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The boundary collocation method for stress intensity factors of cracks at internal boundaries in a multiply stiffened sheet

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ABSTRACT

Stress intensity factors for a crack at a hole in a uniformly stressed, multiply stiffened infinite sheet are determined. The complex variable method is combined with the method of the boundary collocation. The usual closed form complex stress functions are replaced with series expansions containing unknown complex constants. The system equations are reduced by a least squares technique to a series of linear algebraic equations in terms of the unknown attachment forces and the complex constants. The stress intensity factor and the attachment force distribution are determined for several stiffened configurations and are shown to be in agreement with existing solutions.

INTRODUCTION

The Displacement Compatibility Method (DCM), has been used extensively (see citations in ref 1), for the analysis of cracked, reinforced sheets typical of those used in the aircraft industry. Most applications of the DCM, which has proved both efficient and versatile for the determination of attachment forces and stress intensity factors, have been limited to multiply stiffened sheets containing a single crack in the absence of any other boundaries. Cracks may occur at sheet edges or at regions of stress concentration caused by cut-outs near stiffeners. The effect of these boundaries is to increase the stress intensity of the crack thereby reducing its critical length and increasing its growth rate. It is thus desirable to include the effects of the boundaries in the determination of the stress intensity factor if unconservative designs are to be avoided. This has given an incentive to extend the DCM to include finite external boundaries [2] using the Boundary Collocation Method (BCM) [3,4]. In this paper the approach [2] has been modified and its application extended to the analysis of internal boundaries thus enabling it to be applied to the important case of cracks at regions of stress concentration in stiffened sheets. In the previously developed method [2] it was necessary to use an iterative scheme to solve the system equations. By combining the unknown attachment forces with the unknown coefficients in the stress function series the need to solve the system equations

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iteratively is avoided thus reducing the problem to a direct solution with a consequent reduction in solution time.

The present method continues to combine the advantages of the boundary collocation method, developed for finite unstiffened sheets, and the displacement compatibility method, previously used for infinite stiffened sheets, to obtain a method of solution for finite stiffened sheets. The DCM and the BCM are known to be accurate and efficient methods when applied separately and their combination has been shown to be successful for stiffened sheets with external boundaries [2]. The present work is a natural extension of this combination for the important case of stiffened sheets with internal boundaries. The internal boundaries are incorporated by modifying the series expressions for the stress functions so that they can be used to satisfy conditions on the internal boundaries as well as the external boundaries.

THEORETICAL FORMULATION AND SOLUTION OF EQUATIONS

The analysis is based on the complex variable technique due to Muskhelishvili [5] which states that the stress/displacement state within a multiply connected, two dimensional body subjected to in-plane loading may be completely specified in terms of two complex stress functions which can be written in series form. When the elastic body is of arbitrary shape containing internal boundaries, one of which is a straight crack, the complex stress functions can be written in series form such that the traction free conditions on the crack are automatically satisfied. Such functions were used by Newman [3] to obtain the stress intensity factor of a single traction free crack, near boundaries, in a two-dimensional elastic body where the unknown coefficients in the stress function series were found using the BCM. The present work is an extension of the approach [3] to configurations in which the sheet contains reinforcing elements (stiffeners). The method of analysis follows that developed [2] for finite multiply stiffened sheets having external boundaries by replacing the series stress functions with those appropriate to internal boundaries. The system equations are formed by satisfying equilibrium of forces at, and compatibility of displacements between the attachment points, and by satisfying traction boundary conditions on the sheet boundaries. The system equations are solved in a least squares sense for the unknown distribution of attachment forces and the unknown coefficients in the stress function series. In the previously developed method [2] it was necessary to use an iterative scheme to solve the system equations which is avoided in the present method by combining the effects of the attachment forces with the coefficients in the stress function series. The stress intensity factor of the crack is determined from the attachment forces and the stress function coefficients.

CONFIGURATION STUDIED

The configuration to be studied in the present work is shown in Fig. 1. It consists of an infinite sheet containing a hole of radius R centered at the origin of the coordinates ($x = 0, y = 0$). The hole has equal length cracks of length ℓ at each edge along the x axis. The sheet is subjected to a uniform stress σ perpendicular to the crack line.

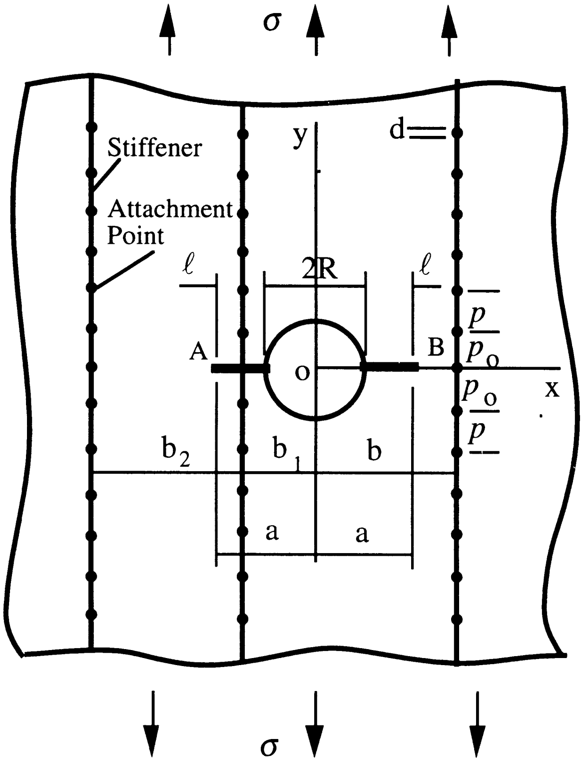

Fig. 1 Symmetrically Cracked Circular Hole in a Uniformly Stressed Stiffened Sheet

In general the sheet is reinforced by arbitrarily spaced stiffeners parallel to the y-axis. The stiffeners are attached to the sheet at discrete points symmetrically either side of the x-axis. The first attachment point is p_O from the x axis and all the other attachment points are p apart . The attachment points are assumed to be represented by localised forces at the center of a rigid insert of diameter d. The sheet has a modulus of elasticity E, Poisson's ratio ν and thickness t. The Young's modulus and area of each stiffener is E_S and A_S respectively. The effect of the in-plane and out of plane bending stiffness of each stiffener is assumed negligible compared to its axial stiffness.

In the present work only a single internal circular boundary is considered although the method can be applied to any internal boundary shape. It is also assumed that the cracks are symmetrically located at the edges of the hole but that the stiffeners are symmetrical only about the x axis. It is further assumed that the ends of the stiffener are sufficiently remote from the crack and any internal boundaries so that the strain in the sheet and stiffener at their ends are identical.

COMPARISION WITH KNOWN SOLUTIONS

Stress Concentration Factor for a hole near a stiffener

The configuration in Fig.2, without cracks, was solved by Bloom [6] using an analytical expression for a point force near a traction free circular hole. This solution did not require collocation of boundary conditions on the hole boundary and thus provides a useful solution for checking the collocation technique used in the present work.

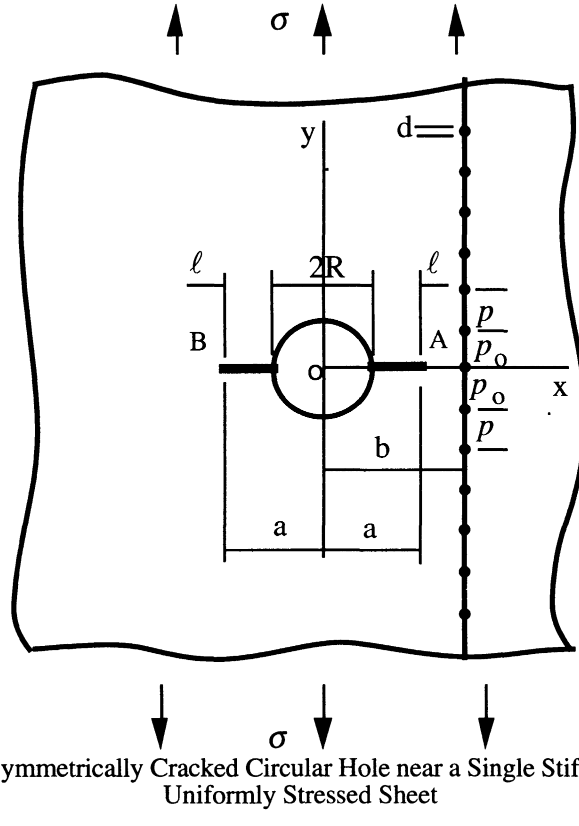

Fig.2 Symmetrically Cracked Circular Hole near a Single Stiffener in a Uniformly Stressed Sheet

The stress concentration factor at the edge of the hole, without cracks, nearest the stiffener in Fig. 2 are shown in Table 1. It can be seen that those determined in the present work using the BCM are in close agreement with the results of Bloom [6].

$\frac{2REt}{A_sE_s}$	**0.5**				**2.0**			
b/R	**1.10**		**1.25**		**1.10**		**1.25**	
p/R	Ref 6	BCM	Ref 6	BCM	Ref 6	BCM	Ref 6	BCM
0.5	1.56	1.58	2.23	2.29	1.97	2.00	2.53	2.59
1.0	1.69	1.72	2.23	2.23	2.14	2.14	2.53	2.54
2.0	2.04	2.07	2.30	2.30	2.46	2.45	2.61	2.61
4.0	2.48	2.48	2.52	2.55	2.76	2.75	2.79	2.79

Table 1 Comparison of stress concentration factors at the edge of a hole near a stiffener ($p_o/p = 0.5, d/p = 0.2, \ell/R = 0$)

Stress Intensity Factors for a Short Crack at a hole near a stiffener

The normalised stress intensity factor $K/(\sigma\sqrt{\pi\ell})$ was also calculated from the stress concentration factor [6] using the known limiting expression for short cracks. These are shown in Table 2 and are compared with the normalised stress intensity factor determined by the BCM with which they are in close agreement. It can also be seen from Table 2 that the normalised stress intensity factors for the crack in the stiffened sheet are all below that for the unstiffened sheet given by $K/(\sigma\sqrt{\pi\ell}) = 3.23$ for $\ell/R = 0.02$. The size of this reduction increases with the amount of stiffening, closer relative stiffener positions and smaller attachment pitches.

$\frac{2REt}{A_sE_s}$	**0.5**				**2.0**			
b/R	**1.10**		**1.25**		**1.10**		**1.25**	
p/R	Short Crack Limit	BCM	Short Crack Limit	BCM	Short Crack Limit	BCM	Short Crack Limit	BCM
0.5	1.75	1.75	2.50	2.56	2.21	2.23	2.84	2.91
1.0	1.90	1.87	2.50	2.48	2.40	2.38	2.84	2.85
2.0	2.29	2.33	2.58	2.59	2.76	2.77	2.93	2.95
4.0	2.78	2.83	2.83	2.91	3.10	3.13	3.13	3.16

Table 2 Comparision of normalised stress intensity factor $K/(\sigma\sqrt{\pi\ell})$ for a short crack ($\ell/R = 0.02$) at the edge of a hole near a stiffener ($p_o/p = 0.5, d/p = 0.2$)

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The results in Tables 1 and 2 were determined for 100 uniformly distributed collocation points on the hole with the maximum power in the stress function series of 20. The number of attachment points was taken to be 5 as more than this did not significantly change the results. A weighting factor of 10 was used for the compatibility equations and unity for the collocation equations.

Comparison of attachment force distributions

The attachment forces for the configuration in Fig. 2 were determined for a long crack at a hole (a/R=2.0) and these are shown in Table 3. They are compared with results determined for a crack without the hole.

b/R	1.1		1.5		1.8	
i	Crack & Hole	Crack only	Crack & Hole	Crack only	Crack & Hole	Crack only
1	-0.182	-0.229	-0.215	-0.211	-0.182	-0.177
2	-0.098	-0.107	-0.085	-0.092	-0.079	-0.085
3	-0.068	-0.065	-0.058	-0.062	-0.061	-0.063
4	-0.059	-0.045	-0.053	-0.047	-0.056	-0.051
5	-0.067	-0.043	-0.070	-0.053	-0.077	-0.061
$\frac{K}{(\sigma\sqrt{\pi a})}$	0.499	0.466	0.368	0.361	0.288	0.275

Table 3 Comparison of normalized attachment forces $P_i/(A_s\sigma_s)$ and nomalised stress intensity factor $K/(\sigma\sqrt{\pi a})$ for a long crack at a hole (a/R=2.00) for various stiffener positions ($p/R = 0.5, p_o/p = 0.5, d/p = 0.2, 2aEt/(A_sE_s) = 1.0$)

It can be seen that when the attachment is remote from the hole $i \geq 3$ the forces are relatively independent of the presence of the hole for all position of the stiffener b/R=1.1, 1.5 and 1.8. For rivets close to the hole $i \leq 2$ the attachment forces are similar for b/R=1.5 and 1.8 but differ significantly for b/R=1.1 when the stiffener is near the hole since in this case the hole is close enough to the stiffener to affect the displacement in the sheet at the attachment points. It can also be seen that the stress intensity factor is reduced more when the stiffener is closer to the crack tip. The results in Table 3 were determined for 100 uniformly distributed collocation points on the hole with the maximum power in the stress function series of 40. A weighting factor of 10 was used for the compatibility equations and unity for the collocation equations.

The attachment forces for the configuration in Fig. 2 with $b/R = 1.1$ have also been determined using the BCM for a short crack at a hole (a/R=1.02). The results are shown in Table 4 where they are compared with those determined for a hole without a crack using the BCM.

i	Hole Only	Crack & Hole $a/R = 1.02$
1	-0.040	-0.042
2	-0.053	-0.053
3	-0.045	-0.045
4	-0.039	-0.039
5	-0.039	-0.039

Table 4 Comparison of normalized attachment forces $P_i/(A_s\sigma_s)$ for a single stiffener ($b/R = 1.1, p/R = 0.5, p_o/p = 0.5, d/p = 0.2, 2aEt/(A_sE_s) = 0.5$) near a hole with or without a short crack($a/R = 1.02$).

It can be seen that the forces for the short crack case are almost identical to those for the hole only. This occurs because the crack is short relative to the hole radius and thus does not significantly change the displacements in the sheet. The results in table 4 were determined for 100 uniformly distributed collocation points on the hole with the maximum power in the stress function series of 20. A weighting factor of 10 was used for the compatibility equations and unity for the collocation equations.

CONCLUSIONS

The BCM has been extended to the determination of stress intensity factors and attachment force distributions in a cracked stiffened sheet containing an internal boundary.

By combining the unknown attachment forces with the constants in the stress function series the need to solve the system equations iteratively can be avoided thus reducing problem to a direct solution of the system equations.

The stress concentration factor of a circular hole near a stiffener in a uniformly stressed sheet has been determined using the BCM and shown to be in agreement with results from a different method.

The stress intensity factors obtained for short crack lengths (a/R=1.02) have been shown to be consistent with existing results.

The attachment forces determined from the BCM for a long crack at a hole have been shown to be consistent with results for a crack only.

The attachment forces determined from the BCM for a short crack at a hole have been shown to be consistent with results for a hole only.

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