## The Calculus of Variations: An Introduction

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## Some Greek Mythology

- Queen Dido of Tyre
- Fled Tyre after the death of her husband
- Arrived at what is present day Libya
- larbas' (King of Libya) offer
_ "Tell them, that this their Queen of theirs may have as much land as she can cover with the hide of an ox."
- What does this have to do with the Calculus of Variations?


## What is the Calculus of Variations

- "Calculus of variations seeks to find the path, curve, surface, etc., for which a given function has a stationary value (which, in physical problems, is usually a minimum or maximum)." (MathWorld Website)
- Variational calculus had its beginnings in 1696 with John Bernoulli
- Applicable in Physics


## Calculus of Variations

- Understanding of a Functional
- Euler-Lagrange Equation
- Fundamental to the Calculus of Variations
- Proving the Shortest Distance Between Two Points
- In Euclidean Space
- The Brachistochrone Problem
- In an Inverse Square Field
- Some Other Applications
- Conclusion of Queen Dido's Story


## What is a Functional?

- The quantity $z$ is called a functional of $f(x)$ in the interval [a,b] if it depends on all the values of $f(x)$ in $[a, b]$.
- Notation

$$
z=\Gamma\left[\begin{array}{c}
f_{a}^{b}(x)
\end{array}\right]
$$

- Example

$$
\Gamma\left[\begin{array}{c}
1 \\
x^{2} \\
0
\end{array}\right]=\int_{0}^{1} \cos \left(x^{2}\right) d x
$$

## Functionals

- The functionals dealt with in the calculus of variations are of the form

$$
\Gamma[f(x)]=\int_{a}^{b} F(x, y(x), \dot{y}(x)) d x
$$

- The goal is to find a $y(x)$ that minimizes $\Gamma$, or maximizes it.
- Used in deriving the Euler-Lagrange equation


## Deriving the Euler-Lagrange Equation

- I set forth the following equation:

$$
y_{\alpha}(x)=y(x)+\alpha g(x)
$$

Where $y_{\alpha}(x)$ is all the possibilities of $y(x)$ that extremize a functional, $y(x)$ is the answer, $\alpha$ is a constant, and $g(x)$ is a random function.


## Deriving the Euler-Lagrange Equation

- Recalling

$$
\Gamma[f(x)]=\int_{a}^{b} F(x, y(x), \dot{y}(x)) d x
$$

- It can now be said that: $\Gamma\left[y_{\alpha}\right]=\int_{a}^{b} F\left(x, y_{\alpha}, \dot{y}_{\alpha}\right) d x$
- At the extremum $\mathrm{y}_{\alpha}=\mathrm{y}_{0}$
$=y$ and

$$
\left.\frac{d \Gamma}{d \alpha}\right|_{\alpha=0}=0
$$

- The derivative of the $\begin{aligned} & \text { Tunctional with respect } \\ & \text { to } \alpha \text { must be evaluated }\end{aligned} \frac{d \Gamma}{d \alpha}=\int_{a}^{b}\left[\frac{\partial}{\partial \alpha} F\left(x, y_{\alpha}, \dot{y}_{\alpha}\right)\right] d x$ and equated to zero


## Deriving the Euler-Lagrange Equation

- The mathematics involved

$$
\begin{aligned}
\frac{d \Gamma}{d \alpha} & =\int_{a}^{b}\left[\frac{\partial}{\partial \alpha} F\left(x, y_{\alpha}, \dot{y}_{\alpha}\right)\right] d x \\
\frac{d \Gamma}{d \alpha} & =\int_{a}^{b}\left[\frac{\partial F}{\partial y_{\alpha}} \frac{\partial y_{\alpha}}{\partial \alpha}+\frac{\partial F}{\partial \dot{y}_{\alpha}} \frac{\partial \dot{y}_{\alpha}}{\partial \alpha}\right] d x
\end{aligned}
$$

- Recalling $y_{\alpha}(x)=y(x)+\alpha g(x)$
- So, we can say

$$
\frac{d \Gamma}{d \alpha}=\int_{a}^{b}\left[\frac{\partial F}{\partial y_{\alpha}} g+\frac{\partial F}{\partial \dot{y}_{\alpha}} \dot{g}\right] d x=\int_{a}^{b} \frac{\partial F}{\partial y_{\alpha}} g d x+\int_{a}^{b} \frac{\partial F}{\partial \dot{y}_{\alpha}} \frac{d g}{d x} d x
$$

## Deriving the Euler-Lagrange Equation

$$
\frac{d \Gamma}{d \alpha}=\int_{a}^{b} \frac{\partial F}{\partial y_{\alpha}} g d x+\int_{a}^{b} \frac{\partial F}{\partial \dot{y}_{\alpha}} \frac{d g}{d x} d x
$$

- Integrate the first part by parts and get

$$
-\int_{a}^{b} g \frac{d}{d x}\left(\frac{\partial F}{\partial \dot{y}_{\alpha}}\right) d x
$$

- So

$$
\frac{d \Gamma}{d \alpha}=\int_{a}^{b} g\left[\frac{\partial F}{\partial y_{\alpha}}-\frac{d}{d x}\left(\frac{\partial F}{\partial \dot{y}_{\alpha}}\right)\right] d x
$$

- Since we stated earlier that the derivative of $\Gamma$ with respect to $\alpha$ equals zero at $\alpha=0$, the extremum, we can equate the integral to zero


## Deriving the Euler-Lagrange Equation

- So

$$
0=\int_{a}^{b} g\left[\frac{\partial F}{\partial y_{0}}-\frac{d}{d x}\left(\frac{\partial F}{\partial \dot{y}_{0}}\right)\right] d x
$$

- We have said that $y_{0}=y, y$ being the extremizing function, therefore

- Since $g(x)$ is an arbitrary function, the quantity in the brackets must equal zero

$$
0=\int_{a}^{b} g\left[\frac{\partial F}{\partial y}-\frac{d}{d x}\left(\frac{\partial F}{\partial \dot{y}}\right)\right] d x
$$

## The Euler-Lagrange Equation

- We now have the Euler-Lagrange Equation

$$
\frac{\partial F}{\partial y}-\frac{d}{d x}\left(\frac{\partial F}{\partial \dot{y}}\right)=0
$$

- When $F=F(y, \dot{y})$, where x is not included, the modified equation is

$$
F-\dot{y} \frac{\partial F}{\partial \dot{y}}=C
$$

## The Shortest Distance Between Two Points on a Euclidean Plane

- What function describes the shortest distance between two points?
- Most would say it is a straight line
- Logically, this is true
- Mathematically, can it be proven?
- The Euler-Lagrange equation can be used to prove this


## Proving The Shortest Distance Between Two Points

- Define the distance to be $s$, so

$$
s=\int d s
$$



- Therefore $s=\int \sqrt{d x^{2}+d y^{2}}$


## Proving The Shortest Distance Between Two Points

- Factoring a $d x^{2}$ inside the square root and taking its square root we obtain

$$
s=\int \sqrt{d x^{2}+d y^{2}} \quad s=\int_{a}^{b} \sqrt{1+\left(\frac{d y}{d x}\right)^{2}} d x
$$

- Now we can let $\dot{y}=\frac{d y}{d x}$
- SO

$$
s=\int_{a}^{b} \sqrt{1+\dot{y}^{2}} d x=\Gamma
$$

## Proving The Shortest Distance Between Two Points

- Since

$$
\Gamma=\int_{a}^{b} \sqrt{1+\dot{y}^{2}} d x
$$

- And we have said that $\Gamma[f(x)]=\int_{a}^{b} F(x, y(x), \dot{y}(x)) d x$
- we see that

$$
F=\sqrt{1+\dot{y}^{2}}
$$

- therefore

$$
\frac{\partial F}{\partial y}=0 \quad \frac{\partial F}{\partial \dot{y}}=\frac{\dot{y}}{\sqrt{1+\dot{y}^{2}}}
$$

## Proving The Shortest Distance Between Two Points

- Recalling the Euler-Lagrange equation

$$
\frac{\partial F}{\partial y}-\frac{d}{d x}\left(\frac{\partial F}{\partial \dot{y}}\right)=0
$$

- Knowing that

$$
\frac{\partial F}{\partial y}=0 \quad \frac{\partial F}{\partial \dot{y}}=\frac{\dot{y}}{\sqrt{1+\dot{y}^{2}}}
$$

- A substitution can be made

$$
-\frac{d}{d x}\left[\frac{\dot{y}}{\sqrt{1+\dot{y}^{2}}}\right]=0
$$

- Therefore the term in brackets must be a constant, since its derivative is 0 .


## Proving The Shortest Distance Between Two Points

- More math to reach the solution

$$
\begin{aligned}
& \frac{\dot{y}}{\sqrt{1+\dot{y}^{2}}}=C \\
& \dot{y}^{2}=C^{2}\left(1+\dot{y}^{2}\right) \\
& \dot{y}^{2}\left(1-C^{2}\right)=C^{2} \\
& \dot{y}^{2}=D \\
& \dot{y}=M
\end{aligned}
$$

# Proving The Shortest Distance Between Two Points 

- Since

$$
\dot{y}=M
$$

We see that the derivative or slope of the minimum path between two points is a constant, $M$ in this case.
The solution therefore is:

$$
y=M x+B
$$

## The Brachistochrone Problem

- Brachistochrone
- Derived from two Greek words
- brachistos meaning shortest
- chronos meaning time
- The problem
- Find the curve that will allow a particle to fall under the action of gravity in minimum time.
- Led to the field of variational calculus
- First posed by John Bernoulli in 1696
- Solved by him and others


## The Brachistochrone Problem

- The Problem restated
- Find the curve that will allow a particle to fall under the action of gravity in minimum time.
- The Solution
- A cycloid
- Represented by the parametric equations

$$
\begin{aligned}
& x= \pm \frac{D}{2}[2 \theta-\sin 2 \theta] \\
& y=\frac{D}{2}[1-\cos 2 \theta]
\end{aligned}
$$

- Cycloid.nb


## The Brachistochrone Problem In an Inverse Square Force Field

- The Problem
- Find the curve that will allow a particle to fall under the action of an inverse square force field defined by $\mathrm{k} / \mathrm{r}^{2}$ in minimum time.

- Mathematically, the force is defined as

$$
F_{r}=-\frac{k}{r^{2}}
$$

# The Brachistochrone Problem In an Inverse Square Force Field 

- Since the minimum time is being considered, an expression for time must be determined

$$
t=\int_{1}^{2} \frac{d s}{v}
$$

- An expression for the velocity $v$ must found and this can be done using the fact that $K E+P E=E$

$$
\frac{1}{2} m v^{2}-\frac{k}{r}=E
$$

## The Brachistochrone Problem In an Inverse Square Force Field

- The initial position $r_{0}$ is known, so the total energy $E$ is given to be $-k / r_{0}$, so

$$
\frac{1}{2} m v^{2}-\frac{k}{r}=-\frac{k}{r_{0}}
$$

An expression can be found for velocity and the desired

$$
\begin{aligned}
& v=\sqrt{\frac{2 k}{m}\left(\frac{1}{r}-\frac{1}{r_{0}}\right)} \\
& t=\sqrt{\frac{m}{2 k}} \int_{1}^{2} \frac{d s}{\sqrt{\left(\frac{1}{r}-\frac{1}{r_{0}}\right)}}
\end{aligned}
$$ expression for time is found

## The Brachistochrone Problem In an Inverse Square Force Field

Determine an
expression for ds


$$
d s^{2}=(d r)^{2}+r^{2}(d \theta)^{2}
$$

## The Brachistochrone Problem In an Inverse Square Force Field

- We continue using a polar coordinate system

$$
\begin{array}{r}
d s^{2}=(d r)^{2}+r^{2}(d \theta)^{2} \\
d s^{2}=(d \theta)^{2}\left[\left(\frac{d r}{d \theta}\right)^{2}+r^{2}\right]
\end{array}
$$

- An expression can be determined for ds to put into the time expression

$$
d s=\sqrt{r^{2}+\dot{r}^{2}} d \theta
$$

## The Brachistochrone Problem In an Inverse Square Force Field

- Here is the term for time $t$

$$
t=\sqrt{\frac{m}{2 k}} \int_{1}^{2} \sqrt{\frac{r r_{0}\left(r^{2}+\dot{r}^{2}\right)}{r_{0}-r}}
$$

- The function $F$ is the term in the integral

$$
F=\sqrt{\frac{r r_{0}\left(r^{2}+\dot{r}^{2}\right)}{r_{0}-r}}
$$

## The Brachistochrone Problem In an Inverse Square Force Field

- Using the modified

Euler-Lagrange equation

$$
F-\dot{r} \frac{\partial F}{\partial \dot{r}}=C
$$

$$
\sqrt{\frac{r r_{0}\left(r^{2}+\dot{r}^{2}\right)}{r_{0}-r}}-\dot{r}^{2} \sqrt{\frac{r r_{0}}{\left(r_{0}-r\right)\left(r^{2}+\dot{r}^{2}\right)}}=C
$$

## The Brachistochrone Problem In an Inverse Square Force Field

- More math involved in finding an integral to be solved

$$
\sqrt{\frac{r\left(r^{2}+\dot{r}^{2}\right)}{r_{0}-r}}-\dot{r}^{2} \sqrt{\frac{r}{\left(r_{0}-r\right)\left(r^{2}+\dot{r}^{2}\right)}}=D
$$

$$
\frac{r^{2} \sqrt{r}}{\sqrt{\left(r_{0}-r\right)\left(r^{2}+\dot{r}^{2}\right)}}=D
$$

$$
\frac{r^{5}}{\left(r_{0}-r\right)\left(r^{2}+\dot{r}^{2}\right)}=G
$$

## The Brachistochrone Problem In an Inverse Square Force Field

- Reaching the integral

$$
\dot{r}=\frac{d r}{d \theta}= \pm \sqrt{\frac{r^{5}-r^{2} G\left(r_{0}-r\right)}{G\left(r_{0}-r\right)}}
$$

- Solving the integral for $r(\Theta)$

$$
\int \sqrt{\frac{G\left(r_{0}-r\right)}{r^{5}-r^{2} G\left(r_{0}-r\right)}} d r= \pm \int d \theta
$$ finds the equation for the path that minimizes the time.

## The Brachistochrone Problem In an Inverse Square Force Field

- Challenging Integral to Solve
- Brachistochrone.nb
- Where to then?
- Use numerical methods to solve the integral
- Consider using elliptical coordinates
- Why Solve this?
- Might apply to a cable stretched out into space to transport supplies


## Some Other Applications

- The Catenary Problem
- Derived from Greek for "chain"
- A chain or cable supported at its end to hang freely in a uniform gravitational field
- Turns out to be a hyperbolic cosine curve
- Derivation of Snell's Law


$$
n_{1} \sin \theta_{i}=n_{2} \sin \theta_{2}
$$

## Conclusion of Queen Dido's Story

- Her problem was to find the figure bounded by a line which has the maximum area for a given perimeter
- Cut ox hide into infinitesimally small strips
- Used to enclose an area
- Shape unknown
- City of Carthage
- Isoperimetric Problem
- Find a closed plane curve of a given perimeter which encloses the greatest area
- Solution turns out to be a semicircle or circle


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