# The CFC approximation in uniformly and differentially rotating relativistic stars*

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**Abstract**

Many astrophysically relevant systems have been studied, in the recent years, by means of the conformal flatness condition (CFC) yielding very satisfactory results as far as the quality of CFC is concerned. However, this does not nullify the need for constant evaluation of the CFC approximation in various different systems. In this work, we study the quality of CFC for the case of single, differentially rotating, relativistic stars. In addition, we verify its excellent performance for the case of uniform rotation. We use the numerical scheme KEH, as it is implemented in the computational code RNS. Necessary changes are made to the code, in order to allow for calculations in the CFC approximation. Our results are very encouraging, as even for the fastest rotating models, deviation from full general relativity is around 5%.

## 1 Introduction

Evaluation of the conformal flatness condition (CFC) approximation in various physical systems is crucial in order to better understand the limits within which it can be applied, as well as the magnitude of the error one is making by choosing to apply the CFC approximation in a physical problem instead of full general relativity. Cook, Shapiro and Teukolsky (CST) [3, 5, 4] tested the scheme developed by Komatsu, Eriguchi and Hachisu (KEH) [8, 9] and used it in order to test the CFC approximation for the case of single,

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uniformly rotating, relativistic stars yielding very encouraging results [6]. In this work, we study the quality of CFC for the case of differential rotation and we verify its excellent performance for the case of uniform rotation.

Following KEH the line element for a stationary, rotating, axisymmetric star in equilibrium is given by

$$ds^2 = -e^{\gamma+\rho}dt^2 + e^{\gamma-\rho}r^2 \sin^2\theta(d\phi - \omega dt)^2 + e^{2\mu}(dr^2 + r^2 d\theta^2) \; , \qquad (1)$$

where $\gamma$, $\rho$, $\omega$ and $\mu$ are metric potentials depending only on $r$ and $\theta$. We assume that the stellar matter behaves as a perfect fluid and that the equation of state obeys the polytropic relation

$$p = K\rho^{1+\frac{1}{N}} \; , \qquad (2)$$

where $\rho$ is the rest mass density, $K$ the polytropic constant and $N$ the polytropic index. For the case of differential rotation we adopt the same rotation law as in KEH, namely

$$F(\Omega) = A^2(\Omega_c - \Omega) \; , \qquad (3)$$

where, $A$ is a positive constant that determines the length scale over which the angular velocity changes within the star and $\Omega_c$ is the angular velocity at the center of the configuration.

## 2 Method – Basic equations

Beginning from the line element expression in the $3+1$ formalism [1, 2, 7] and using the basic assumption of the CFC approximation, $\gamma_{ab} = \psi^4 n_{ab}$, together with the fact that for an axisymmetric star in spherical coordinates $\beta^\phi$ is the only non-zero component of the shift vector $\beta^\alpha$, the line element in the CFC approximation is written as

$$\begin{aligned} ds^2 &= -\alpha^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt) \Rightarrow \\ ds^2 &= -\alpha^2 dt^2 + \psi^4(dr^2 + r^2 d\theta^2) + \psi^4 r^2 \sin^2\theta(d\phi + \beta^\phi dt)^2 \; . \end{aligned} \qquad (4)$$

Comparing the above expression to the form of the line element in the KEH scheme, i.e relation (1), we get

$$\alpha = e^{(\gamma+\rho)/2} \; , \qquad \psi = e^{\mu/2} = e^{(\gamma-\rho)/4} \; , \qquad \beta^\phi = -\omega \; .$$

From the second relation above we obtain

$$\mu = \frac{\gamma - \rho}{2} \; . \qquad (5)$$

Using the above relation in the standard KEH scheme instead of the differential equation for $\mu$, we impose the CFC on our solution.

Various physical quantities are calculated for each model both in CFC and in full GR. As a diagnostic of the quality of the CFC we use the quantity

$$\Delta c = \frac{\mu_{\text{full GR}} - \mu_{\text{CFC}}}{\mu_{\text{CFC}}} = \frac{\mu - \frac{\gamma - \rho}{2}}{\mu} \ . \tag{6}$$

In addition, relative differences between CFC and full GR are calculated for every physical quantity. The above procedure is applied in the sequences of models that appear in Table I of [10], which is reproduced here in Tables 1 and 2 for convenience.

Sequences A and B consist of differentially rotating models, whereas sequences AU and BU of uniformly rotating models. Configurations in sequences A and AU have constant rest mass of $M_0 = 1.506 M_\odot$ and configurations in sequences B and BU have constant central mass density of $\rho_c = 1.28 \times 10^{-3}$ or equivalently constant central energy density of $\epsilon_c = 1.444 \times 10^{-3}$. In all models a polytropic equation of state has been assumed with $N = 1$ and $K = 100$. We note that in addition to the models provided by [10], we located two additional models that rotate even faster, namely model A12 with a polar to equatorial radius ratio of 0.25 and model B13 with $r_p/r_e = 0.34$.

# 3 Results

The outcome of our tests for the different models constructed is very encouraging. Concerning the physical quantities that we calculated, the relative differences between CFC and full GR (Figures 1, 2, 3 and 4) were around or well below $10^{-2}$ in most cases. We also present the quantity $\Delta c$ as a function of the CST variable $s$ (Figures 5a, 5b, 6a and 6b) and also in the $x - z$ plane (Figures 7a, 7b, 8a and 8b) for the fastest rotating models of each sequence. We note that the CST variable $s$ was introduced [3] as an improvement upon the original KEH method via the transformation

$$r = r_e \frac{s}{1 - s} \ , \tag{7}$$

in order to map radial infinity to $s = 1$. In the above expression, $r_e$ is the coordinate equatorial radius. This choice improves accuracy in calculations of radial integrals and the boundary conditions are satisfied exactly. The polar to equatorial radius ratio, $r_p/r_e$, is used to indicate how fast a certain configuration is rotating. The most extreme case is the model B13 but even in that case the maximum value of $\Delta c$ is around 6%.

# 4 Conclusions

The CFC approximation appears to be a robust method to study systems that exhibit differential rotation if the demands for accuracy are not particularly strict, i.e if one can cope with a maximum error of around 5%. In most

Table 1: Sequences A and AU: Differentially and uniformly rotating equilibrium models, of constant rest mass $M_0 = 1.506 M_\odot$.

| Model | $\epsilon_c$ $(\times 10^{-3})$ | $r_p/r_e$ | M | $T/\lvert W\rvert$ $(\times 10^{-1})$ | $\Omega_c$ $(\times 10^{-2})$ | $\Omega_e$ $(\times 10^{-2})$ | $R_e$ |
|---|---|---|---|---|---|---|---|
| A0 | 1.444 | 1.0 | 1.40013 | 0.0000 | 0.000 | 0.000 | 9.58505 |
| A1 | 1.300 | 0.930 | 1.40439 | 0.1770 | 2.012 | 0.756 | 10.0107 |
| A2 | 1.187 | 0.875 | 1.40726 | 0.3264 | 2.575 | 0.975 | 10.3969 |
| A3 | 1.074 | 0.820 | 1.40934 | 0.4853 | 2.940 | 1.124 | 10.8403 |
| A4 | 0.961 | 0.762 | 1.41359 | 0.6639 | 3.193 | 1.233 | 11.3714 |
| A5 | 0.848 | 0.703 | 1.41775 | 0.8580 | 3.337 | 1.302 | 12.0047 |
| A6 | 0.735 | 0.643 | 1.42087 | 1.0692 | 3.381 | 1.336 | 12.7723 |
| A7 | 0.622 | 0.579 | 1.42758 | 1.3111 | 3.340 | 1.337 | 13.7516 |
| A8 | 0.509 | 0.513 | 1.43343 | 1.5801 | 3.199 | 1.301 | 15.0115 |
| A9 | 0.396 | 0.444 | 1.43822 | 1.8842 | 2.952 | 1.223 | 16.6970 |
| A10 | 0.283 | 0.370 | 1.44913 | 2.2362 | 2.605 | 1.101 | 19.0370 |
| A11 | 0.170 | 0.294 | 1.45891 | 2.5966 | 2.187 | 0.945 | 21.9181 |
| A12 | 0.110 | 0.250 | 1.45743 | 2.7431 | 1.965 | 0.859 | 23.3215 |
| | | | | | | | |
| AU0 | 1.444 | 1.0 | 1.40013 | 0.0000 | 0.000 | 0.000 | 9.58505 |
| AU1 | 1.300 | 0.919 | 1.40431 | 0.1963 | 1.296 | 1.296 | 10.1924 |
| AU2 | 1.187 | 0.852 | 1.40756 | 0.3653 | 1.656 | 1.656 | 10.7895 |
| AU3 | 1.074 | 0.780 | 1.41115 | 0.5502 | 1.888 | 1.888 | 11.5566 |
| AU4 | 0.961 | 0.698 | 1.41523 | 0.7548 | 2.028 | 2.028 | 12.6441 |
| AU5 | 0.863 | 0.575 | 1.41997 | 0.9542 | 2.084 | 2.084 | 14.9350 |

Table 2: Sequences B and BU: Differentially and uniformly rotating equilibrium models, of constant central rest mass density $\rho_c = 1.28 \times 10^{-3}$ or equivalently constant central enrgy density $\epsilon_c$.

| Model | $\epsilon_c$ ($\times 10^{-3}$) | $r_p/r_e$ | M | $T/\lvert W\rvert$ ($\times 10^{-1}$) | $\Omega_c$ ($\times 10^{-2}$) | $\Omega_e$ ($\times 10^{-2}$) | $R_e$ |
|---|---|---|---|---|---|---|---|
| B0 | 1.444 | 1.0 | 1.40013 | 0.0000 | 0.000 | 0.000 | 9.58505 |
| B1 | 1.444 | 0.950 | 1.43676 | 0.1249 | 1.800 | 0.666 | 9.74695 |
| B2 | 1.444 | 0.900 | 1.47789 | 0.2573 | 2.574 | 0.994 | 9.92100 |
| B3 | 1.444 | 0.849 | 1.52572 | 0.4004 | 3.200 | 1.163 | 10.1127 |
| B4 | 1.444 | 0.800 | 1.57840 | 0.5462 | 3.728 | 1.342 | 10.3113 |
| B5 | 1.444 | 0.750 | 1.64041 | 0.7041 | 4.227 | 1.504 | 10.5290 |
| B6 | 1.444 | 0.700 | 1.71268 | 0.8718 | 4.707 | 1.651 | 10.7617 |
| B7 | 1.444 | 0.650 | 1.79773 | 1.0500 | 5.186 | 1.789 | 11.0071 |
| B8 | 1.444 | 0.600 | 1.89887 | 1.2393 | 5.684 | 1.921 | 11.2590 |
| B9 | 1.444 | 0.550 | 2.02030 | 1.4396 | 6.233 | 2.052 | 11.5026 |
| B10 | 1.444 | 0.500 | 2.16685 | 1.6497 | 6.890 | 2.192 | 11.7049 |
| B11 | 1.444 | 0.450 | 2.33646 | 1.8673 | 7.780 | 2.359 | 11.7834 |
| B12 | 1.444 | 0.400 | 2.53257 | 2.0721 | 9.119 | 2.584 | 11.6435 |
| B13 | 1.444 | 0.340 | 2.71188 | 2.2769 | 11.93 | 3.010 | 10.9479 |
| | | | | | | | |
| BU0 | 1.444 | 1.0 | 1.40013 | 0.0000 | 0.000 | 0.000 | 9.58505 |
| BU1 | 1.444 | 0.95 | 1.43191 | 0.1199 | 1.075 | 1.075 | 9.82995 |
| BU2 | 1.444 | 0.90 | 1.46626 | 0.2437 | 1.508 | 1.508 | 10.1049 |
| BU3 | 1.444 | 0.85 | 1.50354 | 0.3701 | 1.829 | 1.829 | 10.4171 |
| BU4 | 1.444 | 0.80 | 1.54353 | 0.4975 | 2.084 | 2.084 | 10.7760 |
| BU5 | 1.444 | 0.75 | 1.58545 | 0.6232 | 2.290 | 2.290 | 11.1947 |
| BU6 | 1.444 | 0.70 | 1.62754 | 0.7419 | 2.452 | 2.452 | 11.6926 |
| BU7 | 1.444 | 0.65 | 1.66575 | 0.8439 | 2.569 | 2.569 | 12.2998 |
| BU8 | 1.444 | 0.60 | 1.69174 | 0.9104 | 2.633 | 2.633 | 13.0664 |
| BU9 | 1.444 | 0.58 | 1.69550 | 0.9198 | 2.642 | 2.642 | 13.4354 |

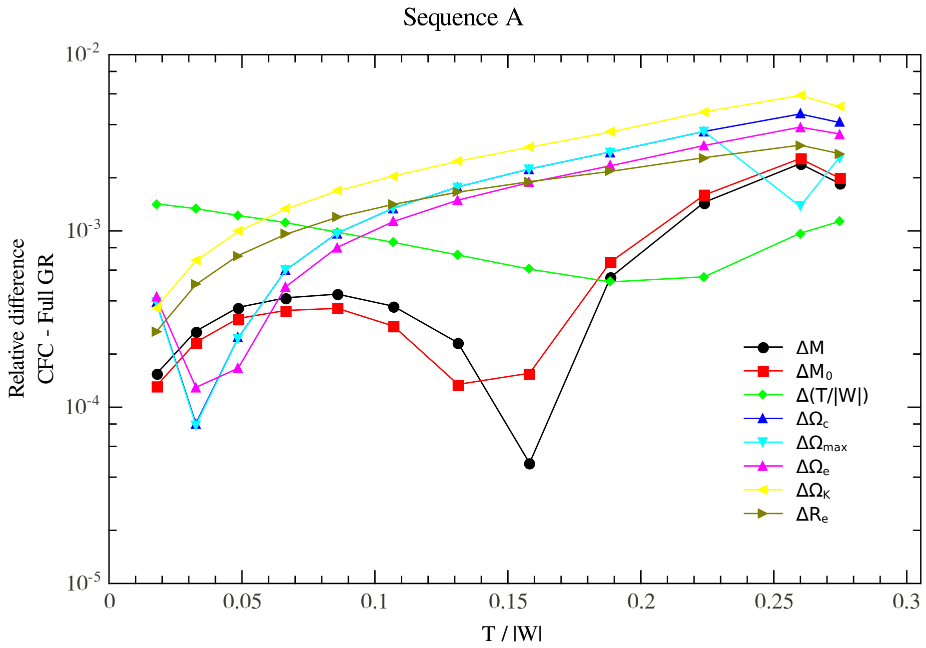



Figure 1: Sequence A: Relative differences between full GR and CFC approximation for all physical quantities.

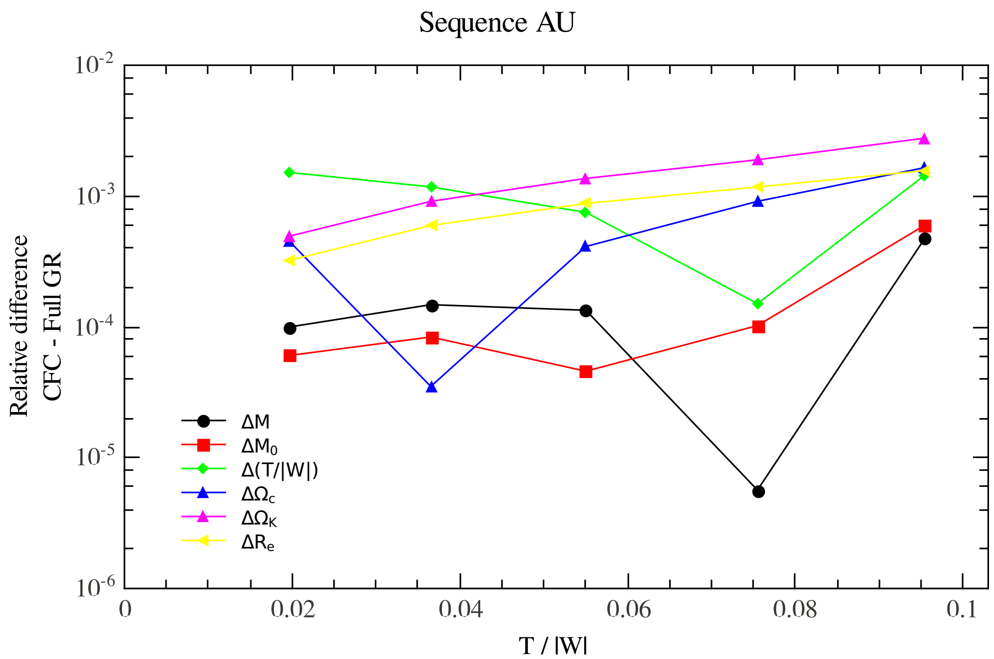



Figure 2: Sequence AU: Relative differences between full GR and CFC approximation for all physical quantities.

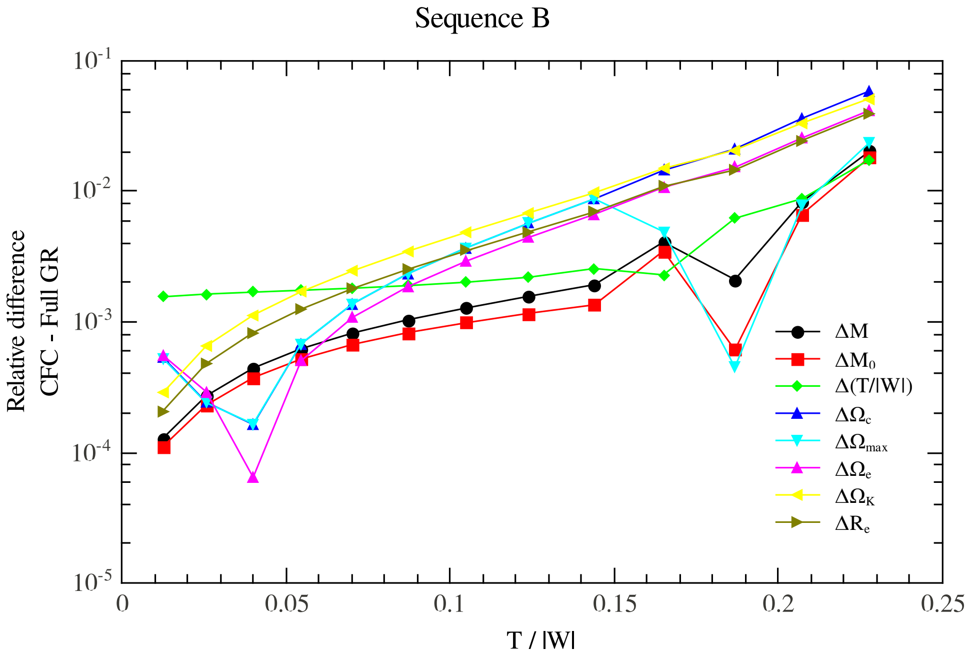



Figure 3: Sequence B: Relative differences between full GR and CFC approximation for all physical quantities.

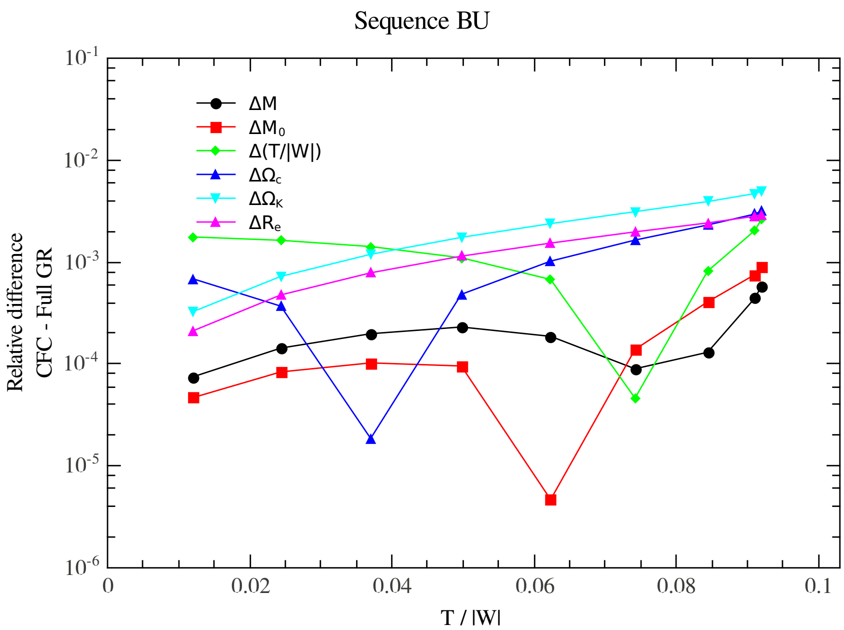
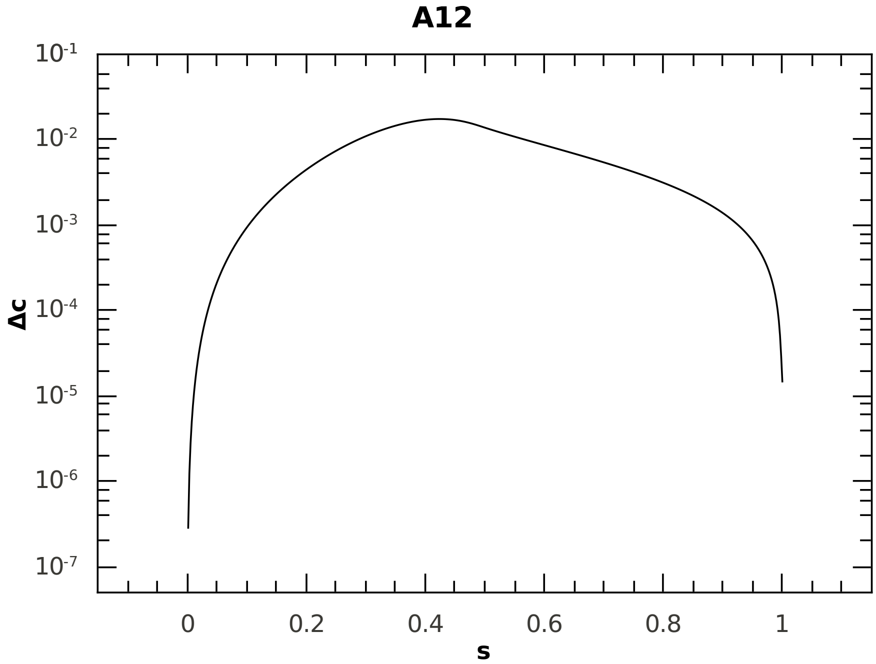
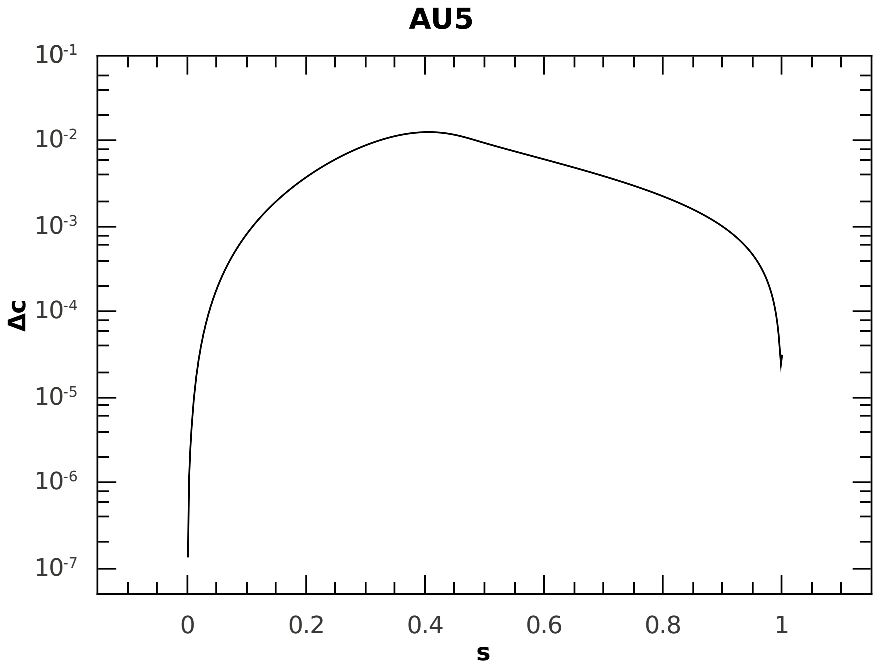



Figure 4: Sequence BU: Relative differences between full GR and CFC approximation for all physical quantities.

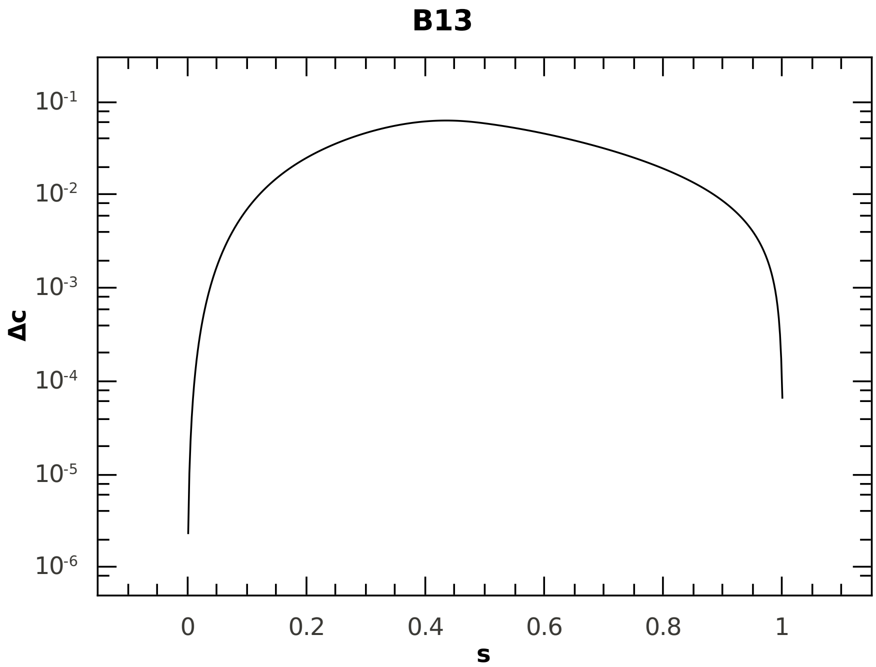



(a) Model A12, $r_p/r_e = 0.25$

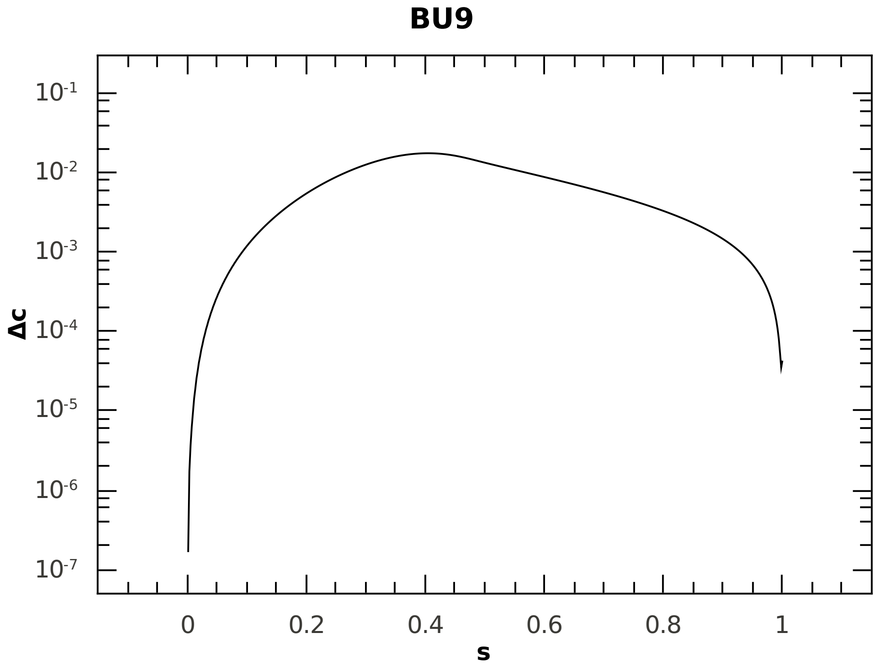



(b) Model AU5, $r_p/r_e = 0.575$

Figure 5: Relative difference $\Delta c$ calculated on the equatorial plane for the fastest rotating model of each sequence.

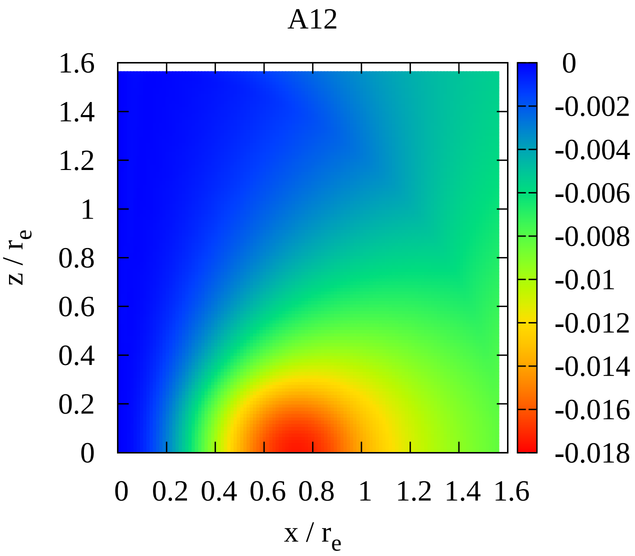



(a) Model B13, $r_p/r_e = 0.34$

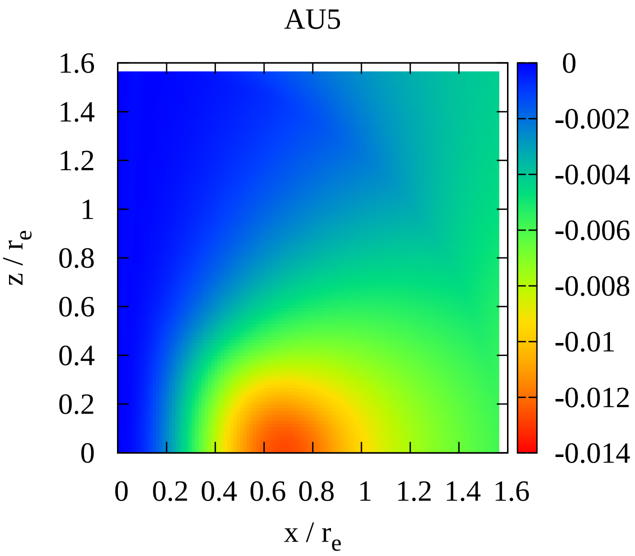



(b) Model BU9, $r_p/r_e = 0.58$

Figure 6: Relative difference $\Delta c$ calculated on the equatorial plane for the fastest rotating model of each sequence.

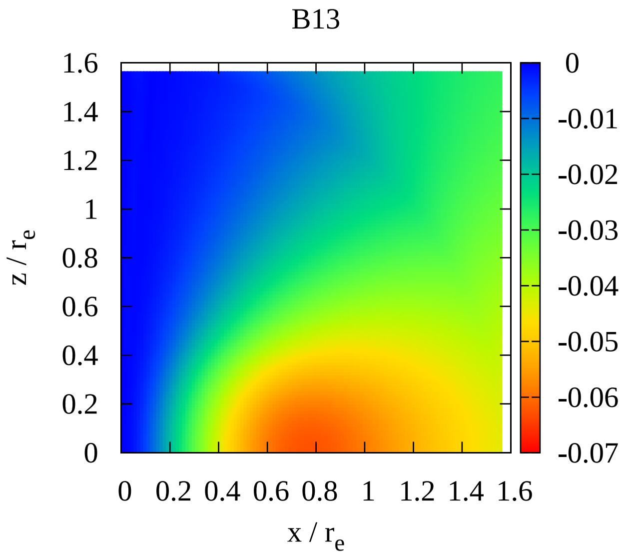



(a) Model A12, $r_p/r_e = 0.25$

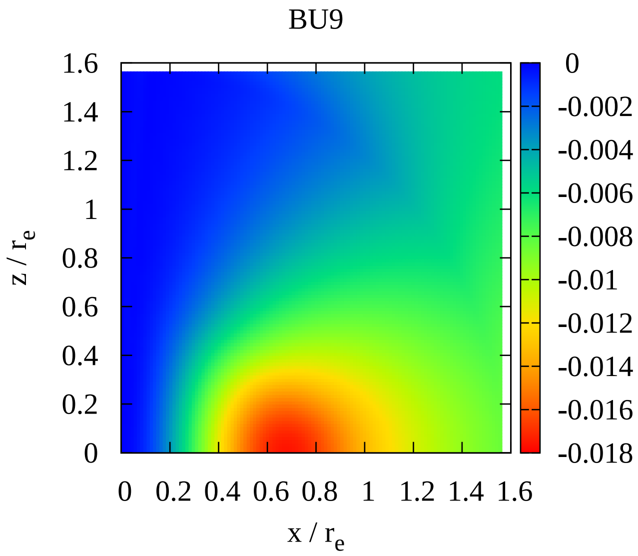



(b) Model AU5, $r_p/r_e = 0.575$

Figure 7: Relative difference $\Delta c$ on the $x - z$ plane for the fastest rotating model of each sequence.




(a) Model B13, $r_p/r_e = 0.34$




(b) Model BU9, $r_p/r_e = 0.58$

Figure 8: Relative difference $\Delta c$ on the $x-z$ plane for the fastest rotating model of each sequence.

cases the errors encountered are *significantly* lower and one should examine the fastest rotating models to observe the maximum error mentioned above. In sequence A, all the relative differences calculated are below 1% and $\Delta c$ for the fastest rotating model of the sequence remains below 2%. In sequence B larger deviations are observed, however only the relative differences for the angular velocities approach the maximum value of 6%. The diagnostic $\Delta c$ for the fastest rotating model of sequence B indicates that the maximum deviation from full GR is only around 6%. This result should provide added confidence in choosing the CFC approximation as a possible candidate to tackle an astrophysically relevant problem that involves differential rotation.

As far as the case of uniform rotation is concerned, we verified that the CFC approximation works particularly well. For every physical quantity that was evaluated, the corresponding relative difference between CFC and full GR never exceeded 1%, staying mainly in the $10^{-4}$ to $10^{-3}$ range. In addition, the diagnostic $\Delta c$ for the fastest uniformly rotating models remained under 2%.

Future directions for this project include the calculation of the Bach tensor for every configuration in order to add another diagnostic in our study of the CFC approximation. If the CFC approximation is a valid method for studying systems that rotate differentially, then the Bach tensor should vanish or otherwise be close to zero. We expect that the calculation of the Bach tensor will not alter significantly the already produced results but will further strengthen CFC as a satisfactory approximation of full GR.

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