# The Calculus of Variations: An Introduction

#### By Kolo Sunday Goshi

## **Some Greek Mythology**

- Queen Dido of Tyre
  - Fled Tyre after the death of her husband
  - Arrived at what is present day Libya
- Iarbas' (King of Libya) offer
  - "Tell them, that this their Queen of theirs may have as much land as she can cover with the hide of an ox."
- What does this have to do with the Calculus of Variations?

## What is the Calculus of Variations

- "Calculus of variations seeks to find the path, curve, surface, etc., for which a given function has a stationary value (which, in physical problems, is usually a minimum or maximum)." (MathWorld Website)
- Variational calculus had its beginnings in 1696 with John Bernoulli
- Applicable in Physics

## **Calculus of Variations**

- Understanding of a Functional
- Euler-Lagrange Equation
  - Fundamental to the Calculus of Variations
- Proving the Shortest Distance Between Two Points
  - In Euclidean Space
- The Brachistochrone Problem
  - In an Inverse Square Field
- Some Other Applications
- Conclusion of Queen Dido's Story

## What is a Functional?

- The quantity z is called a functional of f(x) in the interval [a,b] if it depends on all the values of f(x) in [a,b].
- Notation

$$z = \Gamma \left[ f \left( x \right) \right]_{a}$$

- Example

$$\Gamma\left[x^{2}_{0}\right] = \int_{0}^{1} \cos\left(x^{2}\right) dx$$

## **Functionals**

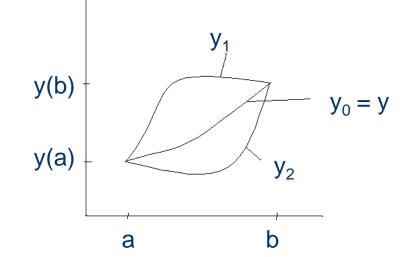
• The functionals dealt with in the calculus of variations are of the form

$$\Gamma\left[f(x)\right] = \int_{a}^{b} F(x, y(x), \dot{y}(x)) dx$$

- The goal is to find a y(x) that minimizes Γ, or maximizes it.
- Used in deriving the Euler-Lagrange equation

• I set forth the following equation:  
$$y_{\alpha}(x) = y(x) + \alpha g(x)$$

Where  $y_{\alpha}(x)$  is all the possibilities of y(x) that extremize a functional, y(x) is the answer,  $\alpha$  is a constant, and g(x) is a random function.



- Recalling  $\Gamma[f(x)] = \int_{a}^{b} F(x, y(x), \dot{y}(x)) dx$
- It can now be said that:  $\Gamma[y_{\alpha}] = \int_{a}^{b} F(x, y_{\alpha}, \dot{y}_{\alpha}) dx$
- At the extremum y<sub>α</sub> = y<sub>0</sub>
  = y and

$$\left.\frac{d\Gamma}{d\alpha}\right|_{\alpha=0} = 0$$

 The derivative of the functional with respect to α must be evaluated and equated to zero

$$\frac{d\Gamma}{d\alpha} = \int_{a}^{b} \left[ \frac{\partial}{\partial \alpha} F(x, y_{\alpha}, \dot{y}_{\alpha}) \right] dx$$

• The mathematics

involved

$$\frac{d\Gamma}{d\alpha} = \int_{a}^{b} \left[ \frac{\partial}{\partial \alpha} F(x, y_{\alpha}, \dot{y}_{\alpha}) \right] dx$$
$$\frac{d\Gamma}{d\alpha} = \int_{a}^{b} \left[ \frac{\partial F}{\partial y_{\alpha}} \frac{\partial y_{\alpha}}{\partial \alpha} + \frac{\partial F}{\partial \dot{y}_{\alpha}} \frac{\partial \dot{y}_{\alpha}}{\partial \alpha} \right] dx$$

- Recalling  $y_{\alpha}(x) = y(x) + \alpha g(x)$
- So, we can say

$$\frac{d\Gamma}{d\alpha} = \int_{a}^{b} \left[ \frac{\partial F}{\partial y_{\alpha}} g + \frac{\partial F}{\partial \dot{y}_{\alpha}} \dot{g} \right] dx = \int_{a}^{b} \frac{\partial F}{\partial y_{\alpha}} g dx + \int_{a}^{b} \frac{\partial F}{\partial \dot{y}_{\alpha}} \frac{dg}{dx} dx$$

$$\frac{d\Gamma}{d\alpha} = \int_{a}^{b} \frac{\partial F}{\partial y_{\alpha}} g dx + \int_{a}^{b} \frac{\partial F}{\partial \dot{y}_{\alpha}} \frac{dg}{dx} dx$$

Integrate the first part by parts and get

$$-\int_{a}^{b}g\frac{d}{dx}\left(\frac{\partial F}{\partial \dot{y}_{\alpha}}\right)dx$$

• So

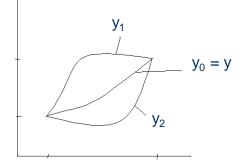
$$\frac{d\Gamma}{d\alpha} = \int_{a}^{b} g \left[ \frac{\partial F}{\partial y_{\alpha}} - \frac{d}{dx} \left( \frac{\partial F}{\partial \dot{y}_{\alpha}} \right) \right] dx$$

 Since we stated earlier that the derivative of Γ with respect to α equals zero at α=0, the extremum, we can equate the integral to zero

• So

$$0 = \int_{a}^{b} g \left[ \frac{\partial F}{\partial y_{0}} - \frac{d}{dx} \left( \frac{\partial F}{\partial \dot{y}_{0}} \right) \right] dx$$

 We have said that y<sub>0</sub> = y, y being the extremizing function, therefore



 Since g(x) is an arbitrary function, the quantity in the brackets must equal zero

$$0 = \int_{a}^{b} g \left[ \frac{\partial F}{\partial y} - \frac{d}{dx} \left( \frac{\partial F}{\partial \dot{y}} \right) \right] dx$$

## **The Euler-Lagrange Equation**

We now have the Euler-Lagrange Equation

$$\frac{\partial F}{\partial y} - \frac{d}{dx} \left( \frac{\partial F}{\partial \dot{y}} \right) = 0$$

• When  $F = F(y, \dot{y})$ , where x is not included, the modified equation is

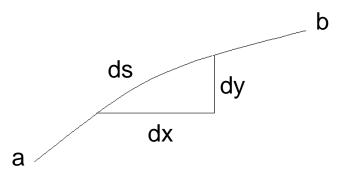
$$F - \dot{y}\frac{\partial F}{\partial \dot{y}} = C$$

## The Shortest Distance Between Two Points on a Euclidean Plane

- What function describes the shortest distance between two points?
  - Most would say it is a straight line
    - Logically, this is true
    - Mathematically, can it be proven?
- The Euler-Lagrange equation can be used to prove this

• Define the distance to be s, so

 $s = \int ds$ 



• Therefore  $s = \int \sqrt{dx^2 + dy^2}$ 

 Factoring a dx<sup>2</sup> inside the square root and taking its square root we obtain

$$s = \int \sqrt{dx^2 + dy^2} \qquad s = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

• Now we can let  $\dot{y} = \frac{dy}{dx}$ 

• SO 
$$s = \int_{a}^{b} \sqrt{1 + \dot{y}^{2}} dx = \Gamma$$

• Since 
$$\Gamma = \int_{a}^{b} \sqrt{1 + \dot{y}^{2}} dx$$

• And we have said that  $\Gamma[f(x)] = \int_{a}^{b} F(x, y(x), \dot{y}(x)) dx$ 

• we see that 
$$F = \sqrt{1 + \dot{y}^2}$$

$$\frac{\partial F}{\partial y} = 0 \qquad \qquad \frac{\partial F}{\partial \dot{y}} = \frac{\dot{y}}{\sqrt{1 + \dot{y}^2}}$$

• therefore

 Recalling the Euler-Lagrange equation

$$\frac{\partial F}{\partial y} - \frac{d}{dx} \left( \frac{\partial F}{\partial \dot{y}} \right) = 0$$

• Knowing that

 $\frac{\partial F}{\partial y} = 0 \qquad \frac{\partial F}{\partial \dot{y}} = \frac{\dot{y}}{\sqrt{1 + \dot{y}^2}}$ 

• A substitution can be made

$$-\frac{d}{dx}\left[\frac{\dot{y}}{\sqrt{1+\dot{y}^2}}\right] = 0$$

• Therefore the term in brackets must be a constant, since its derivative is 0.

More math to reach the solution

$$\frac{\dot{y}}{\sqrt{1+\dot{y}^2}} = C$$
$$\dot{y}^2 = C^2 \left(1+\dot{y}^2\right)$$
$$\dot{y}^2 \left(1-C^2\right) = C^2$$
$$\dot{y}^2 = D$$
$$\dot{y} = M$$

$$\dot{y} = M$$

We see that the derivative or slope of the minimum path between two points is a constant, *M* in this case.

The solution therefore is:

$$y = Mx + B$$

## **The Brachistochrone Problem**

#### • Brachistochrone

- Derived from two Greek words
  - brachistos meaning shortest
  - chronos meaning time
- The problem
  - Find the curve that will allow a particle to fall under the action of gravity in minimum time.
    - Led to the field of variational calculus
- First posed by John Bernoulli in 1696
  - Solved by him and others

## **The Brachistochrone Problem**

#### The Problem restated

- Find the curve that will allow a particle to fall under the action of gravity in minimum time.
- The Solution
  - A cycloid
  - Represented by the parametric equations

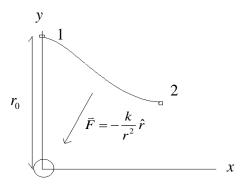
$$x = \pm \frac{D}{2} [2\theta - \sin 2\theta]$$
$$y = \frac{D}{2} [1 - \cos 2\theta]$$



#### • The Problem

- Find the curve that will allow a particle to fall under the action of an inverse square force field defined by k/r<sup>2</sup> in minimum time.
- Mathematically, the force is defined as

$$F_r = -\frac{k}{r^2}$$



 Since the minimum time is being considered, an expression for time must be determined

$$t = \int_{1}^{2} \frac{ds}{v}$$

 An expression for the velocity v must found and this can be done using the fact that KE + PE = E

$$\frac{1}{2}mv^2 - \frac{k}{r} = E$$

 The initial position r<sub>0</sub> is known, so the total energy E is given to be -k/r<sub>0</sub>, so

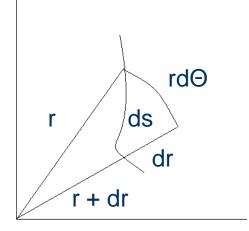
$$\frac{1}{2}mv^2 - \frac{k}{r} = -\frac{k}{r_0}$$

An expression can be found for velocity and the desired expression for time is found

$$v = \sqrt{\frac{2k}{m}} \left(\frac{1}{r} - \frac{1}{r_0}\right)$$

$$t = \sqrt{\frac{m}{2k}} \int_{1}^{2} \frac{ds}{\sqrt{\left(\frac{1}{r} - \frac{1}{r_{0}}\right)}}$$





$$ds^2 = \left(dr\right)^2 + r^2 \left(d\theta\right)^2$$

• We continue using a polar coordinate system

$$ds^2 = \left(dr\right)^2 + r^2 \left(d\theta\right)^2$$

$$ds^{2} = \left(d\theta\right)^{2} \left[ \left(\frac{dr}{d\theta}\right)^{2} + r^{2} \right]$$

• An expression can be determined for *ds* to put into the time expression

 $ds = \sqrt{r^2 + \dot{r}^2} d\theta$ 

• Here is the term for time *t* 

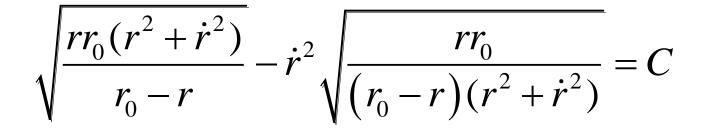
$$t = \sqrt{\frac{m}{2k}} \int_{1}^{2} \sqrt{\frac{rr_{0}(r^{2} + \dot{r}^{2})}{r_{0} - r}}$$

• The function F is the term in the integral

$$F = \sqrt{\frac{rr_0(r^2 + \dot{r}^2)}{r_0 - r}}$$

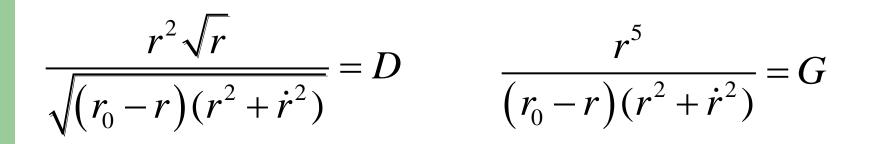
 Using the modified Euler-Lagrange equation

$$F - \dot{r}\frac{\partial F}{\partial \dot{r}} = C$$



 More math involved in finding an integral to be solved

$$\sqrt{\frac{r(r^2 + \dot{r}^2)}{r_0 - r}} - \dot{r}^2 \sqrt{\frac{r}{(r_0 - r)(r^2 + \dot{r}^2)}} = D$$



• Reaching the integral

$$\dot{r} = \frac{dr}{d\theta} = \pm \sqrt{\frac{r^5 - r^2 G(r_0 - r)}{G(r_0 - r)}}$$

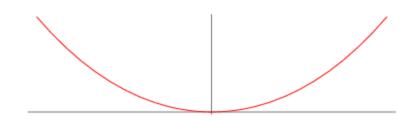
$$\int \sqrt{\frac{G(r_0 - r)}{r^5 - r^2 G(r_0 - r)}} dr = \pm \int d\theta$$

 Solving the integral for r(Θ) finds the equation for the path that minimizes the time.

- Challenging Integral to Solve
  - Brachistochrone.nb
- Where to then?
  - Use numerical methods to solve the integral
  - Consider using elliptical coordinates
- Why Solve this?
  - Might apply to a cable stretched out into space to transport supplies

## **Some Other Applications**

- The Catenary Problem
  - Derived from Greek for "chain"
  - A chain or cable supported at its end to hang freely in a uniform gravitational field
  - Turns out to be a hyperbolic cosine curve
- Derivation of Snell's Law



$$n_1 \sin \theta_i = n_2 \sin \theta_2$$

## **Conclusion of Queen Dido's Story**

- Her problem was to find the figure bounded by a line which has the maximum area for a given perimeter
- Cut ox hide into infinitesimally small strips
  - Used to enclose an area
  - Shape unknown
  - City of Carthage
- Isoperimetric Problem
  - Find a closed plane curve of a given perimeter which encloses the greatest area
  - Solution turns out to be a semicircle or circle

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