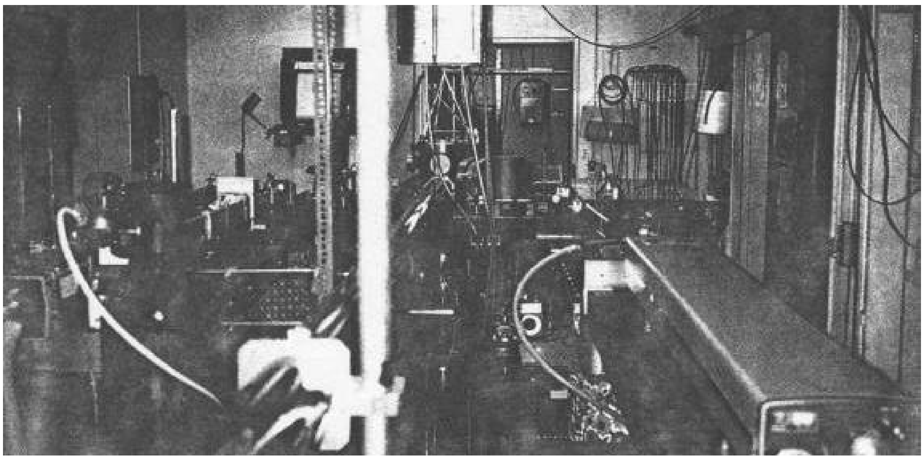


Figure 1 - An EPR apparatus.

The experimental setup consist of two detector, **A** and **B**, and a source of something ("particles" or whatever) **C**. To start a run, the experimenter pushes the button on **C**; something passes from **C** to both detectors. Shortly after the button is pushed each detector flashes one of its lights. Putting a brick between the source and one of the detectors prevents that detectors from flashing, and moving the detectors farther away from the source increases the delay between when the button is pushed and when the lights flash. The switch settings on the detectors vary randomly from one run to another. Note that there are no connections between the three parts of the apparatus, other than via whatever it is that passes from **C** to **A** and **B**.

The photo below shows a realization of such an experiment in the laboratory of Alain Aspect in Orsay, France. In the center of the lab is a vacuum chamber where individual calcium atoms are excited by the two lasers visible in the picture. The re-emitted photons travel 6 meters through the pipes to be detected by a two-channel polarizer.



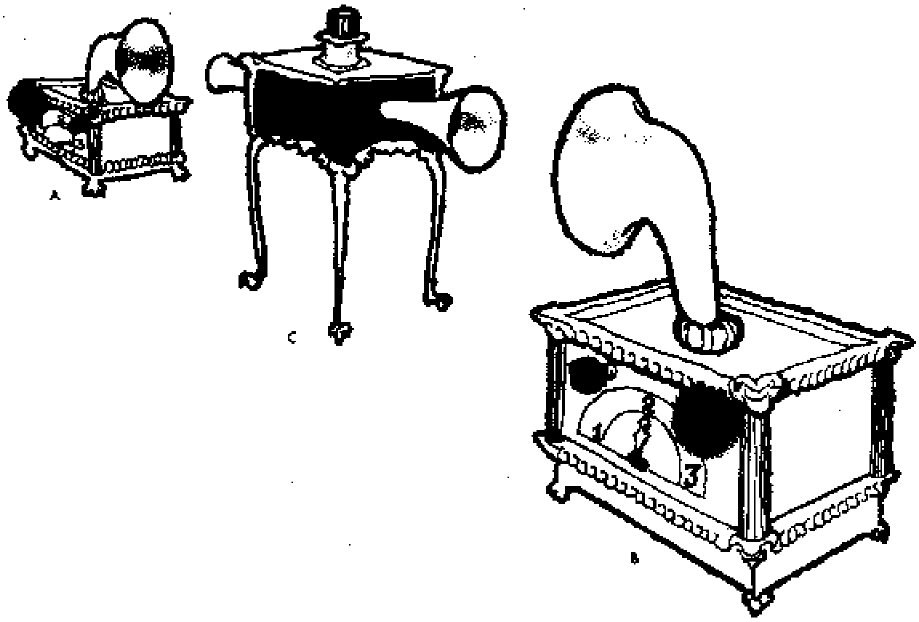


Figure 2 - The result of a run.
Shortly after the experimenter pushed the button on the source in figure 1, the detectors flash one lamp each. The experimenter records the switch settings and the colors of the lamps and then repeats the experiment. Here, for example, the record reads **32RG** –the switches are in positions 3 and 2 and the lamps flashed R and G, respectively.

31RR	12GR	23GR	13RR	33RR	12RR	22RR	32RG	13GG
22GG	23GR	33RR	13GG	31RG	31RR	33RR	32RG	32RR
31RG	33GG	11RR	12GR	33GG	21GR	21RR	22RR	31RG
33GG	11GG	23RR	32GR	12GR	12RG	11GG	31RG	21GR
12RG	13GR	22GG	12RG	33RR	31GR	21RR	13GR	23GR

Figure 3 – Data produced by the apparatus.
This is a fragment of an enormous set of data generated by many, many runs: each entry shows the switch settings and the colors of the lights that flashed for a run. The switch settings are changed randomly from run to run.

31RR	12GR	23GR	13RR	**33RR**	12RR	**22RR**	32RG	13GG
22GG	23GR	**33RR**	13GG	31RG	31RR	**33RR**	32RG	32RR
31RG	**33GG**	**11RR**	12GR	**33GG**	21GR	21RR	**22RR**	31RG
33GG	11GG	23RR	32GR	12GR	12RG	**11GG**	31RG	21GR
12RG	13GR	**22GG**	12RG	**33RR**	31GR	21RR	13GR	23GR

Figure 4 – Switches set the same.
The data of figure 3, but highlighted to pick out those runs in which both detectors had the same switch settings as they flashed. Note that in such runs the lights always flash the same colors.

31**RR**	12**GR**	23**GR**	13**RR**	33**RR**	12**RR**	22**RR**	32**RG**	13**GG**
22**GG**	23**GR**	33**RR**	13**GG**	31**RG**	31**RR**	33**RR**	32**RG**	32**RR**
31**RG**	33**GG**	11**RR**	12**GR**	33**GG**	21**GR**	21**RR**	22**RR**	31**RG**
33**GG**	11**GG**	23**RR**	32**GR**	12**GR**	12**RG**	11**GG**	31**RG**	21**GR**
12**RG**	13**GR**	22**GG**	12**RG**	33**RR**	31**GR**	21**RR**	13**GR**	23**GR**

Figure 5 – Switches set any way.
The data of figure 3, but highlighted to emphasize only the colors of the lights that flashed in each run, no matter how the switches were set when the lights flashed. Note that the pattern of colors is completely random.

Typical data from a large number of runs are shown in **figure 3**. There are just two relevant features:

I) **If one examines only those runs in which the switches have the same setting (figure 4), then one finds that the lights always flash the same colors.**

II) **If one examines all runs , without any regard to how the switches are set (figure 5), then one finds that the pattern of flashing is completely random. In particular, half the time the lights flash the same colors, and half the time different colors.**

That is all there is to the gedanken demonstration.
Should you be bothered by these data unless you have rocks in your head ?

How could it work ?

Consider only those runs in which the switches had the same setting when the particles went through the detectors. In all such runs the detectors flash the same colors. If they could communicate, it would be child's play to make the detectors flash the same colors when their switches had the same setting, but they are completely unconnected. Nor can they have been preprogrammed always to flash the same colors, regardless of what is going on, because the detectors are observed to flash different colors in at least some of those runs in which their switches are differently set, and the switch settings are independent random events.
How, then, are we to account for the first feature of the data? No problem at all. Born, in fact, in a letter of May 1948, offers[5] such an explanation to Einstein:

"It seems to me that your axiom of the 'independence of spatially separated objects A and B' is not as convincing as you make out. It does not take into account the fact of coherence; objects far apart in space which have a common origin need not be independent. I believe that this cannot be denied and simply has to be accepted. Dirac has based his whole book on this."

In our case the detectors are triggered by particles that have a common origin at the source **C**. It is then easy to dream up any number of explanations for the first feature of the data.
Suppose, for example, that what each particle encounters as it enters its detector is a target (**figure 6**) divided into eight regions, labeled RRR, RRG, RGR, RGG, GRR, GRG, GGR, and GGG. Suppose each detector is wired so that if a particle lands in the GRG bin, the detector flips into a mode in which the light flashes G if the switch is set to 1, R if it is set to 2, and G if it is set to 3; RGG leads to a mode with R for 1 and G for 2 and 3, and so on. We can then easily account for the fact that the lights always flash the same colors when the switches have the same settings by assuming that in each run the source always fires its particles into bins with the same labels.
Evidently this is not the only way. One could imagine that particles come in eight varieties: cubes, spheres, tetrahedra,… All settings produce R when a cube is detected, a sphere results in R for settings 1 and 2, G for setting 3, and so forth. The first feature of the data is then accounted for if the two particles produced by the source in each run are always both of the same variety.
Common to all such explanations is the requirement that each particle should, in one way or another, carry to its detector a set of instructions for how it is to flash for *each* of the three possible switch settings, and that in *any* run of the experiment both particles should carry the same instruction sets:

I) A set of instructions that covers *each* of the three possible settings is required because there is no communication between the source and the detectors other than the particles themselves. In runs in which the switches have the same setting, the particles cannot know whether that setting will be 11, 22, or 33. For the detectors always to flash the same colors when the switches have the same setting, the particles must carry instructions that specify colors for each of the three possibilities.

II) The absence of communication between source and detectors also requires that the particles carry such instruction sets in *every* run of the experiment –even those in which the switches end up with different settings- because the particles always have to be prepared: any run may turn out to be one in which the switches end up with the same settings.

This generic explanation is pictured schematically in **figure 7**.

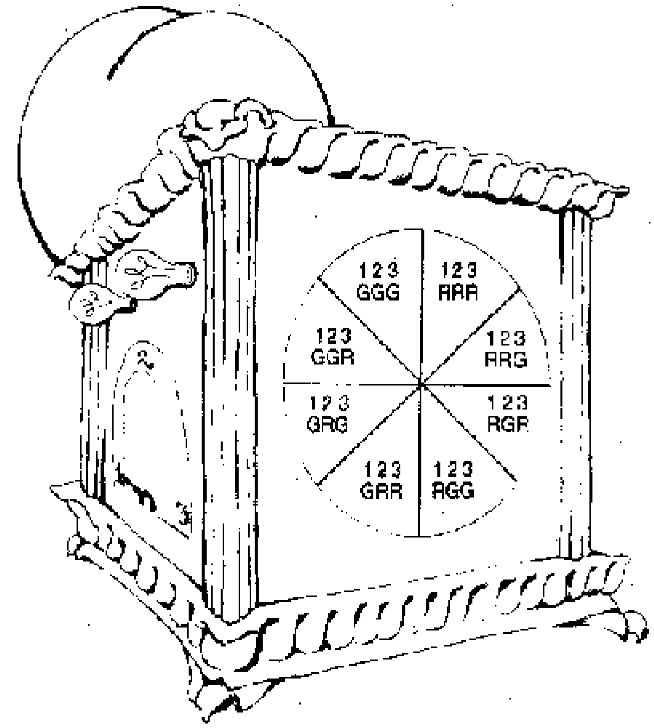


Figure 6 – Model of a detector to produce data like those in figure 4.
Particles from the source fall with equal probability into any of the eight bins; for each bin the color flashed depends on the switch as indicated on the back of the box.

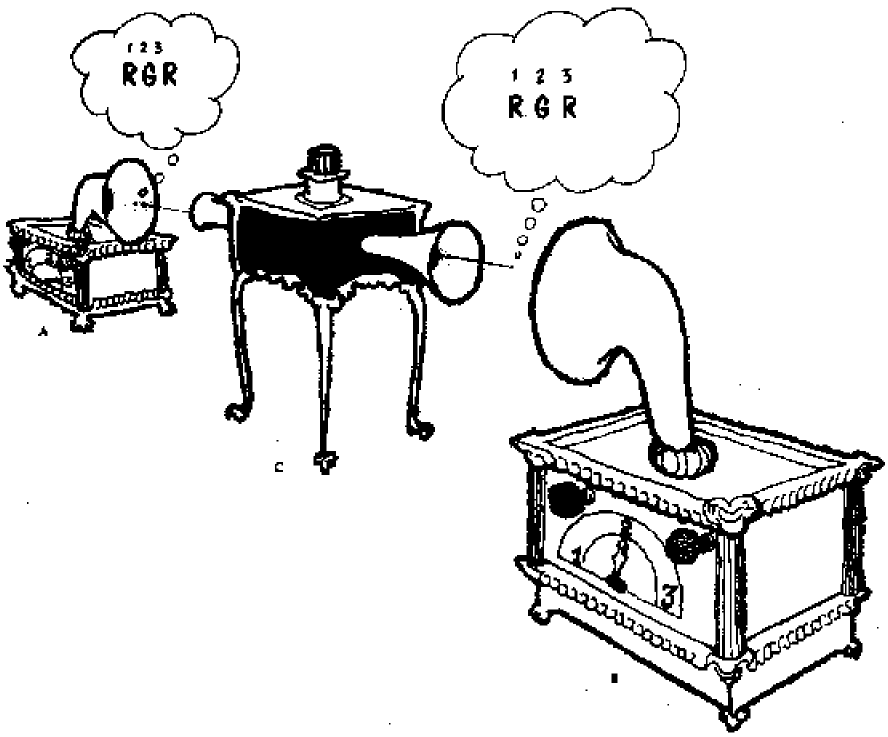


Figure 7 – Instruction sets.
To guarantee that the detectors of figure 6 flash the same color when the switches are set the same, the two particles must in one way or another carry instruction sets specifyng how their detectors are to flash for each possible switch setting. The results of any one run reveal nothing about the instructions beyond the actual data; so in this case, for example, the first instruction (1R) is "something one cannot know anything about", and I've only guessed at it, assuming that "it exists all the same".

Alas, this explanation –the only one, I maintain, that someone not steeped in quantum mechanics will ever be able to come up with (though it is an entertaining game to challenge people to try)- is untenable.

It is inconsistent with the second feature of the data: **There is no conceivable way to assign such instruction sets to the particles from one run to the next that can account for the fact that in all runs taken together, without regard to how the switches are set, the same colors flash half the time.**

Pause to note that we are about to show that "something one cannot know anything about" –the third entry in an instruction set- cannot exist. For even if instruction sets did exist, one could never learn more than two of the three entries (revealed in those runs where the switches ended up with two different settings). Here is the argument.

Consider a particular instruction set, for example, RRG. Should both particles be issued the instruction set RRG, then the detectors will flash the same colors when the switches are set to 11, 22, 33, 12, or 21; they will flash different colors for 13, 31, 23, or 32.

Because the switches at each detector are set randomly and independently, each of these nine cases is equally likely, so the instruction set RRG will result in the same colors flashing ***5/9*** of the time.

Evidently the same conclusion holds for the sets RGR, GRR, GGR, GRG and RGG, because the argument uses only the fact that one color appears twice and the other once. All six such instructions sets also result in the same colors flashing ***5/9*** of the time.

But the only instruction sets left are RRR and GGG, and these each result in the same colors flashing *all* of the time.

Therefore if instructions sets exist, the same colors will flash in at least ***5/9*** of all the runs, regardless of how the instruction sets are distributed from one run of the demonstration to the next.

This is Bell's theorem (also known as Bell's inequality) for the gedanken demonstration.

But in the actual gedanken demonstration the same colors flash only ½ the time.

The data described above violate this Bell's inequality, and therefore there can be no instruction sets.

If you don't already know how the trick is done, may I urge you, before reading how the gedanken demonstration works, to try to invent some other explanation for the first feature of the data that does not introduce connections between the three parts of the apparatus or prove to be incompatible with the second feature.

One way to do it

Here is one way to make such a device:

Let the source produce two particles of spin ½ in the singlet state, flying apart toward the two detectors. (Granted, this is not all that easy to do, but in the Orsay experiments described below, the same effect is achieved with correlated photons).

Each detector contains a Stern-Gerlach magnet, oriented along one of three directions ($a^{(1)}$, $a^{(2)}$, or $a^{(3)}$), perpendicular to the line of flight of the particles, and separated by 120°, as indicated in **figure 8**.

The three settings of the switch determine which orientation is used. The light on one detector flashes red or green, depending on whether the particle is deflected toward the north (spin up) or south (spin down) pole of the magnet as it passes between them; the other detector uses the opposite color convention.

That's it. Clearly there are no connections between the source and the detectors or between the two detectors. We can nevertheless account for the data as follows:

When the switches have the same setting, the spins of both particles are measured along the same direction, so the lights will always flash the same colors if the measurements along the same direction always yield opposite values. But this is an immediate consequence of the structure of the spin singlet state, which has the form:

$$|\psi\rangle = (1/\sqrt{2})\ [\ |+\ -\rangle - |-\ +\rangle\ ] \qquad \textbf{(1)}$$

independent of the direction of the spin quantization axis, and therefore yields +– or –+ with equal probability, but never ++ or –– ,whenever the two spins are measured along any common direction.

To establish the second feature of the data, note that the product m_1m_2 of the results of the two spin measurements (each of which can have the values +½ or –½) will have the value –¼ when the lights flash the same colors and +¼ when they flash different colors. We must therefore show that the product vanishes when averaged over all the nine distinct pairs of orientations the two Stern-Gerlach magnets can have.

For a given pair of orientation, $a^{(i)}$ and $a^{(j)}$, the mean value of this product is just the expectation value in the state ψ of the corresponding product of (commuting) hermitian observables $a^{(i)} \cdot S^{(1)}$ and $a^{(j)} \cdot S^{(2)}$.

Thus the second feature of the data requires:

$$0 = \sum_{ij} \langle\psi|\, [a^{(i)} \cdot S^{(1)}][a^{(j)} \cdot S^{(2)}]\, |\psi\rangle \qquad \textbf{(2)}$$

But equation 2 is an immediate consequence of the linearity of quantum mechanics, which lets one take the sums inside the matrix element, and the fact that the three unit vectors around an equilateral triangle sum to zero:

$$\sum_i a^{(i)} = \sum_j a^{(j)} = 0 \qquad \textbf{(3)}$$

This completely accounts for the data. It also unmasks the gedanken demonstration as a simple embellishment of Bohm's version of the EPR experiment. If we kept only runs in which the switches had the same setting, we would have precisely the Bohm-EPR experiment. The assertion that instruction sets exist is then blatant quantum-theoretic nonsense, for it amounts to the insistence that each particle has stamped on it in advance the outcome of the measurements of three different spin components corresponding to noncommuting observables $S \cdot a^{(i)}$, i=1,2,3. According to EPR, this is merely a limitation of the quantum-theoretic formalism, because instruction sets are the only way to account for the first feature of the data.

Bell's analysis adds to the discussion those runs in which the switches have different settings, extracts the second feature of the data as a further elementary prediction of quantum mechanics, and demonstrates that any set of data exhibiting this feature is incompatible with the existence of the instruction sets apparently required by the first feature, quite independently of the formalism used to explain the data, and quite independently of any doctrines of quantum theology.

The experiments

The experiments of Aspect and his colleagues at Orsay confirm that the quantum-theoretic predictions for this experiment are in fact realized, and that the conditions for observing the results of the experiment can in fact be achieved. (A distinguished colleague once told me that the answer to the EPR paradox was that correlations in the singlet state could never be maintained over macroscopic distances –that anything, even the passage of a cosmic ray in the next room, would disrupt the correlations enough to destroy the effect).

In these experiments the two spin ½ particles are replaced by a pair of photons and the spin measurements become polarization measurements.

The photon pairs are emitted by calcium atoms in a radiative cascade after suitable pumping by lasers. Because the initial and final atomic states have J=0, quantum theory predicts (and experiment confirms) that the photons will be found to have the same polarizations (lights flashing the same colors in the analogous gedanken experiment) if they are measured along the same direction –feature number 1.

But if the polarizations are measured at 120° angles, then theory predicts (and experiment confirms) that they will be the same only a quarter of the time [$¼ = \cos^2 (120°)$].

This is precisely what is needed to produce the statistics of feature number 2 of the gedanken demonstration: the randomly set switches end up with the same setting (same polarizations measured) 1/3 of the time, so in all runs the same colors will flash 1/3 x 1 + 2/3 x (1/4) = ½ the time.

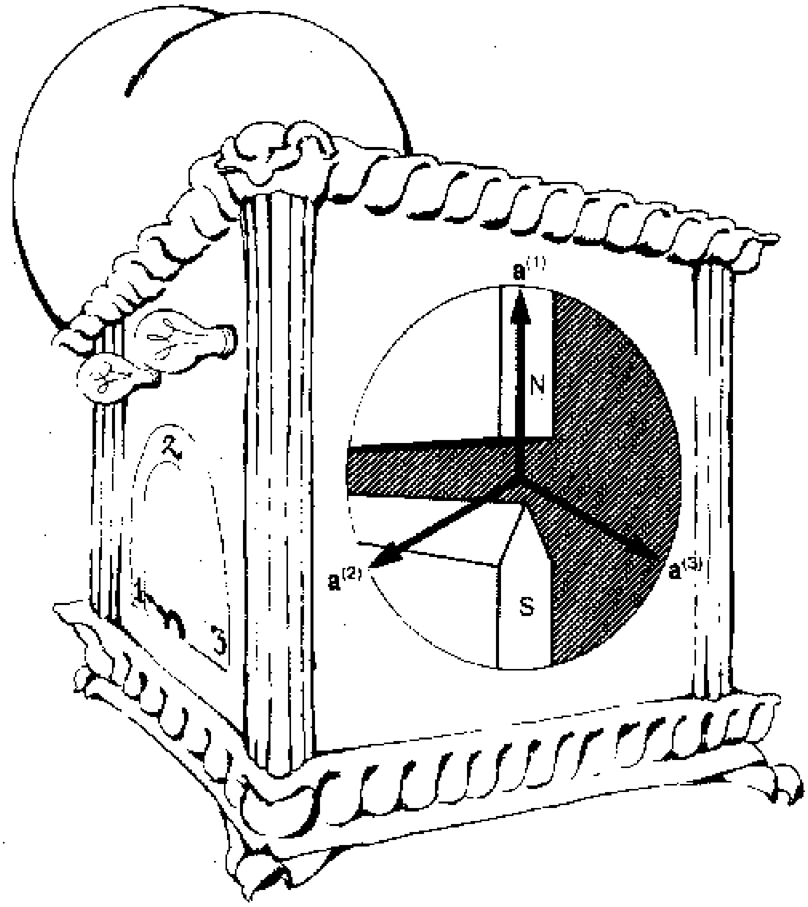


Figure 8 – A realization of the detector to produce the data of figure 3.
The particles have a magnetic moment and can be separated into “spin up” and “spin down” particles by the Stern-Gerlach magnet inside the detector. Setting the switch to positions 1,2, or 3 rotates the north pole of the magnet along the coplanar unit vectors $a^{(1)}$, $a^{(2)}$, or $a^{(3)}$, separated by 120°. The vector sum of the three unit vectors is, of course, zero. The switch positions on the two detectors correspond to the same orientations of the magnetic field. One detector flashes red for spin up, green for spin down; the other uses the opposite color convention.

The people in Orsay were interested in a somewhat modified version of Bell’s argument in which the angles of greatest interest were multiples of 22.5°, but they collected data for many different angles, and, except for EPR specialists, the conceptual differences between the two cases are minor.[16]

There are some remarkable features to these experiments. The two polarization analyzers were placed as far as 13 meters apart without producing any noticeable change in the results, thereby closing the loophole that the strange quantum correlations might somehow diminish as the distance between regions **A** and **B** grew to macroscopic proportions. At such separations it is hard to imagine that a polarization measurement of photon #1 could, in any ordinary sense of the term, “disturb” photon #2.

Indeed, at these large separations, a hypothetical disturbance originating when one photon passed through its analyzer could only reach the other analyzer in time to affect the outcome of the second polarization measurement if it traveled at a superluminal velocity.

In the third paper of the Orsay group’s series, bizarre conspiracy theories are dealt a blow by an ingenious mechanism for rapidly switching the directions along which the polarizations of each photon are measured.

Each photon passes to its detector through a volume of water that supports an ultrasonic standing wave. Depending on the instantaneous amplitude of the wave, the photon either passes directly into a polarizer with one orientation or is Bragg reflected into another with a different orientation.

The standing waves that determine the choice of orientation at each detector are independently driven and have frequencies so high that several cycles take place during the light travel time from one detector to the other. (This corresponds to a refinement of the gedanken demonstration in which, to be absolutely safe, the switches are not given their random settings until *after* the particles have departed from their common source).

What does it mean ?

What is one to make of all this? Are there “spooky actions at a distance” ?
A few years ago I received the text of a letter from the executive director of a California think-tank to the Under-Secretary of Defense for Research and Engineering, alerting him to the EPR correlations:

“If in fact we can control the faster-than-light nonlocal effect, it would be possible…to make an untappable and unjammable command-control-communication system at very high bit rates for use in the submarine fleet. The important point is that since there is no ordinary electromagnetic signal linking the encoder with the decoder in such a hypothetical system, there is nothing for the enemy to tap or jam. The enemy would have to have actual possession of the “black box” decoder to intercept the message, whose reliability would not depend on separation from the encoder nor on ocean or weather conditions….”

Heady stuff indeed! But just what is this nonlocal effect? Using the language of the gedanken demonstration, let us talk about the “N-color” of a particle (N can be 1, 2, or 3) as the color (red or green) of the light that flashes when the particle passes through a detector with its switch set to N.

Because instruction set cannot exist, we know that a particle cannot at the same time carry a definite 1-color, 2-color and 3-color to its detector. On the other hand, for any particular N (say 3), we can determine the 3-color of the particle heading for detector **A** before it gets there by arranging things so that the other particle first reaches detector **B**, where its 3-color is measured.

If the particle at **B** was 3-colored red, the particle at **A** will turn out to be 3-colored red, and green at **B** means green at **A**.

Three questions now arise:

I) Did the particle at **A** have its 3-color prior to the measurement of the 3-color of the particle at **B**? The answer cannot be yes, because, *prior* to the measurement of the 3-color at **B**, it is altogether possible that the roll of the dice at **B** or the whim of the **B**-operator will result in the 2-color or the 1-color being measured at **B** instead. Barring the most paranoid of conspiracy theories, “prior to the measurement of the 3-color at **B**” is indistinguishable from “prior to the measurement of the 2- (or 1-) color at **B**”. If the 3-color already existed, so also must the 2- and 1-colors have existed. But instruction sets (which consist of a specification of the 1-, 2-, and 3-colors) do not exist.

II) Is the particle at **A** 3-colored red *after* the measurement at **B** shows the color red? The answer is surely yes, because under these circumstances it is invariably a particle that will cause the detector at **A** to flash red.

III) Was something (the value of its 3-color) transmitted to the particle at **A** as a result of the measurement at **B**?

Orthodox quantum metaphysicians would, I believe, say no, nothing has changed at **A** as the result of the measurement at **B**; what has changed is our knowledge of the particle at **A**. (Somewhat more spookily, they might object to the naive classical assumption of localizability or separability implicit in the phrases “at **A**” and “at **B**”).

This seems very sensible and very reassuring: N-color does not characterize the particle at all, but only what we know about the particle. But does that last sentence sound as good when “particle” is changed to “photon” and “N-color” to “polarization”? And does it really help you to stop wondering why the lights always flash the same colors when the switches have the same settings?

What is clear is that if there is spooky action at a distance, then, like other spooks, it is absolutely useless except for its effect, begnin or otherwise, on our state of mind.

For the statistical pattern of red and green flashes at detector **A** is entirely random, however the switch is set at detector **B**. Whether the particles arriving at **A** all come with definite 3-colors (because the switch at B was stuck at 3) or definite 2-colors (because the switch was stuck at 2) or no colors at all (because there was a brick in front of the detector at **B**) –all this has absolutely no effect on the statistical distribution of colors observed at **A**. The manifestation of this “action at a distance” is revealed only through a comparison of the data independently gathered at **A** and at **B**.

This is a most curious state of affairs, and while it is wrong to suggest that EPR correlations will replace sonar, it seems to me something is lost by ignoring them or shrugging them off.

The EPR experiment is as close to magic as any physical phenomenon I know of, and magic should be enjoyed. Whether there is physics to be learned by pondering it is less clear. The most elegant answer I have found[17] to this last question comes from one of the great philosophers of our time, whose view of the matter I have taken the liberty of quoting in the form of the poetry it surely is:

We have always had a great deal of difficulty
understanding the world view
that quantum mechanics represents.

At least I do,
because I'm an old enough man
that I havent' got to the point
that this stuff is obvious to me.

Okay, I still get nervous with it….

You know how it always is,
every new idea,
it takes a generation or two
until it becomes obvious
that there's no real problem.

I cannot define the real problem,

therefore I suspect there's no real problem,
but I'm not sure
there's no real problem.

Nobody in the 50 years since Einstein, Podolsky and Rosen has ever put it better than that.

[Some of the views expressed above were developed in the course of occasional technical studies of EPR correlations supported by the National Science Foundation under grant No. DMR 83-14625.]

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