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THE ABSOLUTE ARITHMETIC CONTINUUM AND THE UNIFICATION OF ALL NUMBERS GREAT AND SMALL*

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The mathematical continuum, like number, consists of mere possibility ...

G. W. Leibniz
Die philosophischen Schriften
Band II, p. 475

Abstract. In his monograph *On Numbers and Games*, J. H. Conway introduced a real-closed field containing the reals and the ordinals as well as a great many less familiar numbers including $-\omega$, $\omega/2$, $1/\omega$, $\sqrt{\omega}$ and $\omega-\pi$ to name only a few. Indeed, this particular real-closed field, which Conway calls **No**, is so remarkably inclusive that, subject to the proviso that numbers—construed here as members of ordered fields—be individually definable in terms of sets of NBG (von Neumann–Bernays–Gödel set theory with global choice), it may be said to contain "All Numbers Great and Small." In this respect, **No** bears much the same relation to ordered fields that the system $\mathbb{R}$ of real numbers bears to Archimedean ordered fields.

In Part I of the present paper, we suggest that whereas $\mathbb{R}$ should merely be regarded as constituting an arithmetic continuum (modulo the Archimedean axiom), **No** may be regarded as a sort of absolute arithmetic continuum (modulo NBG), and in Part II we draw attention

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to the unifying framework **No** provides not only for the reals and the ordinals but also for an array of non-Archimedean ordered number systems that have arisen in connection with the theories of non-Archimedean ordered algebraic and geometric systems, the theory of the rate of growth of real functions and nonstandard analysis.

In addition to its inclusive structure as an ordered field, the system **No** of surreal numbers has a rich algebraico-tree-theoretic structure—a simplicity hierarchical structure—that emerges from the recursive clauses in terms of which it is defined. In the development of **No** outlined in the present paper, in which the surreals emerge vis-à-vis a generalization of the von Neumann ordinal construction, the simplicity hierarchical features of **No** are brought to the fore and play central roles in the aforementioned unification of systems of numbers great and small and in some of the more revealing characterizations of **No** as an absolute continuum.

Contents

Introduction

"*Bridging the gap between the domains of discreteness and of continuity*, or between arithmetic and geometry is a central, presumably even *the* central problem of the foundations of mathematics." So wrote Abraham Fraenkel, Yehoshua Bar-Hillel and Azriel Lévy in their mathematico-philosophical classic *Foundations of Set Theory* [1973, pp. 211–212]. Cantor and Dedekind of course believed they had bridged the gap with the creation of their *arithmetico-set theoretic continuum* of real numbers, and for roughly a century now (despite a host of constructivist, predicativist and infinitesimalist challenges)[1] it remains one of the central tenets of standard mathematical philosophy that indeed they had. In accordance with this view the *geometric linear continuum* is assumed to be isomorphic with the arithmetic continuum, the axioms of geometry being so selected to ensure this would be the case. Given the Archimedean nature of the real number system, once this assumption is adopted we have the classic result of standard mathematical philosophy that infinitesimals are superfluous to the analysis of the structure of a continuous straight line.

More than two decades ago, however, we began to suspect that while the Cantor–Dedekind theory succeeds in bridging the gap between the domains of arithmetic and of standard Euclidean geometry, it only reveals a glimpse of a far richer theory of continua that not only allows for infinitesimals but leads to a vast generalization of portions of Cantor's theory of the infinite, a generalization that also provides a setting for Abraham Robinson's infinitesimal approach to analysis as well as for the non-Cantorian theories of the infinite (and infinitesimal) pioneered by Giuseppe Veronese [1891; 1894], Tullio Levi-Civita [1892–1893; 1898], David Hilbert [1899; 1971] and Hans Hahn [1907] in connection with their work on non-Archimedean ordered algebraic and geometric systems, and by Paul du Bois-Reymond [1870–1871; 1875; 1877; 1882], Otto Stolz [1883; 1885], Felix Hausdorff [1907; 1909; 1914] and G. H. Hardy [1910; 1912] in connection with their work on the rate of growth of real functions, theories that have been enjoying a resurgence in interest in recent decades. Central to our theory is J. H. Conway's theory of *surreal numbers* [1976; 2001], and the present author's amplifications and generalizations thereof, and other contributions thereto [Ehrlich 1988; 1989; 1989a; 1992; 1994a; 2001; 2001 (with van den Dries); 2002a; 2011].

In a number of earlier works [Ehrlich 1987; 1989a; 1992; 2005; forthcoming 1], we suggested that *whereas the real number system should be regarded as constituting an arithmetic continuum modulo the Archimedean axiom, the system of surreal numbers* (*henceforth*, **No**) *may be regarded as a sort of absolute arithmetic continuum modulo NBG* (von Neumann–Bernays–Gödel set theory with global choice).[2] In Part I of the present paper we will outline some of the properties of the system of surreal numbers we believe lend

[1] For a survey of many of the constructivist, predicativist and infinitesimalist challenges together with numerous references, see the author's [2005]; for the historical background of several of the infinitesimalist challenges, see the author's [2006] and [forthcoming 2]; and for additional references to infinitesimalist approaches not referred to in the just-mentioned works, see the author's [2007].

[2] In a related work [Ehrlich 2010], we also introduce a formal replacement for the extension of the classical linear continuum sketched by Charles Sanders Peirce [1897; 1898; 1898a; 1900] at the turn of the twentieth century, and show that by limiting **No** to its substructure consisting of its finite and infinitesimal members, one obtains a model of this *Peircean linear continuum*, as we call it.

credence to this mathematico-philosophical thesis, and in Part II we will draw attention to the unifying framework the surreals provide for the reals and the ordinals as well as for the various other sorts of systems of numbers great and small alluded to above.

In the construction of **No** presented below, which differs markedly from Conway's construction, the surreals emerge vis-à-vis a generalization of the von Neumann ordinal construction. In accordance with this construction, each surreal number x emerges as an *orderly partition* (L, R) (see §3.1 for definition) of the set of all surreal numbers that are *simpler than* x (i.e., that are predecessors of x) in the full binary surreal number tree $\langle \mathbf{No}, <_s \rangle$. Although L and R are defined independently of the lexicographic (total) ordering $<$ that is defined on $\langle \mathbf{No}, <_s \rangle$, they are found to coincide with the sets of all surreal numbers that are *simpler than x and less than x* and *simpler than x and greater than x*, respectively, in the lexicographically ordered full binary surreal number tree $\langle \mathbf{No}, <, <_s \rangle$. **No**'s ordinals (whose arithmetic in **No** is different than the familiar Cantorian arithmetic) are identified with the members of the "rightmost" branch of $\langle \mathbf{No}, <, <_s \rangle$, i.e., the unique initial subtree of **No** that is a well-ordered proper class with respect to the lexicographic order on **No**, **No**'s system of reals is identified with the unique initial subtree of **No** that is a Dedekind complete ordered field, and the remaining number systems that are the focus of the second part of the paper likewise emerge as *initial substructures* of **No**—substructures of **No** that are initial subtrees of **No**. Central to their emergence as initial substructures is the fact that the sums and products of any two members of **No** are defined as the *simplest* elements of **No** consistent with **No**'s intended structure as an ordered group and an ordered field, respectively (see §4 below). It is this *simplicity hierarchical structure*—or *s-hierarchical structure*, as we call it [Ehrlich 2001; 1994]—together with **No**'s fullness as a binary tree that underwrites the theorems that are central to the portrayal of **No** as a *unifying, s-hierarchical absolute arithmetic continuum* presented below.

Figure 1 below offers a glimpse of the some of the early stages of the recursive unfolding of this s-hierarchical continuum, where (as the notation suggests) ω is the least infinite ordinal, $-\omega$ is the additive inverse of ω, $1/\omega$, is the multiplicative inverse of ω, $\omega/2$ is ω divided by 2, and so on. In §5, we will present Conway's characterization of the recursive unfolding in **No** of the classical arithmetic continuum; for a complete characterization of the recursive unfolding of the s-hierarchical absolute arithmetic continuum, more generally, see the author's [2011].

Throughout the paper the underlying set theory is assumed to be NBG, and as such by *class* we mean set or proper class, the latter of which, in virtue of the axioms of foundation and global choice, always has the "cardinality" of the class *On* of all ordinals. For details on formalizing the theory of surreal numbers in NBG, which is a conservative extension of ZFC, we refer

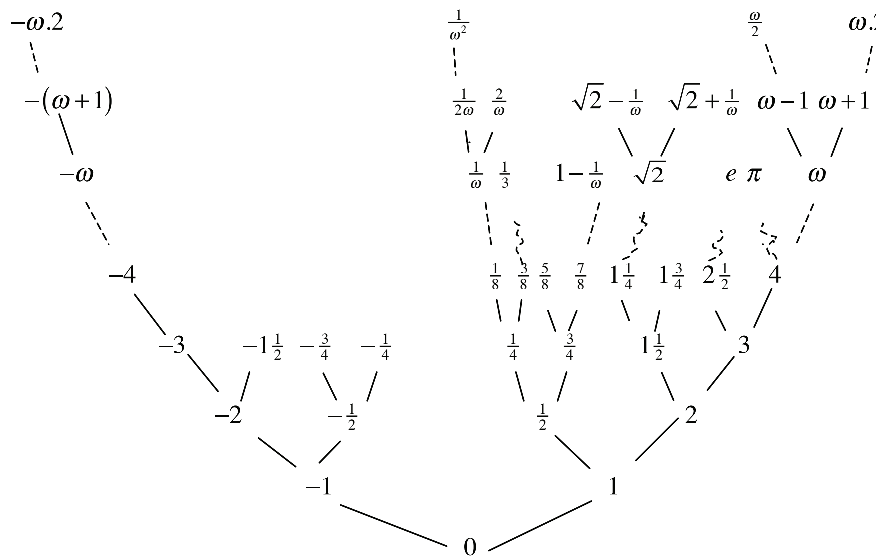

FIGURE 1

the reader to [Ehrlich 1989]. Readers seeking additional background in the theory of surreal numbers more generally may consult the author's [2001 and forthcoming 1] and the works of Conway [1976; 2001], Gonshor [1986] and Alling [1987].

Part I. The absolute arithmetic continuum

§1. All numbers great and small. In addition to the reals and the ordinals, the system of surreal numbers embraces a wide array of less familiar numbers, including $-\omega$, $1/\omega$, $\omega/2$, e^{ω}, $\log\omega$, $\sin(1/\omega)$ and

$$\sqrt[3]{\omega+1}-\frac{\pi}{\omega}$$

to name only a few.[3] Indeed, this particular real-closed field, which following Conway we call **No**, is so remarkably inclusive that, subject to the proviso that numbers—construed here as members of ordered fields—be individually

[3] Following Conway [2001, p. 228], we write e^{ω} for $\exp\omega$, where exp is the well-behaved exponentiation for surreal numbers discovered by Martin Kruskal and developed (using Kruskal's "hints") by Harry Gonshor [Gonshor 1986, Ch. 10]; by log we mean the inverse of exp; and by $\sin(1/\omega)$ we mean $\sum_{n=0}^{\infty}(-1)^n\cdot(1/\omega)^{2n+1}/(2n+1)!$, which is the familiar generalization of the sin function for infinitesimals applied to $1/\omega$. For exponentials and logarithms of surreal numbers, see [Gonshor 1986, Ch. 10] and [van den Dries and Ehrlich 2001]; for restricted trigonometric functions of surreal numbers, see [Alling 1987, 7.5], and for restricted analytic functions of surreal numbers more generally, see §8 of the present paper and [van den Dries and Ehrlich 2001].

definable in terms of sets of NBG, it may be said to contain *all numbers great and small*! In this respect, **No** bears much the same relation to ordered fields that the system $\mathbb{R}$ of real numbers bears to Archimedean ordered fields. This can be made precise by saying that:

THEOREM 1 (Ehrlich 1988; 1989; 1989a; 1992). *Whereas* $\mathbb{R}$ *is* (*up to isomorphism*) *the unique* homogeneous universal Archimedean ordered field, **No** *is* (*up to isomorphism*) *the unique* homogeneous universal ordered field.[4]

Since there is a multitude of real-closed ordered fields, it is natural to inquire if, like $\mathbb{R}$, it is possible to distinguish **No** (to within isomorphism) from the remaining real-closed ordered fields by appealing solely to its order. As we shall now see, the following definition, where the notation "$L < R$" indicates that every member of L precedes every member of R, enables one to do just that.

DEFINITION 1 (Ehrlich 1987). An ordered class $\langle A, < \rangle$ will be said to be an *absolute linear continuum* if for all sub*sets* L and R of A where $L < R$ there is a $y \in A$ such that $L < \{y\} < R$.[5]

An absolute linear continuum $\langle A, < \rangle$ is both *absolutely dense* in the sense that for each pair of nonempty subsets L and R of A where $L < R$, there is a $y \in A$ such that $L < \{y\} < R$, and *absolutely extensive* in the sense that given any (possibly empty) subset L of A there are members a and b of A that are respectively smaller than and greater than every member of L. In fact, since in the definition of an absolute linear continuum L or R may be empty, one can readily show that an ordered class is an absolute linear continuum if and only if it has both of the just-stated properties. Accordingly, since every element of an ordered class must either lie between two of its nonempty subclasses or be greater than or less than every member of some (possibly empty) subclass, these conditions collectively ensure that *absolute linear continua have no order-theoretic limitations that are definable in terms of sets of standard set theory*.

In his *Contributions to the Founding of the Theory of Transfinite Numbers* [1895, §11], Cantor provided a non-metrical characterization of a closed

[4]For the purpose of this paper, an ordered field (Archimedean ordered field) A is said to be *homogeneous universal* if it is *universal*—every ordered field (Archimedean ordered field) whose universe is a class of NBG can be embedded in A—and it is *homogeneous*—every isomorphism between subfields of A whose universes are sets can be extended to an automorphism of A. Since model theorists frequently use the above italicized terms in more general senses (cf. [Jónsson 1960] and [Morley and Vaught 1962]), in the model-theoretic settings of the author's [1989] and [1992] the terms *absolutely homogeneous universal*, *absolutely universal*, and *absolutely homogeneous* are respectively employed in their steads.

[5]In [Ehrlich 1988; 1992; 1994] and a number of other works, we refer to absolute linear continua as "η_{On}-orderings" since they extend to proper classes Hausdorff's [1907; 1914] idea of an η_α*-ordering*, that is, an ordered set A such that for all subsets L and R of A where $L < R$ and $|L|, |R| < \aleph_\alpha$ there is a member of A lying strictly between those of L and those of R.

interval of the *classical linear continuum* and showed that a closed interval of $\mathbb{R}$ is (up to isomorphism) the unique such structure. From the latter one readily obtains the familiar categorical characterization of the ordered set $\mathbb{R}$ itself. The following is the analog of the latter result for absolute linear continua.

THEOREM 2 (Ehrlich 1988). $\langle \mathbf{No}, < \rangle$ *is* (*up to isomorphism*) *the unique absolute linear continuum*.

Unlike the ordered field of reals, however, the ordered field of surreal numbers is not distinguished (up to isomorphism) from other ordered fields by its structure as an ordered class. Indeed, there are infinitely many pairwise non-isomorphic ordered fields that are absolute linear continua. Happily, however, what one can prove is

THEOREM 3 (Ehrlich 1988). **No** *is* (*up to isomorphism*) *the unique real-closed ordered field that is an absolute linear continuum*.

An ordered field is real-closed if and only if it admits no extension to a more inclusive ordered field that results from supplementing the field with solutions to polynomial equations with coefficients in the field. Accordingly, in virtue of Theorem 3, **No** is not only devoid of set-theoretically defined order-theoretic limitations, it is devoid of algebraic limitations as well; moreover, to within isomorphism, it is the unique ordered field that is devoid of both types of limitations or '*holes*', as they might more colloquially be called. That is, **No** *not only exhibits all possible algebraic and set-theoretically defined order-theoretic gradations consistent with its structure as an ordered field, it is to within isomorphism the unique such structure that does*. It is this together with Theorem 1 and a number of closely related results (see [Ehrlich 1992, forthcoming 1]) that naturally suggest that **No** may be regarded as an absolute arithmetic continuum (modulo NBG). However, as we alluded to above, to fully appreciate the nature of this absolute continuum one must appeal to its algebraico-tree-theoretic structure. This is the subject of the remaining three sections of Part I of the paper.

§2. No's simplicity hierarchical structure. In addition to its inclusive structure as an ordered field, **No** has a rich algebraico-tree-theoretic structure that emerges from the recursive clauses in terms of which it is defined. This s-hierarchical structure, as was mentioned above, depends upon **No**'s structure as a lexicographically ordered full binary tree and arises from the fact that the sums and products of any two members of the tree are the simplest possible elements of the tree consistent with **No**'s structure as an ordered group and an ordered field, respectively, it being understood that x is simpler than y just in case x is a predecessor of y in the tree.

Among the striking s-hierarchical features of **No** is that much as the surreal numbers emerge from the empty set of surreal numbers by means of a transfinite recursion that provides an unfolding of the entire spectrum of numbers great and small (modulo the aforementioned provisos), the recursive process of defining **No**'s arithmetic in turn provides an unfolding of the entire spectrum of ordered fields in such a way that an isomorphic copy of every such system either emerges as an initial subtree of **No** or is contained in a theoretically distinguished instance of such a system that does. In particular:

THEOREM 4 (Ehrlich 2001). *Every real-closed ordered field is isomorphic to a recursively defined initial subfield of* **No**.

Closely related to this is

THEOREM 5 (Ehrlich 2001). *Every divisible ordered abelian group is isomorphic to a recursively defined initial subgroup of* **No**.

The proof of Theorem 5 in [Ehrlich 2001] makes use of the following closely related s-hierarchical result in conjunction with the fact that the trivial divisible ordered abelian group $\{0\}$ is an initial subgroup of **No**.

THEOREM 6 (Ehrlich 2001). *If A is a divisible initial subgroup of* **No**, *and a is the simplest element of* **No** *that fills a cut in A, then the divisible subgroup of* **No** *generated by $A \cup \{a\}$ is itself an initial subgroup of* **No**.

An analogous proof of Theorem 4 could have been given in [Ehrlich 2001] using the following analog of Theorem 6 for real-closed ordered fields in conjunction with the fact that **No**'s subfield of real algebraic numbers is a real-closed initial subfield of **No**.

THEOREM 7. *If A is a real-closed initial subfield of* **No**, *and a is the simplest element of* **No** *that fills a cut in A, then the real-closed subfield of* **No** *generated by $A \cup \{a\}$ is an initial subfield of* **No**.

The proof of Theorem 4 employed in [Ehrlich 2001], however, is quite different, being based on Theorem 5, results of Mourgues and Ressayre (and Delon) [1991; 1993] and a theorem [2001: Theorem 18] that provides necessary and sufficient conditions for an ordered field to be isomorphic to an initial subfield of **No**. However, since it will be convenient in §8 to have Theorem 7 at hand and since Theorem 7 brings the embedding theory for **No** into greater harmony with the classical embedding theory for real-closed ordered fields, a proof of the theorem will be provided in the Appendix to the paper.[6]

Another striking s-hierarchical feature of **No** is the following generalization of the Cantor normal form theorem.

THEOREM 8 (Conway 1976; Ehrlich 2001, §3.1). *Every surreal number can be assigned a canonical "proper name" (or normal form) that is a reflection of*

[6]Following the presentation of this paper at the International Congress on Nonstandard Methods and Applications in Mathematics in Pisa in 2006, Antongiulio Fornasiero informed the author that Theorem 7 follows from results he established in his unpublished Ph.D. thesis [2004], results he has since stated without proof in [2006, p. 140: Lemmas 7.5 and 7.6]. However, since we believe our proof is of independent interest, we include it here.

its characteristic s-hierarchical properties. These Conway names, *as we call them, are expressed as formal sums of the form* $\sum_{\alpha<\beta} \omega^{y_\alpha} \cdot r_\alpha$ *where* β *is an ordinal,* $(y_\alpha)_{\alpha<\beta}$ *is a strictly decreasing sequence of surreals, and* $(r_\alpha)_{\alpha<\beta}$ *is a sequence of nonzero real numbers. Every such expression is in fact the Conway name of some surreal number, the Conway name of an ordinal being just its* Cantor normal form.

Since Conway names of surreal numbers are reflections of their characteristic s-hierarchical features, one can appeal to the algebraico-tree-theoretic structure of **No** to obtain revealing answers to the following two questions that are motivated by **No**'s structure as a full binary tree (see [Ehrlich 2011] for details):

(i) Given the Conway name of a surreal number, what are the Conway names of its two immediate successors?

(ii) Given the Conway names of the members of a chain of surreal numbers of limit length, what is the Conway name of the immediate successor of the chain?

Moreover, since every real-closed ordered field is isomorphic to an initial subfield of **No**, the answers provided to (i) and (ii) above not only shed light on the recursive unfolding of **No**, but on the recursive unfolding in **No** of real-closed ordered fields more generally, as the depiction of the early stages of the recursive unfolding of the surreal number tree in Figure 1 above only begins to show.

As we shall see in the second part of the paper, the just-mentioned s-hierarchical features of **No** play significant roles in the unification of systems of numbers great and small referred to above. Before turning to these matters, however, we will first introduce the surreal number tree in a manner that underscores the author's contention that the theory of surreal numbers may be naturally regarded as a vast generalization of portions of Cantor's theory of the infinite, and then characterize **No** as an ordered field together with its s-hierarchical structure.

§3. The surreal number tree. In Conway's construction of surreal numbers, a surreal number is an equivalence class consisting of an entire proper class of equivalent representations, which makes it impossible to collect the surreal numbers thus construed into a legal class of NBG. If one wishes, one can carry out Conway's construction in the equiconsistent set theory of Ackermann (cf. [Ehrlich 1994, p. 246]),[7] but in NBG the surreals must

[7] Ackermann's set theory is a conservative extension of ZF having classes as well as sets (cf. [Fraenkel, Bar-Hillel and Lévy l973, pp. 148–153; Lévy 1976, pp. 207–212]). Following Fraenkel, Bar-Hillel and Lévy, here we include the *axiom of foundation* (for sets) among the axioms of Ackermann's set theory. Since Ackermann [1956] did not do so, some authors distinguish between A and A^*, where by A they mean Ackermann's original axioms (or some equivalent set thereof) and by A^* they mean what we have called Ackermann's set theory (cf.

be "cut down to size". This can be done most easily by identifying each number with a canonical representative of each of Conway's equivalence classes, or by identifying each number with a canonical subset of each class. For the purpose of formalization, Conway proposes the latter approach, making use of the set-theoretic rank function [1976, p. 65], but even Conway admits that any such attempt to formalize his construction in NBG "destroys a lot of its symmetry" [1976, p. 65].[8] By contrast, we have introduced two alternative constructions that employ canonical representatives from Conway's construction: one is based on cuts due to Cuesta Dutari [1954; 1958–1959] that generalize the familiar cuts of Dedekind [Ehrlich 1988; Alling and Ehrlich 1986 and 1987], and the other is a generalization of the von Neumann ordinal construction [Ehrlich 1994; 2002a]. In the present discussion, we introduce the surreals via the latter approach using a recursive construction reminiscent of Conway's. Besides being simpler than Conway's construction, this construction places the surreal number tree at the center of the theory of surreal numbers, where the s-hierarchical structure of **No** suggests it belongs.[9]

3.1. Construction. Von Neumann defines an ordinal as a transitive set that is well ordered by the membership relation. As a result, for von Neumann, an ordinal emerges as the set of all of its predecessors in the 'long' though rather trivial binary tree $\langle On, \epsilon\rangle$ of ordinals ordered by membership.[10] So, for example, 0 is identified with $\varnothing$, 1 is identified with $\{\varnothing\} = \{0\}$, 2 is

[Lévy and Vaught 1961; Reinhardt 1970]). In A^*, unlike in NBG, besides the class V of all sets, there exists the power class $P(V)$ of all subclasses of V, the power class $P(P(V))$ of all subclasses of $P(V)$ and so on [Lévy and Vaught 1961, p. 1061; also see Fraenkel, Bar-Hillel and Lévy l973, p. 153 and Lévy l976, p. 212].

[8] As Conway mentions [1976, p. 65], the difficulties of formalizing his approach in NBG can be sidestepped by employing sign-expansions of surreal numbers (see [Conway 1976, pp. 29–30; Ehrlich 2011, pp. 5–6]) as the surreal numbers themselves (cf. [Gonshor 1986] and [Ehrlich 2001]). However, in addition to severing the theory of surreal numbers from Conway's theory of games, the sign-expansion approach presupposes the availability of the ordinals, the latter of which strikes this author as an aesthetic blemish on a theory of all numbers great and small.

[9] By contrast, three chapters after introducing the surreal numbers Conway shows that the inductively defined ordered class of surreal numbers can be given the structure of lexicographically ordered full binary tree, but this structure plays a limited role in Conway's treatment. What does play a role in Conway's treatment is a *birthday* function that maps each surreal number to the level of recursion at which it is created as well as the weaker notion of *simplicity*: x is *simpler than* y, if x was born prior to y. A surreal number's birthday corresponds to the tree-rank of the surreal number in our development.

In private conversations with the author, Conway expressed regret for having placed the emphasis on a simpler than relation defined in terms of the birthday map as opposed to the predecessor relation in the tree. For additional comments on this matter, see [Ehrlich 1994, p. 257: Note 1] and [Ehrlich 2001, p. 1232: Note 2].

[10] The structure $\langle On, \epsilon\rangle$ is not often described as a binary tree since its structure as a tree is indistinguishable from its familiar structure as a well-ordered class. In the theory of surreal

identified with $\{\varnothing, \{\varnothing\}\} = \{0, 1\}$ and so on. As we mentioned above, in our construction of surreal numbers, which generalizes von Neumann's construction, each surreal number x emerges as a characteristic ordered pair (L, R) of sets of surreal numbers whose union is the set of all surreal numbers that are simpler than x (i.e., that are predecessors of x) in the full binary surreal number tree $\langle \mathbf{No}, <_s \rangle$.

The recursive approach we employ requires one to have an appropriate ordered pair (L, R) at hand before it can be identified as a surreal number. To ensure the availability of the requisite class of set-theoretical entities, we follow Conway in first introducing the class of *games*.

Construction of Games

If L and R are any two sets of games, then there is a game (L, R). All games are constructed in this way.

Beginning with $L = \varnothing$ and $R = \varnothing$, where $\varnothing$ is the empty set (of games), the above construction leads to an entire proper class of games, $(\varnothing, \varnothing)$ being the first game constructed. The closure condition "All games are constructed in this way" simply ensures that everything that is a game arises in the specified way from sets of previously constructed games.

Notational Convention

If $x = (L, R)$ is a game, then following Conway, L is said to be the set of *left options* of x and R is said to be the set of *right options* of x. We will write L_x for the set of left options of x, R_x for the set of right options of x, and (L_x, R_x) for x itself.

To recursively extract the surreal numbers from the class of games we make use of the following terminology from [Ehrlich 2002a].

Definitions

A game x is said to be *simpler than* a game $y = (L_y, R_y)$, written $x <_s y$, if $x \in L_y$ or $x \in R_y$; a chain of games (i.e., a class of games totally ordered by $<_s$) is said to be *ancestral* if it is closed under the simpler than relation, i.e., x is a member of the chain if y is a member of the chain and $x <_s y$; and a partition L, R of an ancestral chain of games is said to be *orderly*, if $L_x \subseteq L$ and $R_x \subseteq R$ for each element $x = (L_x, R_x)$ of the chain.

The members of the class **No** of surreal numbers are now identified by means of the following simple construction in which the clause "there is a surreal number (L, R)" signifies that *the game* (L, R) *is a surreal number*.

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numbers, however, one must distinguish between its binary tree structure, on the one hand, and its structure as a totally ordered class, on the other hand. See, however, Corollary 1.

Construction of Surreal Numbers

If L, R is an orderly partition of an ancestral chain of surreal numbers, then there is a surreal number (L, R). All surreal numbers are constructed in this way.

In accordance with this construction, $(\varnothing, \varnothing)$ is the simplest surreal number, since even before any game has been identified as a surreal number there is an orderly partition $L = \varnothing, R = \varnothing$ of the *empty ancestral chain* $\varnothing$ of surreal numbers. Moreover, once the surreal number $(\varnothing, \varnothing)$ has been constructed, the orderly partitions $\varnothing, \{(\varnothing, \varnothing)\}$ and $\{(\varnothing, \varnothing)\}, \varnothing$ of the ancestral chain of surreal numbers consisting solely of $(\varnothing, \varnothing)$ give rise to the surreal numbers $(\varnothing, \{(\varnothing, \varnothing)\})$ and $(\{(\varnothing, \varnothing)\}, \varnothing)$, respectively, and so on (see Theorem 9 below for details).

If one wishes, one could develop the theory of surreal numbers making use of the von Neumann ordinals, which are already at hand. On the other hand, if one wants to develop the theory of ordinals within the theory of surreal numbers, *as seems befitting a theory of all numbers great and small*, one must first identify "our" ordinals.

Isolation of the Ordinals

A surreal number (L, R) will be said to be an *ordinal* if $R = \varnothing$. By *On* we mean the class of ordinals so defined. For all ordinals $x = (L_x, \varnothing)$ and $y = (L_y, \varnothing)$, x will be said to be *less than* y, written $x <_{On} y$, if $L_x \subset L_y$.

In accordance with the above definition, $(\varnothing, \varnothing)$ is an ordinal, $(\{(\varnothing, \varnothing)\}, \varnothing)$ is an ordinal, $(\{(\varnothing, \varnothing), (\{(\varnothing, \varnothing)\}, \varnothing)\}, \varnothing)$ is an ordinal, and so on. Indeed, in our approach, the just-cited ordinals are ultimately identified with the finite ordinals $0 = (\varnothing, \varnothing)$, $1 = (\{0\}, \varnothing)$ and $2 = (\{0, 1\}, \varnothing)$, respectively.

That the ordered class $\langle On, <_{On} \rangle$ of ordinals so defined has all of the requisite properties possessed by any of the more familiar constructs so-called follows from

Proposition 1. *There is a one-to-one order preserving correspondence between the ordered class $\langle On, <_{On} \rangle$ and the ordered class of von Neumann ordinals.*

As the reader will recall, a *tree* $\langle A, <_A \rangle$ is a partially ordered class such that for each $x \in A$, the class $\{y \in A : y <_A x\}$ of *predecessors* of x, written $\mathrm{pr}_A(x)$, is a set well ordered by $<_A$. The *tree-rank* of $x \in A$, written $\rho_A(x)$, is the ordinal corresponding to the well-ordered set $\langle \mathrm{pr}_A(x), <_A \rangle$; the αth *level* of A is $\langle x \in A : \rho_A(x) = \alpha \rangle$; and a *root* of A is a member of the zeroth level. If $x, y \in A$, then y is said to be an *immediate successor* of x if $x <_A y$ and $\rho_A(y) = \rho_A(x) + 1$; and if $(x_\alpha)_{\alpha < \beta}$ is a chain in A (of *length* β), then y is said to be an *immediate successor of the chain* if $x_\alpha <_A y$ for all $\alpha < \beta$ and $\rho_A(y)$ is the least ordinal greater than the tree-ranks of the members of the chain. A tree $\langle A, <_A \rangle$ is said to be *binary* if each member of A has at most two immediate successors and every chain in A of limit length has at most one immediate successor. If every member of A has two immediate successors and every chain in A of limit length (including the empty chain) has an immediate successor, then the binary tree $\langle A, <_A \rangle$ is said to be *full*.

THEOREM 9. *$\langle \mathbf{No}, <_s \rangle$ is a full binary tree. In particular, $(\varnothing, \varnothing)$ is the root of the tree; if $x = (L_x, R_x)$ is a surreal number, then $(L_x, \{x\} \cup R_x)$ and $(L_x \cup \{x\}, R_x)$ are the immediate successors of x; and if $(x_\alpha)_{\alpha<\beta}$ is a chain of surreal numbers of infinite limit length, then $(\bigcup_{\alpha<\beta} L_{x_\alpha}, \bigcup_{\alpha<\beta} R_{x_\alpha})$ is the immediate successor of the chain.*

There are three mutually exclusive and collectively exhaustive relations that distinct surreal numbers x and y may bear to one another with respect to simplicity: either $x <_s y$, $y <_s x$, or x is *incomparable with* y (i.e., $x \not<_s y$ and $y \not<_s x$). The components of the following definition specify when $x < y$ for these respective cases.

THE RULE OF ORDER

For all surreal numbers $x = (L_x, R_x)$ and $y = (L_y, R_y)$, $x < y$ if and only if $x \in L_y$ or $y \in R_x$ or $R_x \cap L_y \neq \varnothing$.

It is not difficult to show that $\langle \mathbf{No}, < \rangle$ is totally ordered by the Rule of Order ([Ehrlich 1994, p. 246: Theorem 2.1]. Moreover, it is evident that, if $x = (L_x, R_x)$ is a surreal number, then $(L_x, \{x\} \cup R_x)$ is the immediate successor of x less than x and $(L_x \cup \{x\}, R_x)$ is the immediate successor of x greater than x; and if $(x_\alpha)_{\alpha<\beta}$ is a chain of surreal numbers of infinite limit length, then $(\bigcup_{\alpha<\beta} L_{x_\alpha}, \bigcup_{\alpha<\beta} R_{x_\alpha})$, which is the immediate successor of the chain, is always greater than the members of $\bigcup_{\alpha<\beta} L_{x_\alpha}$ and less than the members of $\bigcup_{\alpha<\beta} R_{x_\alpha}$. Using this in conjunction with definition of the simpler than relation, one may obtain the aforementioned generalization of the idea underlying von Neumann's ordinal construction.

THEOREM 10. *For each surreal number x,*

$$x = \left(L_{s(x)}, R_{s(x)}\right)$$

where

$$L_{s(x)} = \{a \in No\colon a <_s x \ \ \& \ \ a < x\}$$

and

$$R_{s(x)} = \{a \in No\colon a <_s x \ \ \& \ \ x < a\}.$$

A maximal subclass of a tree $\langle A, <_A \rangle$ well ordered by $<_A$ is called a *branch*. By appealing to Theorem 10 together with the definition of **No**'s ordinals and the corresponding ordering thereof, we also have

COROLLARY 1. **No**'*s ordinals are the members of the rightmost branch of* $\langle \mathbf{No}, <, <_s \rangle$, *i.e.*, the unique initial subtree of **No** that is a proper class well ordered by $<$. *Each ordinal* α *is the member of the branch of tree-rank* α. *In particular*

$$\alpha = \left(L_{s(\alpha)}, \varnothing\right).$$

Moreover, for all ordinals α *and* β, $\alpha <_{On} \beta$ *if and only if* $\alpha <_s \beta$ *if and only if* $\alpha < \beta$.

An *initial subtree* of a tree $\langle A, <_A \rangle$ is a subclass A' of A with the induced order such that for each $x \in A'$, $\mathrm{pr}_{A'}(x) = \mathrm{pr}_A(x)$. Using the axiom of global choice (or simply the axiom of choice, if A is a set) a tree may be shown to be binary if and only if it is isomorphic to an initial subtree of the *canonical full binary tree* $\langle B, <_B \rangle$, where B is the class of all sequences of 0s and 1s indexed over some ordinal and $x <_B y$ signifies that x is a proper initial subsequence of y [5, p. 216].[11] Moreover, as is well known, $\langle B, <_B \rangle$ (as well as every initial subtree thereof) can be totally ordered (lexicographically) in accordance with the definition:

$$(x_\alpha)_{\alpha<\mu} <_{lex(B)} (y_\alpha)_{\alpha<\sigma} \text{ if and only if } x_\beta = y_\beta \text{ for all } \beta < \text{ some } \delta,$$
$$\text{but } x_\delta < y_\delta, \text{ it being understood that } 0 < \textit{undefined} < 1.$$

The resulting structure $\langle B, <_{lex(B)}, <_\beta \rangle$ is called the *lexicographically ordered canonical full binary tree*.

In the theory of surreal numbers, however, it is more convenient work with the following representation independent characterization of a lexicographically order binary tree.

DEFINITION 2 (Ehrlich 2001). A binary tree $\langle A, <_A \rangle$ together with a total ordering $<$ defined on A is said to be *lexicographically ordered* if for all $x, y \in A$, x is incomparable with y if and only if x and y have a common predecessor lying between them (i.e., there is a $z \in A$ such that $z <_A x$, $z <_A y$ and either $x < z < y$ or $y < z < x$).

The appellation "lexicographically ordered" is motivated by the fact that there is a unique order preserving isomorphism between a binary tree that is ordered in this sense and an initial subtree of $\langle B, <_{lex(B)}, <_B \rangle$ [Ehrlich 2001: Theorem 1, p. 1234]. As one would expect from the following result, in the case of $\langle \mathbf{No}, <, <_s \rangle$ the isomorphism is onto.

THEOREM 11. $\langle \mathbf{No}, <, <_s \rangle$ *is a lexicographically ordered full binary tree.*

PROOF. Since $\langle \mathbf{No}, <, <_s \rangle$ is an ordered full binary tree, it remains to show $\langle \mathbf{No}, <, <_s \rangle$ is lexicographically ordered. Let $x, y \in \mathbf{No}$. In virtue of the Rule of Order and Theorem 10: $x < y$ if and only if $x \in L_{S(y)}$ or $y \in R_{S(x)}$or $R_{S(x)} \cap L_{S(y)} \neq \varnothing$. But if x is incomparable with y, then $R_{S(x)} \cap L_{S(y)} \neq \varnothing$,

[11] In writings on surreal numbers, it is commonplace to employ sequences of $-$ signs and $+$ signs rather than 0s and 1s to represent canonical binary trees.

in which case x and y have a common predecessor lying between them, and if x is comparable with y, then $x \in L_{S(y)}$ or $y \in L_{S(x)}$, in which case x and y cannot have a common predecessor lying between them. Indeed, if $x \in L_{S(y)}$, then $\rho_{No}(z) > \rho_{No}(x)$ for any $z \in L_{S(y)}$ for which $x < z$, which implies $z \notin R_{S(x)}$; and if $y \in R_{S(x)}$, then $\rho_{No}(z) > \rho_{No}(y)$ for any $z \in R_{S(x)}$ for which $z < y$, which implies $z \notin L_{S(y)}$. $\dashv$

§4. The s-hierarchical ordered field of surreal numbers. Central to the algebraico-tree-theoretic development of the theory of surreal numbers is the following consequence of **No**'s structure as a lexicographically ordered full binary tree: if L and R are two sub*sets* of **No** for which every member of L precedes every member of R $(L < R)$, there is a *simplest* member of **No** lying between the members of L and the members of R [Ehrlich 2001, pp. 1234–1235; 2011: Proposition 2.4]. Co-opting notation introduced by Conway, the simplest member of **No** lying between the members of L and the members of R is denoted by the expression

$$\{L \mid R\}.$$

It is not difficult to show that each surreal number x is the simplest member of **No** lying between its predecessors on the left and its predecessors on the right, i.e.,

$$x = \left\{L_{s(x)} \mid R_{s(x)}\right\}.$$

Using this representation, the algebraico-tree-theoretic formulation of the central theorem in the theory of surreal numbers may be stated as follows.

Theorem 12 (Conway 1976; Ehrlich 2001). *$\langle \mathbf{No}, +, \cdot, <, <_s \rangle$ is an ordered field when $+$, $-$ and $\cdot$ are defined by recursion as follows where x^L, x^R, y^L and y^R are understood to range over the members of $L_{s(x)}$, $R_{s(x)}$, $L_{s(y)}$ and $R_{s(y)}$, respectively.*[12]

Definition of $x + y$.

$$x + y = \{x^L + y, x + y^L \mid x^R + y, x + y^R\}.$$

Definition of $-x$.

$$-x = \{-x^R \mid -x^L\}.$$

[12] In the following definitions of $+$, $-$ and $\cdot$, the set-theoretic brackets that enclose the sets of "right-sided members" and the sets of "left-sided members" are omitted (in accordance with custom).

Definition of xy.

$$xy = \{x^L y + xy^L - x^L y^L, x^R y + xy^R - x^R y^R \\ \mid x^L y + xy^R - x^L y^R, x^R y + xy^L - x^R y^L\}.$$

Although the algebraico-tree-theoretic definitions of sums, products and additive inverses of surreal numbers are apt to appear rather cryptic to readers unfamiliar with the theory of surreal numbers, they have natural interpretations. To begin with, in virtue of the nature of the representations of x and y, we have

$$x^L < x < x^R \text{ and } y^L < y < y^R$$

for all x^L, x^R, y^L and y^R. And this in conjunction with elementary algebra of ordered fields implies that if **No** is to be an ordered field $x + y$ must lie between the sums on the left and right of $x + y$ in the above definition of $x + y$ and that xy must lie between the arithmetical expressions on the left and right of xy in the above definition of xy [Ehrlich, 1994a, pp. 252–253; 2001, p. 1236; Alling 1987, pp. 133–139]. Accordingly, since $x + y$ and xy must lie between the arithmetic expressions on the left and right in their respective recursive definitions, these definitions respectively require that $x + y$ and xy be the simplest member of the surreal number tree so situated. The constraint on additive inverses, which is a consequence of the definition of addition [Ehrlich 2001, p. 1237], ensures that the portion of the surreal number tree less than 0 is (in absolute value) a mirror image of the portion of the surreal number tree greater than 0, 0 being the simplest element of, as well as the unique root in, the surreal number tree.

With the above algebraico-tree-theoretic version of Conway's theorem in mind, in [Ehrlich 2001, p. 1236] we defined an *s-hierarchical ordered field* as an ordered field with a lexicographically ordered binary tree structure (in the sense of Definition 2) whose sums and products satisfy the above conditions on sums and products, with no supposition that the sums and products be recursively defined as they are in **No**. The s-hierarchical subfields of an s-hierarchical ordered field A coincide with the initial subfields of A [Ehrlich 2001, p. 1236], and when, as in the case of **No**, the arithmetic is recursively defined, *the sums and products of the elements* of A (*and of elements of the initial subfields of A more generally*) *get defined just as soon as there is sufficient previously defined ordered algebraico-tree theoretic information to do so*.

Besides being (up to isomorphism) the unique Dedekind complete ordered field, $\mathbb{R}$ is (up to isomorphism) the unique universal and the unique maximal (or non-extensible), Archimedean ordered field, the condition of maximality or of non-extensibility being Hilbert's [1900] classical continuity condition. Analogs of these results also hold for $\mathbb{R}$ considered as an s-hierarchical ordered field. More importantly, however, s-hierarchical analogs of these classical characterization theorems also hold for **No**.

If A is an s-hierarchical ordered field, then A is said to be *universal* if every s-hierarchical ordered field is isomorphic to an initial subfield of A;[13] A is said to be *maximal* (or *non-extensible*) if there is no s-hierarchical ordered field that properly contains A as an initial subfield; and A is be said to be *complete* if $\{L \mid R\}$ exists whenever L and R are subsets of A for which $L < R$.

THEOREM 13 (Ehrlich 2001). *Let A be an s-hierarchical ordered field. A is complete if and only if A is universal if and only if A is maximal if and only if A is isomorphic to* **No**.

Whereas Theorems 1 and 3 may be said to characterize **No** as an absolute arithmetic continuum, Theorem 13 may be said to characterize **No** as an *s-hierarchical absolute arithmetic continuum*. In the remainder of the paper we will turn our attention to the unifying capacity of this remarkable s-hierarchical structure.

Part II. The unification of systems of numbers great and small

Throughout the 17th and 18th centuries, talk of infinitesimal line segments and numbers to measure them was commonplace in discussions of the calculus. However, as a result of the conceptual difficulties that arose from these conceptions their role became more subdued in the 19th-century calculus discussions and was eventually "banished" therefrom. However, whereas most late 19th- and pre-Robinsonian 20th-century mathematicians banished infinitesimals from the calculus, they were by no means banished from mathematics.[14] Indeed, between the early 1870s and the appearance of Abraham Robinson's work on *nonstandard analysis* in 1961 there emerged a large, diverse, technically deep and philosophically pregnant body of consistent (non-Archimedean) mathematics of the infinitely large and the infinitely small. Unlike nonstandard analysis, which is primarily concerned with providing a treatment of the calculus making use of infinitesimals, the bulk of the former work is either concerned with the rate of growth of real-valued functions or with geometry and the concepts of number and of magnitude, or grew out of the natural evolution of such discussions.

[13] The definition of universal introduced above is equivalent to, thought not identical with, the definition of universal employed in [Ehrlich 2001]. The equivalence of the two notions is evident in virtue of Definition 8 and Lemma 2 of [Ehrlich 2001].

[14] Nevertheless, there is a widespread misconception in the literature that infinitesimals were indeed banished from late 19th- and pre-Robinsonian 20th-century mathematics. For in-depth discussions of the roots of this misconception, see the author's [2006] and [forthcoming 2].

Following a brief discussion of **No**'s containment of the reals, we will turn our attention to some of the relations that exists between **No** and the non-Archimedean ordered fields that have emerged from the three just-mentioned bodies of work.

§5. The containment of the reals. Although **No** contains an entire proper class of isomorphic copies of the ordered field of real numbers, only one of them is an initial subfield of **No**. In writings on the surreal numbers, the latter subfield is identified as **No**'s subfield of real numbers, the set of whose members, in accordance with our construction, may be defined as follows.

Definition 3. Let $\mathbb{D}$ be the set of all surreal numbers having finite tree-rank and further let

$$\mathbb{R} = \mathbb{D} \cup \{(L, R) \in \mathbf{No}\colon L \text{ and } R \text{ are infinite subsets of } \mathbb{D}\}.$$

Except for inessential changes, the following result regarding the structure of $\mathbb{R}$ is due to Conway [1976, pp. 23–25].

Proposition 2. $\mathbb{R}$ (*with* $+$, $-$, $\cdot$ *and* $<$ *defined* à la **No**) *is isomorphic to the ordered field of real numbers defined in any of the more familiar ways*, $\mathbb{D}$ *being* **No***'s ring of dyadic rationals* (*i.e., rationals of the form* $m/2^n$ *where* m *and* n *are integers*); *moreover,* $n = \{0, \dots, n-1 \mid \varnothing\}$ *and* $-n = \{\varnothing \mid -(n-1), \dots 0\}$ *for each positive integer* n, $0 = \{\varnothing \mid \varnothing\}$, *and the remainder of the dyadics are the arithmetic means of their left and right predecessors of greatest tree-rank*.

It is not difficult to see that

$$\mathbb{R} - \mathbb{D} = \{\{L \mid R\} \in \mathbf{No}\colon (L, R) \text{ is a Dedekind gap in } \mathbb{D}\}$$

and, hence, that the members of $\mathbb{R} - \mathbb{D}$ have tree-rank ω. Accordingly, since every ordered field contains a copy of the ring of dyadic rationals and every Archimedean ordered field is a subfield of $\mathbb{R}$, in virtue of the above we have

Proposition 3 (Ehrlich 2001). *Every Archimedean ordered field is isomorphic to exactly one initial subfield of* **No**, *the latter being an initial subfield of* $\mathbb{R}$.

§6. Non-Archimedean ordered number fields inspired by non-Archimedean geometry. Following Wallis's and Newton's incorporation of directed segments into Cartesian geometry, it became loosely understood that given a unit segment AB of a line L of a classical Euclidean space, the collection of directed segments of L emanating from A including the degenerate segment AA itself constitutes an Archimedean ordered field with AA and AB the additive and multiplicative identities of the field and addition and multiplication of segments suitably defined. These ideas were made precise by Giuseppe Veronese [1891; 1894] and David Hilbert [1899] in their works on the foundations of geometry, from which the modern conceptions of Archimedean and non-Archimedean ordered fields emerged.

Unlike the analytic constructions of Hilbert, Veronese's constructions of non-Archimedean ordered fields were synthetic, though he did represent the line segments that emerged from his constructions using a loosely defined, complicated system of numbers consisting of finite and transfinite series of the form

$$\infty_1^{y_1} r_1 + \infty_1^{y_2} r_2 + \infty_1^{y_3} r_3 + \dots$$

where $r_1, r_2, r_3, \dots$ are real numbers, and $\infty_1^{y_1}, \infty_1^{y_2}, \infty_1^{y_3}, \dots$ is a sequence of *units*, each of which is infinitesimal relative to the preceding units, ∞_1 being the number (of "infinite order 1" [1891, p. 101]) introduced by Veronese to represent the infinitely large line segment whose existence is postulated by his "hypothesis on the existence of bounded infinitely large segments" [1891, p. 84]. Veronese's number system was provided an analytic foundation by Tullio Levi-Civita [1892–1893/1954, 1898/1954], who therewith provided the first analytic constructions of non-Archimedean ordered fields. Among the array of non-Archimedean ordered fields that emerged from Levi-Civita's work are isomorphic copies of the *ordered fields of Laurent power series with coefficients in subfields of the reals and exponents in ordered abelian groups*, ordered fields that continue to play prominent roles in the foundations of geometry.

Building on the work of Levi-Civita, Hans Hahn [1907] constructed non-Archimedean ordered number fields having properties that generalize the familiar continuity properties of Dedekind [Ehrlich 1997] and Hilbert [Ehrlich 1995; 1997a], and he demonstrated (vis-à-vis his embedding theorem for ordered abelian groups)[15] that his number systems provide a panorama of the finite, infinite and infinitesimal numbers that can enter into a non-Archimedean theory of continua based on the concept of an ordered field [Ehrlich 1995, 1997, 1997a]. This idea was later brought into sharper focus when it was demonstrated that *every ordered field could be embedded in a suitable Hahn field*.[16]

[15] For concise, modern proofs of Hahn's embedding theorem, see, for example, [Hausdorff 1914, pp. 194–207], [Clifford 1954] and [Fuchs 1963, pp. 56–61]. Like Hausdorff's [1909], which is discussed in §8, Hausdorff's elegant proof of Hahn's embedding theorem has been largely overlooked in the literature. For additional references and historical perspective, see [Ehrlich 1995].

[16] As we explained in [Ehrlich 1995, p. 187], this theorem has a long and complicated history that makes it difficult to attribute it to any single author. However, by the early 1950s, as a result of the work of Kaplansky [1942], it appears to have assumed the status of a "folk theorem" among knowledgeable field theorists. Still, it appears to be Paul Conrad who published the first explicit statement of the result [1954, p. 328], and Conrad (together with Dauns) [1969; Theorem II] later isolated, and provided a detailed proof-sketch of, a revealing formulation of the theorem. Since that time many alternative proofs, variations, and strengthenings have appeared including those of Priess-Crampe [1973; 1983, pp. 62–64, 124], Priess-Crampe and von Chossey [1975], Rayner [1975], Mourgues and Ressayre [1993], van den Dries [1991] and Dales and Wooden [1996].

Hahn's celebrated theorem on ordered fields is given by

HAHN 1 [1907]. *Let $\mathbb{R}$ be the ordered field of real numbers and Γ be a nontrivial ordered abelian group.*

(i) *The collection, $\mathbb{R}((t^\Gamma))$, of all series*

$$\sum_{\alpha<\beta} t^{\gamma_\alpha} \cdot r_\alpha$$

where $\{\gamma_\alpha : \alpha < \beta \in On\}$ is a (possibly empty) strictly increasing sequence of members of Γ and $\{r_\alpha : \alpha < \beta\}$ is a sequence of members of $\mathbb{R} - \{0\}$ is a non-Archimedean ordered field when the order is defined lexicographically and sums and products are defined termwise, it being understand that $t^\gamma \cdot t^\kappa = t^{\gamma+\kappa}$.[17]

Moreover:

(ii) *The structure, $\mathbb{R}((t^\Gamma))_\lambda$, that results by limiting the construction of $\mathbb{R}((t^\Gamma))$ to those series where β is less than a given uncountable initial number λ (i.e., an uncountable infinite cardinal λ) is likewise a non-Archimedean ordered field.*

With HAHN 1 Hahn provided an implicit prescription for creating all numbers great and small (modulo NBG). On the other hand, it was only with **No** (considered as an s-hierarchical ordered field) that the full potential for the generation and systemization of these numbers was realized. To state the central s-hierarchical relation between Hahn's classical systems and **No** the following terminology is required.

Henceforth, by $\mathbb{R}((t^\Gamma))_{On}$ we mean the ordered field (with order, addition, and multiplication defined *à la* Hahn), consisting of all series of the form

$$\sum_{\alpha<\beta} t^{\gamma_\alpha} \cdot r_\alpha$$

where $\{\gamma_\alpha : \alpha < \beta \in On\}$ is a (possibly empty) strictly increasing sequence of elements of an ordered abelian group Γ and $r_\alpha \in \mathbb{R} - \{0\}$ for each $\alpha < \beta$ where $\mathbb{R}$ is the ordered field of reals. If Γ is a set, then $\mathbb{R}((t^\Gamma))_{On}$ is just the Hahn field $\mathbb{R}((t^\Gamma))$, and otherwise $\mathbb{R}((t^\Gamma))_{On}$ is an ordered field of power *On*. An element x of $\mathbb{R}((t^\Gamma))_{On}$ is said to be a *truncation* (*proper truncation*) of

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Kaplansky's above-cited paper on maximally valued fields draws heavily on the valuation-theoretic classics of Krull [1932] and Ostrowski [1935]. No attempt to provide the historical background for the non-Archimedean mathematics that permeates the theory of surreal numbers would be complete if it fails to take into account these three papers and the theory of valuations more generally. We intend to do this on another occasion. For the time being, we direct the reader to [Alling 1987], where a valuation-theoretic analysis of the theory of surreal numbers is provided.

[17]Hahn [1907] follows Veronese in using strictly decreasing sequences of members of Γ. Thus, whereas t is infinite for Hahn, t is infinitesimal for us. In both approaches, the latter of which is more commonly used in the literature today, the successive terms in the sequence $t^{\gamma_0}, t^{\gamma_1}, t^{\gamma_2}, \ldots$ are infinitesimal relative to the preceding terms.

$\sum_{\alpha<\beta} t^{\gamma_\alpha} \cdot r_\alpha \in \mathbb{R}((t^\Gamma))_{On}$ if $x = \sum_{\alpha<\sigma} t^{\gamma_\alpha} \cdot r_\alpha$ for some $\sigma \leq \beta$ ($\sigma < \beta$); if every truncation of a member of a subclass of $\mathbb{R}((t^\Gamma))_{On}$ is itself a member of the subclass, the subclass is said to be *truncation closed*. Roughly speaking, a truncation of x is an approximation of x, increasingly longer truncations being better approximations. Finally, a subfield F of $\mathbb{R}((t^\Gamma))_{On}$ is said to be *cross sectional* if $\{t^\gamma : \gamma \in \Gamma\} \subset F$. Roughly speaking, each t^γ serves as *the unit of measure* of the γth *Archimedean class* of $\mathbb{R}((t^\Gamma))_{On}$. Thus, roughly speaking, a subfield F of $\mathbb{R}((t^\Gamma))_{On}$ is cross sectional if it contains all the canonical units contained in $\mathbb{R}((t^\Gamma))_{On}$.

In addition to stating the relation between Hahn fields and **No**, the following result provides necessary and sufficient conditions for an ordered field to be isomorphic to an initial subfield of **No**.

Theorem 14 (Ehrlich 2001). *Let Γ be an ordered abelian group that is an initial subgroup of* **No**.

(i) *There is an isomorphism of ordered fields from $\mathbb{R}((t^\Gamma))_{On}$ onto an initial subfield of No that sends*

$$\sum_{\alpha<\beta} t^{\gamma_\alpha} \cdot r_\alpha$$

to the surreal number having Conway name

$$\sum_{\alpha<\beta} \omega^{-\gamma_\alpha} \cdot r_\alpha.$$

(ii) *For the special case where $\Gamma = \mathbf{No}$, the just-said isomorphism maps $\mathbb{R}((t^{\mathbf{No}}))_{On}$ onto* **No**.

(iii) *The result expressed in* (*i*) *holds, more generally, for any truncation closed, cross sectional subfield of $\mathbb{R}((t^\Gamma))_{On}$; moreover, every initial subfield A of* **No** *is the image of precisely one such isomorphism* (*where Γ is an initial subgroup of* **No** *that depends on A*).

In subsequent portions of the paper we will to appeal to a number of distinguished initial subfields of **No**. We will draw this section of the paper to a close with a decomposition theorem for **No** that collects them and a host of other closely related initial subfields together in a revealing manner. To formulate the result we require a construct that emerges from the first part of the following simple consequence of Theorem 14.

Corollary 2. (i) *Let Γ be an initial subgroup of* **No** *and λ be an uncountable initial number $\leq On$. There is a unique initial subfield of* **No** *containing $\{\omega^\gamma : \gamma \in \Gamma\}$ that is isomorphic to $\mathbb{R}((t^\Gamma))_\lambda$. Henceforth, these initial subfields, which are obtained by replacing each formal sum $\sum_{\alpha<\beta} t^{\gamma_\alpha} \cdot r_\alpha$ with the corresponding Conway name $\sum_{\alpha<\beta} \omega^{-\gamma_\alpha} \cdot r_\alpha$, will be denoted $\mathbb{R}((\tau^\Gamma))_\lambda$, where $\tau = \omega^{-1}$.* (ii) *If A is an initial subfield of* **No** *and Γ is the initial subgroup of*

No *consisting of* $\{\gamma : \omega^\gamma \in A\}$, *then* A *is a truncation closed, cross sectional subfield of* $\mathbb{R}((\tau^\Gamma))_{On}$. *If* Γ *is a set, then* $\mathbb{R}((\tau^\Gamma))_{On}$ *is isomorphic to* $\mathbb{R}((t^\Gamma))$ *and will be called* the Hahn field in **No** induced by A.

THEOREM 15 (van den Dries and Ehrlich 2001). *Let* $\mathbf{No}(\lambda)$ *be the members of* **No** *of tree-rank* $< \lambda \leq On$. *Then* $\mathbf{No}(\lambda)$ *is an ordered field* (*indeed, a real-closed ordered field*) *if and only if* λ *is an* ε*-number* (*i.e.,* $\lambda = \omega^\lambda$). *Furthermore, if* λ *is an* ε*-number, then* $\mathbf{No}(\lambda) = \mathbb{R}((\tau^{\mathbf{No}(\lambda)}))_\lambda$ *just in case* λ *is a regular initial number.*[18]

§7. Ordinals and omnific integers in initial subfields of No. In their writings on the foundations of geometry, Veronese and Levi-Civita proclaimed that their infinite numbers are founded on hypotheses other than those employed by Cantor. With this in mind, Veronese observed in his letter to Cantor, dated November 13, 1890

> [i]n ... [my number system] ... there is not a first infinite number .... This is the essential difference between your number ω and my number ∞_1. [Meschkowski and Nilson 1991, p. 329]

Such proclamations, however, did not persuade Cantor of the cogency of non-Archimedean number systems, and in his letter to Wilhelm Killing dated April 5, 1895 Cantor expressed the matter thus:

> Of his *infinitely large numbers* ... [Veronese] ... says that they are introduced *on other hypotheses* than mine. But mine depend upon *absolutely no hypotheses*, but are immediately derived from the natural concept of set. They are just as necessary and free from arbitrariness as the finite whole numbers. [Dauben 1979, p. 351: Note 79]

While Cantor rejected the legitimacy of non-Archimedean numbers systems, he certainly shared Veronese's and Levi-Civita's beliefs that there is no place for infinite ordinals in their number systems or in number systems, more generally, having additive and/or multiplicative structures that are commutative. After all, according to Cantor,

> the laws governing them [i.e., the Cantorian laws governing ordinals] can be derived from immediate inner intuition with apodictic certainty [Cantor 1883 in 1932, p. 170; Cantor 1883 in Ewald 1996, p. 886].

The view that Cantor's operations on ordinals are the only legitimate such operations was first challenged by Hessenberg [1906, pp. 591–594] with his introduction of the *natural sums of ordinals* (written in Cantor normal

[18]Prominent among the regular initial numbers that are ε-numbers are the successor cardinals and the (strongly) inaccessible cardinals $> \omega$. *On* is the only inaccessible cardinal $> \lambda$ whose existence can be proved in NBG.

form), and this was followed two decades later by Hausdorff's [1927, pp. 68–69; 1957, pp. 80–81] introduction of the corresponding *natural products of ordinals*, which he generously attributed to Hessenberg. Carruth [1942] later showed that the natural sums and products of Hessenberg and Hausdorff are in fact distinguished special cases of a wide class of sums and products of ordinals that likewise could be called "natural", but the terminology of Hessenberg and Hausdorff has stuck. However, it appears to have been the Italian geometer Federigo Enriques, in his *Sui numeri non archimedei e su alcune loro interpretazioni* (On non-Archimedean Numbers and Some of their Interpretations) [1911, pp. 90–96; also see 1912, pp. 472–478; 1924, pp. 367–372] who first explicitly noted that a non-Archimedean ordered field can be a natural vehicle for embedding a commutative semiring of finite and infinite ordinals. Indeed, without mention of Cantor normal forms or natural sums and products of ordinals, Enriques essentially showed that $\mathbb{R}((\tau^{\mathbb{Z}}))$, considered as an ordered field of formal power series (without its surreal connotations), is an extension of the lexicographically ordered semiring of ordinals $\alpha < \omega^{\omega}$ (written in Cantor normal form) with sums and products defined naturally. This all but forgotten work appeared almost four decades before Roman Sikorski's seminal paper *On an Ordered Algebraic Field* [1948], the work to which contemporary order-algebrists usually trace the insight that lexicographically ordered semirings of ordinals with sums and products defined naturally can be embedded in ordered fields.

However, with the exception of special cases, such as those identified by Enriques and Sikorski (also see [Klaua 1994] and [Ehrlich 2001, p. 1256]), there are rarely natural embeddings of semirings of ordinals into ordered fields that distinguish themselves from the plethora of possible embeddings. In the case of initial subfields of **No**, however, this is not the case, as the following amplification of our earlier remarks on the containment of ordinals in **No** makes clear.[19]

Let $On(A)$ be the class of ordinals in an initial subfield A of **No**. Then A will be said to be α-*Archimedean* if and only if α is the height of $On(A)$ (considered as an initial subtree of A).

As is evident from the following result, the idea of an α-Archimedean, initial subfield of **No** is a natural s-hierarchical adaptation and generalization of the idea of an Archimedean subfield of **No**.

Proposition 4 (Ehrlich 2001). *If A is an initial subfield of* **No**, *then A is α-Archimedean if and only if for all $a, b \in A$ where $a > b > 0$ there is an $\eta < \alpha$ such that $\eta b > a$; furthermore, A is Archimedean if and only if A is ω-Archimedean.*

[19] For a more general discussion in terms of s-hierarchical ordered groups and fields, see [Ehrlich 2001, §5].

An ordinal α is said to be *additively indecomposable* if $\beta +_c \gamma < \alpha$ whenever $\beta, \gamma < \alpha$ (where $+_c$ is the Cantorian sum). Moreover, if $0 < \alpha \leq On$, then α is additively indecomposable if and only if $\alpha = \omega^{\varphi}$ for some ordinal $\varphi \leq On$.

The following is a generalization for initial subfields of **No** of a compilation of results established by Conway for **No**.

THEOREM 16 (Ehrlich 2001). *Let A be an initial subfield of* **No**. *Then A is ω^{φ}-Archimedean for some nonzero indecomposable ordinal $\varphi \leq On$. If A is ω^{φ}-Archimedean, then A contains a canonical cofinal, ordered subsemiring $On(A)$ of ordinals consisting of all surreal numbers $x < \omega^{\varphi}$ such that $x = \sum_{\alpha<n} \omega^{\varphi_\alpha} \cdot a_\alpha$ for some finite descending sequence $(\varphi_\alpha)_{\alpha<n}$ of ordinals $< \varphi$ and some sequence $(a_\alpha)_{\alpha<n}$ of finite ordinals > 0; $On(A)$ in turn is contained in a discrete, canonical subring $Oz(A)$ of A—the* omnific integer part *of A—consisting of all members of A of the form $\sum_{\alpha<\beta} \omega^{\varphi_\alpha} \cdot a_\alpha$ where $\varphi_\alpha \geq 0$ for all $\alpha < \beta$ and a_α is an integer whenever $\varphi_\alpha = 0$; for each $x \in A$ there is a $z \in Oz(A)$ such that $z \leq x \leq z + 1$.*

As an addendum to Theorem 16 we note that, if A is an initial subfield of **No**, then the ring $\mathbb{Z}$ of integers is an initial subring of $Oz(A)$, which in turn is an initial subring of A [Ehrlich 2011, Note 2, pp. 3–4]; moreover, $\mathbb{Z} = Oz(A)$ if and only if A is Archimedean.

§8. Paul du Bois-Reymond's *Infinitärcalcül* and its aftermath. Although interest in the rates of growth of real functions is already found in Euler's *De infinities infinitis gradibus tam infinite magnorum quam infinite parvorum* (On the infinite degrees of infinity of the infinitely large and infinitely small) [1778], their systematic study was first undertaken by Paul du Bois-Reymond. The groundwork for his theory was laid out in his paper *Sur la grandeur relative des infinis des functions* (On the relative size of the infinities of functions) [1870–1871] and developed in more than a dozen other works (cf. [1875; 1877; 1882]).[20] Much of the motivation for du Bois-Reymond's *Infinitärcalcül* (calculus of infinities) that is still of interest to researchers today is encapsulated by the following remarks with which G. H. Hardy begins his important monograph on du Bois-Reymond's system.

> The notions of the 'order of greatness' or 'order of smallness' of a function $f(n)$ of a positive integral variable n, when n is 'large', or of a function $f(x)$ of a continuous variable x, when x is 'large' or 'small' or 'nearly equal to a', are important even in the most elementary stages of mathematical analysis. We learn there that x^2 tends to infinity with x, and moreover that x^2 tends to infinity *more rapidly than* x, i.e., that the ratio x^2/x tends to infinity also;

[20] For a complete list of du Bois-Reymond's writings on his *Infinitärcalcül* and a survey of the contents thereof, see [Fisher 1981].

> and that x^3 tends to infinity more rapidly than x^2, and so on indefinitely. We are thus led to the idea of a 'scale of infinity' (x^n) formed by the functions $x, x^2, x^3, \ldots, x^n, \ldots$. This scale may be supplemented and to some extent completed by the interpolation of non-integral powers of x. But there are functions whose rates of increase cannot be measured by any of the functions of our scale, even when thus completed. Thus $\log x$ tends to infinity more slowly, and e^x more rapidly, than *any* power of x; and $x/(\log x)$ tends to infinity more slowly than x, but more rapidly than any power of x less than the first.
>
> As we proceed further in analysis, and come into contact with its modern developments, such as the theory of Fourier's series, the theory of integral functions, or the theory of singular points of analytic functions in general, the importance of these ideas becomes greater and greater. It is the systematic study of them, the investigation of general theorems concerning them and ready methods of handling them, that is the subject of Paul du Bois-Reymond's *Infinitärcalcül* or 'calculus of infinities'. [Hardy 1910, pp. 1–2]

Du Bois-Reymond erects his *Infinitärcalcül* primarily on families of increasing functions from $\mathbb{R}^+ = \{x \in \mathbb{R}\colon x > 0\}$ to $\mathbb{R}^+$ such that for each function f of a given family, $\lim\limits_{x\to\infty} f(x) = +\infty$, and for each pair of functions f and g of the family, $0 \leq \lim\limits_{x\to\infty} f(x)/g(x) \leq +\infty$. He assigns to each such function f a so-called *infinity*, and defines an ordering on the infinities of such functions by stipulating that for each pair of such functions f and g:

$f(x)$ has an infinity *greater than* that of $g(x)$, if $\lim\limits_{x\to\infty} f(x)/g(x) = \infty$;

$f(x)$ has an infinity *equal to* that of $g(x)$, if $\lim\limits_{x\to\infty} f(x)/g(x) = a \in \mathbb{R}^+$;

$f(x)$ has an infinity *less than* that of $g(x)$, if $\lim\limits_{x\to\infty} f(x)/g(x) = 0$.

In accordance with this scheme, the infinities of the following functions

$$\ldots, e^{e^x}, e^x, \ldots, x^n, \ldots, x^3, x^2, x, x^{1/2}, x^{1/3}, \ldots, x^{1/n}, \ldots, \ln x, \ln(\ln x), \ldots$$

increase as we move from right to left. Moreover, as the comparative graphs of several of these functions illustrate (see Figure 2), given any two functions f and g from a family of the just-said kind, $f(x)$ has a greater infinity than $g(x)$ if $f(x) > g(x)$ for all $x >$ some x_0.

Unfortunately, du Bois-Reymond was not always as clear as one would hope about the precise contents of the families of functions with which he was concerned; nor did he make any real use of arithmetic operations on the infinities arising from such families and attempt thereby to bring algebra to his infinities as Otto Stolz [1883; 1885] and others later would. On the

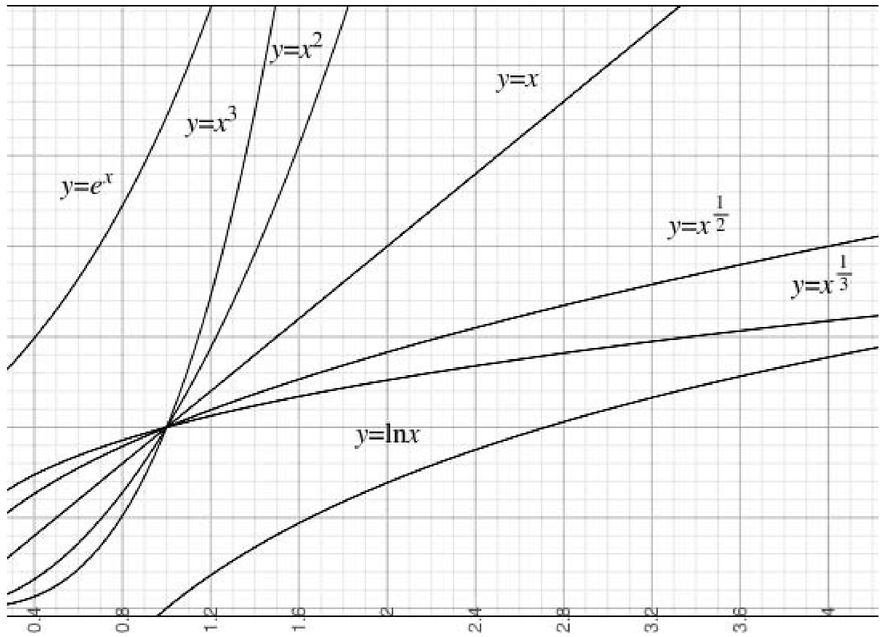

FIGURE 2

other hand, he did establish a number of *order-theoretic* results regarding his "infinities", including those referred to by Hardy above, that provided intimations of their inability to be adequately represented by ordered sets of real numbers and, hence, by ordered subsets of Archimedean ordered groups.

It was the just-said intimations along with arithmetic hints contained in du Bois-Reymond's [1877] that Otto Stolz [1883] seized upon when he established for the first time the existence of a non-Archimedean ordered algebraic system (see [Ehrlich 2006] for historical details).

Stolz [1883, pp. 506–507] considers the set of all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ for which $\lim_{x\to\infty} f(x) = +\infty$ formed by means of finite combinations of the operations $+, -, \cdot$ and $\div$ from positive rational powers of the functions $x, \ln x, \ln(\ln x), \ldots, e^x, e^{e^x}, \ldots$ where $\ln x$ is the natural logarithm of x and e is the base of the natural logarithm. Following du Bois-Reymond, Stolz assigns to each such function f an *infinity*—which he denotes"$\mathfrak{A}(f)$"—and defines an ordering on the infinities of such functions in the manner specified above. To complete the construction, Stolz defines addition and subtraction of the infinities by the rules:

$$\mathfrak{A}(f) + \mathfrak{A}(g) = \mathfrak{A}(f \cdot g)$$
$$\mathfrak{A}(f) - \mathfrak{A}(g) = \mathfrak{A}(f/g), \text{ if } \mathfrak{A}(f) > \mathfrak{A}(g),$$

and shows that the resulting structure (with equivalence classes of functions with equal infinities taken as the elements) is the positive cone of a non-Archimedean, divisible ordered abelian group, where $(1/n)\mathfrak{A}(f) = \mathfrak{A}(\sqrt[n]{f})$. Stolz further observes that one can supplement the above structure with an "ideal element" $\mathfrak{A}(1)$—*the order of finitude*—to serve as an additive identity, and that one can also consider *orders of vanishing* or *orders of infinitesimalitude* by taking into account functions f for which $\lim_{x\to\infty} f(x) = 0$.

While du Bois-Reymond usually restricted his investigations to families of the above said kind, on occasion he mistakenly assumed that each pair of increasing functions f and g from $\mathbb{R}^+$ to $\mathbb{R}^+$ for which $\lim_{x\to\infty} f(x) = +\infty$ and $\lim_{x\to\infty} g(x) = +\infty$ could be compared infinitarily in the manner described above. This led him to postulate the existence of an all-inclusive ordering of the infinities of such real functions—an *infinitary pantachie*, as he called it [1882, p. 220].[21] Such a pantachie, according to du Bois-Reymond, would provide a conception of a numerical linear continuum richer than that of Cantor and Dedekind.

However, having demonstrated (as Stolz [1879, p. 232] and Pincherle [1884, p. 742] had before him) that there could be no such all-inclusive ordering of the infinities of such functions, Georg Cantor proclaimed: "the '*infinitary pantachie*,' of du Bois-Reymond, belongs in the wastebasket *as nothing but paper numbers!*" [1895, p. 107]. Hausdorff, by contrast, suggested, "[t]here is no reason to reject the entire theory because of the possibility of incomparable functions as G. Cantor has done" [1907, p. 107], and in its place undertook the study of *maximally inclusive sets of pairwise comparable real functions*, each of which, retaining du Bois-Reymond's term, he calls an *infinitary pantachie* or a *pantachie* for short. This led him to his well-known investigation of η_1-orderings (and η_α-orderings more generally) [1907], and to the following less well-known theorem.

HAUSDORFF 1 (Hausdorff 1907; 1909). *Infinitary pantachies exist. If P is an infinitary pantachie, then P is an* η_1*-ordering of power* $2^{\aleph_0}$; *in fact, P is* (*up to isomorphism*) *the unique* η_1*-ordering of power* $\aleph_1$, *assuming* (*the Continuum Hypothesis*) CH.

Hausdorff's theorem is actually more general than our previous remarks suggest. For in addition to modifying du Bois-Reymond's conception of an infinitary pantachie, Hausdorff redirected du Bois-Reymond's investigation by investigating *numerical sequences* rather than continuous functions (although, see below), deleting the monotonicity assumption, and replacing the infinitary rank ordering with the *final rank ordering* (which was illustrated above). That is, Hausdorff redirected du Bois-Reymond's investigation to

[21] Du Bois-Reymond explains that his adjective "pantachie" derives from the Greek words for "everywhere".

the study of subsets of the set $\mathcal{B}$ of all numerical sequences

$$A = (a_1, a_2, a_3, \ldots, a_n, \ldots)$$

in which the a_n are real numbers, and he defines the "final ordering" on $\mathcal{B}$ (and subsets thereof) by the conditions $A < B$ if *eventually* $a_n < b_n$, $A = B$ if *eventually* $a_n = b_n$, $A > B$ if *eventually* $a_n > b_n$, and $A \parallel B$ (i.e., A is *incomparable* with B) in all other cases, where "eventually" means for all values of n with the exception of a finite number, thus for all $n \geq$ some n_0 [1909 in 2005, p. 276].[22] Hausdorff, who bases his theory on representative elements of the equivalences classes of eventually equal numerical sequences rather than on the equivalence classes themselves, calls a subset $\mathfrak{B}$ of $\mathcal{B}$ totally ordered by the final order a *pantachie* if it is not properly contained in another subset of $\mathcal{B}$ totally ordered by the final order.

However, as Hausdorff emphasizes, the above result is also applicable to pantachies consisting solely of continuous functions since *each pantachie of numerical sequences is order isomorphic to a pantachie of only continuous functions, and vice versa* [1907, p. 112; 1907 in 2005, p. 134]. For, as Hausdorff observes: "a *continuous* function $\alpha(x)$ can be associated with each numerical sequence A in an infinity of ways so that the infinitary relations are preserved. The most obvious way is to define a piecewise linear function by $\alpha(x) = a_n(n+1-x) + a_{n+1}(x-n)$ for $n \leq x \leq n+1$, i.e., by connecting the points with rectangular coordinates (n, a_n) by a line graph." And conversely, one may map each continuous function $f(x)$ to a cofinal sequence $f(1), f(2), \ldots, f(n), \ldots$ [1907, pp. 111–112; 1907 in 2005, pp. 133–134].

In his investigation of 1907, Hausdorff also raises the question of the existence of a pantachie that is algebraically a field, but he only makes partial headway in providing an answer. However, in 1909 he returned to the problem and provided a stunning positive answer. Indeed, beginning with the ordered set of numerical sequences of the form $(r, r, r, \ldots, r, \ldots)$ where r is a rational number, and utilizing what appears to be the very first algebraic application of his maximal principle, Hausdorff proves the following little-known, remarkable result.

HAUSDORFF 2 (1909). *There is a pantachie of numerical sequences, henceforth $\mathbb{H}_p$, that is an ordered field, whose field operations are given by*

$$A + B = (a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n, \ldots)$$
$$A - B = (a_1 - b_1, a_2 - b_2, \ldots, a_n - b_n, \ldots)$$
$$AB = (a_1 b_1, a_2 b_2, \ldots, a_n b_n, \ldots)$$

[22] Strictly speaking, in his [1907] unlike his [1909] and [1914, pp. 189–194], Hausdorff only considers numerical sequences in which the a_n are positive real numbers. However, the proof of the above theorem carries over to the more general case. Moreover, in his [1907], unlike his later works, he uses the term "finally" instead of "eventually" in the definitions of $<$, $>$, $=$ and $\parallel$.

$$A/B = (a_1/b_1, a_2/b_2, \dots, a_n/b_n, \dots),$$

where $A+B$, $A-B$, AB and A/B are defined up to final equality. Any such pantachie is, in fact, a real-closed ordered field.

Writing before Artin and Schrier [1926], Hausdorff of course does not refer to $\mathbb{H}_p$ as real closed; but he essentially establishes $\mathbb{H}_p$ is real-closed by showing it is the union of a chain of ordered fields, each of which admits no algebraic extension to a more inclusive ordered field. Moreover, recognizing the real-closed nature of $\mathbb{H}_p$ was not the only forward looking feature of Hausdorff 2. Indeed, as J. M. Plotkin observes in his Introduction to his recent translation of Hausdorff's paper: "the resulting structure would be recognized today by model theorists as the reduced power of the real field $\mathbb{R}$ over the index set $\mathbb{N}$ modulo the filter $Cof(\mathbb{N})$ of cofinite subsets of $\mathbb{N}$" [Plotkin 2005, p. 269].

The following result, whose proof uses the fact that $\mathbb{H}_p$ is a real-closed field that is an η_1-ordering of power $2^{\aleph_0}$, brings to the fore the relation between real-closed pantachies and **No**.

Theorem 17. *Let $\mathbb{H}_p$ be an ordered field that is a pantachie. Then $\mathbb{H}_p$ is isomorphic with an initial subfield of* **No** *extending* $\mathbf{No}(\omega_1)$; *in fact, assuming* CH, $\mathbb{H}_p$ *is isomorphic to* $\mathbf{No}(\omega_1)$ *and, hence, to* $\mathbb{R}((\tau^{\mathbf{No}(\omega_1)}))_{\omega_1}$.

Proof. By trivial variations in the argument employed in [Ehrlich 1988: Lemma 1 and p. 15], $\mathbf{No}(\omega_1)$ is a real-closed field that is an η_1-ordering of power $2^{\aleph_0}$. By Theorem 15, $\mathbf{No}(\omega_1) = \mathbb{R}((\tau^{\mathbf{No}(\omega_1)}))_{\omega_1}$ which, by Corollary 2, is isomorphic to $\mathbb{R}((t^{\mathbf{No}(\omega_1)}))_{\omega_1}$. But Esterle [1977: Section 3], extending the classical work of Alling [1962], has shown that every real-closed field that is an η_1-ordering contains an isomorphic copy of $\mathbb{R}((t^{\mathbf{No}(\omega_1)}))_{\omega_1}$. Accordingly, by employing the classical embedding theory for real-closed ordered fields in conjunction with sufficiently many applications of Theorem 7, the canonical isomorphism from $\mathbb{R}((t^{\mathbf{No}(\omega_1)}))_{\omega_1}$ onto $\mathbb{R}((\tau^{\mathbf{No}(\omega_1)}))_{\omega_1}$ can readily be extended to an isomorphism of a real-closed extension of $\mathbb{R}((t^{\mathbf{No}(\omega_1)}))_{\omega_1}$ onto an initial subfield of **No** that is an extension of

$$\mathbb{R}((\tau^{\mathbf{No}(\omega_1)}))_{\omega_1} = \mathbf{No}(\omega_1).$$

To complete the proof it only remains to recall the classical result of Erdös, Gillman and Henriksen [1955: also see Gillman and Jerison 1961: Theorem 13.13] that there is (up to isomorphism) a unique real-closed ordered field that is an η_1-ordering of power $\aleph_1$, if CH is the case. ⊣

Much as Hausdorff realized that Hausdorff 1 is also applicable to pantachies of continuous functions, he was aware the same is true of Hausdorff 2 (with the field operations appropriately construed). Apparently unaware of Hausdorff's work, Boshernitzan [1981], working with equivalence classes rather than with representative elements, essentially rediscovered Hausdorff 2 for the case of continuous real-valued functions $f(x)$

defined for sufficiently large x. In Boshernitzan's terminology, there are maximal *B-fields* (which, in virtue of his [1981: Proposition 3.4], are real-closed). Essentially being pantachies in Hausdorff's sense, maximal B-fields fall under the umbrella of Theorem 17.

One year after Hausdorff published HAUSDORFF 2, G. H. Hardy published the monograph on du-Bois-Reymond's theory from which we quoted above. The express purpose of the monograph is "to bring the *Infinitärcalcül* up to date, stating explicitly and proving carefully a number of general theorems the truth of which Du Bois-Reymond seems to have tacitly assumed" [1910: Preface]. For this purpose Hardy identified, and provided a systematic analysis of, a class of *logarithmico-exponential* functions–*L-functions* as he calls them—consisting of "*real one-valued functions defined, for all values of x greater than some definite value, by a finite combination of the ordinary algebraic operations* (viz. $+$, $-$, $\cdot$, $\div$ and $\sqrt[n]{\ }$) *and the functional symbols* $\log(\dots)$ and $e^{(\dots)}$, *operating on the variable x and on real constants*" [Hardy 1910; 1924, p. 17]. Drawing inspiration from Hardy's work on L-functions, Bourbaki [1951, pp. 107–126; 1976] introduced the idea of a *Hardy field* and developed the basic theory thereof. According to Aschenbrenner and van den Dries, "[t]his theory is the modern incarnation of ideas on "Orders of Infinity" originating with Du Bois-Reymond and put on a firm foundation by Hardy" [2000, p. 309].

Bourbaki bases the theory on *germs of functions* at ∞. More specifically, let f and g be real valued functions defined on intervals of the form $\{x \in \mathbb{R}\colon x > a\}$ for some $a \in \mathbb{R}$. For example, $\{(x, x^3)\colon x > 2\}$ and $\{(x, x + 7)\colon x > -4\}$ are real-valued functions of this form. f and g are said to have the same *germ* at ∞, written $\overline{f} = \overline{g}$, if eventually $f(x) = g(x)$. A *Hardy Field H* is a set of germs at ∞ that is closed under differentiation ($\overline{f}' \in H$ whenever $\overline{f} \in H$, where f' is the derivative of f) that forms a field under the component-wise operations $\overline{f} + \overline{g} = \overline{f + g}$ and $\overline{f} \cdot \overline{g} = \overline{f \cdot g}$, where $f + g$ and $f \cdot g$ are the corresponding operations on functions. H admits a relational extension to an ordered field in accordance with the definition: $\overline{f} > \overline{g}$ if $\overline{f - g}$ is ultimately positive for $\overline{f}, \overline{g} \in H$. Henceforth, we will not distinguish notationally between a function and its germ.

A Hardy field H will be said to be a *real-closed logarithmico-exponential Hardy field* if it is a real-closed extension of $\mathbb{R}(x)$ such that $\exp(f) \in H$ if $f \in H$, and $\log(f) \in H$ if $f \in H$ and $f > 0$, where $\exp(f)$ and $\log(f)$ denote the germs of $\exp \circ f$ and $\log \circ f$, respectively. As is well known, every Hardy field admits an extension to a real-closed logarithmico-exponential Hardy field (see, for example, [Kuhlmann 2001: Theorem 6.17]).[23]

[23]What we call a real-closed logarithmico-exponential Hardy field, Kuhlmann [2000, p. 94] simply calls an exponential Hardy field.

Following Bourbaki [1951, pp. 113–114], let **H** be the Hardy field of germs of L-functions described above. Also let **LE** be the smallest real-closed logarithmico-exponential Hardy field containing **H**.[24] Then **LE** is the smallest real-closed logarithmico-exponential Hardy field in the sense that every real-closed logarithmico-exponential Hardy field contains **LE**.

Theorem 19 below expresses a critical relation between **LE** and **No**. We now set the stage for proving the theorem by introducing some terminology and recalling some further properties of the $\mathbf{No}(\lambda)$'s introduced in Theorem 15.

For each $m \in \mathbb{N}$, let $\mathbb{R}\{X_1, \dots, X_m\}$ be the ring of all real power series in $X_1, \dots, X_m$ that converge in a neighborhood of $I^m = [-1, 1]^m$. Moreover, for $f \in \mathbb{R}\{X_1 \dots, X_m\}$ let $\widetilde{f}\colon \mathbb{R}^m \to \mathbb{R}$ be given by

$$\widetilde{f}(x) = \begin{cases} f(x), & \text{for } x \in I^m \\ 0, & \text{for } x \notin I^m \end{cases}.$$

Following van den Dries, Macintyre and Marker (D-M-M) [1994], the $\widetilde{f}$'s so defined are called *restricted analytic functions*. Let $\mathbb{R}_{\mathrm{an}}$ be the reals with its natural L_{an}-structure, where L_{an} is the language of ordered rings $\{<, 0, 1, +, -, \cdot\}$ supplemented by a new function symbol for each function $\widetilde{f}$, and further let $\mathbf{No}_{\mathrm{an}}$ be the relational extension of **No** that arises by extending **No** to an L_{an}-structure as in [van den Dries and Ehrlich 2001]. Also let $\mathbb{R}_{\mathrm{an,exp}}$ be the reals with its natural $L_{\mathrm{an,exp}}$-structure, where $L_{\mathrm{an,exp}}$ is L_{an} supplemented by a new function symbol for exponentiation. And, finally, let e^x denote (with minor abuse of notation) both the exponential function in the reals and the well-behaved exponential function on the surreals mentioned in §1 that extends the familiar operation on real numbers [Gonshor 1986, Chapter 10].

THEOREM 18 (van den Dries and Ehrlich 2001). *Let λ be an ε-number. Then the field $\mathbf{No}(\lambda)$ is closed under exponentiation, and under taking logarithms of positive elements. In fact, the field $\mathbf{No}(\lambda)$ equipped with the restricted analytic functions and exponentiation induced by* **No** *is an elementary substructure of* $(\mathbf{No}_{\mathrm{an}}, e^x)$ *and an elementary extension of* $(\mathbb{R}_{\mathrm{an}}, e^x)$.

Now suppose ε_0 is the least ε-number, i.e., let ε_0 be the least ordinal greater than $\omega, \omega^{\omega}, \omega^{\omega^{\omega}}, \dots$.

THEOREM 19. *There is a unique ordered field embedding h of* **LE** *into* $\mathbf{No}(\varepsilon_0)$ *defined by the conditions*: $h(r) = r$ *for all* $r \in \mathbb{R}$, $h(x) = \omega$ *and* $h(e^f) = e^{h(f)}$

[24]In his [1910, p. 20; 1924, p. 17], Hardy mentions a possible extension of his system **H** of L-functions, which he adopts in his [1912]. **LE**, which is characterized in a number of different though equivalent ways in the literature (cf. [Boshernitzan 1981, p. 235], [van den Dries, Macintyre and Marker 1997, p. 426] and [Kuhlmann 2001, pp. 94, 111]), is sometimes generously attributed to Hardy. However, van den Dries has shown the author a proof that **LE** *properly contains* both **H** and Hardy's extension thereof.

for all $f \in \mathbf{LE}$. *Moreover, the image of* $\mathbf{LE}$ *is an initial subtree of* $\mathbf{No}(\varepsilon_0)$, *and the image of the sequence* $x, e^x, e^{e^x}, e^{e^{e^x}}, \ldots$ *is cofinal with* $\mathbf{No}(\varepsilon_0)$. *In particular,* $h(x) = \omega$, $h(e^x) = e^\omega = \omega^\omega$, $h(e^{e^x}) = e^{e^\omega} = \omega^{\omega^\omega}$, $h(e^{e^{e^x}}) = e^{e^{e^\omega}} = \omega^{\omega^{\omega^\omega}}, \ldots$.

Proof. (Existence): The proof is an adaptation of an argument of D-M-M [1997]. Let $(\mathbf{No}(\varepsilon_0)_{\mathrm{an}}, e^x)$ be the field $\mathbf{No}(\varepsilon_0)$ equipped with the restricted analytic functions and exponentiation induced by $\mathbf{No}$. By Theorem 18, $\mathbf{No}(\varepsilon_0)_{\mathrm{an}}, e^x)$ is an elementary extension of $(\mathbb{R}_{\mathrm{an}}, e^x)$. Let $\mathcal{H}_{\mathbf{LE}}$ be the smallest real-closed subfield of $\mathbf{No}(\varepsilon_0)$ containing $\mathbb{R}(\omega)$ and closed under log and exp, and further let $\mathcal{H}_{\mathrm{an,exp}}$ be the smallest elementary submodel of $(\mathbf{No}(\varepsilon_0)_{\mathrm{an}}, e^x)$ containing $\mathbb{R}(\omega)$. Clearly $\mathbb{R}(\omega) \subset \mathcal{H}_{\mathbf{LE}} \subset \mathcal{H}_{\mathrm{an,exp}}$. Now let $H(\mathbb{R}_{\mathrm{an,exp}})$ be the field of germs at ∞ of the functions $f \colon \mathbb{R} \to \mathbb{R}$ definable in $\mathbb{R}_{\mathrm{an,exp}}$. In virtue of D-M-M's [1994 §5], there is a well-defined $L_{\mathrm{an,exp}}$-isomorphism from $H(\mathbb{R}_{\mathrm{an,exp}})$, which is a Hardy field, onto $\mathcal{H}_{\mathrm{an,exp}}$ that sends the germ of the identity function x to ω [1997, p. 426]. The (exponential) ordered field embedding h in the statement of the theorem is simply the restriction of the $L_{\mathrm{an,exp}}$-map to $\mathbf{LE}$.

(Uniqueness): Extend the language $L_{\mathrm{an,exp}}$ by new function symbols, one for each semialgebraic function from $\mathbb{R}^n$ to $\mathbb{R}$. Let this extended language be L', and construe $\mathbf{LE}$ and $\mathbf{No}$ as L'-structures (extending the L'-structure $\mathbb{R}$) in the obvious way. Then $\mathbf{LE}$ is generated as an L'-structure over $\mathbb{R}$ by (the germ of) x, that is, each element of $\mathbf{LE}$ is given by an L'-term $t(x)$. Any embedding h as defined above is an L'-homomorphism and thus maps any element $t(x)$ in $\mathbf{LE}$ to $t(\omega)$ which shows that h is unique.

(Initial Subtree): By Corollary 2 (ii), $\mathbf{No}(\varepsilon_0)_{\mathrm{an}}, e^x)$ is a truncation closed, cross-sectional subfield of the Hahn field in $\mathbf{No}$ induced by $\mathbf{No}(\varepsilon_0)$. Moreover, by essentially the same argument D-M-M employ to prove that their $H_{\mathbf{LE}}$ and $H_{\mathrm{an,exp}}$ are truncation closed subfields of the ordered power series field $\mathbb{R}((t))^{\mathbf{LE}}$ [1997: Corollary 3.9, p. 425], $\mathcal{H}_{\mathbf{LE}}$ and $\mathcal{H}_{\mathrm{an,exp}}$ are truncation closed subfields of $\mathbf{No}(\varepsilon_0)$ with its natural power series structure. This together with the fact that $\mathbb{R}(\omega) \subset \mathcal{H}_{\mathbf{LE}} \subset \mathcal{H}_{\mathrm{an,exp}}$ implies that $\mathcal{H}_{\mathbf{LE}}$ and $\mathcal{H}_{\mathrm{an,exp}}$ are truncation closed, cross sectional subfields of the Hahn fields in $\mathbf{No}$ induced by $\mathcal{H}_{\mathbf{LE}}$ and $\mathcal{H}_{\mathrm{an,exp}}$ respectively; and so, by Theorem 14, $\mathcal{H}_{\mathbf{LE}}$ and $\mathcal{H}_{\mathrm{an,exp}}$ are initial subfields (of $\mathbf{No}$ and hence) of $\mathbf{No}(\varepsilon_0)$. Finally, the cofinality portion of the theorem is an immediate consequence of the definitions of h and $\mathbf{No}(\varepsilon_0)$ together with Theorems 10.11 and 10.14 of [Gonshor 1986]. ⊣

LE includes the germs of the aforementioned functions employed by Stolz in his groundbreaking investigation of 1883. Using this and the fact that every real-closed logarithmico-exponential Hardy field contains **LE**, it is a simple matter to prove the following result that expresses the historically important relation between Stolz's orders of infinity of functions and the divisible ordered abelian group of Archimedean classes of real-closed logarithmico-exponential Hardy fields.

PROPOSITION 5. *Let H be a real-closed logarithmico-exponential Hardy field. Stolz's system of orders of infinity is isomorphic to the strictly positive cone of a subgroup of the value group* (*i.e., the divisible ordered abelian group of Archimedean classes*) *of H.*

Every Hardy field can be extended to a maximal (non-extensible) Hardy field. Maximal Hardy fields are real-closed logarithmico-exponential Hardy fields having other nice properties [Sjödin 1971; Robinson 1972; Boshernitzan 1981]. Whether maximal Hardy fields are pantachies or even η_1-orderings does not appear to have been explored. We suspect this is a consequence of the fact that the modern algebraic works on orders of infinity, beginning with the writings of Hardy, have lost sight of Hausdorff's classical investigations.[25] On the other hand, Sjödin [1971: Lemma 4, p. 221 and Theorem 6, p. 231] has shown that maximal Hardy fields contain no countable cofinal subset, and this coupled with the fact that maximal Hardy fields are ordered fields, further implies that maximal Hardy fields have no countable coinitial subset and that for each field member x, $\{y : y < x\}$ ($\{y : y > x\}$) has no countable cofinal (coinitial) subset. Accordingly, whether or not maximal Hardy fields are η_1-orderings turns on whether or not they have (ω, ω^*)-gaps. While we suspect they do not, we know of no proof of this. Furthermore, by Hausdorff's construction, any maximal Hardy field that is not a pantachie can be extended to a real-closed pantachie. However, while the functions contained in the set-theoretic difference of the two structures could be continuous (and perhaps even differentiable), the resulting pantachie could not be closed under differentiation.

§9. Hyperreal number systems. In the early 1960s Abraham Robinson [1961, 1966] made the momentous discovery that among the real-closed extensions of the reals there are number systems that can provide the basis for a consistent and entirely satisfactory nonstandard approach to analysis based on infinitesimals. Robinson motivated his work with the following words, which make clear that he was well aware of the non-Archimedean contributions of his forerunners.

> It is our main purpose to show that these models [i.e., number systems] provide a natural approach to the age-old problem

[25] An important exception is Abraham Robinson's [1972]. However, outside of Robinson's passing remark to the effect that in [Hausdorff 1909] "Felix Hausdorff made significant contributions" to du Bois-Reymond's theory, no indication of Hausdorff's contributions is provided.

Kurt Gödel also appreciated the significance of Hausdorff's work on pantachies, though his interest was largely set theoretic. For Gödel's interest in pantachies and the relevance pantachies have for contemporary set theory, see [Kanamori 2007, §8].

> of producing a calculus involving infinitesimal (infinitely small) and infinitely large quantities. As is well known, the use of infinitesimals, strongly advocated by Leibnitz and unhesitatingly accepted by Euler fell into disrepute after the advent of Cauchy's methods which put Mathematical Analysis on a firm foundation. Accepting Cauchy's standards of rigor, later figures in the domain of non-archimedean quantities concerned themselves only with small fragments of the edifice of Mathematical Analysis. We mention only du Bois-Reymond's Calculus of infinities [1875] and Hahn's work on non-archimedean fields [1907] which in turn were followed by the theories of Artin–Schreier [1926] and, returning to analysis, of Hewitt [1948] and Erdös, Gillman and Henriksen [1955]. Finally, a recent and rather successful effort at developing a calculus of infinitesimals is due to Schmieden and Laugwitz [1958] whose number system consists of infinite sequences of rational numbers. The drawback of this system is that it includes zero-divisors and that it is only partially ordered. In consequence, many classical results of the Differential and Integral calculus have to be modified to meet the changed circumstance. [1961/1979, p. 4]

Being ordered fields, Robinson's number systems do not have the just-cited consequences of the number system of Schmieden and Laugwitz. By analogy with Thoralf Skolem's [1934] *nonstandard model of arithmetic*, a number system from which Robinson drew inspiration, Robinson called his totally ordered number systems *nonstandard models of analysis*. These number systems, which are now more often called *hyperreal number systems* [Keisler 1976, 1994], may be characterized as follows: let $\langle \mathbb{R}, S \colon S \in \mathfrak{F} \rangle$ be a relational structure where $\mathfrak{F}$ is the set of all finitary relations defined on $\mathbb{R}$ (including all functions). Furthermore, let ${}^*\mathbb{R}$ be a proper extension of $\mathbb{R}$ and for each n-ary relation $S \in \mathfrak{F}$ let ${}^*S$ be an n-ary relation on ${}^*\mathbb{R}$ that is an extension of S. The structure $\langle {}^*\mathbb{R}, \mathbb{R}, {}^*S \colon S \in \mathfrak{F} \rangle$ is said to be a hyperreal number system if it satisfies the *transfer principle*: every n-tuple of real numbers satisfies the same first-order formulas in $\langle \mathbb{R}, S \colon S \in \mathfrak{F} \rangle$ as it satisfies in $\langle {}^*\mathbb{R}, \mathbb{R}, {}^*S \colon S \in \mathfrak{F} \rangle$.

The existence of hyperreal number systems is a consequence of the compactness theorem of first-order logic and there are a number of algebraic techniques that can be employed to construct such a system. One commonly used technique is the *ultrapower* construction (cf. Keisler [1976, pp. 48–57]; Goldblatt [1998, chapter 3]), though not all hyperreal number systems can be obtained this way. By results of H. J. Keisler [1963; 1976, pp. 58–59], however, every hyperreal number system is isomorphic to a *limit ultrapower*.

Since every real-closed ordered field is isomorphic to an initial subfield of **No**, the underlying ordered field of any hyperreal number system is likewise isomorphic to an initial subfield of **No**. For example, the familiar ultrapower construction of a hyperreal number system as a quotient ring of $\mathbb{R}^{\mathbb{N}}$ (modulo a given nonprincipal ultrafilter on $\mathbb{N}$) is isomorphic to $\mathbf{No}(\omega_1) = \mathbb{R}((\tau^{\mathbf{No}(\omega_1)}))_{\omega_1}$ assuming CH, insofar as any such quotient ring of $\mathbb{R}^{\mathbb{N}}$ is a real-closed field that is an η_1-ordering of power $2^{\aleph_0}$ [Goldblatt 1998, p. 33]. Similarly, if we assume there is an uncountable inaccessible cardinal, ω_α being the least, then $\mathbf{No}(\omega_\alpha) = \mathbb{R}((\tau^{\mathbf{No}(\omega_\alpha)}))_{\omega_\alpha}$ is isomorphic to the underlying ordered field in the hyperreal number system employed by Keisler in his *Foundations of Infinitesimal Calculus* [1976].

In the remainder of this section we will see that $\mathbf{No} = \mathbb{R}((\tau^{\mathbf{No}}))_{On}$ itself can be employed as the underlying ordered field in what may be naturally described as the maximal hyperreal number system in NBG. For this purpose, we begin with Keisler's well-known foundation for Robinson's Theory of Infinitesimals presented in his just-cited work and in the Epilog of the 2nd edition of his corresponding text [1986].

Keisler's Axioms for Hyperreal Number Systems

Axiom A. $\mathbb{R}$ is a complete ordered field.

Axiom B. $\mathbb{R}^*$ is a proper ordered field extension of $\mathbb{R}$.

Axiom C (Function Axiom). For each function f of n variables there is a corresponding hyperreal function f^* of n variables, called the *natural extension* of f. The field operations of $\mathbb{R}^*$ are the natural extensions of the field operations of $\mathbb{R}$.

Axiom D (Solution Axiom). If two systems of formulas [finite sets of equations or inequalities between terms] have exactly the same real solutions, they have exactly the same hyperreal solutions.

Commenting on these axioms, Keisler observes:

> The real numbers are the unique complete ordered field. By analogy, we would like to uniquely characterize the hyperreal structure $\langle \mathbb{R}, \mathbb{R}^*, * \rangle$ by some sort of completeness property. However, we run into a set-theoretic difficulty; there are structures $\mathbb{R}^*$ of arbitrary large cardinal number which satisfy Axioms A–D, so there cannot be a largest one. Two ways around this difficulty are to make $\mathbb{R}^*$ a proper class rather than a set, or to put a restriction on the cardinal number of $\mathbb{R}^*$. We use the second method because it is simpler. [Keisler 1976, p. 59]

Central to Keisler's solution to the uniqueness problem is the idea of a *saturated* hyperreal number system. A hyperreal number system $\langle {}^*\mathbb{R}, \mathbb{R}, {}^*S : S \in \mathfrak{F} \rangle$ is said to be κ-*saturated* if any set of formulas with constants from ${}^*\mathbb{R}$ of power less than κ is satisfiable whenever it is finitely satisfiable. If κ

is the power of ${}^*\mathbb{R}$, the hyperreal number system is said to be saturated. Although there is a wide range of hyperreal number systems in ZFC that are saturated to varying degrees of power less than the power of ${}^*\mathbb{R}$, saturated hyperreal number systems cannot be proved to exist in ZFC. In virtue of classical results from the theory of saturated models, however, there is (up to isomorphism) a unique saturated hyperreal number system of power κ whenever $\kappa > 2^{\aleph_0}$ and either κ is (strongly) inaccessible or the generalized continuum hypothesis (GCH) holds at κ (i.e., $\kappa = \aleph_{\alpha+1} = 2^{\aleph_\alpha}$ for some α). So, for example, by supplementing ZFC with the assumption of the existence of an uncountable inaccessible cardinal, one can obtain uniqueness (up to isomorphism) by limiting attention to saturated hyperreal number systems having the least such power.

With the above in mind, Keisler sets the stage to overcome the uniqueness problem by introducing the following axiom, and then proceeds to prove the subsequent theorem.

Axiom E (Saturation Axiom). Let S be a set of equations and inequalities involving real functions, hyperreal constants, and variables, such that S has a smaller cardinality than $\mathbb{R}^*$. If every finite subset of S has a hyperreal solution, then S has a hyperreal solution.

Keisler 1 [1976]. *There is up to isomorphism a unique structure* $\langle \mathbb{R}, \mathbb{R}^*, * \rangle$ *such that Axioms A–E are satisfied and the cardinality of* $\mathbb{R}^*$ *is the first uncountable inaccessible cardinal.*

If $\langle \mathbb{R}, \mathbb{R}^*, * \rangle$ satisfies Axioms A–D, then $\mathbb{R}^*$ is of course real-closed. It is also evident that, if $\langle \mathbb{R}, \mathbb{R}^*, * \rangle$ further satisfies Axiom E, then $\mathbb{R}^*$ is an η_α-ordering of power $\aleph_\alpha$, where $\aleph_\alpha$ is the power of $\mathbb{R}^*$. Accordingly, since (in NBG) **No** is (up to isomorphism) the unique real-closed field that is an η_{On}-ordering of power $\aleph_{On}$ (see Theorem 3 and Note 5), $\mathbb{R}^*$ would be isomorphic to **No** in any model of A–E that is a proper class (in NBG).

Motivated by the above, in September of 2002 we wrote to Keisler, reminded him of his idea of making "$\mathbb{R}^*$ a proper class rather than a set", observed that in such a model $\mathbb{R}^*$ would be isomorphic to **No**, and inquired how he had intended to prove the result for proper classes since the proof he employs, which uses a superstructure, cannot be carried out for proper classes in NBG or in any of the most familiar alternative class theories.[26]

In response, Keisler offered the following revealing remarks, which he has graciously granted us permission to reproduce.

[26]The difficulty is that, in addition to **No**, such a proof would require the power class $P(\mathbf{No})$ of all subclasses of **No**, the power class $P(P(\mathbf{No}))$ of all subclasses of subclasses of **No** and so on, none of which exist in NBG.

On the other hand, as we mentioned to Keisler, his proof *can* be extended to the class case in the equiconsistent set theory of Ackermann (see Note 7).

> What I had in mind in getting around the uniqueness problem for the hyperreals in "Foundations of Infinitesimal Calculus" was to work in NBG with global choice (i.e., a class of ordered pairs that well orders the universe). This is a conservative extension of ZFC. I was not thinking of doing it within a superstructure, but just getting four objects $\mathbb{R}$, $\mathbb{R}^*$, $<^*$, $*$ which satisfy Axioms A–E. $\mathbb{R}$ is a set, $\mathbb{R}^*$ is a proper class, $<^*$ is a proper class of ordered pairs of elements of $\mathbb{R}^*$, and $*$ is a proper class of ordered triples (f, x, y) of sets, where f is an n-ary real function for some n, x is an n-tuple of elements of $\mathbb{R}^*$ and y is in $\mathbb{R}^*$. In this setup, $f^*(x) = y$ means that (f, x, y) is in the class $*$. There should be no problem with $*$ being a legitimate entity in NBG with global choice. Since each ordered triple of sets is again a set, $*$ is just a class of sets. I believe that this can be done in an explicit way so that $\mathbb{R}$, $\mathbb{R}^*$, $<^*$, and $*$ are definable in NBG with an extra symbol for a well ordering of V. [Keisler to Ehrlich 10/20/02]

Moreover, in a subsequent letter, Keisler went on to add:

> I did not do it that way because it would have required a longer discussion of the set theoretic background. [Keisler to Ehrlich 5/14/06]

Anticipating, however, that the details of Keisler's proof-plan can indeed be carried out in NBG, we draw the main body of the paper to a close by announcing the following intriguing result.

THEOREM 20. *In NBG there is* (*up to isomorphism*) *a unique structure* $\langle \mathbb{R}, \mathbb{R}^*, * \rangle$ *such that Axioms A–E are satisfied and for which* $\mathbb{R}^*$ *is a proper class*; *moreover, in such a structure* $\mathbb{R}^*$ *is isomorphic to* **No**. *Such a structure is in fact* (*to within isomorphism*) *the unique model of Axioms A–E whose existence can be established in NBG without additional assumptions.*

Appendix: Proof of Theorem 7

Central to our proof of Theorem 7 is Theorem 14 and the following result, the first part of which is due to Mourgues and Ressayre [1993: Lemma 3.4], and the second part of which is a special case of a result of van den Dries [1991: Theorem] (that was rediscovered by Fornasiero [2006]) that both strengthens and generalizes a result of F. Delon that was rediscovered and used to great effect by Mourgues and Ressayre [1991: Lemma 3.5; 1993: Lemma 3.5].

PROPOSITION 6. *Let R be an ordered subfield of* $\mathbb{R}$, Γ *be an ordered abelian group, and A be a truncation closed subfield of* $\mathbb{R}((t^\Gamma))$. *Then*:

I. *If $y \in \mathbb{R}((t^\Gamma)) - A$ has all of its proper truncations in A, then the subfield $A(y)$ of $\mathbb{R}((t^\Gamma))$ is also truncation closed.*

II. *If $R \subset A$, then the algebraic closure of A in $\mathbb{R}((t^{\Gamma}))$ is also truncation closed.*

We also use the simple generalization of Corollary 2 that, if Γ is an initial subgroup of **No** and R is a subfield of $\mathbb{R}$, then there is a unique initial subfield $R((\tau^{\Gamma}))$ of **No** containing $\{\omega^{\gamma} : \gamma \in \Gamma\}$ that is isomorphic to $\mathbb{R}((t^{\Gamma}))$.

Proof of Theorem 7. Let A be a real-closed initial subfield of **No** (whose universe is a set). Then $\Gamma = \{\gamma \in \mathbf{No} : \omega^{\gamma} \in A\}$ is a divisible initial subgroup of **No** and $R = \{r \in \mathbb{R} : r\omega^0 \in A\}$ is a real-closed initial subfield of **No**. Let x be the simplest member of **No** that fills a cut in A. Then all the proper truncations of the Conway name of x are in A. Either $x \in \mathbb{R}((\tau^{\Gamma}))$ or $x \notin \mathbb{R}((\tau^{\Gamma}))$. First suppose $x \in \mathbb{R}((\tau^{\Gamma}))$. Then by Part I of Proposition 6, $A(x)$ is a truncation closed subfield of $\mathbb{R}((\tau^{\Gamma}))$. Now let $\overline{R}$ be the real-closure in **No** of $\{r \in \mathbb{R} : r\omega^0 \in A(x)\}$. Since $A(x) \subseteq \overline{R}((\tau^{\Gamma}))$ and $\overline{R}((\tau^{\Gamma}))$ is a truncation closed subfield of $\mathbb{R}((\tau^{\Gamma}))$, $A(x)$ is a truncation closed subfield of $\overline{R}((\tau^{\Gamma}))$. Moreover, by repeatedly applying Part I of Proposition 6 to a suitable (possibly empty) set of elements of the form $r\omega^0$ where $r \in \overline{R} - R$, the subfield $A(x, \overline{R})$ of $\overline{R}((\tau^{\Gamma}))$ generated by $A(x) \cup \overline{R}$ is likewise seen to be a truncation closed subfield of $\overline{R}((\tau^{\Gamma}))$. But then, by Part II of Proposition 6, so is the real-closure of $A(x, \overline{R})$ in $\overline{R}((\tau^{\Gamma}))$. Moreover, since A is a cross-sectional subfield of $\overline{R}((\tau^{\Gamma}))$, so is the real-closure of $A(x, \overline{R})$ in $\overline{R}((\tau^{\Gamma}))$. Furthermore, since $A(x) \subseteq A(x, \overline{R})$ and the real-closure of $A(x)$ in $\overline{R}((\tau^{\Gamma}))$ contains $A(x, \overline{R})$, the real-closures of $A(x)$ and $A(x, \overline{R})$ in $\overline{R}((\tau^{\Gamma}))$ coincide; from which it follows that the real-closure of $A(x)$ in $\overline{R}((\tau^{\Gamma}))$ is an initial subfield of $\overline{R}((\tau^{\Gamma}))$ and, hence, of **No**. Next suppose $x \notin \mathbb{R}((\tau^{\Gamma}))$. Then either (i) $x = \omega^{y}$ where y is the simplest member of **No** that fills a cut in Γ, the cuts $(\Gamma, \varnothing)$ and $(\varnothing, \Gamma)$ not being excluded, or (ii) $x = \sum_{\alpha<\beta} \omega^{\gamma_\alpha} \cdot r_\alpha \pm \omega^{y}$ where the γ_αs constitute a coinitial subset of Γ and y is the simplest member of **No** that fills the cut $(\varnothing, \Gamma)$. However, since $A(x) = A(\omega^{y})$ when (ii) is the case, (ii) reduces to a special case of (i). Thus, suppose (i) is the case. Then, by Theorem 6, the divisible subgroup Γ' of **No** generated by $\Gamma \cup \{y\}$ is initial. Moreover, since A is a truncation closed subfield of $\mathbb{R}((\tau^{\Gamma'}))$, by Part I of Proposition 6 $A(\omega^{y})$ is a truncation closed subfield of $\mathbb{R}((\tau^{\Gamma'}))$, and by Part II, so is its real-closure in $\mathbb{R}((\tau^{\Gamma'}))$. Moreover, since the real-closure of $A(\omega^{y})$ in $\mathbb{R}((\tau^{\Gamma'}))$ is a cross sectional subfield of $\mathbb{R}((\tau^{\Gamma'}))$ in **No**, by Theorem 14, the real-closure of $A(\omega^{y})$ in $\mathbb{R}((\tau^{\Gamma'}))$ is an initial subfield of $\mathbb{R}((\tau^{\Gamma'}))$ and, hence, of **No**. $\dashv$

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