2016 AUSTRALIAN MATHEMATICAL OLYMPIAD

DAY 1
Tuesday, 9 February 2016
Time allowed: 4 hours
No calculators are to be used.
Each question is worth seven points.

1. Find all positive integers n such that $2^n + 7^n$ is a perfect square.

2. Let ABC be a triangle. A circle intersects side BC at points U and V, side CA at points W and X, and side AB at points Y and Z. The points U, V, W, X, Y, Z lie on the circle in that order. Suppose that $AY = BZ$ and $BU = CV$.

Prove that $CW = AX$.

3. For a real number x, define $\lfloor x \rfloor$ to be the largest integer less than or equal to x, and define $\{x\} = x - \lfloor x \rfloor$.

(a) Prove that there are infinitely many positive real numbers x that satisfy the inequality

$$\{x^2\} - \{x\} > \frac{2015}{2016}.$$

(b) Prove that there is no positive real number x less than 1000 that satisfies this inequality.

4. A *binary sequence* is a sequence in which each term is equal to 0 or 1. We call a binary sequence *superb* if each term is adjacent to at least one term that is equal to 1. For example, the sequence $0, 1, 1, 0, 0, 1, 1, 1$ is a superb binary sequence with eight terms. Let B_n denote the number of superb binary sequences with n terms.

Determine the smallest integer $n \geq 2$ such that B_n is divisible by 20.

The Mathematics Olympiads are supported by the Australian Government Department of Education through the Mathematics and Science Participation Program.

Australian Mathematical Olympiad Committee

A department of the Australian Mathematics Trust

2016 AUSTRALIAN MATHEMATICAL OLYMPIAD

DAY 2
Wednesday, 10 February 2016
Time allowed: 4 hours
No calculators are to be used.
Each question is worth seven points.

5. Find all triples (x, y, z) of real numbers that simultaneously satisfy the equations

$$\begin{aligned} xy + 1 &= 2z \\ yz + 1 &= 2x \\ zx + 1 &= 2y. \end{aligned}$$

6. Let a, b, c be positive integers such that $a^3 + b^3 = 2^c$.

Prove that $a = b$.

7. Each point in the plane is assigned one of four colours.

Prove that there exist two points at distance 1 or $\sqrt{3}$ from each other that are assigned the same colour.

8. Three given lines in the plane pass through a point P.

(a) Prove that there exists a circle that contains P in its interior and intersects the three lines at six points A, B, C, D, E, F in that order around the circle such that $AB = CD = EF$.

(b) Suppose that a circle contains P in its interior and intersects the three lines at six points A, B, C, D, E, F in that order around the circle such that $AB = CD = EF$. Prove that

$$\frac{1}{2}\,\text{area}(\text{hexagon } ABCDEF) \geq \text{area}(\triangle APB) + \text{area}(\triangle CPD) + \text{area}(\triangle EPF).$$

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AUSTRALIAN MATHEMATICAL OLYMPIAD 2016 SOLUTIONS

1. Find all positive integers n such that $2^n + 7^n$ is a perfect square.

Solution 1 (Mike Clapper)

Since $2^1 + 7^1 = 9 = 3^2$, $n = 1$ is a solution. We will now show that it is the only solution.

For $n > 1$, we have $2^n \equiv 0 \pmod 4$. We also have $7^n \equiv (-1)^n \pmod 4$. Since all perfect squares are either congruent to 0 or 1 modulo 4, $2^n + 7^n$ cannot be a perfect square if n is odd and greater than 1. So write $n = 2m$, where m is a positive integer.

We would like to show that 2^n+7^n cannot be a perfect square. Considering this expression modulo 5, we have $2^n + 7^n = 4^m + 49^m \equiv 2 \times (-1)^m \pmod 5$. Therefore, $2^n + 7^n$ is congruent to 2 or 3 modulo 5. On the other hand, all perfect squares are congruent to 0, 1 or 4 modulo 5.

Therefore, $n = 1$ is indeed the only solution to the problem.

Solution 2

As in Solution 1, we prove that $n = 1$ is a solution and that any other can be written as $n = 2m$, where m is a positive integer.

We would like to show that 2^n+7^n cannot be a perfect square. Considering this expression modulo 3, we have $2^n + 7^n = 4^m + 49^m \equiv 2 \times 1^m \equiv 2 \pmod 3$. On the other hand, all perfect squares are congruent to 0 or 1 modulo 3.

Therefore, $n = 1$ is indeed the only solution to the problem.

Solution 3

As in Solution 1, we prove that $n = 1$ is a solution and that any other can be written as $n = 2m$, where m is a positive integer.

We would like to show that $2^n + 7^n$ cannot be a perfect square. This follows from the inequality

$$(7^m)^2 < 2^n + 7^n < (7^m + 1)^2,$$

which means that 2^n+7^n lies between two consecutive perfect squares. The left inequality is obvious since 2^n is positive. The right inequality is also obvious, since it is equivalent to $4^m < 2 \cdot 7^m + 1$.

Therefore, $n = 1$ is indeed the only solution to the problem.

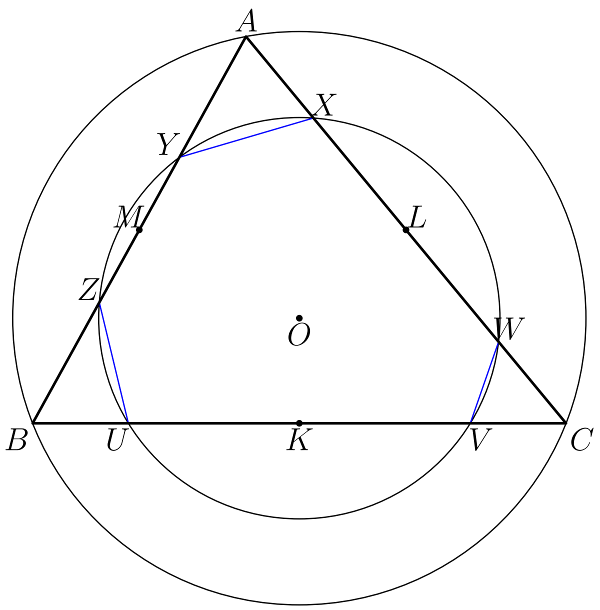
The Mathematics/Informatics Olympiads are supported by the Australian Government Department of Education and Training through the Mathematics and Science Participation Program.

2. Let ABC be a triangle. A circle intersects side BC at points U and V, side CA at points W and X, and side AB at points Y and Z. The points U, V, W, X, Y, Z lie on the circle in that order. Suppose that $AY = BZ$ and $BU = CV$.

   Prove that $CW = AX$.

Solution 1 (Angelo Di Pasquale)

Let O be the centre of the circumcircle of $UVWXYZ$ and let K, L, M be the midpoints of BC, CA, AB, respectively.



Since $AY = BZ$ and $BU = CV$, the points M and K are the midpoints of AB and BC, respectively. Therefore, the perpendicular to AB passing through M is the perpendicular bisector of both the segments AB and YZ. Similarly, the perpendicular to BC passing through K is the perpendicular bisector of both the segments BC and UV. Hence, these two perpendicular bisectors pass through O as well as the circumcentre of triangle ABC. It follows that O is the circumcentre of triangle ABC.

However, since O is the centre of the circumcircle of $UVWXYZ$, we have that OL is perpendicular to WX. Thus, OL is perpendicular to CA. Since O is the circumcentre of triangle ABC, we must have that L is the midpoint of CA. The fact that L is the midpoint of both WX and CA implies that $CW = AX$.

Solution 2 (Angelo Di Pasquale)

Using the power of a point theorem from A, then B, then C, we find that

$$AX \cdot AW = AY \cdot AZ = BZ \cdot BY = BU \cdot BV = CV \cdot CU = CW \cdot CX.$$

Therefore, we have

$$AX \cdot (AX + XW) = CW \cdot (CW + WX) \quad \Rightarrow \quad (AX - CW) \cdot (AX + CW + WX) = 0.$$

Therefore, it must be the case that $CW = AX$.

Solution 3 (Angelo Di Pasquale and Jamie Simpson)

Let M be the midpoint of AB and let O be the centre of the circumcircle of $UVWXYZ$.

Since $AY = BZ$, M is also the midpoint of YZ. But triangle OYZ is isosceles with $OY = OZ$. Therefore, triangle OMZ is congruent to triangle OMY (SSS) and $\angle OMZ = \angle OMY = 90°$. It follows that triangle OMB is congruent to triangle OMA (SAS). Therefore, $OB = OA$ and, by a similar argument, we have $OB = OC$.

We deduce that $OA = OC$, which implies that $\angle OCW = \angle OCA = \angle OAC = \angle OAX$. Since $OW = OX$, we have $\angle OWX = \angle OXW$, which implies $\angle OWC = \angle OXA$. Thus, triangle OWC is congruent to triangle OXA (AAS). From this, we have $CW = AX$, as desired.

3. For a real number x, define $\lfloor x \rfloor$ to be the largest integer less than or equal to x, and define $\{x\} = x - \lfloor x \rfloor$.

   (a) Prove that there are infinitely many positive real numbers x that satisfy the inequality

$$\{x^2\} - \{x\} > \frac{2015}{2016}.$$

   (b) Prove that there is no positive real number x less than 1000 that satisfies this inequality.

Solution 1

(a) We will show that $x = n + \frac{1}{n+1}$ satisfies the inequality for sufficiently large positive integers n.

$$\begin{aligned}\{x^2\} - \{x\} &= \left\{n^2 + \frac{2n}{n+1} + \frac{1}{(n+1)^2}\right\} - \left\{n + \frac{1}{n+1}\right\} \\ &= \left\{n^2 + 2 - \frac{2}{n+1} + \frac{1}{(n+1)^2}\right\} - \frac{1}{n+1} \\ &= \left(1 - \frac{2}{n+1} + \frac{1}{(n+1)^2}\right) - \frac{1}{n+1} \\ &= 1 - \frac{3}{n+1} + \frac{1}{(n+1)^2} \\ &> 1 - \frac{3}{n+1}\end{aligned}$$

Therefore, $x = n + \frac{1}{n+1}$ satisfies the inequality as long as n is a positive integer such that

$$1 - \frac{3}{n+1} > \frac{2015}{2016} \quad \Leftrightarrow \quad n > 3 \times 2016 - 1.$$

(b) Let $x = a + b$, where $a = \lfloor x \rfloor$ and $b = \{x\}$, and consider the following inequalities.

$$\{x^2\} - \{x\} > \frac{2015}{2016} \quad \Rightarrow \quad 1 - b > \frac{2015}{2016} \quad \Rightarrow \quad b < \frac{1}{2016}$$

Now use $b < \frac{1}{2016}$ to deduce the following inequalities.

$$\begin{aligned}\{x^2\} - \{x\} > \frac{2015}{2016} \qquad &\Rightarrow \qquad \{(a+b)^2\} = \{2ab + b^2\} > \frac{2015}{2016} \\ &\Rightarrow \qquad 2ab + b^2 > \frac{2015}{2016} \\ &\Rightarrow \qquad a > \frac{2015}{2016} \cdot \frac{1}{2b} - \frac{b}{2} > \frac{2015}{2016} \cdot \frac{2016}{2} - \frac{1}{2} \cdot \frac{1}{2016} > 1000\end{aligned}$$

Therefore, there is no positive real number x less than 1000 that satisfies the inequality.

Solution 2 (Chaitanya Rao)

Solution to part (a) only.

Let $x = a + 10^{-4}$, where a is an integer. Then

$$\begin{aligned}\{x^2\} - \{x\} &= \{a^2 + 2a10^{-4} + 10^{-8}\} - 10^{-4} \\ &= \{2a10^{-4} + 10^{-8}\} - 10^{-4}.\end{aligned}$$

Now let $a = 4999 + 5000n$ for $n = 0, 1, 2, \ldots$. We find that

$$\begin{aligned}\{x^2\} - \{x\} &= \{0.9998 + n + 10^{-8}\} - 10^{-4} \\ &= 0.9998 + 10^{-8} - 10^{-4} \\ &= 0.9997 + 10^{-8}.\end{aligned}$$

Since $10^4 > 6048$, we have $\frac{3}{10^4} < \frac{3}{6048} = \frac{1}{2016}$. Therefore,

$$\{x^2\} - \{x\} = 0.9997 + 10^{-8} > 1 - \frac{3}{10^4} > 1 - \frac{1}{2016} = \frac{2015}{2016}.$$

Hence, the positive real numbers of the form $x = 4999 + 5000n + 10^{-4}$ for $n = 0, 1, 2, \ldots$ satisfy the inequality.

Solution 3 (Ivan Guo)

Solution to part (a) only.

For convenience, we set $\varepsilon = \frac{1}{2016}$. Using the notation $x = a+b$, where $a = \lfloor x \rfloor$ and $b = \{x\}$, we obtain

$$\{x^2\} = \{a^2 + 2ab + b^2\} = \{2ab + b^2\}.$$

We would like to find (a, b) such that $2ab + b^2 < 1$ and $2ab + b^2 - b > 1 - \varepsilon$. By noting $0 \leq b^2 \leq b$, it suffices to find (a, b) such that $2ab + b < 1$ and $2ab - b > 1 - \varepsilon$. This can be achieved by fixing $2ab = 1 - \frac{\varepsilon}{2}$ and choosing a to be a sufficiently large integer so that $b < \frac{\varepsilon}{2}$.

Solution 4 (Angelo Di Pasquale)

Solution to part (b) only.

Intuitively, we would like $\{x^2\}$ to be just below an integer and $\{x\}$ to be just above an integer. Hence, let $x = a + b$ and $x^2 = m - c$ where a and m are integers and $0 \leq b, c < 1$. For convenience, we also set $\varepsilon = \frac{1}{2016}$.

Since $\{x^2\} = 1 - c$, we require

$$1 - c - b > 1 - \varepsilon \quad \Leftrightarrow \quad b + c < \varepsilon.$$

Note that

$$\begin{aligned} & (a+b)^2 = m - c \\ \Rightarrow \quad & a^2 + 2ab + b^2 + c = m.\end{aligned}$$

However, m is an integer such that $m > a^2$, which implies that $m \geq a^2 + 1$. Therefore,

$$\begin{aligned} & a^2 + 2ab + b^2 + c \geq a^2 + 1 \\ \Rightarrow \quad & 2ab \geq 1 - b^2 - c \geq 1 - (b + c) > 1 - \varepsilon \\ \Rightarrow \quad & a > \frac{1-\varepsilon}{2b} > \frac{1-\varepsilon}{2\varepsilon} = \frac{2015}{2}.\end{aligned}$$

It follows that $x = a + b > 1000$.

Solution 5 (Hans Lausch)

For $n = 1, 2, 3, \ldots$ and $t = 0, 1, 2, \ldots$ and all real numbers x, let $f_{n,t}(x) = (x^2 - (n^2 + t)) - (x - n)$. The functions $f_{n,t}$ for $n = 1, 2, 3, \ldots$ and $t = 0, 1, 2, \ldots, 2n$ and $\sqrt{n^2 + t} \le x < \sqrt{n^2 + t + 1}$ satisfy the equation $f_{n,t}(x) = \{x^2\} - \{x\}$.

As $f_{n,t}$ is an increasing function for $x \ge \frac{1}{2}$, we conclude that for a fixed $0 \le t \le 2n$,

$$\max \left\{ f_{n,t}(x) \,\middle|\, \sqrt{n^2 + t} \le x \le \sqrt{n^2 + t + 1} \right\} = f_{n,t}(\sqrt{n^2 + t + 1}) = 1 - (\sqrt{n^2 + t + 1} - n).$$

For a fixed positive integer n, this is maximal if and only if $t = 0$. So for $n \le x < n + 1$,

$$\max f_{n,t}(x) = f_{n,0}(\sqrt{n^2 + 1}) = 1 - (\sqrt{n^2 + 1} - n).$$

(a) As $\lim\limits_{n \to \infty} \left[1 - (\sqrt{n^2 + 1} - n)\right] = 1$, there exists a positive integer N such that

$$f_{N,0}(\sqrt{N^2 + 1}) = 1 - (\sqrt{N^2 + 1} - N) > \frac{2015}{2016}.$$

Since $f_{N,0}$ is continuous and increasing, it follows that there exists $\delta > 0$ such that all x satisfying $\sqrt{N^2 + 1} - \delta < x < \sqrt{N^2 + 1}$ also satisfy the given inequality.

(b) Note that $\{x^2\} - \{x\} < f_{n,0}(\sqrt{n^2 + 1}) = 1 - (\sqrt{n^2 + 1} - n)$, for $n \le x < n + 1$. Also, we have that $1 - (\sqrt{n^2 + 1} - n)$ is an increasing function of n. Thus, for $x < 1000$, we have $n \le 999$ and

$$\{x^2\} - \{x\} < 1 - (\sqrt{999^2 + 1} - 999) = 1 - \frac{1}{\sqrt{999^2 + 1} + 999} < 1 - \frac{1}{1999} < \frac{2015}{2016}.$$

4. A *binary sequence* is a sequence in which each term is equal to 0 or 1. We call a binary sequence *superb* if each term is adjacent to at least one term that is equal to 1. For example, the sequence $0, 1, 1, 0, 0, 1, 1, 1$ is a superb binary sequence with eight terms. Let B_n denote the number of superb binary sequences with n terms.

   Determine the smallest integer $n \geq 2$ such that B_n is divisible by 20.

Solution 1

The *Fibonacci sequence* $F_0, F_1, F_2, \ldots$ is defined by $F_0 = 0$, $F_1 = 1$, and $F_m = F_{m-1} + F_{m-2}$ for $m \geq 2$. We will prove that

$$\begin{aligned} B_{2m} &= F_{m+1}^2, && \text{for } m \geq 1, \\ B_{2m+1} &= F_m F_{m+3}, && \text{for } m \geq 0. \end{aligned}$$

First, observe that a binary sequence $b_1, b_2, \ldots, b_n$ with $n \geq 2$ is superb if and only if

- $b_2 = b_{n-1} = 1$; and
- there is no $1 \leq k \leq n - 2$ such that $b_k = b_{k+2} = 0$.

So a binary sequence $b_1, b_2, \ldots, b_{2m}$ with an even number of terms is superb if and only if

- $b_2 = b_{2m-1} = 1$;
- $b_4, b_6, \ldots, b_{2m}$ is a binary sequence that does not contain two consecutive terms equal to 0; and
- $b_1, b_3, \ldots, b_{2m-3}$ is a binary sequence that does not contain two consecutive terms equal to 0.

It follows that the number of superb binary sequences $b_1, b_2, \ldots, b_{2m}$ is equal to the number of ways to choose the two binary sequences $b_4, b_6, \ldots, b_{2m}$ and $b_1, b_3, \ldots, b_{2m-3}$, both with $m - 1$ terms, without two consecutive terms equal to 0. We will prove below that the number of binary sequences with k terms that do not contain two consecutive terms equal to 0 is F_{k+2}. Therefore, $B_{2m} = F_{m+1}^2$.

Similarly, a binary sequence $b_1, b_2, \ldots, b_{2m+1}$ with an odd number of terms is superb if and only if

- $b_2 = b_{2m} = 1$;
- $b_4, b_6, \ldots, b_{2m-2}$ is a binary sequence that does not contain two consecutive terms equal to 0; and
- $b_1, b_3, \ldots, b_{2m+1}$ is a binary sequence that does not contain two consecutive terms equal to 0.

It follows that the number of superb binary sequences $b_1, b_2, \ldots, b_{2m+1}$ is equal to the number of ways to choose the two binary sequences $b_4, b_6, \ldots, b_{2m-2}$ and $b_1, b_3, \ldots, b_{2m+1}$, with $m - 2$ terms and $m + 1$ terms respectively, without two consecutive terms equal to 0. We will prove below that the number of binary sequences with k terms that do not contain two consecutive terms equal to 0 is F_{k+2}. Therefore, $B_{2m+1} = F_m F_{m+3}$.

> **Lemma.** The number of binary sequences with k terms that do not contain two consecutive terms equal to 0 is F_{k+2}.

Proof. It is easy to check that the lemma is true for $k = 0, 1, 2, 3$. Now suppose that the lemma is true for $k = n-1$ and $k = n$, where $n \geq 1$. A binary sequence with $n+1$ terms without two consecutive terms equal to 0 must either end in the term 1 or the terms $1, 0$. In the first case, the number of such binary sequences is F_{n+2} by the inductive hypothesis. In the second case, the number of such binary sequences is F_{n+1} by the inductive hypothesis. Therefore, the number of binary sequences with $n+1$ terms that do not contain two consecutive terms equal to 0 is $F_{n+2} + F_{n+1} = F_{n+3}$. So the lemma is true for $k = n+1$ and hence, for all non-negative integers k by induction. □

For B_{2m} to be divisible by 20, we require F_{m+1} to be divisible by 10. Modulo 2, the Fibonacci sequence repeats every three terms as follows.

$$0, 1, 1, 0, 1, 1 \ldots$$

Modulo 5, the Fibonacci sequence repeats every twenty terms as follows.

$$0, 1, 1, 2, 3, 0, 3, 3, 1, 4, 0, 4, 4, 3, 2, 0, 2, 2, 4, 1, 0, 1, 1, \ldots$$

It follows that F_{m+1} is divisible by 10 if and only if $m+1$ is divisible by 15. Therefore, taking $m = 14$ yields $n = 28$ as the smallest even integer greater than 2 such that B_n is divisible by 20.

For B_{2m+1} to be divisible by 20, we require $F_m F_{m+3}$ to be divisible by 20. Since F_m is even if and only if m is divisible by 3, we know that $F_m F_{m+3}$ is divisible by 4 if and only if m is divisible by 3. For $F_m F_{m+3}$ to be divisible by 5, we require m to be divisible by 5 or $m+3$ to be divisible by 5. It follows that $F_m F_{m+3}$ is divisible by 20 if and only if m is divisible by 15 or $m+3$ is divisible by 15. Therefore, taking $m = 12$ yields $n = 25$ as the smallest odd integer greater than 2 such that B_n is divisible by 20.

In conclusion, $n = 25$ is the smallest positive integer greater than 2 such that B_n is divisible by 20.

Solution 2 (Angelo Di Pasquale, Daniel Mathews and Ian Wanless)

Any superb binary sequence X of length $n \geq 6$ takes exactly one of the following forms.

(1) Its middle $n-2$ terms are all 1s.

(2) It is of the form $b_1, b_2, \ldots b_k, 0, \underbrace{1, 1, \ldots, 1}_{n-k-1}$ where $2 \leq k \leq n-3$.

(3) It is of the form $b_1, b_2, \ldots b_k, 0, \underbrace{1, 1, \ldots, 1}_{n-k-1} 0$ where $2 \leq k \leq n-4$.

Note that in (2) and (3), $b_1, b_2, \ldots, b_k$ is a superb sequence if and only if X is.

Observe that (1) yields 4 superb sequences, (2) yields $B_2 + B_3 + \cdots + B_{n-3}$ superb sequences (one for each k), and (3) yields $B_2 + B_3 + \cdots + B_{n-4}$ superb sequences. Therefore,

$$B_n = 4 + 2(B_2 + B_3 + \cdots + B_{n-4}) + B_{n-3}.$$

Replacing n with $n+1$ yields

$$B_{n+1} = 4 + 2(B_2 + B_3 + \cdots + B_{n-3}) + B_{n-2}.$$

Subtracting these two equations yields

$$B_{n+1} - B_n = B_{n-2} + B_{n-3} \qquad \Rightarrow \qquad B_{n+1} = B_n + B_{n-2} + B_{n-3}.$$

By inspection, we find that $B_2 = 1$, $B_3 = 3$, $B_4 = 4$, and $B_5 = 5$.

If we use the recursion $B_{n+1} = B_n + B_{n-2} + B_{n-3}$ to compute the values of $B_i \pmod 4$, we find that, starting from B_2, the sequence cycles $1, 3, 0, 1, 1, 0, \ldots$. Thus, $4 \mid B_i$ if and only if $i \equiv 1 \pmod 3$.

We now use the recursion to compute the values of $B_i \pmod 5$ until we find the first $i \equiv 1 \pmod 3$ for which $5 \mid B_i$. Starting from B_2, the values of the sequence are

$$1, 3, 4, 0, 4, 1, 0, 4, 4, 0, 4, 2, 1, 0, 1, 4, 0, 1, 1, 0, 1, 3, 4, 0,$$

at which point we stop because we have found that $n = 25$.

Solution 3 (Ian Wanless)

We note that each run of 1s in a superb binary sequence has to have length 2 or more. For $n \geq 4$ we partition the sequences counted by B_n into 3 cases.

- Case 1: The first run of 1s has length at least 3.
  In this case, removing one of the 1s in the first run leaves a superb sequence of length $n-1$, and conversely, every such sequence of length $n-1$ can be extended to one of our sequences in a unique way. So there are B_{n-1} sequences in this case.
- Case 2: The first run has length 2 and the first term in the sequence is a 1.
  In this case, the sequence begins 110 and what follows is any one of the B_{n-3} superb sequences of length $n-3$.
- Case 3: The first run has length 2 and the first term in the sequence is a 0.
  In this case, the sequence begins 0110 and what follows is any one of the B_{n-4} superb sequences of length $n-4$.

We conclude that B_n satisfies the recurrence $B_n = B_{n-1} + B_{n-3} + B_{n-4}$ for $n \geq 4$, with initial conditions $B_0 = 1, B_1 = 0, B_2 = 1, B_3 = 3$. From this recurrence, we can calculate the sequence modulo 20 to find it begins

$$1, 0, 1, 3, 4, 5, 9, 16, 5, 19, 4, 5, 9, 12, 1, 15, 16, 9, 5, 16, 1, 15, 16, 13, 9, 0,$$

from which we deduce that the answer is $n = 25$.

Solution 4 (Kevin McAvaney)

All sequences mentioned in this proof are binary.

- Let $A(n)$ be the number of superb sequences with n terms.
- Let $B(n)$ be the number of sequences with n terms that do not contain the strings 000 or 010.
- Let $C(n)$ be the number of sequences with n terms that do not contain the strings 000 or 010 and end in 0.

- Let $D(n)$ be the number of sequences with n terms that do not contain the strings 000 or 010 and end in 1.

The D-sequences end in 11 or 1101 or 11001, so $D(n) = D(n-1) + D(n-3) + D(n-4)$ for $n \geq 5$. The C-sequences end in 110 or 1100, so $C(n) = D(n-2) + D(n-3)$ for $n \geq 4$. For $n \geq 5$, $B(n) = D(n) + C(n) = D(n) + D(n-2) + D(n-3) = D(n+1)$. For $n \geq 7$, the A-sequences have the form $011 \cdots 110$ or $011 \cdots 11$ or $11 \cdots 110$ or $11 \cdots 11$, where the string indicated by dots is a B-sequence. Hence,

$$\begin{aligned} A(n) &= B(n-6) + 2B(n-5) + B(n-4) \\ &= D(n-5) + 2D(n-4) + D(n-3) \\ &= D(n) - D(n-1) + D(n-1) - D(n-2) \\ &= D(n) - D(n-2). \end{aligned}$$

By inspection, we have $D(1) = 1$, $D(2) = 2$, $D(3) = 4$, $D(4) = 6$, and $A(1) = 2$, $A(2) = 1$, $A(3) = 3$, $A(4) = 4$, $A(5) = 5$, $A(6) = 9$. Working modulo 20, we seek the smallest value of $n \geq 7$ for which $D(n) - D(n-2) = 0$. From the following table, we see that it is $n = 25$.

n	$D(n)$
1	1
2	2
3	4
4	6
5	9
6	15
7	5
8	0
9	4

n	$D(n)$
10	4
11	9
12	13
13	1
14	14
15	16
16	10
17	5
18	15

n	$D(n)$
19	1
20	16
21	16
22	12
23	9
24	1
25	9

5. Find all triples (x, y, z) of real numbers that simultaneously satisfy the equations

$$\begin{aligned} xy + 1 &= 2z \\ yz + 1 &= 2x \\ zx + 1 &= 2y. \end{aligned}$$

Solution 1

First, we note that the equations are symmetric in x, y, z, so that permuting a solution to the equations will always yield a solution. Now subtract the first equation from the second to obtain

$$yz - xy = 2x - 2z \qquad \Rightarrow \qquad (y+2)(z-x) = 0 \qquad \Rightarrow \qquad y = -2 \text{ or } z = x.$$

Note that we can similarly deduce that $z = -2$ or $x = y$ as well as $x = -2$ or $y = z$. Let us consider the two cases $y = -2$ and $z = x$ separately.

- *Case 1:* $y = -2$
  The original equations reduce to $-2x + 1 = 2z$, $-2z + 1 = 2x$, and $zx + 1 = -4$. The first of these allows us to write $z = -x + \frac{1}{2}$ and we may substitute this into the third equation to yield

  $$\left(-x + \tfrac{1}{2}\right)x + 1 = -4 \qquad \Rightarrow \qquad 2x^2 - x - 10 = 0 \qquad \Rightarrow \qquad x = \frac{5}{2} \text{ or } x = -2.$$

  Using the fact that $y = -2$ and $z = -x + \frac{1}{2}$, we obtain the solutions $(x, y, z) = (\frac{5}{2}, -2, -2)$ and $(-2, -2, \frac{5}{2})$. By the symmetry observed earlier, we also obtain the solution $(x, y, z) = (-2, \frac{5}{2}, -2)$. It is easy to check that these solutions all satisfy the original equations.
- *Case 2:* $z = x$
  We earlier deduced that $z = -2$ or $x = y$. By the symmetry of the original equations, the case $z = -2$ has already been considered. So it remains to consider when $x = y = z$. In this case, all three equations reduce to the single equation $x^2 + 1 = 2x$, which has the unique solution $x = 1$. Therefore, we obtain the solution $(x, y, z) = (1, 1, 1)$ and it is easy to check that this satisfies the original equations.

Therefore, the only solutions to the equations are given by $(x, y, z) = (\frac{5}{2}, -2, -2)$, $(-2, \frac{5}{2}, -2)$, $(-2, -2, \frac{5}{2})$ and $(1, 1, 1)$.

Solution 2 (Angelo Di Pasquale and Daniel Mathews)

Multiply the first equation by z and rearrange to get

$$2z^2 - z = xyz.$$

Similarly, $2x^2 - x = xyz$ and $2y^2 - y = xyz$. But the quadratic $2w^2 - w = xyz$ has at most two real solutions. So two of x, y, z are equal and we may assume without loss of generality that $z = y$. The equations become

$$xy + 1 = 2y \qquad (1)$$

$$y^2 + 1 = 2x. \qquad (2)$$

From equation (2), we obtain $x = \frac{y^2+1}{2}$. Substiting this into equation (1) yields

$$y^3 - 3y + 2 = 0 \qquad \Leftrightarrow \qquad (y-1)^2(y+2) = 0.$$

If $y = 1$, we find that $x = 1$ and $z = 1$. If $y = -2$, we find that $x = \frac{5}{2}$ and $z = -2$.

Solution 3 (Angelo Di Pasquale)

Put $z = \frac{xy+1}{2}$ into the second and third equations, and tidy up to get

$$xy^2 + y + 2 = 4x \tag{1}$$
$$x^2y + x + 2 = 4y. \tag{2}$$

Solving for x in equation (1) yields $x = -\frac{y+2}{y^2-4}$.

If $y = -2$, then solving equation (2) for x yields $x = -2$ or $x = \frac{5}{2}$. Using the fact that $z = \frac{xy+1}{2}$, we arrive at $(x, y, z) = \left(-2, -2, \frac{5}{2}\right)$ or $\left(\frac{5}{2}, -2, -2\right)$.

If $y \neq -2$, then $x = -\frac{1}{y-2}$. Substituting this into equation (2) yields

$$4y^3 - 18y^2 + 24y - 10 = 0 \qquad \Leftrightarrow \qquad (y-1)^2(2y-5) = 0.$$

Then $y = 1$ leads to $(x, y, z) = (1, 1, 1)$, and $y = \frac{5}{2}$ leads to $(x, y, z) = \left(-2, \frac{5}{2}, -2\right)$.

Solution 4 (Angelo Di Pasquale)

This is a trick for squeezing out a set of three independent equations in terms of the symmetric functions $a = x + y + z$, $b = xy + xz + yz$ and $c = xyz$. Add t to each of the equations, and then multiply the three equations together to get

$$(xy + t + 1)(yz + t + 1)(zx + t + 1) = (2x + t)(2y + t)(2z + t).$$

Expanding this out, substituting in a, b, c for the relevant symmetric expressions in x, y, z, and then writing it as a polynomial in t yields

$$t^2(b + 3 - 2a) + t(ac + 3 - 2b) + c^2 + ac + b + 1 - 8c = 0.$$

Since this is true for all values of t, the above expression must be the zero polynomial. Hence,

$$b + 3 - 2a = 0 \tag{1}$$
$$ac + 3 - 2b = 0 \tag{2}$$
$$c^2 + ac - 8c + b + 1 = 0. \tag{3}$$

Substituting $b = 2a - 3$ from equation (1) into equations (2) and (3) yields

$$a(c-4) = -9 \tag{4}$$
$$c^2 - 8c - 2 + a(c+2) = 0. \tag{5}$$

Multiplying equation (5) by $c - 4$ and using equation (4) yields the following cubic after tidying up.

$$c^3 - 12c^2 + 21c - 10 = 0 \qquad \Leftrightarrow \qquad (c-1)^2(c-10) = 0$$

The case $c = 1$ implies $a = 3$ and $b = 3$. So by Vieta's formulas, x, y, z are the three zeros of the cubic $w^3 - 3w^2 + 3w + 1 = (w - 1)^3$. Therefore, $(x, y, z) = (1, 1, 1)$.

The case $c = 10$ implies $a = -\frac{3}{2}$ and $b = -6$. So by Vieta's formulas, x, y, z are the three zeros of the cubic $w^3 + \frac{3}{2}w^2 - 6w - 10 = (w+2)^2\left(w - \frac{5}{2}\right)$. Therefore, $(x, y, z) = \left(-2, -2, \frac{5}{2}\right)$ and its permutations.

Solution 5 (Alan Offer)

Put $(x, y, z) = (a + 1, b + 1, c + 1)$. Then the given equations become

$$ab + a + b = 2c \tag{1a}$$

$$bc + b + c = 2a \tag{1b}$$

$$ca + c + a = 2b. \tag{1c}$$

Let $A = a + b + c$, $B = ab + bc + ca$ and $C = abc$. Then adding the equations (1a), (1b), (1c) together gives $B + 2A = 2A$, so $B = 0$. Consequently, $f(u) = (u - a)(u - b)(u - c) = u^3 - Au^2 - C$. Also, $a^2 + b^2 + c^2 = A^2 - 2B = A^2$.

Adding a to both sides of equation (1b) and multiplying the result by a gives (together with similar results obtained from equations (1a) and (1c))

$$C + Aa = 3a^2 \tag{2a}$$

$$C + Ab = 3b^2 \tag{2b}$$

$$C + Ac = 3c^2. \tag{2c}$$

Adding these together gives $3C + A^2 = 3(a^2 + b^2 + c^2) = 3A^2$, so $3C = 2A^2$.

Since $f(a) = 0$, we have $a^3 = Aa^2 + C$. Hence, multiplying equation (2a) by a produces $Ca + Aa^2 = 3a^3 = 3Aa^2 + 3C$. Simplified, this becomes (together with similar results obtained from equations (2b) and (2c))

$$Ca = 2Aa^2 + 3C$$

$$Cb = 2Ab^2 + 3C$$

$$Cc = 2Ac^2 + 3C.$$

Adding these together and recalling that $a^2 + b^2 + c^2 = A^2$, we find that $CA = 2A^3 + 9C$. Multiplying by 3 and using the fact that $3C = 2A^2$, this becomes $2A^3 = 6A^3 + 18A^2$, and so $A^2(2A + 9) = 0$. It follows that either $A = 0$ or $A = -\frac{9}{2}$.

If $A = 0$, then $f(u) = u^3$, so $a = b = c = 0$.

If $A = -\frac{9}{2}$, then $2f(u) = 2u^3 + 9u^2 - 27 = (u + 3)^2(2u - 3)$, so two of a, b, c are equal to -3 while the third is equal to $\frac{3}{2}$.

For the original system of equations, this yields the solutions

$$(x, y, z) \in \left\{(1, 1, 1), (\tfrac{5}{2}, -2, -2), (-2, \tfrac{5}{2}, -2), (-2, -2, \tfrac{5}{2})\right\},$$

and substitution verifies that these are indeed solutions.

Solution 6 (Chaitanya Rao)

We consider the three cases $x > y$, $x < y$ and $x = y$.

- Case 1: If $x > y$, the second and third equations lead to $yz+1 > zx+1$ or $z(y-x) > 0$. Since $y - x < 0$ this implies $z < 0$. From the first equation this in turn implies that $xy + 1 < 0$, so x and y are of opposite sign. We conclude that $x > 0 > y$ and $z < 0$.

  By symmetry of the equations, we can use a similar argument to show that if any variable is greater than another, then the third variable must be negative. This means that either of the assumptions $y > z$ or $y < z$ lead to the contradictory statement that $x < 0$, so we have that $x > 0 > y = z$. The given equations then become $xy + 1 = 2y$ and $y^2 + 1 = 2x$. Multiplying the second of these by y and using the first equation gives $y^3 + y = 2xy = 4y - 2$ or $(y - 1)^2(y + 2) = 0$. The only negative root is $y = -2$ and so $x = \frac{y^2+1}{2} = \frac{5}{2}$. Therefore, we have the solution $(x, y, z) = (\frac{5}{2}, -2, -2)$.

- Case 2: If $x < y$, interchange x and y in Case 1 to obtain the solution $(x, y, z) = (-2, \frac{5}{2}, -2)$.

- Case 3: If $x = y$, we proceed similarly to the last part of Case 1, obtaining the equations $xz + 1 = 2x$ and $x^2 + 1 = 2z$, from which $(x - 1)^2(x + 2) = 0$ and so $x = y = 1$ or $x = y = -2$. Hence, $z = \frac{x^2+1}{2}$ is equal to 1 or $\frac{5}{2}$. This gives the solutions $(x, y, z) = (1, 1, 1)$ or $(-2, -2, \frac{5}{2})$.

We end up with four solutions: $(x, y, z) = (\frac{5}{2}, -2, -2), (-2, \frac{5}{2}, -2), (-2, -2, \frac{5}{2})$ and $(1, 1, 1)$. It is easily checked that each of these satisfies the original system of equations.

6. Let a, b, c be positive integers such that $a^3 + b^3 = 2^c$.

   Prove that $a = b$.

Solution 1

Note that a and b must have the same parity. If a and b are even and $a^3 + b^3$ is a power of two, then $(\frac{a}{2})^3 + (\frac{b}{2})^3$ is also a power of two. But since $\frac{a}{2}$ and $\frac{b}{2}$ are positive integers, $(\frac{a}{2})^3 + (\frac{b}{2})^3$ is of the form 2^d, where d is a positive integer. So if there are distinct positive integers whose cubes sum to a power of two, then one can repeatedly divide them by two to obtain distinct positive odd integers whose cubes sum to a power of two.

So suppose now that a and b are odd. Rewrite the equation as $(a + b)(a^2 - ab + b^2) = 2^c$, which implies that there are non-negative integers m and n such that

$$a + b = 2^m$$
$$a^2 - ab + b^2 = 2^n.$$

Since $a^2 - ab + b^2$ is odd, we must have $n = 0$ and it follows that $a + b = 2^c = a^3 + b^3$. However, $a+b \leq a^3+b^3$ with equality if and only if $a = b = 1$. Therefore, the only solution to $a^3 + b^3 = 2^c$ with a and b odd is $(a, b, c) = (1, 1, 1)$. It follows that the only solutions to $a^3 + b^3 = 2^c$ must have $a = b$.

Solution 2 (Angelo Di Pasquale)

Let n be the greatest non-negative integer such that $2^n \mid a$ and $2^n \mid b$. Write$a = 2^n A$ and $b = 2^n B$ for positive integers A and B. Then we have $2^{3n}(A^3 + B^3) = 2^c$, where at least one of A and B is odd. Since $2^{3n} \mid 2^c$, we have $c = 3n + d$ for some non-negative integer d, so $A^3 + B^3 = 2^d$. Since $A, B \geq 1$, we have $d \geq 1$, so $A + B$ is even. Since at least one of A and B is odd, we conclude that both are odd.

So we have $2^d = (A + B)(A^2 - AB + B^2)$. Since $2^d, A + B > 0$, then we also have $A^2 - AB + B^2 > 0$. But $A^2 - AB + B^2$ is odd and a factor of 2^d, so $A^2 - AB + B^2 = 1$.

If $A > B$, then $A^2 - AB + B^2 = A(A - B) + B^2 \geq A + B^2 \geq 2$, so this case does not occur. Similarly, $A < B$ does not occur.

If $A = B$, it follows that $A = B = 1$, and so $a = b$.

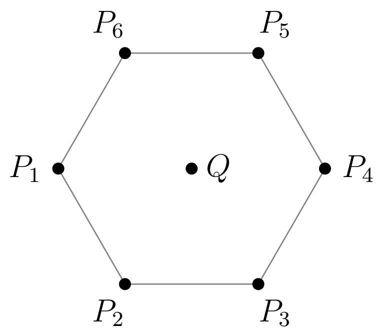
7. Each point in the plane is assigned one of four colours.

   Prove that there exist two points at distance 1 or $\sqrt{3}$ from each other that are assigned the same colour.

Solution 1

Suppose that the four colours are blue, red, yellow, and green. We argue by contradiction and suppose that there do not exist two points at distance 1 or $\sqrt{3}$ from each other that have the same colour.

Pick a point P_1 in the plane and suppose that it is coloured blue, without loss of generality. Construct a regular hexagon $P_1P_2P_3P_4P_5P_6$ with side length 1 and centre Q. Note that the points P_1, P_2, P_6, Q must be coloured differently. So suppose without loss of generality that Q is coloured red, P_2 is coloured yellow, and P_6 is coloured green.

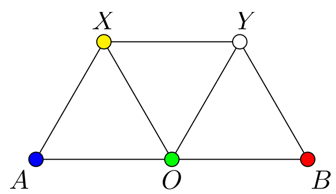


Now note that P_1, P_6, P_5, Q must be coloured differently, which forces P_5 to be yellow. Similarly, P_6, P_5, P_4, Q must be coloured differently, which forces P_4 to be blue. It follows that any point at distance 2 from P_1 must be coloured blue. In other words, there is a circle of radius 2 that is coloured blue. However, there exists a chord on this circle of length 1, which forces two points at distance 1 that are the same colour. This contradicts our original assumption, so it follows that there exist two points at distance 1 or $\sqrt{3}$ from each other that are the same colour.

Solution 2 (Angelo Di Pasquale)

Suppose that the four colours are blue, red, yellow, and green. We argue by contradiction and suppose that there do not exist two points at distance 1 or $\sqrt{3}$ from each other that have the same colour.

Consider an isosceles triangle ABC with $BC = 1$ and $AB = AC = 2$. Since B and C must be different colours, one of them is coloured differently to A. Without loss of generality, A is blue and B is red. Orient the plane so that AB is a horizontal segment.



Let O be the midpoint of AB. Then as $AO = BO = 1$, O is not blue or red. Without loss of generality, O is green. Let X be the point above line AB so that $\triangle AOX$ is equilateral. It is easy to compute that $XB = \sqrt{3}$ and $XA = XO = 1$. Hence, X is not red, blue or green, and must be yellow. Finally, let Y be the point above line AB so that $\triangle BOY$ is equilateral. Then it is easy to compute that $YX = YO = YB = 1$ and $YA = \sqrt{3}$. Hence Y cannot be any of the four colours, giving the desired contradiction.

8. Three given lines in the plane pass through a point P.

   (a) Prove that there exists a circle that contains P in its interior and intersects the three lines at six points A, B, C, D, E, F in that order around the circle such that $AB = CD = EF$.

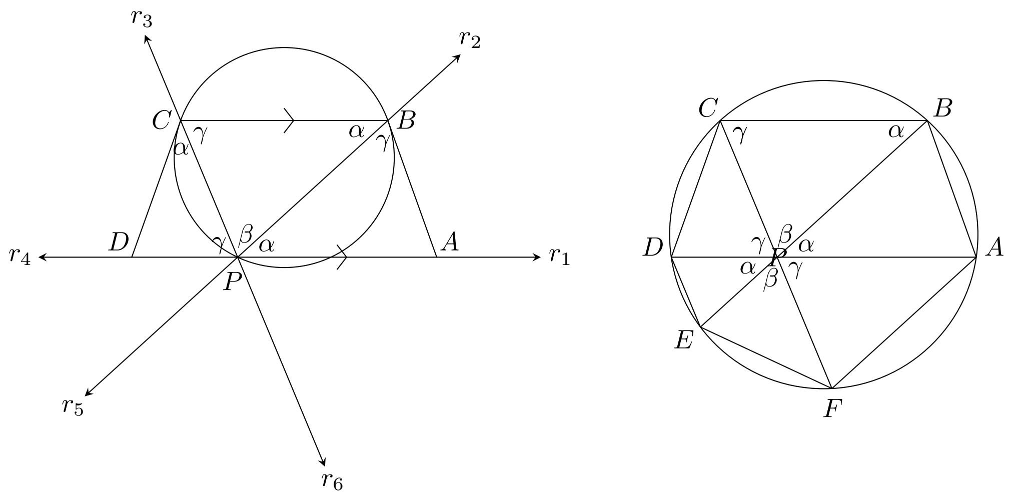
   (b) Suppose that a circle contains P in its interior and intersects the three lines at six points A, B, C, D, E, F in that order around the circle such that $AB = CD = EF$. Prove that

$$\frac{1}{2}\,\text{area(hexagon } ABCDEF) \geq \text{area}(\triangle APB) + \text{area}(\triangle CPD) + \text{area}(\triangle EPF).$$

Solution 1 (Angelo Di Pasquale)

(a) Let r_1, r_2, r_3, r_4, r_5, r_6 be the rays in order emanating from P along the lines. Note that the union of r_1 and r_4 is one of the three given lines. The same holds for r_2 and r_5, as well as for r_3 and r_6.

Let point B be chosen arbitrarily on r_2. Then locate C on r_3 so that $BC \parallel r_1$. Next, let the tangent at B to circle BPC intersect r_1 at A. (If C_1 is any point on the ray CB beyond B, then the tangent at B lies in between the rays BC_1 and BP, and hence it really does intersect r_1, rather than r_4.) Similarly, let the tangent at C to circle BPC intersect r_4 at D. Let $\alpha = \angle APB$, $\beta = \angle BPC$ and $\gamma = \angle CPD$. Then by the alternate segment theorem and the fact that $BC \parallel AD$ we have $\angle DCP = \angle CBP = \alpha$ and $\angle PBA = \angle PCB = \gamma$. Since $\alpha + \beta + \gamma = 180°$ we may use the angle sum in triangles CPD and APB to deduce that $\angle PDC = \angle BAP = \beta$. Hence, $ABCD$ is an isosceles trapezium with $AB = CD$ and $ABCD$ is cyclic.



Let the lines BP and CP intersect circle $ABCD$ for a second time at points E and F, respectively. Note that P lies inside circle $ABCD$ because it lies on segment AD. Thus E is on r_5 and F is on r_6. We have $\angle EDA = \angle EBA = \gamma$. Hence, $\angle EDC = \gamma + \beta = 180° - \alpha = 180° - \angle DCP$ and so $DE \parallel CP$. It follows that $DE \parallel CF$, which implies that $CDEF$ is an isosceles trapezium with $CD = EF$. Hence, circle $ABCDEF$ has the required properties.

(b) As in part (a), let $\alpha = \angle APB = \angle DPE$, $\beta = \angle BPC = \angle EPF$ and $\gamma = \angle CPD = \angle FPA$.

Since $AB = CD$, it follows that $ABCD$ is an isosceles trapezium with $AD \parallel BC$. Hence $\angle CBP = \alpha$ and $\angle PCB = \gamma$. Since $BCEF$ is cyclic we have $\angle PFE = \angle CBP = \alpha$ and $\angle FEP = \angle PCB = \gamma$. Similarly $CF \parallel DE$ and $AF \parallel BE$ which lead to $\angle PBA = \angle EDP = \gamma$, $\angle BAP = \angle PED = \beta$, $\angle DCP = \angle PAF = \alpha$ and $\angle PDC = \angle AFP = \beta$.

Thus triangles PAB, BPC, CDP, PED, FPE and AFP are similar.

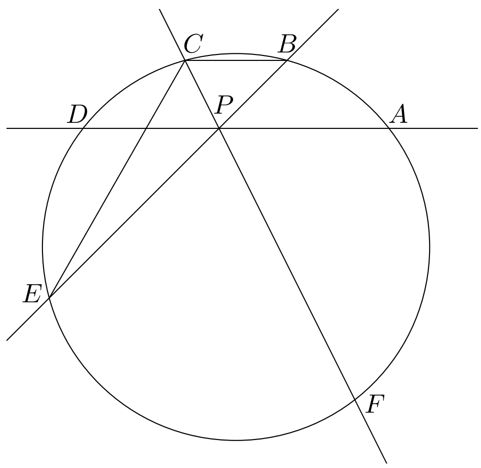
Let $x = BC$, $y = CP$ and $z = BP$. Then since $z : y : x = BP : PC : CB = CD : DP : PC$, we have $DP = \frac{y^2}{x}$ and $EF = CD = \frac{yz}{x}$. Since $z : x = BP : CB = PA : BP$, we have $PA = \frac{z^2}{x}$. Since the ratio of areas of similar figures is the square of the ratio of corresponding lengths we have

$$\begin{aligned}
&|PAB| : |BPC| : |CDP| : |PED| : |FPE| : |AFP| \\
&= PB^2 : BC^2 : CP^2 : PD^2 : FE^2 : AP^2 \\
&= z^2 : x^2 : y^2 : \frac{y^4}{x^2} : \frac{y^2z^2}{x^2} : \frac{z^4}{x^2} \\
&= z^2x^2 : x^4 : x^2y^2 : y^4 : y^2z^2 : z^4.
\end{aligned}$$

The inequality to be proved is equivalent to $|BPC| + |PED| + |AFP| \geq |PAB| + |CDP| + |FPE|$. Thus, it suffices to show that $x^4 + y^4 + z^4 \geq x^2y^2 + y^2z^2 + z^2x^2$. However, this is equivalent to $(x^2 - y^2)^2 + (y^2 - z^2)^2 + (z^2 - x^2)^2 \geq 0$. (Alternatively we may use the rearrangement inequality or the Cauchy–Schwarz inequality.)

Solution 2 (Angelo Di Pasquale)

(a) Let $r_1, r_2, r_3, r_4, r_5, r_6$ be as in Solution 1. Let $B \in r_2$ and $C \in r_3$ be fixed points such that $BC \parallel r_1$. Let E be a variable point on r_5. Consider the family of circles passing through points B, C and E. Let A, D and F be the intersection points of circle BCE with rays r_1, r_4 and r_6, respectively. Then $BC \parallel AD$. Thus $ABCD$ is an isosceles trapezium with $AB = CD$.



Consider the ratio $r = \frac{EF}{AB}$ as E varies on ray r_5. As E approaches P, AB approaches $\min\{BP, CP\}$ while EF approaches 0. Hence, r approaches 0.

As E diverges away from P, $\angle BEC$ approaches 0. Hence $\angle ECF = \angle BPC - \angle BEC$ approaches $\angle BPC$ and $\angle ADB$ approaches 0. Thus, eventually $\angle ECF > \angle ADB$ and so $r > 1$.

Since r varies continuously with E, we may apply the intermediate value theorem to deduce that there is a position for E such that $r = 1$. The circle BCE now has the required property.

(b) As in Solution 1, we deduce that triangles AFP, PAB, BPC are similar. Hence,

$$\begin{aligned} \frac{|AFP|}{|PAB|} \cdot \frac{|BPC|}{|PAB|} &= \frac{AP^2}{PB^2} \cdot \frac{PB^2}{PA^2} = 1 \\ \Rightarrow \qquad |APB| &= \sqrt{|FPA| \cdot |BPC|} \le \frac{1}{2}|FPA| + \frac{1}{2}|BPC|, \end{aligned}$$

where we have used the AM–GM inequality in the last line. Adding this to the two analogously derived inequalities $|CPD| \le \frac{1}{2}|BPC| + \frac{1}{2}|DPE|$ and $|EPF| \le \frac{1}{2}|DPE| + \frac{1}{2}|FPA|$ yields the result.

Solution 3 (Ivan Guo)

Solution to part (b) only.

Similar to Solution 2, it suffices to prove that

$$|APF| + |BPC| \ge 2|APB|,$$

since we can add the analogous inequalities together to get the required result.

Let AF and BC intersect at X. From part (a) of Solution 1, we know that the triangles APF, BPC and XCF are all similar. Furthermore, triangles XAB and APB are congruent. So it suffices to prove that

$$|APF| + |BPC| \ge \frac{1}{2}|XCF|.$$

Since all three triangles are similar, their areas are proportional to the squares of their bases. So we would like to show that

$$FP^2 + PC^2 \ge \frac{1}{2}(FP + PC)^2.$$

This is true since the inequality rearranges to $\frac{1}{2}(FP - PC)^2 \ge 0$.

Solution 4 (Daniel Mathews)

(a) As in Solution 1, label the rays $r_1, r_2, r_3, r_4, r_5, r_6$. Let the angle between rays r_1 and r_2 (respectively, r_2 and r_3, r_3 and r_4) be a (respectively, b, c), so that $a+b+c = 180°$. Construct points A, B, C, D, E, F on $r_1, r_2, r_3, r_4, r_5, r_6$ respectively so that

$$\begin{aligned} PA &= 1 & PD &= \frac{\sin^2 a}{\sin^2 c} \\ PB &= \frac{\sin b}{\sin c} & PE &= \frac{\sin^2 a}{\sin b \sin c} \\ PC &= \frac{\sin a \sin b}{\sin^2 c} & PF &= \frac{\sin a}{\sin b}. \end{aligned}$$

Consider triangle PAB. We have $\angle APB = a$, so $\angle PBA + \angle PAB = b + c$. Moreover, the sine rule yields $\frac{\sin \angle PAB}{\sin \angle PBA} = \frac{PB}{PA} = \frac{\sin b}{\sin c}$. It follows that $\angle PAB = b$ and $\angle PBA = c$. Moreover, we have $\frac{AB}{PA} = \frac{\sin APB}{\sin PBA} = \frac{\sin a}{\sin c}$, so $AB = \frac{\sin a}{\sin c}$.

Similarly, we can compute all the angles in triangles PBC, PCD, PDE, PEF, PFA. We find they are all similar, each with angles a, b, c. We find that $\angle ADC = \angle AFC = 180° - \angle ABC$ and $\angle BED = \angle BAD = 180° - \angle BCD$, so that $ABCDEF$ is cyclic. We also calculate $AB = CD = EF = \frac{\sin a}{\sin c}$. Thus, the circle through $ABCDEF$ satisfies the given conditions.

Moreover, any circle satisfying these conditions has this form once we specify PA to have unit length. For if A, B, C, D, E, F are as required, then we can deduce that AB is parallel to $r_3 r_6$, CD is parallel to $r_2 r_5$, and EF is parallel to $r_1 r_4$. We can then show that all angles must be as found above, and then, by the sine rule, if we set $PA = 1$, then all lengths PA, PB, PC, PD, PE, PF are as in the construction.

(b) Using the lengths and angles constructed above, we can compute the areas of the six triangles $PAB, PBC, PCD, PDE, PEF, PFA$ in terms of $\sin a$, $\sin b$ and $\sin c$. For instance, $2|PAB| = PA.PB \sin a = \frac{\sin b . \sin a}{\sin c}$. Writing $p = \sin a$, $q = \sin b$, $r = \sin c$, we then have

$$\begin{aligned} 2|PAB| &= \frac{pq}{r} & \qquad 2|PDE| &= \frac{p^5}{qr^3} \\ 2|PBC| &= \frac{pq^3}{r^3} & \qquad 2|PEF| &= \frac{p^3}{qr} \\ 2|PCD| &= \frac{p^3 q}{r^3} & \qquad 2|PFA| &= \frac{pr}{q}. \end{aligned}$$

The required inequality can also be written as

$$|PAB| + |PCD| + |PEF| \leq |PBC| + |PDE| + |PFA|,$$

which, after substituting the areas as above, clearing denominators and cancelling common factors, is equivalent to

$$q^2 r^2 + p^2 q^2 + p^2 r^2 \leq q^4 + p^4 + r^4.$$

This inequality follows from the rearrangement inequality or the Cauchy–Schwarz inequality.