The Black-Scholes Model

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Options Markets

The Black-Merton-Scholes-Merton (BMS) model

- Black and Scholes (1973) and Merton (1973) derive option prices under the following assumption on the stock price dynamics,

$$dS_t = \mu S_t dt + \sigma S_t dW_t \quad \text{(explained later)}$$

- The binomial model: Discrete states and discrete time (The number of possible stock prices and time steps are both finite).
- The BMS model: Continuous states (stock price can be anything between 0 and ∞) and continuous time (time goes continuously).
- Scholes and Merton won Nobel price. Black passed away.
- BMS proposed the model for stock option pricing. Later, the model has been extended/twisted to price currency options (Garman&Kohlhagen) and options on futures (Black).
- I treat all these variations as the same concept and call them indiscriminately the BMS model (combine chapters 13&14).

Primer on continuous time process

$$dS_t = \mu S_t dt + \sigma S_t dW_t$$

- The driver of the process is W_t, a Brownian motion, or a Wiener process.
- The process W_t generates a random variable that is normally distributed with mean 0 and variance t, $\phi(0, t)$. (Also referred to as Gaussian.)
- The process is made of independent normal increments $dW_t \sim \phi(0, dt)$.
  - "d" is the continuous time limit of the discrete time difference (Δ).
  - Δt denotes a finite time step (say, 3 months), dt denotes an extremely thin slice of time (smaller than 1 milisecond).
  - It is so thin that it is often referred to as *instantaneous*.
  - Similarly, $dW_t = W_{t+dt} - W_t$ denotes the instantaneous increment (change) of a Brownian motion.
- By extension, increments over non-overlapping time periods are independent: For $(t_1 > t_2 > t_3)$, $(W_{t_3} - W_{t_2}) \sim \phi(0, t_3 - t_2)$ is independent of $(W_{t_2} - W_{t_1}) \sim \phi(0, t_2 - t_1)$.

Properties of a normally distributed random variable

$$dS_t = \mu S_t dt + \sigma S_t dW_t$$

- If $X \sim \phi(0,1)$, then $a + bX \sim \phi(a, b^2)$.
- If $y \sim \phi(m, V)$, then $a + by \sim \phi(a + bm, b^2 V)$.
- Since $dW_t \sim \phi(0, dt)$, the instantaneous price change $dS_t = \mu S_t dt + \sigma S_t dW_t \sim \phi(\mu S_t dt, \sigma^2 S_t^2 dt)$.
- The instantaneous return $\frac{dS}{S} = \mu dt + \sigma dW_t \sim \phi(\mu dt, \sigma^2 dt)$.
  - Under the BMS model, μ is the annualized mean of the instantaneous return — instantaneous mean return.
  - σ^2 is the annualized variance of the instantaneous return — instantaneous return variance.
  - σ is the annualized standard deviation of the instantaneous return — instantaneous return volatility.

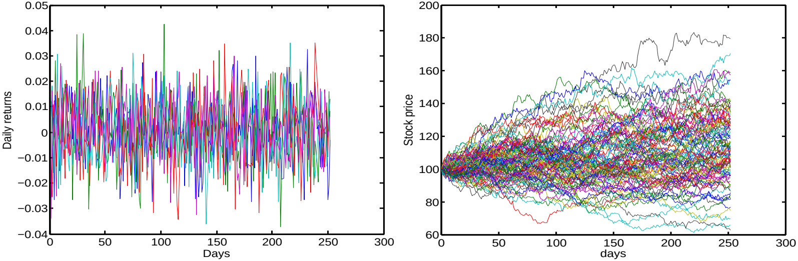
Geometric Brownian motion

$$dS_t/S_t = \mu dt + \sigma dW_t$$

- The stock price is said to follow a geometric Brownian motion.
- μ is often referred to as the drift, and σ the diffusion of the process.
- Instantaneously, the stock price change is normally distributed, $\phi(\mu S_t dt, \sigma^2 S_t^2 dt)$.
- Over longer horizons, the price change is lognormally distributed.
- The log return (continuous compounded return) is normally distributed over all horizons:
  $d \ln S_t = \left(\mu - \frac{1}{2}\sigma^2\right) dt + \sigma dW_t$. (By Ito's lemma)
  - $d \ln S_t \sim \phi(\mu dt - \frac{1}{2}\sigma^2 dt, \sigma^2 dt)$.
  - $\ln S_t \sim \phi(\ln S_0 + \mu t - \frac{1}{2}\sigma^2 t, \sigma^2 t)$.
  - $\ln S_T/S_t \sim \phi\left(\left(\mu - \frac{1}{2}\sigma^2\right)(T-t), \sigma^2(T-t)\right)$.
- Integral form: $S_t = S_0 e^{\mu t - \frac{1}{2}\sigma^2 t + \sigma W_t}$, $\quad \ln S_t = \ln S_0 + \mu t - \frac{1}{2}\sigma^2 t + \sigma W_t$

Simulate 100 stock price sample paths

$$dS_t = \mu S_t dt + \sigma S_t dW_t, \quad \mu = 10\%, \sigma = 20\%, S_0 = 100, t = 1.$$



- Stock with the return process: $d \ln S_t = (\mu - \frac{1}{2}\sigma^2)dt + \sigma dW_t$.
- Discretize to daily intervals $dt \approx \Delta t = 1/252$.
- Draw standard normal random variables $\varepsilon(100 \times 252) \sim \phi(0, 1)$.
- Convert them into daily log returns: $R_d = (\mu - \frac{1}{2}\sigma^2)\Delta t + \sigma\sqrt{\Delta t}\varepsilon$.
- Convert returns into stock price sample paths: $S_t = S_0 e^{\sum_{d=1}^{252} R_d}$.

The key idea behind BMS

- The option price and the stock price depend on the same underlying source of uncertainty.
- The Brownian motion dynamics imply that if we slice the time thin enough (dt), it behaves like a binominal tree.
- Reversely, if we cut Δt small enough and add enough time steps, the binomial tree converges to the distribution behavior of the geometric Brownian motion.
  - Under this thin slice of time interval, we can combine the option with the stock to form a riskfree portfolio.
  - Recall our hedging argument: Choose Δ such that $f - \Delta S$ is riskfree.
  - The portfolio is riskless (under this thin slice of time interval) and must earn the riskfree rate.
  - Magic: μ does not matter for this portfolio and hence does not matter for the option valuation. Only σ matters.
    - We do not need to worry about risk and risk premium if we can hedge away the risk completely.

Partial differential equation

- The hedging argument mathematically leads to the following partial differential equation:

$$\frac{\partial f}{\partial t} + (r-q)S\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} = rf$$

  - At nowhere do we see μ. The only free parameter is σ (as in the binominal model).

- Solving this PDE, subject to the terminal payoff condition of the derivative (e.g., $f_T = (S_T - K)^+$ for a European call option), BMS can derive analytical formulas for call and put option value.
  - Similar formula had been derived before based on distributional (normal return) argument, but μ (risk premium) was still in.
  - The realization that option valuation does not depend on μ is big. Plus, it provides a way to hedge the option position.

The BMS formulae

$$\begin{array}{rcl} c_t &=& S_t e^{-q(T-t)} N(d_1) - K e^{-r(T-t)} N(d_2), \\ p_t &=& -S_t e^{-q(T-t)} N(-d_1) + K e^{-r(T-t)} N(-d_2), \end{array}$$

where

$$\begin{array}{rcl} d_1 &=& \frac{\ln(S_t/K) + (r-q)(T-t) + \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}}, \\ d_2 &=& \frac{\ln(S_t/K) + (r-q)(T-t) - \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}} = d_1 - \sigma\sqrt{T-t}. \end{array}$$

Black derived a variant of the formula for futures (which I like better):

$$c_t = e^{-r(T-t)}\left[F_t N(d_1) - K N(d_2)\right],$$

with $d_{1,2} = \frac{\ln(F_t/K) \pm \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}}$.

- Recall: $F_t = S_t e^{(r-q)(T-t)}$. Use forward price F_t to accommodate various carrying costs/benefits.
- Once I know call value, I can obtain put value via put-call parity: $c_t - p_t = e^{-r(T-t)}\left[F_t - K_t\right]$.

Cumulative normal distribution

$$c_t = e^{-r(T-t)}\left[F_t N(d_1) - KN(d_2)\right], \quad d_{1,2} = \frac{\ln(F_t/K) \pm \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}}$$

- $N(x)$ denotes the cumulative normal distribution, which measures the probability that a normally distributed variable with a mean of zero and a standard deviation of 1 ($\phi(0,1)$) is less than x.
- See tables at the end of the book for its values.
- Most software packages (including excel) has efficient ways to computing this function.
- Properties of the BMS formula:
  - As S_t becomes very large or K becomes very small, $\ln(F_t/K) \uparrow \infty$, $N(d_1) = N(d_2) = 1$. $c_t = e^{-r(T-t)}\left[F_t - K\right]$.
  - Similarly, as S_t becomes very small or K becomes very large, $\ln(F_t/K) \uparrow -\infty$, $N(-d_1) = N(-d_2) = 1$. $p_t = e^{-r(T-t)}\left[-F_t + K\right]$.

Options on what?

Why does it matter?

- As long as we assume that the underlying security price follows a geometric Brownian motion, we can use (some versions) of the BMS formula to price European options.
- Dividends, foreign interest rates, and other types of carrying costs may complicate the pricing formula a little bit.
- A simpler approach: Assume that the underlying futures/forwards price (of the same maturity of course) process follows a geometric Brownian motion.
- Then, as long as we observe the forward price (or we can derive the forward price), we do not need to worry about dividends or foreign interest rates — They are all accounted for in the forward pricing.
- Know how to price a forward, and use the Black formula.

Implied volatility

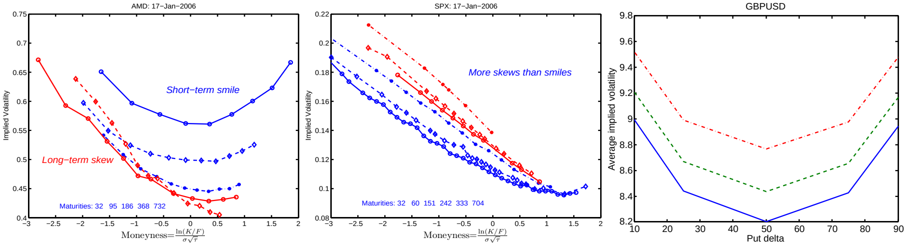
$$c_t = e^{-r(T-t)}\left[F_t N(d_1) - KN(d_2)\right], \quad d_{1,2} = \frac{\ln(F_t/K) \pm \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}}$$

- Since F_t (or S_t) is observable from the underlying stock or futures market, (K, t, T) are specified in the contract. The only unknown (and hence free) parameter is σ.
- We can estimate σ from time series return. (standard deviation calculation).
- Alternatively, we can choose σ to match the observed option price — implied volatility (IV).
- There is a one-to-one, monotonic correspondence between prices and implied volatilities.
  - As long as the option price does not allow arbitrage against cash, there exists a solution for a positive implied volatility that can match the price.
- Traders and brokers often quote implied volatilities rather than dollar prices.
  - More stable; more informative; excludes arbitrage
- The BMS model says that $IV = \sigma$. In reality, the implied volatility calculated from different options (across strikes, maturities, dates) are usually different.

Violations of BMS assumptions

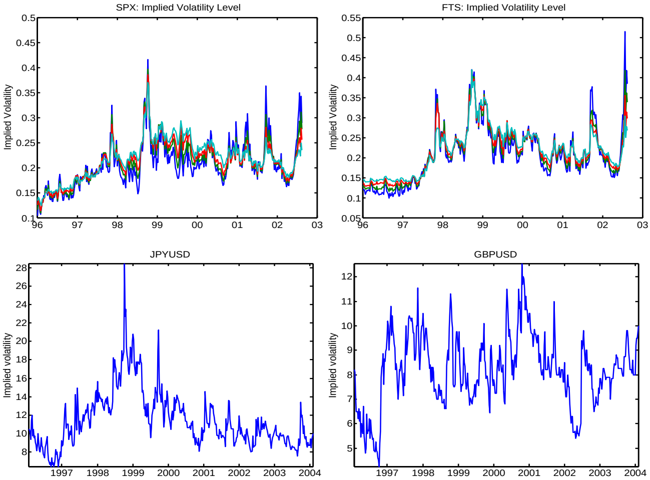
- The BMS model says that $IV = \sigma$. In reality, the implied volatility calculated from different options (across strikes, maturities, dates) are usually different.
- These difference indicates that in reality the security price dynamics differ from the BMS assumptions:
- *Jumps*: BMS assume that the security price moves by a small amount (diffusion) over a short time interval. In reality, sometimes the market can jump by a large amount in an instant.
  - With jumps, returns are no longer normally distributed, but tend to have fatter tails, and sometimes can be asymmetric (skewed).
  - Implied volatility at different strikes will be different.
- *Stochastic volatility*: The volatility σ of a security is not constant, but varies randomly over time, and can be correlated with the return move.
  - Implied volatilities will change over time.
  - Stochastic volatility also induces return non-normality.
  - Return-volatility correlation induces return distribution asymmetry.
- Other sources of variations such as credit risk for individual stock and emerging market currency, crash risk for aggregate market index...

Implied volatility smiles and skews



- Plots of option implied volatilities across different strikes at the same maturity often show a smile or skew pattern, reflecting deviations from the return normality assumption.
- A smile implies that the probability of reaching the tails of the distribution is higher than that from a normal distribution. $\Rightarrow$ Fat tails, or (formally) leptokurtosis.
- A negative skew implies that the probability of downward movements is higher than that from a normal distribution. $\Rightarrow$ Negative skewness in the distribution.

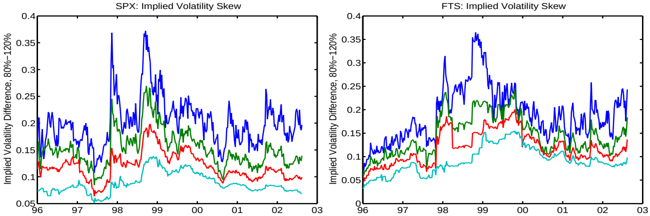
Stochastic volatility on stock indexes and currencies



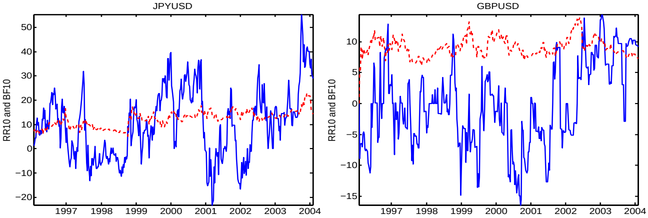
At the-money option implied volatilities vary strongly over time, higher during crises and recessions.

Stochastic skewness on stock indexes and currencies

Implied volatility spread between 80% and 120% strikes



10-delta call minis 10-delta put implied volatility



Return skewness also varies over time.

Second-generation option pricing models

- Second-generation option pricing models strive to add new features to capture the observed implied volatility behaviors
  - *Jumps*: BMS uses Brownian motion to capture continuous price movements, second-generation models use a more general class of processes called *Lévy process* to capture both continuous and discontinuous movements.
  - *Stochastic volatility*: MBS assumes constant volatility for the Brownian motion, second-generation models allow the intensity of the Lévy processes to vary stochastically over time
    - Use the concept of *time change* to capture the mapping between calendar clock and business (activity) clock
- *The doctoral class* provides guidance on how to design models based on observed features and how to price options under newly designed models.

Summary

- Understand the basic properties of normally distributed random variables.
- Map a stochastic process to a random variable.
- Understand the link between BMS and the binomial model.
- Memorize the BMS formula (any version).
- Understand forward pricing and link option pricing to forward pricing.
- Can go back and forth with the put-call parity conditions, lower and upper bounds, either in forward or in spot notation.
- Understand the general implications of the implied volatility plots.