The Circular Unitary Ensemble and the Riemann zeta function: the microscopic landscape

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Abstract

We show in this paper that after proper scalings, the characteristic polynomial of a random unitary matrix converges almost surely to a random analytic function whose zeros, which are on the real line, form a determinantal point process with sine kernel. Our scaling is performed at the so-called "microscopic" level, that is we consider the characteristic polynomial at points which are of order $1/n$ distant. We draw several consequences from our result. On the random matrix theory side, we obtain the limiting distribution for ratios of characteristic polynomials where the points are evaluated at points of the form $\exp(2i\pi\alpha/n)$. We also give an explicit expression for the (dependence) relation between two different values of the characteristic polynomial on the microscopic scale. On the number theory side, inspired by the Keating-Snaith philosophy, we conjecture some new limit theorems for the Riemann zeta function at the stochastic process level as well as some alternative approach to the conjecture by Goldston, Montgomery and Gonek for the moments of the logarithmic derivative of the Riemann zeta function. We prove our main random matrix theory result in the framework of virtual isometries to circumvent the fact that the rescaled characteristic polynomial does not even have a moment of order one, hence making the classical techniques of random matrix theory difficult to apply.

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1 Introduction

A major breakthrough in the so called random matrix approach in number theory is the seminal paper of Keating and Snaith [KS00], where they conjecture that the characteristic polynomial of a random unitary matrix, restricted to the unit circle, is a good and accurate model to predict the value distribution of the Riemann zeta function on the critical line. In particular, using this philosophy, they were able to conjecture the exact asymptotics of the moments of the Riemann zeta function, a result which was considered to be out of reach with classical tools form analytic number theory. One simple and naive explanation for the success of the characteristic polynomial as a random model to the Riemann zeta function comes from Montgomery's conjecture that asserts that the zeros of the Riemann zeta function on the critical line (after rescaling) statistically behave like the eigenangles (after rescaling) of large random unitary matrices. Moreover the limiting point process obtained from the eigenvalues is a determinantal point process with the sine kernel. A natural question which then naturally arose in the community was the existence of a random analytic function with zeros which from a determinantal point process with the sine kernel and which would be obtained as a limiting object from characteristic polynomials. As we shall see below, the sequence of characteristic polynomials of random unitary matrices of growing dimensions does not converge. We shall nonetheless prove that after a proper rescaling in "time" (the characteristic polynomial can be viewed as a stochastic process with parameter $z \in \mathbb{C}$, and we shall consider the characteristic polynomial at the scale z/n) and space (that is we normalize with the value of the characteristic polynomial at 1), this sequence converges locally uniformly on compact subsets of the complex plane to a random analytic function with the desired property. The convergence will be proved to occur almost surely, thanks to the use of virtual isometries introduced in [BNN12]. The basic idea behind virtual isometries is that of coupling the different dimensions of the unitary groups $U(n)$ together in such a way that marginal distribution on each $U(n)$, for fixed n, is the Haar measure. This strong convergence will in turn imply the weak convergence of the same objects. But since our rescaled characteristic polynomials do not even have a moment of order one, proving the weak convergence as stated in Theorem 1.2 with classical methods does not seem to be an easy task. On the other hand combining some of the fine estimates on the eigenvalues from [MNN13] which make strong use of the coupling from virtual isometries and some other classical estimates on sine kernel determinantal point processes is enough to establish almost sure convergence. We shall see that our main limit theorem has several far reaching consequences:

1. On the random matrix theory side, we shall be able to characterize the limit of ratios of characteristic polynomials evaluated at points of the form $\frac{\alpha}{n}$ for $\alpha \in \mathbb{C}$. We shall also give a description of the dependence between the log of the characteristic polynomial evaluated at various points distant of $\frac{\alpha}{n}$. Ratios of characteristic polynomials of random matrices are relevant objects which have been extensively studied in recent years, for instance in relation with quantum chaotic systems or analytic number theory (see [BS06], [CS07], [CFZ08], [BG06]), using a wide range of techniques (e.g. classical analysis, representation theory or supersymmetry methods). To the best of our knowledge the problem of characterizing the limiting object on the microscopic scale has never been addressed or solved before.[1] We shall also derive the limiting object for the rescaled logarithmic derivative of the characteristic polynomial at the microscopic scale. This limiting object was not known before; in fact, after we prove the convergence of the properly rescaled logarithmic derivative, we can give an alternative combinatorial identity for its moments through the formulas in [CFZ08] or [CS08]. However we shall see that there exist explicit formulas for the moments using only the correlation functions of the sine kernel determinantal point process (though it should be noted that the computations become very heavy beyond the second complex moments).

2. On the number theory side, we shall state some conjectures relating our limiting random analytic function to the Riemann zeta function: our scaling amounts to eliminating the contribution of prime numbers to keep only those of the Riemann zeros and thus obtain a limiting object whose zeros form a sine kernel determinantal point process, in agreement with the GUE conjecture. We shall also relate the logarithmic derivative of our limiting function to recent conjectures of Goldston, Gonek and Montgomery [GGM01] on the second moment of the logarithmic derivative of the Riemann zeta function. We shall be able to provide a very general conjecture on the logarithmic derivative of the Riemann zeta function in agreement with the predictions obtained in [GGM01] and in [FGLL13].

[1] This question was asked to A.N. by Alexei Borodin in a private communication.

Moreover we shall see that the logarithmic derivative of our random analytic function is of interest in its own and relate it to a Gaussian field on the mesoscopic scale: such objects, for more general point processes, have been recently studied by Aizenmann and Warzel in [AW13] and our results can be viewed as a complement to the results obtained in there.

In the sequel, we introduce the main objects and notation and state our main theorem.

1.1 The characteristic polynomial of random unitary matrices and the number theory connections

It is a well known fact in the theory of random unitary matrices that, when properly rescaled, the eigenvalues converge to a determinantal point process with sine kernel:

Proposition 1.1. *Let E_n denote the set of eigenvalues taken in $(-\pi, \pi]$ and multiplied by $n/2\pi$ of a random unitary matrix of size n following the Haar measure. Let us also define, for $y \neq y'$,*

$$K(y, y') = \frac{\sin[\pi(y' - y)]}{\pi(y' - y)}$$

and

$$K(y, y) = 1.$$

Let E_∞ be a determinantal sine-kernel process, i.e. a point process such that for all $r \in \{1, \dots, n\}$, and for all Borel measurable and bounded functions F with compact support from $\mathbb{R}^r$ to $\mathbb{R}$,

$$\mathbb{E}\left(\sum_{x_1 \neq \cdots \neq x_r \in E_\infty} F(x_1, \dots, x_r)\right) = \int_{\mathbb{R}^r} F(y_1, \dots, y_r)\rho_r(y_1, \dots, y_r)dy_1 \dots dy_r,$$

where

$$\rho_r(y_1, \dots, y_r) = \det((K(y_j, y_k))_{1 \leq j,k \leq r}).$$

Then, the point process E_n converges to E_∞ in the following sense: for all Borel measurable bounded functions f with compact support from $\mathbb{R}$ to $\mathbb{R}$,

$$\sum_{x \in E_n} f(x) \underset{n \to \infty}{\longrightarrow} \sum_{x \in E_\infty} f(x),$$

where the convergence above holds in law.

We now recall basic facts about the Riemann zeta function (the reader can find more details in classical textbooks such as [Tit86]). The Riemann zeta function is defined, for $\Re(s) > 1$, by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

It has a meromorphic continuation to the whole complex plane with a single pole at 1. It also satisfies a functional equation which we can be stated as follows:

$$\pi^{-s/2}\Gamma(s/2)\zeta(s) = \pi^{(s-1)/2}\Gamma((1-s)/2)\zeta(1-s),$$

and

$$\zeta(s) = \chi(s)\zeta(1-s),$$

where

$$\chi(1-s) = \chi(s)^{-1} = 2(2\pi)^{-s}\Gamma(s)\cos(\pi s/2).$$

The non-trivial zeros of the zeta function are denoted by $\rho = \sigma + it$, where $0 < \sigma < 1$. The Riemann hypothesis is the assertion that all non trivial zeros satisfy $\sigma = 1/2$ and hence all non trivial zeros are of the form $\rho = 1/2 + it$, with $t \in \mathbb{R}$. If we assume the Riemann hypothesis, then the zeros come in conjugate pairs and we note the zeros in the upper half-plane as $1/2 + i\gamma_j$, where $0 < \gamma_1 \leq \gamma_2 \leq \cdots$. One can count the number of such zeros up to some height T:

$$N(T) := \#\{j;\ 0 \leq \gamma_j \leq T\} = \frac{T}{2\pi}\log\frac{T}{2\pi e} + O(\log T).$$

The connection to random matrix theory was conjectured by Montgomery in [Mon73]: it is conjectured that the rescaled zeros of the zeta function $\tilde{\gamma} := \gamma/(2\pi)\log\gamma$ (this rescaling is done in order to obtain an average spacing of order 1) satisfy the same limit theorem as the one given in Proposition 1.1 for the rescaled eigenvalues of random unitary matrices (in fact the conjecture was initially stated for the pair correlation and then extended to all correlations by Rudnik and Sarnak in [RS96]; see the recent paper of Conrey and Snaith [CS14] for a detailed account and new methods).

Another major insight came with the work of Keating and Snaith ([KS00]) where they use the characteristic polynomial of random unitary matrices to model the value distribution of the Riemann zeta function on the critical line (i.e. the family $\{\zeta(1/2 + it),\ t \geq 0\}$) to make spectacular predictions on the moments of the Riemann zeta function. In particular, in [KS00] they computed the moments of the characteristic polynomial of a random unitary matrix following the Haar measure. They deduced that the characteristic polynomial asymptotically behaves like a log-normal distributed random variable when the dimension n goes to infinity: more precisely, its logarithm, divided by $\sqrt{\log n}$, tends to a complex Gaussian random variable Z such that $\mathbb{E}[Z] = \mathbb{E}[Z^2] = 0$ and $\mathbb{E}[|Z|^2] = 1$. This result has been generalized in Hughes, Keating and O'Connell [HKO01], where the authors proved the asymptotic independence of the characteristic polynomial taken at different fixed points. A question which then naturally arises concerns the behavior of the characteristic polynomial at points which vary with the dimension and which are sufficiently close to each other in order to avoid asymptotic independence. The scale we consider in the present paper is the average spacing of the eigenangles of a unitary matrix in dimension n, i.e. $2\pi/n$. More precisely, let $(U_n)_{n\geq 1}$ be a sequence of matrices, U_n being Haar-distributed in $U(n)$, and let Z_n be the characteristic polynomial of U_n:

$$Z_n(X) = \det\left(\mathrm{Id} - U_n^{-1}X\right) = \det\left(\mathrm{Id} - U_n^{*}X\right).$$

For a given $z \in \mathbb{C}$, we consider the value of Z_n at the two points 1 and $e^{2iz\pi/n}$, whose distance is equivalent to $2\pi|z|/n$ when n goes to infinity. We know that the law of $Z_n(1)$ can be approximated by the exponential of a gaussian variable of variance $\log n$, so it does not converge when n goes to infinity: the same is true for $Z_n(e^{2iz\pi/n})$. In order to obtain a convergence in law, it is then natural to consider the ratio $Z_n(e^{2iz\pi/n})/Z_n(1)$, which has order of magnitude 1 and which is well-defined as soon as 1 is not an eigenvalue of U_n, an event occurring almost surely.

If we consider all the values of z together, we obtain a random entire function ξ_n, defined by

$$\xi_n(z) = \frac{Z_n(e^{2iz\pi/n})}{Z_n(1)}.$$

We will prove that this function has a limiting distribution when n goes to infinity. More precisely, one of the main results of this article is the following:

Theorem 1.2. *In the space of continuous functions from $\mathbb{C}$ to $\mathbb{C}$, endowed with the topology of uniform convergence on compact sets, the random entire function ξ_n converges in law to a limiting entire function ξ_∞. The zeros of ξ_∞ are all real and form a determinantal sine-kernel point process, i.e. for all $r \geq 1$, the r-point correlation function ρ_r corresponding to this point process is given, for all $x_1, \dots, x_r \in \mathbb{R}$, by*

$$\rho_r(x_1, \dots, x_r) = \det\left(\frac{\sin[\pi(x_j - x_k)]}{\pi(x_j - x_k)}\right)_{1 \leq j,k \leq r}.$$

Taking a finite number of points $z_1, \dots, z_p \in \mathbb{C}$, we see in particular that the joint law of the mutual ratios of $Z_n(e^{2i\pi z_1/n}), \dots, Z_n(e^{2i\pi z_p/n})$ converges when n goes to infinity. Now one can hope to gain new insights on the behaviour of ratios of characteristic polynomials on this microscopic scale. More precisely, let us define:

$$R(\alpha_1, \cdots, \alpha_r; \beta_1, \cdots, \beta_r) := \frac{Z_n(e^{2i\alpha_1\pi/n}) \cdots Z_n(e^{2i\alpha_r\pi/n})}{Z_n(e^{2i\beta_1\pi/n}) \cdots Z_n(e^{2i\beta_r\pi/n})}, \tag{1}$$

where $r \in \mathbb{N}$ and $\alpha_j \in \mathbb{C}$, $\beta_j \in \mathbb{C}$, for all $1 \leq j \leq r$. Ratios such as (1), on the macroscopic scale (i.e. without the $1/n$ in the arguments) have been extensively studied in random matrix theory for different random matrix ensembles, e.g. the GUE by Borodin and Strahov in [BS06] or in the CUE case by Conrey, Framer and Zirnbauer ([CFZ08]), by Conrey and Snaith ([CS07]) or Bump and Gamburd ([BG06]). In all cases, one considers the expectation of the ratios and the n-limit of this expression. But the n-limit of $R(\alpha_1, \cdots, \alpha_r; \beta_1, \cdots, \beta_r)$ had remained an open problem. In fact, we shall prove a strong version (i.e. with almost sure convergence) of Theorem 1.2 which will immediately yield the n-limit of $R(\alpha_1, \cdots, \alpha_r; \beta_1, \cdots, \beta_r)$ as well as some central limit theorem for the vector $(\log Z_n(e^{2i\pi z/n}), \log Z_n(1))$. The almost sure convergence is established through the machinery of virtual isometries that we recall in the next paragraph.

1.2 Virtual isometries and almost sure convergence

In order to prove Theorem 1.2, we will define the sequence $(U_n)_{n\geq 1}$ of unitary matrices in a common probability space, with a coupling chosen in such a way that an almost sure convergence occurs. An interest of this method is that it is more convenient to deal with pointwise convergence than with convergence in law when we work on a functional space. Moreover, the coupling gives a powerful way to track the sequence $(\xi_n)_{n\geq 1}$ of holomorphic function, and a deterministic link between this sequence and the limiting function ξ_∞.

Besides it is important to stress that the moments method, which is a classical technique in random matrix theory, is impossible to implement. Indeed the random function at hand ξ_n does not have any integer moment when evaluated on circle, which makes the use of the formulas on moments of ratios in [BG06] and [CFZ08] difficult to use. For example, in Theorem 3 of the article [BG06], one clearly sees the divergence of ratios, as the evaluation points get close to 1.

The coupling we consider here corresponds to the notion of *virtual isometries*, as defined by Bourgade, Najnudel and Nikeghbali in [BNN12]. The sequence $(U_n)_{n\geq 1}$ can be constructed in the following way:

1. One considers a sequence $(x_n)_{x\geq 1}$ of independent random vectors, x_n being uniform on the unit sphere of $\mathbb{C}^n$.

2. Almost surely, for all $n \geq 1$, x_n is different from the last basis vector e_n of $\mathbb{C}^n$, which implies that there exists a unique $R_n \in U(n)$ such that $R_n(e_n) = x_n$ and $R_n - I_n$ has rank one.

3. We define $(U_n)_{n\geq 1}$ by induction as follows: $U_1 = x_1$ and for all $n \geq 2$,

$$U_n = R_n \begin{pmatrix} U_{n-1} & 0 \\ 0 & 1 \end{pmatrix}.$$

It has already been proven in [BHNY08] that with this construction, U_n follows, for all $n \geq 1$, the Haar measure on $U(n)$. From now on, we always assume that the sequence $(U_n)_{n\geq 1}$ is defined with this coupling.

For each value of n, let $\lambda_1^{(n)}, \dots, \lambda_n^{(n)}$ be the eigenvalues of U_n, ordered counterclockwise, starting from 1: they are almost surely pairwise distinct and different from 1. If $1 \leq k \leq n$, we denote by $\theta_k^{(n)}$ the argument of $\lambda_k^{(n)}$, taken in the interval $(0, 2\pi)$: $\theta_k^{(n)}$ is the k-th strictly positive eigenangle of U_n. If we consider all the eigenangles of U_n, taken not only in $(0, 2\pi)$ but in the whole real line, we get a (2π)-periodic set with n points in each period. If the eigenangles are indexed increasingly by $\mathbb{Z}$, we obtain a sequence

$$\dots < \theta_{-1}^{(n)} < \theta_0^{(n)} < 0 < \theta_1^{(n)} < \theta_2^{(n)} < \dots,$$

for which $\theta_{k+n}^{(n)} = \theta_k^{(n)} + 2\pi$ for all $k \in \mathbb{Z}$.

It is also convenient to extend the sequence of eigenvalues as a n-periodic sequence indexed by $\mathbb{Z}$, in such a way that for all $k \in \mathbb{Z}$,

$$\lambda_k^{(n)} = \exp\left(i\theta_k^{(n)}\right).$$

With the notation above, the following holds:

Theorem 1.3 (Theorem 7.3 in [MNN13]). *Almost surely, the point process*

$$\left(y_k^{(n)} := \frac{n}{2\pi}\theta_k^{(n)},\ k \in \mathbb{Z}\right)$$

converges pointwise to a determinantal sine-kernel point process $(y_k,\ k \in \mathbb{Z})$. And moreover, almost surely, the following estimate holds for all $\varepsilon > 0$:

$$\forall k \in [-n^{\frac{1}{4}}, n^{\frac{1}{4}}],\ y_k^{(n)} = y_k + O_\varepsilon\left((1+k^2)n^{-\frac{1}{3}+\varepsilon}\right)$$

Remark 1.4. *The implied constant in O_ε is random: more precisely, it may depend on the sequence $(U_m)_{m\geq 1}$ and on ε. However, it does not depend on k and n.*

We are now able to state the main convergence result of the paper.

Theorem 1.5. *Almost surely and uniformly on compact subsets of $\mathbb{C}$, we have the convergence:*

$$\xi_n(z) \stackrel{n\to\infty}{\longrightarrow} \xi_\infty(z) := e^{i\pi z} \prod_{k\in\mathbb{Z}} \left(1 - \frac{z}{y_k}\right)$$

Here, the infinite product is not absolutely convergent. It has to be understood as the limit of the following product, obtained by regrouping the factors two by two:

$$\left(1 - \frac{z}{y_0}\right) \prod_{k\geq 1} \left[\left(1 - \frac{z}{y_k}\right)\left(1 - \frac{z}{y_{-k}}\right)\right],$$

which is absolutely convergent.

This theorem immediately implies Theorem 1.2, provided that ξ_∞ is entire and that the zeros of ξ_∞ are exactly given by the sequence $(y_k)_{k\in\mathbb{Z}}$.

The first point is a direct consequence of the fact that ξ_∞ is the uniform limit on compact sets of the sequence of entire functions $(\xi_n)_{n\geq 1}$, and the second point is a consequence of the fact that the k-th factor of the absolutely convergent product above vanishes at y_k and y_{-k} and only at these points.

Now, thanks to the a.s. convergence, we can state the following corollaries.

Corollary 1.6. *Let $r \in \mathbb{N}$ and $\alpha_j \in \mathbb{C}$, $\beta_j \in \mathbb{C}$ but $\beta_j \notin (y_k)_{k\in\mathbb{Z}}$, for all $1 \leq j \leq r$. Then the following convergence holds a.s. as $n \to \infty$:*

$$R(\alpha_1, \cdots, \alpha_r; \beta_1, \cdots, \beta_r) := \frac{Z_n(e^{2i\alpha_1\pi/n}) \cdots Z_n(e^{2i\alpha_r\pi/n})}{Z_n(e^{2i\beta_1\pi/n}) \cdots Z_n(e^{2i\beta_r\pi/n})} \to \frac{\xi_\infty(\alpha_1) \cdots \xi_\infty(\alpha_r)}{\xi_\infty(\beta_1) \cdots \xi_\infty(\beta_r)}$$

Since the convergence in Theorem 1.5 holds almost surely in the space of holomorphic functions, we immediately obtain:

Corollary 1.7. *We have a.s. as $n \to \infty$:*

$$\frac{2i\pi}{n} \frac{Z_n'(e^{2i\pi z/n})}{Z_n(1)} \to \xi_\infty'.$$

The next corollary involves the logarithm of Z_n. The determination of this logarithm is the only one such that $\log Z_n$ vanishes at 0 (recall that $Z_n(0) = 1$), and which is continuous on the following maximal simply connected domain

$$\mathcal{D} := \mathbb{C} \setminus \left\{ r e^{i\theta_k^{(n)}} | k \in \mathbb{Z}, r \geq 1 \right\}.$$

Note that for all $z \in \mathcal{D}$, we have:

$$\log Z_n(z) = \sum_{k=1}^{n} \log \left(1 - \frac{z}{\lambda_k^{(n)}} \right),$$

where the principal branch of the logarithm is considered.

Corollary 1.8. *Let $z \in \mathbb{C}$. The following convergence holds in law as $n \to \infty$*

$$\left(\frac{\log Z_n(e^{2i\pi z/n})}{\sqrt{(1/2)\log n}}, \frac{\log Z_n(1)}{\sqrt{(1/2)\log n}} \right) \to (\mathcal{N}, \mathcal{N})$$

where $\mathcal{N}$ stands for a standard complex Gaussian random variable.

Proof. One checks that

$$\log Z_n(e^{2i\pi z/n}) - \log Z_n(1) = \log \xi_n(z),$$

where $\log \xi_n$ is the unique determination of the logarithm, vanishing at 0, and continuous in the domain

$$\mathcal{D}'_n := \mathbb{C} \setminus \left\{ y_k^{(n)} - iu | k \in \mathbb{Z}, u \geq 0 \right\}.$$

Let $\log \xi_\infty$ be the similar determination of the logarithm of ξ_∞. Let us fix $z \in \mathbb{C}$, $t > 0$ such that $z + it$ has strictly positive imaginary part, and let L be the line consisting of the two segments from 0 to $z + it$ and from $z + it$ to z. We also suppose that the random functions $(\xi_n)_{n \geq 1}$ and ξ_∞ are coupled in such a way that almost surely, ξ_n tends to ξ_∞ uniformly on compact sets of $\mathbb{C}$. Almost surely, for n large enough, 0 and $\Re z$ are not zeros of ξ_n and one deduces that L is included in $\mathcal{D}'_n$. Hence,

$$\log \xi_n(z) = \int_L \frac{\xi'_n(s)}{\xi_n(s)} ds$$

and

$$\log \xi_\infty(z) = \int_L \frac{\xi'_\infty(s)}{\xi_\infty(s)} ds.$$

Now, (ξ_n, ξ'_n) tends to $(\xi_\infty, \xi'_\infty)$ uniformly on L. Moreover, ξ_∞ is continuous and non-vanishing on the compact set L, which implies that $|\xi_\infty|$, and then $|\xi_n|$ for n large enough, are bounded away from zero on L. Hence, ξ'_n/ξ_n tends to ξ'_∞/ξ_∞ uniformly on L, and then $\log \xi_n(z)$ tends to $\log \xi_\infty(z)$. We deduce that

$$\frac{\log Z_n(e^{2i\pi z/n})}{\sqrt{(1/2)\log n}} - \frac{\log Z_n(1)}{\sqrt{(1/2)\log n}} \underset{n\to\infty}{\longrightarrow} 0$$

almost surely with the coupling above, and then in probability. Since we already know that the second term of the difference tends in law to $\mathcal{N}$, we are done.

□

Remark 1.9. *This central limit theorem is consistent with the predictions of [FK14] on the correlation of the log of the characteristic polynomial taken at two points distant of* $1/n$*.*

Remark 1.10. *A similar result would hold for a finite number of points.*

Remark 1.11. *From Corollary 1.7 one can also deduce a joint central limit theorem for the log of the derivative of the characteristic polynomial at* $e^{2i\pi z/n}$ *and the log of the characteristic polynomial at* 1*.*

We can eventually easily derive the limiting random analytic function for the logarithmic derivative:

Corollary 1.12. *We have almost surely, for all* $z \notin \{y_k,\ k \in \mathbb{Z}\}$ *:*

$$\frac{2i\pi}{n} \frac{Z_n'(e^{2i\pi z/n})}{Z_n(e^{2i\pi z/n})} \underset{n\to\infty}{\longrightarrow} \frac{\xi_\infty'(z)}{\xi_\infty(z)},$$

where

$$\frac{\xi_\infty'(z)}{\xi_\infty(z)} = i\pi + \sum_{k\in\mathbb{Z}} \frac{1}{z-y_k} =: i\pi + \frac{1}{z-y_0} + \sum_{k=1}^{\infty}\left(\frac{1}{z-y_k} + \frac{1}{z-y_{-k}}\right).$$

Hence, for all $\alpha_1, \dots, \alpha_r \notin \{y_k,\ k \in \mathbb{Z}\}$*,*

$$\left(\frac{2i\pi}{n}\right)^r \frac{Z_n'(e^{2i\alpha_1\pi/n})}{Z_n(e^{2i\alpha_1\pi/n})} \frac{Z_n'(e^{2i\alpha_2\pi/n})}{Z_n(e^{2i\alpha_2\pi/n})} \cdots \frac{Z_n'(e^{2i\alpha_r\pi/n})}{Z_n(e^{2i\alpha_r\pi/n})} \underset{n\to\infty}{\longrightarrow} \frac{\xi_\infty'(\alpha_1)}{\xi_\infty(\alpha_1)} \cdots \frac{\xi_\infty'(\alpha_r)}{\xi_\infty(\alpha_r)}.$$

1.3 Outline of the paper

The proof of Theorem 1.5 will be made in several steps in Section 2 , using estimates on the argument of Z_n, stated in Section 2.1, and estimates on the renormalized eigenangles $y_k^{(n)}$, stated in Section 2.2. In Section 3, we prove some properties of the limiting random function ξ_∞, we compute the moments of order 1 and 2 of its logarithmic derivative, and we state some related conjectures on the behavior of the Riemann zeta function in the neighborhood of the critical line. In Section 4, we prove that in a sense which can be made precise, the fluctuations of the determinantal sine-kernel process, viewed at a scale tending to infinity, converge in law to a blue noise, i.e. a noise whose spectral density is proportional to the frequency. In relation with this convergence, we show that the fluctuations of ξ_∞'/ξ_∞, viewed at a large scale, tend to a Gaussian process on $\mathbb{C}\backslash\mathbb{R}$, whose convariance structure is explicitly computed. This covariance is consistent with the computation of the two first moments of ξ_∞'/ξ_∞.

2 Proof of Theorem 1.5

2.1 On the argument of the characteristic polynomial

In this section, we study the argument of Z_n, in order to deduce estimates on the deviation of $y_k^{(n)}$ from k.

Here, we define the argument as the imaginary part of $\log Z_n$, with the determination of the logarithm given in the previous section.

The next proposition gives a link between the number of eigenvalues of U_n in a given arc of circle, and the variation of the argument of Z_n along this arc. The derivation is relatively standard and we shall not reproduce a proof here (see [Hug01], p. 35-36. or [BHNN13], proof of Proposition 2.2).

Proposition 2.1. *Consider A and B two points on the unit circle. Note $\widehat{AB}$ for the arc joining A and B counterclockwise. Denote by $\ell\left(\widehat{AB}\right)$ the length of the arc and $N\left(\widehat{AB}\right)$ the number of zeros of Z_n in the arc. We assume that A and B are not zeros of Z_n. Then:*

$$N\left(\widehat{AB}\right) = \frac{n\ell\left(\widehat{AB}\right)}{2\pi} - \frac{1}{\pi}\left[\Im\log\left(Z_n(B)\right) - \Im\log\left(Z_n(A)\right)\right].$$

Remark 2.2. *This shows that the imaginary part of the determination of the logarithm $\Im\log Z_n(z)$ increases with speed $n/2$ and jumps by $-\pi$ when encountering a zero.*

Corollary 2.3. *Let $k \in \mathbb{Z}$, and let $\varepsilon > 0$ be small enough so that there are no eigenangle of U_n in $[0, \varepsilon]$ and $(\theta_k^{(n)}, \theta_k^{(n)} + \varepsilon]$. Then:*

$$k = y_k^{(n)} - \frac{1}{\pi}\Im\left(\log\left(Z_n(e^{i(\theta_k^{(n)}+\varepsilon)})\right) - \log\left(Z_n(e^{i\varepsilon})\right)\right)$$

Proof. Notice first that if k is increased by n, $\theta_k^{(n)}$ increases by 2π, $y_k^{(n)}$ increases by n, $\lambda_k^{(n)} = e^{i\theta_k^{(n)}}$ does not change, and the assmption made on ε remains the same. Hence, in the equality we want to prove, the right-hand side and the left-hand side both increase by n, which implies that it is sufficient to show the corollary for $1 \leq k \leq n$. If these inequalities are satisfied, let us choose, in the previous proposition, $A = e^{i\varepsilon}$ and $B = e^{i(\theta_k^{(n)}+\varepsilon)}$. Then we note that

$$N\left(\widehat{AB}\right) = k,$$

and

$$\frac{n\ell\left(\widehat{AB}\right)}{2\pi} = \frac{n\theta_k^{(n)}}{2\pi} = y_k^{(n)},$$

which proves the corollary. □

This corollary shows that it is equivalent to control the argument of Z_n, and the distance between k and $y_k^{(n)}$. In the remaining of this section, we give some explicit bounds on the distribution of $\Im\log(Z_n)$ on the unit circle.

Proposition 2.4. *For all $x > 0$, one has*

$$\mathbb{P}\left(\left|\Im\left(\log Z_n(1)\right)\right| \geq x\right) \leq 2\exp\left(-\frac{x^2}{C + \log n}\right),$$

where $C > 0$ is a universal constant.

Remark 2.5. *In the proof below, we prove that one can take* $C = \frac{\pi^2}{6} + 1$.

Proof. Let us note

$$X_n = \Im\left(\log Z_n(1)\right)$$

Thanks to the formula (1.1) in [BHNY08]:

$$\forall \lambda \in \mathbb{R}, \mathbb{E}\left(e^{\lambda X_n}\right) = \prod_{k=1}^{n} \frac{\Gamma\left(k\right)^2}{\Gamma\left(k + \frac{i\lambda}{2}\right)\Gamma\left(k - \frac{i\lambda}{2}\right)}$$

Let us start with the standard Chernoff bound:

$$\forall \lambda > 0, \mathbb{P}\left(X_n \geq x\right) \leq e^{-\lambda x}\mathbb{E}\left(e^{\lambda X_n}\right).$$

Now, using the infinite product formula for the Gamma function:

$$\forall z \in \mathbb{C}, \frac{1}{\Gamma(z)} = e^{\gamma z} z \prod_{j=1}^{\infty}\left(1 + \frac{z}{j}\right)e^{-z/j},$$

$$\begin{aligned}
\mathbb{E}\left(e^{\lambda X_n}\right) &= \prod_{k=1}^{n} \frac{\Gamma\left(k\right)^2}{\Gamma\left(k + \frac{i\lambda}{2}\right)\Gamma\left(k - \frac{i\lambda}{2}\right)} \\
&= \prod_{k=1}^{n}\left(\frac{k^2 + \frac{\lambda^2}{4}}{k^2}\prod_{j=1}^{\infty}\frac{\left(1 + \frac{k+\frac{i\lambda}{2}}{j}\right)\left(1 + \frac{k-\frac{i\lambda}{2}}{j}\right)}{\left(1 + \frac{k}{j}\right)^2}\right) \\
&= \prod_{k=1}^{n}\left(\frac{k^2 + \frac{\lambda^2}{4}}{k^2}\prod_{j=1}^{\infty}\frac{\left(j + k + \frac{i\lambda}{2}\right)\left(j + k - \frac{i\lambda}{2}\right)}{\left(j+k\right)^2}\right) \\
&= \prod_{k=1}^{n}\prod_{j=0}^{\infty}\frac{\left(j+k\right)^2 + \frac{\lambda^2}{4}}{\left(j+k\right)^2} \\
&= \prod_{k=1}^{n}\prod_{j=0}^{\infty}\left(1 + \frac{\lambda^2}{4\left(j+k\right)^2}\right) \\
&\leq \exp\left(\sum_{k=1}^{n}\sum_{j=0}^{\infty}\frac{\lambda^2}{4\left(j+k\right)^2}\right) \\
&= \exp\left(\frac{\lambda^2}{4}\sum_{k=1}^{n}\sum_{j=k}^{\infty}\frac{1}{j^2}\right) \\
&\leq \exp\left(\frac{\lambda^2}{4}\sum_{k=1}^{n}\left(\frac{1}{k^2} + \int_k^{\infty}\frac{dt}{t^2}\right)\right) \\
&= \exp\left(\frac{\lambda^2}{4}\sum_{k=1}^{n}\left(\frac{1}{k^2} + \frac{1}{k}\right)\right) \\
&\leq \exp\left(\frac{\lambda^2}{4}\left(\frac{\pi^2}{6} + 1 + \log n\right)\right)
\end{aligned}$$

Eventually for $C = \frac{\pi^2}{6} + 1$, we obtain

$$\mathbb{P}\left(X_n \geq x\right) \leq \min_{\lambda > 0} e^{-\lambda x + \frac{\lambda^2}{4}(C + \log n)}.$$

The minimum is reached for $\lambda = \frac{2x}{C + \log n}$, giving us the bound:

$$\mathbb{P}\left(\Im\left(\log Z_n(1)\right) \geq x\right) \leq \exp\left(-\frac{x^2}{C + \log n}\right).$$

The desired bound is obtained from the symmetry of $\Im\left(\log Z_n(1)\right)$, as eigenvalues are invariant in law under conjugation:

$$\begin{aligned}
&\mathbb{P}\left(\left|\Im\left(\log Z_n(1)\right)\right| \geq x\right) \\
=&\mathbb{P}\left(\Im\left(\log Z_n(1)\right) \geq x\right) + \mathbb{P}\left(-\Im\left(\log Z_n(1)\right) \geq x\right) \\
=&2\mathbb{P}\left(\Im\left(\log Z_n(1)\right) \geq x\right)
\end{aligned}$$

□

We deduce the following estimate on the maximum of the argument of Z_n on the unit circle:

Proposition 2.6. *Almost surely:*

$$\sup_{|z|=1, z \in \mathcal{D}} \left|\Im \log Z_n(z)\right| = O\left(\log n\right)$$

More precisely, for any $D > \sqrt{2}$:

$$\exists n_0 \in \mathbb{N}, \forall n \geq n_0, \sup_{|z|=1, z \in \mathcal{D}} \left|\Im \log Z_n(z)\right| \leq D \log n$$

which means that almost surely:

$$\limsup_n \frac{1}{\log n} \sup_{|z|=1, z \in \mathcal{D}} \left|\Im \log Z_n(z)\right| \leq \sqrt{2}$$

Proof. Consider n regularly spaced points on the circle, say:

$$x_{k,n} := e^{i\frac{2\pi k}{n}}, \quad k = 0, 1, 2, \ldots, n-1,$$

and the events:

$$A_{k,n} := \left\{\left|\Im \log Z_n\left(x_{k,n}\right)\right| \geq D \log n\right\}$$

Because the law of the spectrum of U_n is invariant under rotation, all the events $A_{k,n}$ have the same probability for different k's. Moreover, thanks to the previous Chernoff bound:

$$n\mathbb{P}\left(A_{0,n}\right) \leq 2n \exp\left(-\frac{D^2(\log n)^2}{C + \log n}\right)$$

$$\leq 2n\exp\left(-D^2\left(\log n - C\right)\right)$$
$$\leq 2e^{D^2C}n^{1-D^2}$$

Hence:

$$\sum_{n=1}^{\infty}\sum_{k=1}^{n}\mathbb{P}\left(A_{k,n}\right) = \sum_{n=1}^{\infty} n\mathbb{P}\left(A_{0,n}\right) < \infty$$

The Borel-Cantelli lemma ensures that, almost surely:

$$\exists n_0 \in \mathbb{N}, \forall n \geq n_0, \forall k, \quad \left|\Im \log Z_n\left(x_{k,n}\right)\right| \leq D\log n$$

Now consider a point $z = e^{i\theta} \in \mathcal{D}$. For fixed n, it lies on the arc between $x_{k,n}$ and $x_{k+1,n}$ for a certain k. Because

$$\theta \mapsto \Im \log Z_n(e^{i\theta})$$

is piece-wise linear, increasing with speed $n/2$ and only jumping by $-\pi$, we have:

$$\Im \log Z_n(e^{i\theta}) \leq \Im \log Z_n(x_{k,n}) + \frac{n}{2}\left(\theta - \frac{2\pi k}{n}\right) \leq \Im \log Z_n(x_{k,n}) + \pi$$

In the other direction, we have

$$\Im \log Z_n(e^{i\theta}) \geq \Im \log Z_n(x_{k+1,n}) - \frac{n}{2}\left(\frac{2\pi(k+1)}{n} - \theta\right) \geq \Im \log Z_n(x_{k+1,n}) - \pi$$

So that, almost surely:

$$\exists n_0 \in \mathbb{N}, \forall n \geq n_0, \forall z \in \mathcal{D}, \quad \left|\Im \log Z_n\left(z\right)\right| \leq \pi + D\log n$$

The more precise estimate $\left|\Im \log Z_n\left(z\right)\right| \leq D\log n$ follows after replacing D by $D' \in (\sqrt{2}, D)$ in the previous computation and considering n_0 large enough so that $\pi < (D - D')\log n$. □

2.2 Precise estimates for the eigenvalues of virtual isometries

The following estimate will reveal crucial for the proof of Theorem 1.5.

Proposition 2.7. *Almost surely and uniformly in n and k:*

$$y_k^{(n)} = k + O\left(\log(2 + |k|)\right)$$

In fact, if $y_k^{(n)}$ is replaced by y_k ($n \to \infty$), this estimate is already easily deduced from existing literature (for example [MM13], [Sos02]):

Lemma 2.8. *Almost surely:*

$$\forall k \in \mathbb{Z}, y_k = k + O\left(\log(2 + |k|)\right)$$

Proof. Consider a sine-kernel process y_k. For $A > 0$ and $a < b$, let $X_{[a,b]}$ be the number of particles y_k in $[a, b]$, and let $X_A := X_{[0,A]}$. Thanks to Proposition 2 in [MM13] (which is by the way also a standard result in the theory of point processes), which can be applied to the sine-kernel process, X_A is a sum of independent Bernoulli random variables. As in Corollary 4 in [MM13], we can deduce, using the Bernstein inequality that

$$\forall t > 0, \mathbb{P}\left(|X_A - A| \geq t\right) \leq 2\exp\left(-\min\left(\frac{t^2}{4\operatorname{Var}(X_A)}, \frac{t}{2}\right)\right).$$

An estimate for the variance is proved by Costin and Lebowitz [CL95] (see also Soshnikov [Sos02]):

$$\operatorname{Var}(X_A) = \frac{1}{\pi^2}\log A + O(1)$$

Therefore, for all $D > 0$,

$$\mathbb{P}\left(|X_A - A| \geq D\log A\right) \leq 2\exp\left(-(\log A)\min\left(\frac{D^2\pi^2}{4 + O(1/\log A)}, \frac{D}{2}\right)\right).$$

For $D > 2$, and A large enough, $D^2\pi^2/[4 + O(1/\log A)] > D/2$, which implies:

$$\mathbb{P}\left(|X_A - A| \geq D\log A\right) \leq 2\exp\left(-(\log A)(D/2)\right) = 2A^{-D/2}.$$

This quantity is summable for positive integer values of A. By Borel-Cantelli's lemma, we deduce that almost surely, for $A \in \mathbb{N}$:

$$X_A = A + O\left(\log(2 + |A|)\right).$$

From the inequality

$$X_{[0,\lfloor A\rfloor]} \leq X_{[0,A]} \leq X_{[0,\lceil A\rceil]},$$

we deduce that the estimate remains true for all $A \geq 0$. Taking $A = y_k$ for $k > 0$ proves the proposition for positive indices. With the same argument one handles the negative ones. □

In order to prove Proposition 2.7, we will also need the two lemmas:

Lemma 2.9. *Almost surely:*

$$\forall k \in \mathbb{Z}, y_k^{(n)} = k + O\left(\log n\right)$$

Proof. This is an immediate consequence of Corollary 2.3 and Proposition 2.6. □

Lemma 2.10. *For every $0 < \eta < \frac{1}{6}$, there exists $\varepsilon > 0$ such that, almost surely:*

$$\forall k \in [-n^\eta, n^\eta], y_k^{(n)} = y_k + O\left(n^{-\varepsilon}\right)$$

Proof. Since $k \in [-n^{1/4}, n^{1/4}]$, we can apply Theorem 1.3, which gives, for all $\delta > 0$,

$$y_k^{(n)} = y_k + O_\delta\left((1 + k^2)n^{-\frac{1}{3}+\delta}\right).$$

Since $k = O(n^\eta)$,

$$y_k^{(n)} = y_k + O_\delta\left(n^{2\eta - \frac{1}{3} + \delta}\right),$$

which, by taking

$$\delta = \frac{1}{6} - \eta > 0,$$

gives the desired result, for

$$\varepsilon = -2\eta + \frac{1}{3} - \delta = 2\delta - \delta = \delta > 0.$$

□

Proof of Proposition 2.7. In the range $|k| \geq n^{1/7}$, it is a consequence of Lemma 2.9. In the range $|k| < n^{1/7}$, it is a consequence of Lemmas 2.8 and 2.10 (for $\eta = 1/7$). □

2.3 Infinite product representation of the ratio and its convergence

First, let us express ξ_n in function of the renormalized eigenangles of U_n.

Proposition 2.11. *One has*

$$\xi_n(z) = e^{i\pi z} \prod_{k \in \mathbb{Z}} \left(1 - \frac{z}{y_k^{(n)}}\right),$$

where the infinite product has to be understood as the limit of the product from $k = -A$ to $k = A$ when the integer A goes to infinity.

Proof.

$$\begin{aligned}
\xi_n(z) &= \frac{Z_n\left(\exp(\frac{i2\pi z}{n})\right)}{Z_n(1)} \\
&= \prod_{k=1}^{n} \frac{1 - \frac{\exp(\frac{i2\pi z}{n})}{\lambda_k^{(n)}}}{1 - \frac{1}{\lambda_k^{(n)}}} \\
&= \prod_{k=1}^{n} \frac{1 - \exp(\frac{i2\pi z}{n} - i\theta_k^{(n)})}{1 - \exp\left(-i\theta_k^{(n)}\right)} \\
&= \prod_{k=1}^{n} \frac{\exp(\frac{i2\pi z}{2n} - \frac{1}{2}i\theta_k^{(n)})}{\exp(-\frac{1}{2}i\theta_k^{(n)})} \frac{\exp(-\frac{i2\pi z}{2n} + \frac{1}{2}i\theta_k^{(n)}) - \exp\left(-\frac{1}{2}i\theta_k^{(n)} + \frac{i2\pi z}{2n}\right)}{\exp\left(\frac{1}{2}i\theta_k^{(n)}\right) - \exp\left(-\frac{1}{2}i\theta_k^{(n)}\right)} \\
&= \prod_{k=1}^{n} \exp(\frac{i\pi z}{n}) \frac{\sin\left(\frac{\pi z}{n} - \frac{1}{2}\theta_k^{(n)}\right)}{\sin\left(-\frac{1}{2}\theta_k^{(n)}\right)}
\end{aligned}$$

$$= \exp(i\pi z) \prod_{k=1}^{n} \frac{\sin\left(\frac{1}{2}\theta_k^{(n)} - \frac{\pi z}{n}\right)}{\sin\left(\frac{1}{2}\theta_k^{(n)}\right)}$$

Now, the standard product formula for the sine function can be written as follows:

$$\forall \alpha \in \mathbb{C}, \sin(\alpha) = \alpha \lim_{A\to\infty} \prod_{0<|j|\leq A} \left(1 - \frac{\alpha}{\pi j}\right).$$

We then have:

$$\begin{aligned}
\xi_n(z) &= \exp(i\pi z) \prod_{k=1}^{n} \left(\frac{\frac{1}{2}\theta_k^{(n)} - \frac{\pi z}{n}}{\frac{1}{2}\theta_k^{(n)}} \lim_{A\to\infty} \prod_{0<|j|\leq A} \frac{1 - \frac{\frac{1}{2}\theta_k^{(n)} - \frac{\pi z}{n}}{\pi j}}{1 - \frac{\frac{1}{2}\theta_k^{(n)}}{\pi j}} \right) \\
&= \exp(i\pi z) \prod_{k=1}^{n} \left(\left(1 - \frac{z}{y_k^{(n)}}\right) \lim_{A\to\infty} \prod_{0<|j|\leq A} \left(1 - \frac{z}{nj + y_k^{(n)}}\right) \right) \\
&= \exp(i\pi z) \prod_{k=1}^{n} \lim_{A\to\infty} \prod_{0\leq|j|\leq A} \left(1 - \frac{z}{nj + y_k^{(n)}}\right)
\end{aligned}$$

Using the periodicity of the eigenangles, we have:

$$y_{k+jn}^{(n)} = jn + y_k^{(n)},$$

and then

$$\xi_n(z) = \exp(i\pi z) \lim_{A\to\infty} \prod_{1-nA\leq k\leq n+nA} \left(1 - \frac{z}{y_k^{(n)}}\right).$$

Now, for $B \geq 2n$, $A \geq 2$ integers such that $An \leq B \leq An + n - 1$, the product of $1 - \frac{z}{y_k^{(n)}}$ from $1 - nA$ to $n + nA$ and the product from $-B$ to B differ by at most $2n$ factors, which are all $1 + O(|z|/y_{nA}^{(n)}) + O(|z|/|y_{1-nA}^{(n)}|) = 1 + O(|z|/nA)$. The quotient between these two products is then well-defined and $\exp[O(|z|/A)] = \exp[O(n|z|/B)]$ for B large enough, which implies that it tends to one when B goes to infinity. Hence,

$$\xi_n(z) = \exp(i\pi z) \lim_{B\to\infty} \prod_{-B\leq k\leq B} \left(1 - \frac{z}{y_k^{(n)}}\right).$$

□

We are now ready to prove Theorem 1.5.

Proof of theorem 1.5. Thanks to the estimate from Proposition 2.7:

$$y_k^{(n)} = k + O\left(\log(2 + |k|)\right)$$

We have that, for $k \geq 1$ and z in a compact K:

$$\begin{aligned}\left(1 - \frac{z}{y_k^{(n)}}\right)\left(1 - \frac{z}{y_{-k}^{(n)}}\right) &= 1 - z\frac{O(\log(2+|k|))}{k^2} + O\left(\frac{|z|^2}{k^2}\right) \\ &= 1 + \frac{O_K\left(\log(2+|k|)\right)}{k^2}\end{aligned}$$

Hence:

$$\xi_n(z) = e^{i\pi z} \prod_{k \in \mathbb{Z}} \left(1 - \frac{z}{y_k^{(n)}}\right)$$

is a sequence of entire functions uniformly bounded on compact sets. Therefore, by Montel's theorem uniform convergence on compact sets is implied by pointwise convergence. Let us then focus on proving pointwise convergence.

Fix $A \geq 2$. Let us prove that:

$$\prod_{|k| \leq A} \left(1 - \frac{z}{y_k^{(n)}}\right) - \prod_{k \in \mathbb{Z}} \left(1 - \frac{z}{y_k^{(n)}}\right) = O_K\left(\frac{\log A}{A}\right), \tag{2}$$

$$\prod_{|k| \leq A} \left(1 - \frac{z}{y_k}\right) - \prod_{k \in \mathbb{Z}} \left(1 - \frac{z}{y_k}\right) = O_K\left(\frac{\log A}{A}\right). \tag{3}$$

Here, the infinite products are, as before, the limits of the products from $-B$ to B for B going to infinity. Note that the existence of the infinite product involving y_k is an immediate consequence of the absolute convergence of the product

$$\left(1 - \frac{z}{y_0}\right) \prod_{k \geq 1} \left[\left(1 - \frac{z}{y_k}\right)\left(1 - \frac{z}{y_{-k}}\right)\right],$$

stated in Theorem 1.5, and following from the estimate:

$$\left(1 - \frac{z}{y_k}\right)\left(1 - \frac{z}{y_{-k}}\right) = 1 - z\frac{O(\log(2+|k|))}{k^2} + O\left(\frac{|z|^2}{k^2}\right) = 1 + \frac{O_K\left(\log(2+|k|)\right)}{k^2}.$$

We now prove (2): a proof of (3) is simply obtained by removing the indices n. We have:

$$\prod_{|k| \geq A} \left(1 - \frac{z}{y_k^{(n)}}\right) = 1 + O_K\left(\sum_{k \geq A} \frac{\log(2+|k|)}{k^2}\right) = 1 + O_K\left(\frac{\log A}{A}\right)$$

and

$$\prod_{|k| \leq A} \left(1 - \frac{z}{y_k^{(n)}}\right) = O_K(1)$$

Therefore:

$$\prod_{|k| \leq A} \left(1 - \frac{z}{y_k^{(n)}}\right) - \prod_{k \in \mathbb{Z}} \left(1 - \frac{z}{y_k^{(n)}}\right)$$

$$
\begin{aligned}
&= \prod_{|k|\le A}\left(1-\frac{z}{y_k^{(n)}}\right)\left(1-\prod_{|k|>A}\left(1-\frac{z}{y_k^{(n)}}\right)\right)\\
&= \prod_{|k|\le A}\left(1-\frac{z}{y_k^{(n)}}\right)\left(1-\left(1+O_K\left(\frac{\log A}{A}\right)\right)\right)\\
&=O_K\left(\frac{\log A}{A}\right)
\end{aligned}
$$

Because errors are uniform in n, this is saying:

$$
\sup_n \left|\prod_{|k|\le A}\left(1-\frac{z}{y_k^{(n)}}\right)-\prod_{k\in\mathbb{Z}}\left(1-\frac{z}{y_k^{(n)}}\right)\right| \xrightarrow[A\to\infty]{} 0
$$

Now:

$$
\begin{aligned}
&\left|\prod_{k\in\mathbb{Z}}\left(1-\frac{z}{y_k^{(n)}}\right)-\prod_{k\in\mathbb{Z}}\left(1-\frac{z}{y_k}\right)\right|\\
\le &\left|\prod_{|k|\le A}\left(1-\frac{z}{y_k^{(n)}}\right)-\prod_{|k|\le A}\left(1-\frac{z}{y_k}\right)\right|\\
&+\left|\prod_{k\in\mathbb{Z}}\left(1-\frac{z}{y_k^{(n)}}\right)-\prod_{|k|\le A}\left(1-\frac{z}{y_k^{(n)}}\right)\right|\\
&+\left|\prod_{k\in\mathbb{Z}}\left(1-\frac{z}{y_k}\right)-\prod_{|k|\le A}\left(1-\frac{z}{y_k}\right)\right|\\
\le &\left|\prod_{|k|\le A}\left(1-\frac{z}{y_k^{(n)}}\right)-\prod_{|k|\le A}\left(1-\frac{z}{y_k}\right)\right|+O_K\left(\frac{\log A}{A}\right)
\end{aligned}
$$

Hence, as $y_k^{(n)} \to y_k$ pointwise:

$$
\limsup_{n\to\infty}\left|\prod_{k\in\mathbb{Z}}\left(1-\frac{z}{y_k^{(n)}}\right)-\prod_{k\in\mathbb{Z}}\left(1-\frac{z}{y_k}\right)\right| = O_K\left(\frac{\log A}{A}\right)
$$

Taking $A \to \infty$ completes the proof. □

3 Properties of the limiting function ξ_∞, its logarithmic derivative and the number theory connection

In this section, we establish some properties of ξ_∞ and then link ξ_∞ and its logarithmic derivative to the Riemann zeta function.

3.1 The order of ξ_∞ as an entire function

We first start with a simple statement on the order of ξ_∞ as an entire function:

Proposition 3.1. *Almost surely, ξ_∞ is of order 1. More precisely, the exists a.s. a random $C > 0$, such that for all $z \in \mathbb{C}$.*

$$|\xi_\infty(z)| \leq e^{C|z|\log(2+|z|)}.$$

On the other hand, there exists a.s. a random $c > 0$ such that for all $x \in \mathbb{R}$,

$$|\xi_\infty(ix)| \geq ce^{c|x|}.$$

Proof. We have:

$$\left(1 - \frac{z}{y_k}\right)\left(1 - \frac{z}{y_{-k}}\right) = 1 - z\frac{O(\log(2+|k|))}{k^2} + O\left(\frac{|z|^2}{k^2}\right)$$

with errors being uniform in z and $k \geq 1$. We distinguish between three regimes for $k \in \mathbb{Z}$ different from zero: $|k| \geq e^{|z|}$, $|z| \leq |k| < e^{|z|}$, $1 \leq |k| < |z|$. In the first regime,

$$\left(1 - \frac{z}{y_k}\right)\left(1 - \frac{z}{y_{-k}}\right) = 1 + O\left(\frac{|z|(\log(2+|k|))}{k^2}\right),$$

which implies

$$\begin{aligned}
\left|\prod_{k \geq e^{|z|}} \left(1 - \frac{z}{y_k}\right)\left(1 - \frac{z}{y_{-k}}\right)\right| &\leq \exp\left(O\left(|z| \sum_{k \geq e^{|z|}} \frac{\log(2+k)}{k^2}\right)\right) \\
&= \exp\left(O\left(|z| \sum_{k \geq e^{|z|}} k^{-3/2}\right)\right) \\
&= \exp\left(O\left(|z|e^{-|z|/2}\right)\right) = O(1).
\end{aligned}$$

In the second regime,

$$\log(2+|k|) \leq \log(e^{|z|}+2) \leq \log(3e^{|z|}) \leq |z|+2,$$

and then

$$\left(1 - \frac{z}{y_k}\right)\left(1 - \frac{z}{y_{-k}}\right) = 1 + O\left(\frac{|z|(|z|+2)}{k^2}\right),$$

which implies

$$\left|\prod_{|z| \leq k < e^{|z|}} \left(1 - \frac{z}{y_k}\right)\left(1 - \frac{z}{y_{-k}}\right)\right| \leq \exp\left(O\left(|z|(|z|+2) \sum_{k \geq |z| \vee 1} \frac{1}{k^2}\right)\right) = \exp O(|z|).$$

Finally, in the third regime, we have, since $|y_k/k|$ is a.s. bounded from below,

$$1 - \frac{z}{y_k} = 1 + O(|z/k|),$$

which in turn implies

$$\left|\prod_{1\le k<|z|}\left(1-\frac{z}{y_k}\right)\left(1-\frac{z}{y_{-k}}\right)\right|\le \exp\left(O\left(|z|\sum_{1\le k<|z|}(1/k)\right)\right)=\exp O\left(|z|\log(2+|z|)\right).$$

Since

$$\left|1-\frac{z}{y_0}\right|\le \exp(|z|/y_0)=\exp O(|z|),$$

we deduce by combining the three regimes, the following upper bound:

$$|\xi_\infty(z)|\le \exp O\left(|z|\log(2+|z|)\right).$$

In order to prove the lower bound, we first use the equality:

$$|\xi_\infty(ix)|^2=\prod_{k\in\mathbb{Z}}\left(1+\frac{x^2}{y_k^2}\right).$$

Since $|y_k|=O(|k|)$ for $k\neq 0$, we deduce that there exists a random $c>0$ such that

$$|\xi_\infty(ix)|^2\ge\prod_{k\neq 0}\left(1+\frac{x^2}{ck^2}\right),$$

and then

$$|\xi_\infty(ix)|\ge\prod_{k\ge 1}\left(1+\frac{x^2}{ck^2}\right)=\frac{\sinh(\pi x/\sqrt{c})}{\pi x/\sqrt{c}},$$

which shows the lower bound given in the proposition.

□

3.2 The logarithmic derivative ξ_∞ and conjectures related to the Riemann zeta function

Now we state a conjecture which relates the random function ξ_∞ to the behavior of the zeta function close to the critical line:

Conjecture 3.2. *Let U be a uniform random variable on $[0,1]$ and $T>0$ a real parameter going to infinity. Our random limiting function should be related to the renormalized zeta function with randomized argument. We conjecture the following convergence in law, uniformly in the parameter z on every compact set:*

$$\left(\frac{\zeta\left(\frac{1}{2}+iTU-\frac{i2\pi z}{\log T}\right)}{\zeta\left(\frac{1}{2}+iTU\right)};z\in\mathbb{C}\right)\overset{T\to\infty}{\longrightarrow}(\xi_\infty(z);z\in\mathbb{C})$$

By taking logarithmic derivatives, it is natural also to conjecture the following convergence

$$\left(\frac{-i2\pi}{\log T}\frac{\zeta'}{\zeta}\left(\frac{1}{2}+iTU-\frac{i2\pi z}{\log T}\right);z\in\mathbb{C}\right)\overset{T\to\infty}{\longrightarrow}\left(\frac{\xi'_\infty}{\xi_\infty}(z);z\in\mathbb{C}\right)$$

on compact sets bounded away from the real line.

This conjecture is supported by the following lemma:

Lemma 3.3. *We have, for $z \notin \mathbb{R}$,*

$$\frac{\xi_\infty'}{\xi_\infty}(z) = i\pi + \sum_{k\in\mathbb{Z}} \frac{1}{z-y_k} =: i\pi + \frac{1}{z-y_0} + \sum_{k=1}^{\infty}\left(\frac{1}{z-y_k} + \frac{1}{z-y_{-k}}\right),$$

and when the random variable U is fixed:

$$\frac{-i2\pi}{\log T}\frac{\zeta'}{\zeta}\left(\frac{1}{2} + iTU - \frac{i2\pi z}{\log T}\right) = i\pi + \sum_{\tilde{\gamma}} \frac{1}{z-\tilde{\gamma}} + o(1)$$

where $\tilde{\gamma}$ are the non-trivial zeros of the Riemann zeta function centered around $\frac{1}{2} + iUT$ and *renormalized so that their average spacing around the origin is $\mathcal{O}(1)$. More precisely:*

$$\tilde{\gamma} := \frac{-\log T}{2\pi i}\left(\rho - \frac{1}{2} - iUT\right)$$

with ρ a zero of ζ. The infinite sum on $\tilde{\gamma}$ has to be understood as follows:

$$\sum_{\tilde{\gamma}} \frac{1}{z-\tilde{\gamma}} = \frac{1}{z-\tilde{\gamma}_0} + \sum_{k=1}^{\infty}\left(\frac{1}{z-\tilde{\gamma}_k} + \frac{1}{z-\tilde{\gamma}_{-k}}\right),$$

where $(\tilde{\gamma}_k)_{k\in\mathbb{Z}}$ are ordered by increasing real part, increasing imaginary part if they have the same real part, and counted with multiplicity.

Remark 3.4. *The absolute convergence of the last sum can be easily deduced from the classical estimate, for $A > 2$, on the number of nontrivial zeros $N(A)$ with imaginary part in $[0, A]$, or in $[-A, 0]$:*

$$N(A) = \varphi(A) + O(\log A),$$

for

$$\varphi(A) = \frac{A}{2\pi}\log\left(\frac{A}{2\pi e}\right).$$

Indeed, all the ways to number the renormalized zeros $\tilde{\gamma}$ consistently with the statement of the lemma are deduced from each other by translation of the indices, and for any such numbering one checks that

$$\tilde{\gamma}_k = \mathrm{sgn}(k)\frac{\log T}{2\pi}\varphi^{(-1)}(|k|) + O(\log(2+|k|)),$$

where $\varphi^{(-1)}$ is the inverse of the bijection from $[2\pi e, \infty)$ to $\mathbb{R}_+$, induced by φ. The implicit constant depends on T, U and the precise numbering of the zeros, but not on k. This estimate is sufficient to ensure the convergence of the last series in the lemma, when one takes into account that $\varphi^{(-1)}(k) \geq k/\log k$ for all $k \geq 2$. The sum of the series does not depend on the numbering of the $\tilde{\gamma}$'s, since any translation of the indices change the partial sums by a bounded number of terms, which tend to zero. Note that the $\tilde{\gamma}$'s are all real if and only if the Riemann hypothesis is satisfied.

Proof. The convergence of the first series in the lemma is easily deduced from the estimate in Proposition 2.7. The partial sums are the logarithm derivatives of the corresponding partial products associated to ξ_∞. Since uniform convergence on compact sets of non-vanishing holomorphic functions implies the corresponding convergence of the logarithmic derivative, we get the part of the lemma related to ξ'_∞/ξ_∞. For the formula involving ζ, we start by the Hadamard product formula for the zeta function:

$$\forall s \in \mathbb{C}\backslash\{1\}, \zeta(s) = \pi^{s/2} \frac{\prod_\rho \left(1 - \frac{s}{\rho}\right)}{2(s-1)\Gamma\left(1 + \frac{s}{2}\right)}.$$

The product has to be computed by grouping pairs of conjugate non-trivial zeros of zeta. Hence, for s not a zero nor a pole:

$$\frac{\zeta'}{\zeta}(s) = \frac{1}{2}\log\pi + \sum_\rho \frac{1}{s-\rho} - \frac{1}{s-1} - \frac{1}{2}\frac{\Gamma'}{\Gamma}\left(1 + \frac{s}{2}\right)$$

Take $s = \frac{1}{2} + iTU - \frac{i2\pi z}{\log T}$ with $T \to \infty$ and use the asymptotics $\frac{\Gamma'}{\Gamma}\left(1 + \frac{s}{2}\right) = \log T + O(1)$. The error is uniform in z on compact sets away from the real line. Then:

$$\begin{aligned}\frac{-i2\pi}{\log T}\frac{\zeta'}{\zeta}\left(\frac{1}{2} + iTU - \frac{i2\pi z}{\log T}\right) &= \frac{-i2\pi}{\log T}\sum_\rho \frac{1}{s-\rho} + \frac{i2\pi}{\log T}\frac{1}{2}(\log T + O(1)) + o(1)\\ &= i\pi + \frac{-i2\pi}{\log T}\sum_\rho \frac{1}{-\frac{i2\pi z}{\log T} - \left(\rho - \frac{1}{2} - iUT\right)} + o(1)\end{aligned}$$

Here, all the sums on ρ are obtained by grouping pairs of conjugate values of ρ. Writing the last sum in function of the sequence $(\tilde{\gamma}_k)_{k\in\mathbb{Z}}$ gives

$$\frac{-i2\pi}{\log T}\frac{\zeta'}{\zeta}\left(\frac{1}{2} + iTU - \frac{i2\pi z}{\log T}\right) = i\pi + \sum_{k=1}^{\infty}\left(\frac{1}{z - \tilde{\gamma}_{a+k}} + \frac{1}{z - \tilde{\gamma}_{a+1-k}}\right) + o(1),$$

where a depends only on the way to number the $\tilde{\gamma}_k$'s. Changing the partial sums by at most $2|a| + 1$ terms, all tending to zero, gives the partial sums of the series in the lemma. □

Our formulation can be easily related to the GUE conjectures [RS96], which is the natural extension of Montgomery's conjecture [Mon73] on pair correlations. Indeed, the previous lemma gives a good heuristic of Conjecture 3.2: since the randomized and renormalized zeros $\tilde{\gamma}$ are expected to behave like a sine kernel point process, the two expressions should match in law when $T \to \infty$. It is interesting to notice that the term $i\pi$ in the expression of ζ'/ζ is due to the "Archimedian" gamma factor in the Hadamard product of ζ. With the same renormalization corresponding to the average spacing of the zeros, we get the same term for the logarithmic derivative of the characteristic polynomial of the CUE.

We will now compute the two first moments of $\frac{\xi'_\infty}{\xi_\infty}$, which will naturally give a conjecture on the corresponding moments of $\frac{\zeta'}{\zeta}$. A particular case of our conjecture is in fact equivalent to the pair correlation conjecture under Riemann hypothesis, thanks to a paper by Goldston, Gonek and Montgomery [GGM01]. One should also note that recently Farmer, Gonek, Lee and Lester obtain in [FGLL13] an equivalent formulation, with different methods, for the moments of the logarithmic derivative of the Riemann zeta function in terms of the correlation functions of the sine kernel: the objects that are introduced there are different but our formulation is essentially the same as theirs. The main difference is that we propose to consider directly a random meromorphic function which follows from a conjecture for the ratios of the zeta function itself (in particular there is no more n-limit to consider on the random matrix side) and that the logarithmic derivative ξ'_∞/ξ_∞ seems to carry some spectral interpretation (see the last section and the reference there to the recent work by Aizenman and Warzel [AW13]).

A first useful technical tool is provided in the following proposition.

Proposition 3.5. *Almost surely, for all $z \notin \{y_k, k \in \mathbb{Z}\}$,*

$$\frac{\xi'_\infty(z)}{\xi_\infty(z)} = i\pi + \lim_{A\to\infty} \sum_{|y_k|<A} \frac{1}{z - y_k}.$$

Moreover, if we denote

$$\sum_{|y_k|\geq A} \frac{1}{z-y_k} := \frac{\xi'_\infty(z)}{\xi_\infty(z)} - i\pi - \sum_{|y_k|<A} \frac{1}{z-y_k},$$

then for any compact set K, bounded away from $\mathbb{R}$, and for all $p > 1$, there exists an absolute constant $C_{p,K}$ such that:

$$\forall A > 2, \sup_{z\in K} \mathbb{E}\left(\left|\sum_{|y_k|\geq A} \frac{1}{z-y_k}\right|^p\right)^{\frac{1}{p}} \leq C_{p,K} \frac{\log A}{\sqrt{A}}$$

Proof. By Lemma 2.8, there exists almost surely $C > 0$ such that

$$|y_k - k| \leq C(\log(2 + |k|))$$

for all $k \in \mathbb{Z}$. In order to prove the first part of the proposition, it is sufficient to show that almost surely, for all $z \notin \{y_k, k \in \mathbb{Z}\}$,

$$\left(i\pi + \sum_{|y_k|<A} \frac{1}{z-y_k}\right) - \left(\sum_{|k|<A-C\log(2+A)} \frac{1}{z-y_k} + i\pi\right) \underset{A\to\infty}{\longrightarrow} 0.$$

Indeed, the second term of the difference is already known to converge to $\xi'_\infty(z)/\xi_\infty(z)$. Now, $|k| < A - C\log(2 + A)$ implies that

$$|y_k| \leq |k| + C\log(2 + |k|) \leq |k| + C\log(2 + A) < A,$$

and then we have to show

$$\sum_{|k|\geq A-C\log(2+A), |y_k|<A} \frac{1}{z-y_k} \underset{A\to\infty}{\longrightarrow} 0.$$

Since $|y_k| \geq |k| - C\log(2+|k|)$, it is sufficient to prove

$$\sum_{|k|\geq A - C\log(2+A), |k|-C\log(2+|k|)<A} \frac{1}{|z-y_k|} \underset{A\to\infty}{\longrightarrow} 0.$$

Now, this convergence holds since for C, z and $(y_k)_{k\in\mathbb{Z}}$ fixed, the number of terms of the sum is $O(\log A)$ when A goes to infinity, and all the terms are $O(1/A)$.

Let $\alpha > 1$ to be fixed later. From the convergence above, we can write

$$\begin{aligned}\mathbb{E}\left(\left|\sum_{|y_k|\geq A} \frac{1}{z-y_k}\right|^p\right)^{\frac{1}{p}} &= \mathbb{E}\left(\left|\sum_{l\in\mathbb{N}} \sum_{A\vee l^\alpha \leq |y_k| < A\vee(l+1)^\alpha} \frac{1}{z-y_k}\right|\right)^{\frac{1}{p}} \\ &\leq \sum_{l\in\mathbb{N}} \mathbb{E}\left(\left|\sum_{A\vee l^\alpha \leq |y_k| < A\vee(l+1)^\alpha} \frac{1}{z-y_k}\right|^p\right)^{\frac{1}{p}}\end{aligned}$$

Now, for every $z\in K$, and l such that $l^\alpha \geq \sup_{z\in K}|z|+1$,

$$l^\alpha \leq |y_k| < (l+1)^\alpha \Rightarrow \left|\frac{1}{z-y_k} + \frac{1}{(1+l)^\alpha \operatorname{sgn} y_k}\right| \ll \frac{1}{(1+l)^{\alpha+1}},$$

the implicit constant depending only on K. Indeed,

$$\begin{aligned}|z-y_k-(1+l)^\alpha \operatorname{sgn} y_k| &\leq |z| + |y_k - (1+l)^\alpha \operatorname{sgn} y_k| \leq |z| + ||y_k| - (1+l)^\alpha| \\ &\leq O_K(1) + (1+l)^\alpha - l^\alpha = O_K((1+l)^{\alpha-1}),\end{aligned}$$

and

$$|z-y_k||(1+l)^\alpha \operatorname{sgn} y_k| \geq (l^\alpha - |z|)(1+l)^\alpha \gg_K (1+l)^{2\alpha},$$

since from $l^\alpha \geq \sup_{z\in K}|z|+1$, we get

$$l^\alpha - |z| \geq l^\alpha\left(\sup_{z\in K}|z|+1\right)^{-1} \geq 2^{-\alpha}\left(\sup_{z\in K}|z|+1\right)^{-1}(1+l)^\alpha.$$

Hence, for an interval I, if:

$$X_I := \operatorname{Card}\{k\in\mathbb{Z} | y_k \in I\}$$

is the number of points from the sine kernel that fall in I, then for A large enough:

$$\begin{aligned}&\mathbb{E}\left(\left|\sum_{|y_k|\geq A} \frac{1}{z-y_k}\right|^p\right)^{\frac{1}{p}} \\ &\ll \sum_{l\in\mathbb{N}}\left(\mathbb{E}\left(\left|\sum_{A\vee l^\alpha \leq |y_k| < A\vee(l+1)^\alpha} \frac{1}{(1+l)^\alpha \operatorname{sgn} y_k}\right|^p\right)^{\frac{1}{p}} + \mathbb{E}\left(\left|\sum_{A\vee l^\alpha \leq |y_k| < A\vee(l+1)^\alpha} \frac{1}{(1+l)^{\alpha+1}}\right|^p\right)^{\frac{1}{p}}\right)\end{aligned}$$

$$
\begin{aligned}
&\leq \sum_{l\in\mathbb{N}} \frac{1}{(1+l)^\alpha}\mathbb{E}\left(\left|X_{[A\vee l^\alpha, A\vee(l+1)^\alpha]} - X_{[-A\vee(l+1)^\alpha, -A\vee l^\alpha]}\right|^p\right)^{\frac{1}{p}}\\
&\quad + \sum_{l\in\mathbb{N}} \frac{1}{(1+l)^{\alpha+1}}\mathbb{E}\left(\left|X_{[A\vee l^\alpha, A\vee(l+1)^\alpha]} + X_{[-A\vee(l+1)^\alpha, -A\vee l^\alpha]}\right|^p\right)^{\frac{1}{p}}\\
&\ll \sum_{l\in\mathbb{N}} \frac{1}{(1+l)^\alpha}\mathbb{E}\left(\left|X_{[0,A\vee(l+1)^\alpha - A\vee l^\alpha]} - \mathbb{E}\left(X_{[0,A\vee(l+1)^\alpha - A\vee l^\alpha]}\right)\right|^p\right)^{\frac{1}{p}}\\
&\quad + \sum_{l\in\mathbb{N}} \frac{1}{(1+l)^{\alpha+1}}\mathbb{E}\left(\left|X_{[0,A\vee(l+1)^\alpha - A\vee l^\alpha]}\right|^p\right)^{\frac{1}{p}}
\end{aligned}
$$

The last step uses the symmetry and translation invariance of the sine kernel process. Here, we can be more generous in our estimate and write:

$$
\begin{aligned}
&\frac{1}{(1+l)^{\alpha+1}}\mathbb{E}\left(\left|X_{[0,A\vee(l+1)^\alpha - A\vee l^\alpha]}\right|^p\right)^{\frac{1}{p}}\\
\leq &\frac{1}{(1+l)^\alpha}\mathbb{E}\left(\left|X_{[0,A\vee(l+1)^\alpha - A\vee l^\alpha]} - \mathbb{E}\left(X_{[0,A\vee(l+1)^\alpha - A\vee l^\alpha]}\right)\right|^p\right)^{\frac{1}{p}}\\
&+ \frac{1}{(1+l)^{\alpha+1}}\mathbb{E}\left(X_{[0,A\vee(l+1)^\alpha - A\vee l^\alpha]}\right)
\end{aligned}
$$

Hence:

$$
\begin{aligned}
\mathbb{E}\left(\left|\sum_{|y_k|\geq A}\frac{1}{z-y_k}\right|^p\right)^{\frac{1}{p}} \ll &\sum_{l\in\mathbb{N}} \frac{1}{(1+l)^\alpha}\mathbb{E}\left(\left|X_{[0,A\vee(l+1)^\alpha - A\vee l^\alpha]} - \mathbb{E}\left(X_{[0,A\vee(l+1)^\alpha - A\vee l^\alpha]}\right)\right|^p\right)^{\frac{1}{p}}\\
&+ \frac{1}{(1+l)^{\alpha+1}}\mathbb{E}\left(X_{[0,A\vee(l+1)^\alpha - A\vee l^\alpha]}\right)
\end{aligned}
$$

Recall that for $B>0$:

$$
\mathbb{E}\left(X_{[0,B]}\right) = B
$$

Hence:

$$
\sum_{l\in\mathbb{N}} \frac{1}{(1+l)^{\alpha+1}}\mathbb{E}\left(X_{[0,A\vee(l+1)^\alpha - A\vee l^\alpha]}\right) \ll \sum_{l\geq A^{\frac{1}{\alpha}}-1} \frac{(1+l)^{\alpha-1}}{(1+l)^{\alpha+1}} \ll A^{-\frac{1}{\alpha}}
$$

On the other hand, because $X_{[0,B]} - B$ have tails bounded by a Gaussian density:

$$
\begin{aligned}
\mathbb{E}\left(\left|X_{[0,B]} - B\right|^p\right) = &p\int_0^\infty t^p \mathbb{P}\left(\left|X_{[0,B]} - B\right| \geq t\right)\frac{dt}{t}\\
\leq &2p\int_0^\infty t^p \exp\left(-\min\left(\frac{t^2}{4\,\mathrm{Var}(X_{[0,B]})}, \frac{t}{2}\right)\right)\frac{dt}{t}\\
\leq &2p\int_0^\infty t^p \exp\left(-\frac{t^2}{4\,\mathrm{Var}(X_{[0,B]})}\right)\frac{dt}{t} + 2p\int_0^\infty t^p \exp\left(-\frac{t}{2}\right)\frac{dt}{t}\\
\ll &\left(\mathrm{Var}(X_{[0,B]})^{\frac{1}{2}p} + 1\right)
\end{aligned}
$$

where the implicit constant depends on p. Therefore:

$$\mathbb{E}\left(\left|X_{[0,B]}-B\right|^{p}\right)^{\frac{1}{p}} \ll \operatorname{Var}(X_{[0,B]})^{\frac{1}{2}}+1$$

In the end:

$$\begin{aligned}
&\mathbb{E}\left(\left|\sum_{|y_k|\geq A}\frac{1}{z-y_k}\right|^{p}\right)^{\frac{1}{p}} \\
\ll & A^{-\frac{1}{\alpha}}+\sum_{l\geq A^{\frac{1}{\alpha}}-1}\frac{1}{(1+l)^{\alpha}}\mathbb{E}\left(\left|X_{[0,(l+1)^{\alpha}\vee A-l^{\alpha}\vee A]}-\mathbb{E}\left(X_{[0,(l+1)^{\alpha}\vee A-l^{\alpha}\vee A]}\right)\right|^{p}\right)^{\frac{1}{p}} \\
\leq & A^{-\frac{1}{\alpha}}+\sum_{l\geq A^{\frac{1}{\alpha}}-1}\frac{1}{(1+l)^{\alpha}}\left(\operatorname{Var}(X_{[0,(l+1)^{\alpha}\vee A-l^{\alpha}\vee A]})^{\frac{1}{2}}+1\right)
\end{aligned}$$

The estimate:

$$\operatorname{Var}(X_I) \ll 1+\log|I|$$

allows to conclude the proof:

$$\begin{aligned}
\mathbb{E}\left(\left|\sum_{|y_k|\geq A}\frac{1}{z-y_k}\right|^{p}\right)^{\frac{1}{p}} \ll & A^{-\frac{1}{\alpha}}+\sum_{l\geq A^{\frac{1}{\alpha}}-1}\frac{1}{(1+l)^{\alpha}}\left(1+\log\left(1+l\right)\right) \\
\ll & A^{-\frac{1}{\alpha}}+A^{\frac{1}{\alpha}-1}\log A,
\end{aligned}$$

when we take the optimal exponent $\alpha=2$. □

Corollary 3.6.

$$\lim_{A\to\infty}\mathbb{E}\left(\left|\sum_{|y_k|\geq A}\frac{1}{z-y_k}\right|^{p}\right)=0$$

and:

$$\forall z\notin\mathbb{R},\forall p\geq 1,\frac{\xi_\infty'}{\xi_\infty}(z)\in L^p$$

This corollary allow to compute the moments of ξ_∞'/ξ_∞ by first restricting the infinite sums to the y_k's between $-A$ and A, and then by letting $A\to\infty$. More precisely, for all fixed $z_1,z_2,\dots,z_p\notin\mathbb{R}$,

$$\forall p\geq 1,\frac{\xi_\infty'}{\xi_\infty}(z_1)\dots\frac{\xi_\infty'}{\xi_\infty}(z_p)\in L^p$$

and

$$\mathbb{E}\left(\frac{\xi_\infty'}{\xi_\infty}(z_1)\dots\frac{\xi_\infty'}{\xi_\infty}(z_p)\right)=\lim_{A\to\infty}\mathbb{E}\left(\prod_{j=1}^{p}\left(i\pi+\sum_{|y_k|<A}\frac{1}{z_j-y_k}\right)\right).$$

The last quantity can be computed thanks to the sine kernel correlation functions of order less or equal than p, on the segment $[-A,A]$. We will now perform the computation of the two first moments.

Remark 3.7. *Before proceeding we should mention that since we have been able to prove the convergence of the rescaled logarithmic derivative of the characteristic polynomial to $\frac{\xi'_\infty}{\xi_\infty}$, we should also be able to obtain an alternative expression for the moments using the formulas in [CS08] for the moments of ratios of the logarithmic derivative of the characteristic polynomial. Although the combinatorial expressions there provide closed formulas, we do not find them easier to handle than the method we have described above. As we shall see it below, the formulas for the second moments are already very involved.*

First moment $M_1(z), z \notin \mathbb{R}$:

$$\begin{aligned}
M_1(z) &:= \mathbb{E}\left(\frac{\xi'_\infty}{\xi_\infty}(z)\right) \\
&= i\pi + \lim_{A\to\infty} \mathbb{E}\left(\sum_{|y_k|\le A} \frac{1}{z-y_k}\right) \\
&= i\pi + \lim_{A\to\infty} \int_{[-A,A]} dy \frac{\rho_1(y)}{z-y} \\
&= i\pi\left(1 - \operatorname{sgn}\left(\Im(z)\right)\right) \\
&= i2\pi \mathbb{1}_{\{\Im(z)<0\}}
\end{aligned}$$

Second moment $M_2(z,z'); z, z' \notin \mathbb{R}$: Let us first assume that z and z' have not the same real part, in particular $z_1 \neq z_2$. One has:

$$\begin{aligned}
M_2(z,z') &:= \mathbb{E}\left(\frac{\xi'_\infty}{\xi_\infty}(z)\frac{\xi'_\infty}{\xi_\infty}(z')\right) \\
&= -\pi^2 + \pi^2\left(\operatorname{sgn}\left(\Im(z)\right) + \operatorname{sgn}\left(\Im(z')\right)\right) + \mathbb{E}\left(\sum_{k,l} \frac{1}{z-y_k}\frac{1}{z'-y_l}\right) \\
&= -\pi^2 + \pi^2\left(\operatorname{sgn}\left(\Im(z)\right) + \operatorname{sgn}\left(\Im(z')\right)\right) + \lim_{A\to\infty} \mathbb{E}\left(\sum_{|y_k|,|y_l|\le A} \frac{1}{z-y_k}\frac{1}{z'-y_l}\right)
\end{aligned}$$

Moreover:

$$\mathbb{E}\left(\sum_{|y_k|,|y_l|\le A} \frac{1}{z-y_k}\frac{1}{z'-y_l}\right) = \int_{[-A,A]} \frac{dy}{(z-y)(z'-y)} + \int_{[-A,A]^2} \frac{dy_1 dy_2 \left(1 - S(y_1-y_2)^2\right)}{(z-y_1)(z'-y_2)},$$

where

$$S(x) = \frac{\sin(\pi x)}{\pi x}$$

The first integral corresponds to the indices $k = l$ while the second integral corresponds to $k \neq l$. The former is handled by a partial fraction decomposition (recall that $z \neq z'$):

$$\lim_{A\to\infty} \int_{[-A,A]} \frac{dy}{(z-y)(z'-y)} = i\pi \frac{\operatorname{sgn}\left(\Im(z)\right) - \operatorname{sgn}\left(\Im(z')\right)}{z-z'}$$

The second integral can be written as $I_1 - I_2$, where

$$I_1 = \int_{[-A,A]^2} \frac{dy_1 dy_2}{(z-y_1)(z'-y_2)},$$

and

$$I_2 = \int_{[-A,A]^2} \frac{S(y_1-y_2)^2}{(z-y_1)(z'-y_2)} dy_1 dy_2.$$

One has immediately

$$\lim_{A\to\infty} I_1 = \lim_{A\to\infty} \left(\int_{[-A,A]} \frac{dy}{z-y} \right) \left(\int_{[-A,A]} \frac{dy}{z'-y} \right) = -\pi^2 \operatorname{sgn}(\Im(z)) \operatorname{sgn}(\Im(z')).$$

For fixed z and z', the integral I_2 is dominated by

$$\begin{aligned}
&\int_{\mathbb{R}^2} \frac{1}{(1+|y_1|)(1+|y_2|)[1+(y_1-y_2)^2]} dy_1 dy_2 \\
&\leq \frac{1}{2} \int_{\mathbb{R}^2} \frac{1}{1+(y_1-y_2)^2} \left(\frac{1}{(1+|y_1|)^2} + \frac{1}{(1+|y_2|)^2} \right) dy_1 dy_2 \\
&= \int_{\mathbb{R}} \frac{dy}{1+y^2} \int_{\mathbb{R}} \frac{du}{(1+|u|)^2} < \infty.
\end{aligned}$$

Hence,

$$\lim_{A\to\infty} I_2 = \int_{\mathbb{R}^2} \frac{S(y_1-y_2)^2}{(z-y_1)(z'-y_2)} dy_1 dy_2,$$

where the last integral is absolutely convergent. The change of variable $u = y_2$, $v = y_1 - y_2$ gives

$$\lim_{A\to\infty} I_2 = \int_{\mathbb{R}} dv S(v)^2 \int_{\mathbb{R}} \frac{du}{(z-u-v)(z'-u)}.$$

The integral in u can again be computed by a partial fraction decomposition, and one gets

$$\int_{\mathbb{R}} \frac{du}{(z-u-v)(z'-u)} = i\pi \frac{\operatorname{sgn}(\Im(z)) - \operatorname{sgn}(\Im(z'))}{z-z'-v}.$$

Note that since z and z' are assumed to have different imaginary parts, the denominator does not vanish. One then has

$$\lim_{A\to\infty} I_2 = i\pi \left[\operatorname{sgn}(\Im(z)) - \operatorname{sgn}(\Im(z')) \right] \int_{\mathbb{R}} \frac{S(v)^2}{z-z'-v} dv,$$

where

$$\begin{aligned}
\int_{\mathbb{R}} \frac{S(v)^2}{z-z'-v} dv &= \frac{1}{4\pi^2} \int_{\mathbb{R}} \frac{2 - e^{2i\pi v} - e^{-2i\pi v}}{v^2(z-z'-v)} dv. \\
&= \frac{1}{4\pi^2} \int_{\mathbb{R}} \frac{1 - e^{2i\pi v} + 2i\pi v}{v^2(z-z'-v)} dv + \frac{1}{4\pi^2} \int_{\mathbb{R}} \frac{1 - e^{-2i\pi v} - 2i\pi v}{v^2(z-z'-v)} dv,
\end{aligned}$$

In the two last integrals, the integrands are bounded near zero and dominated by $1/v^2$ at infinity, and then the integrals are absolutely convergent. Moreover, the integrands can be extended to meromorphic functions of v, with the unique pole $v = z - z'$. Note that because of the addition of the terms $\pm 2i\pi v$, there is no pole at $v = 0$. In the first integral, if we replace $\mathbb{R}$ by the contour given by the union of $(-\infty, -R]$, $[-R, -R + iR]$, $[-R + iR, R + iR]$, $[R + iR, R]$ and (R, ∞), the modified integral tends to zero when R goes to infinity. One deduces that the initial integral is equal to $2i\pi$ times the sum of the residues of the integrand at the poles in the upper half plane:

$$\frac{1}{4\pi^2} \int_{\mathbb{R}} \frac{1 - e^{2i\pi v} + 2i\pi v}{v^2(z - z' - v)} dv = \frac{1 - e^{2i\pi(z-z')} + 2i\pi(z - z')}{2i\pi(z - z')^2} \mathbb{1}_{\Im(z-z')>0}$$

Changing v in $-v$ and exchanging z and z', we deduce

$$\frac{1}{4\pi^2} \int_{\mathbb{R}} \frac{1 - e^{-2i\pi v} - 2i\pi v}{v^2(z - z' - v)} dv = -\frac{1 - e^{-2i\pi(z-z')} - 2i\pi(z - z')}{2i\pi(z - z')^2} \mathbb{1}_{\Im(z-z')<0},$$

and by adding the equalities:

$$\int_{\mathbb{R}} \frac{S(v)^2}{z - z' - v} dv = \frac{\operatorname{sgn}(\Im(z - z')) \left(1 - e^{2i\pi(z-z') \operatorname{sgn}(\Im(z-z'))}\right)}{2i\pi(z - z')^2} + \frac{1}{z - z'}.$$

By noting that

$$i\pi \left[\operatorname{sgn}(\Im(z)) - \operatorname{sgn}\left(\Im(z')\right)\right] \operatorname{sgn}\left(\Im(z - z')\right) = 2i\pi \mathbb{1}_{\Im(z)\Im(z')<0},$$

we deduce

$$\lim_{A\to\infty} I_2 = \frac{1 - e^{2i\pi(z-z') \operatorname{sgn}(\Im(z-z'))}}{(z - z')^2} \mathbb{1}_{\Im(z)\Im(z')<0} + i\pi \frac{\operatorname{sgn}(\Im(z)) - \operatorname{sgn}(\Im(z'))}{z - z'}.$$

Hence,

$$\begin{aligned}\lim_{A\to\infty} (I_1 - I_2) = -\pi^2 \operatorname{sgn}(\Im(z)) \operatorname{sgn}\left(\Im(z')\right) - \frac{1 - e^{2i\pi(z-z') \operatorname{sgn}(\Im(z-z'))}}{(z - z')^2} \mathbb{1}_{\Im(z)\Im(z')<0} \\ - i\pi \frac{\operatorname{sgn}(\Im(z)) - \operatorname{sgn}(\Im(z'))}{z - z'},\end{aligned}$$

and

$$\begin{aligned}\lim_{A\to\infty} \mathbb{E}\left(\sum_{|y_k|,|y_l|\leq A} \frac{1}{z - y_k} \frac{1}{z' - y_l}\right) &= -\pi^2 \operatorname{sgn}(\Im(z)) \operatorname{sgn}\left(\Im(z')\right) \\ &\quad - \frac{1 - e^{2i\pi(z-z') \operatorname{sgn}(\Im(z-z'))}}{(z - z')^2} \mathbb{1}_{\Im(z)\Im(z')<0}.\end{aligned}$$

Hence

$$M_2(z, z') = -4\pi^2 \mathbb{1}_{\Im(z)<0, \Im(z')<0} - \frac{1 - e^{2i\pi(z-z') \operatorname{sgn}(\Im(z-z'))}}{(z - z')^2} \mathbb{1}_{\Im(z)\Im(z')<0}.$$

This formula has been proven for $\Im(z) \neq \Im(z')$. It remains true without this assumption. Indeed, the L^2 convergence of $i\pi + \sum_{|y_k|\leq A} \frac{1}{z-y_k}$ towards $\xi'(z)/\xi(z)$ for $A \to \infty$ has been proven uniformly in compact sets away from the real line. Since the joint moments of the former quantity are easily proven to be continuous, one deduces that M_2 is continuous with respect to $z, z' \notin \mathbb{R}$.

Second moment with a conjugate $\tilde{M}_2(z,z'); z,z' \notin \mathbb{R}$: Let us now define

$$\tilde{M}_2(z,z') := \mathbb{E}\left(\frac{\xi'_\infty}{\xi_\infty}(z)\overline{\frac{\xi'_\infty}{\xi_\infty}(z')}\right)$$

Since

$$\overline{\frac{\xi'_\infty}{\xi_\infty}(z')} = -2i\pi + \frac{\xi'_\infty}{\xi_\infty}(\overline{z'}),$$

one gets

$$\tilde{M}_2(z,z') = M_2(z,\overline{z'}) - 2i\pi M_1(z),$$

and then

$$\tilde{M}_2(z,z') = 4\pi^2 \mathbb{1}_{\Im(z)<0,\Im(z')<0} - \frac{1-e^{2i\pi(z-\overline{z'})\operatorname{sgn}(\Im(z-\overline{z'}))}}{(z-\overline{z'})^2}\mathbb{1}_{\Im(z)\Im(z')>0}.$$

In particular, we get the L^2 norm:

$$\mathbb{E}\left(\left|\frac{\xi'_\infty}{\xi_\infty}(z)\right|^2\right) = 4\pi^2\mathbb{1}_{\Im(z)<0} + \frac{1-e^{-4\pi|\Im(z)|}}{4\Im^2(z)}.$$

As a consequence of the previous computation, if our conjecture is true and moments are also controlled then:

Conjecture 3.8.

$$\begin{aligned}
&\lim_{T\to\infty}\frac{1}{\log^2 T}\mathbb{E}\left(\frac{\zeta'}{\zeta}\left(\frac{1}{2}+iUT+\frac{a}{\log T}\right)\frac{\zeta'}{\zeta}\left(\frac{1}{2}+iUT+\frac{a'}{\log T}\right)\right)\\
=&\mathbb{1}_{\Re(a)<0,\Re(a')<0} - \frac{1-e^{-(a'-a)\operatorname{sgn}\Re(a'-a)}}{(a-a')^2}\mathbb{1}_{\Re(a)\Re(a')<0}\\
&\lim_{T\to\infty}\frac{1}{\log^2 T}\mathbb{E}\left(\frac{\zeta'}{\zeta}\left(\frac{1}{2}+iUT+\frac{a}{\log T}\right)\overline{\frac{\zeta'}{\zeta}\left(\frac{1}{2}+iUT+\frac{a'}{\log T}\right)}\right)\\
=&\mathbb{1}_{\Re(a)<0,\Re(a')<0} + \frac{1-e^{-\left(a+\overline{a'}\right)\operatorname{sgn}\Re(a+\overline{a'})}}{\left(a+\overline{a'}\right)^2}\mathbb{1}_{\Re(a)\Re(a')>0}
\end{aligned}$$

Remark 3.9. *In Lemma 3.3, we see that there is a correspondance between a and $-2i\pi z$ in this conjecture and the computations just above. This explains the signs of the terms involved in the conjecture, and the fact the imaginary parts of z and z' are replaced by the real parts of a and a'.*

For $a = a'$, one recovers the first statement of theorem 3 in [GGM01], which is equivalent to the pair correlation conjecture under Riemann hypothesis. Higher moments formulas are also expected to be equivalent to the convergence of higher correlation functions of ζ zeros towards the corresponding correlations for the sine-kernel process.

4 Mesoscopic fluctuations and blue noise

The function $\frac{\xi'_\infty}{\xi_\infty}(z) - i\pi$ studied in the pervious section was recently considered by Aizenman and Warzel in [AW13]. They prove that for any $z \in \mathbb{R}$, the value of this function follows the Cauchy distribution: in fact, their result applies to more general point processes than the sine kernel process. In the present paper, we deal with the same function but away from the real line. In this section we shall view this function in the framework of linear statistics and will study its fluctuations on a mesoscopic level. It is may be worth noting here that $\frac{\xi'_\infty}{\xi_\infty}(z) - i\pi$ also has a spectral interpretation: informally, it is the trace of the resolvent of the (unbounded) random Hermitian operator whose spectrum consists exactly of the points $(y_k)_{k\in\mathbb{Z}}$ that we constructed in [MNN13]. This interpretation is informal since the series corresponding to the resolvant is not absolutely convergent.

For $s \geq 0$, we consider the Sobolev space:

$$H^s := \left\{ f \in L^2(\mathbb{R}, \mathbb{C}) \;\middle|\; \int_{\mathbb{R}} \left|\hat{f}(k)\right|^2 \left(1 + |k|^2\right)^s dk \right\},$$

where the Fourier transform of f is normalized as follows:

$$\hat{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x) e^{-ikx} dx.$$

We then call *blue noise* a Gaussian family of centered variables indexed by $H^{1/2}$, denoted $(\mathcal{B}(f))_{f\in H^{1/2}}$, such that $f \mapsto \mathcal{B}(f)$ is linear, $\mathcal{B}(f)$ is a.s. real if f is a real-valued function, and

$$\mathbb{E}[|\mathcal{B}(f)|^2] = \frac{1}{2\pi} \int_{\mathbb{R}} |k| |\hat{f}(k)|^2 dk.$$

The covariance structure of $\mathcal{B}$ is then:

$$\mathbb{E}[\mathcal{B}(f)\mathcal{B}(g)] = \frac{1}{2\pi} \int_{\mathbb{R}} |k| \hat{f}(k) \hat{g}(-k) dk,$$

$$\mathbb{E}[\mathcal{B}(f)\overline{\mathcal{B}(g)}] = \frac{1}{2\pi} \int_{\mathbb{R}} |k| \hat{f}(k) \overline{\hat{g}(k)} dk.$$

Similarly as for the Brownian motion, we can take the notation:

$$\int_{\mathbb{R}} f(t) d\mathcal{B}_t := \mathcal{B}(f).$$

Now, for any function $f \in L^1(\mathbb{R}, \mathbb{C}) \cap L^2(\mathbb{R}, \mathbb{C})$, we have

$$\begin{aligned} \mathbb{E}\left[\left(\sum_{k\in\mathbb{Z}} |f(y_k)|\right)^2\right] &= \int_{\mathbb{R}} |f|^2 + \int_{\mathbb{R}^2} (1 - S^2(x-y)) |f(x)| |f(y)| dx dy \\ &\leq \int_{\mathbb{R}} |f|^2 + \left(\int_{\mathbb{R}} |f|\right)^2 < \infty, \end{aligned}$$

and then

$$X_f := \sum_{k\in\mathbb{Z}} f(y_k) - \int f$$

is well-defined as a square-integrable random variable. As we will see in Corollary 4.3, X_f can also be defined as a square-integrable random variable as soon as $f \in H^{1/2}$, even if f is not integrable.

In this section, we examine the behavior of $\sum_{k\in\mathbb{Z}} f\left(\frac{y_k}{L}\right) - L\int f$ as $L \to \infty$ for suitable functions f:

Theorem 4.1. *If $(y_k; k \in \mathbb{Z})$ is a sine kernel point process, there is a blue noise $\mathscr{B}$ such that*

$$\left(X_{f\left(\frac{\cdot}{L}\right)}\right)_{f\in H^{1/2}} \stackrel{L\to\infty}{\longrightarrow} \left(\int f(t)d\mathscr{B}_t\right)_{f\in H^{1/2}},$$

the convergence holding in law for finite-dimensional marginals.

In Subsection 4.2, we analyse the asymptotic behavior of the Stieltjes transform of the sine kernel process. To that endeavor, we apply the result to the complex-valued functions $f_z(t) = \frac{1}{z-t}$.

4.1 The sine kernel from afar

We will need an intermediate proposition:

Proposition 4.2 (Adapted from Soshnikov [Sos00])**.** *If f is a smooth, real-valued function with compact support and if the p-th cumulant of X_f is denoted $C_p(f)$, then we have:*

$$C_1(f) = 0$$

$$\left|C_2(f) - \frac{1}{2\pi}\int \left|\hat{f}(k)\right|^2 |k|dk\right| \ll \int |k| \left|\hat{f}(k)\right|^2 \mathbb{1}_{|k|\geq\pi} dk$$

$$\forall p \geq 3, \left|C_p(f)\right| \ll_p \int_{k_1+\cdots+k_p=0} \mathbb{1}_{|k_1|+\cdots+|k_p|>2\pi}|k_1| \left|\hat{f}(k_1)\dots\hat{f}(k_p)\right| dk$$

where in the previous equation, dk stands for the Lebesgue measure on the hyperplane $\{k_1+\cdots+k_p=0\}$.

Proof. The first equality is immediate. Now, since $y_k^{(n)} = \frac{n}{2\pi}\theta_k^{(n)}$ converges almost surely to y_k, X_f is the almost sure limit of:

$$\begin{aligned} X_{n,f} &:= \sum_{k\in\mathbb{Z}} f\left(\frac{n}{2\pi}\theta_k^{(n)}\right) - \int f \\ &= \sum_{k=1}^n \left(-\frac{1}{n}\int f + \sum_{l\in\mathbb{Z}} f\left(\frac{n}{2\pi}\theta_k^{(n)} + nl\right)\right) \\ &= \sum_{k=1}^n \psi_n\left(\theta_k^{(n)}\right), \end{aligned}$$

where ψ_n is the sequence of 2π-periodic functions with zero mean:

$$\psi_n(\theta) = -\frac{1}{n}\int f + \sum_{l\in\mathbb{Z}} f\left(\frac{n}{2\pi}\theta + nl\right)$$

If $\hat{f}$ is the Fourier transform of f, the Fourier coefficients

$$\left(c_k(\psi_n) := \frac{1}{2\pi}\int_0^{2\pi} \psi_n(\theta)e^{-ik\theta}d\theta; k\in\mathbb{Z}\right)$$

of ψ_n are given by:

$$c_0(\psi_n) = 0,$$
$$\forall k\in\mathbb{Z}^*, c_k(\psi_n) = \frac{\sqrt{2\pi}}{n}\hat{f}\left(\frac{2\pi k}{n}\right).$$

If $C_{p,n}(f)$ is the p-th cumulant of $X_{n,f}$, thanks to the main combinatorial lemma and lemma 1 in [Sos00], we have:

$$C_{1,n}(f) = 0$$
$$\left|C_{2,n}(f) - \frac{2\pi}{n}\sum_{k\in\mathbb{Z}}\frac{|k|}{n}\hat{f}\left(\frac{2\pi k}{n}\right)\hat{f}\left(-\frac{2\pi k}{n}\right)\right| \ll \frac{1}{n}\sum_{|k|>\frac{1}{2}n}\frac{|k|}{n}\hat{f}\left(\frac{2\pi k}{n}\right)\hat{f}\left(-\frac{2\pi k}{n}\right)$$
$$\forall p\geq 3, \left|C_{p,n}(f_n)\right| \ll_p \frac{1}{n^{p-1}}\sum_{\substack{k_1+\cdots+k_p=0\\ |k_1|+\cdots+|k_p|>n}}\frac{|k_1|}{n}\left|\hat{f}\left(\frac{2\pi k_1}{n}\right)\ldots\hat{f}\left(\frac{2\pi k_p}{n}\right)\right|$$

As $\hat{f}$ decays at infinity faster than any power, we recognize three converging Riemann sums. The first one is:

$$\frac{2\pi}{n}\sum_{k\in\mathbb{Z}}\frac{|k|}{n}\hat{f}\left(\frac{2\pi k}{n}\right)\hat{f}\left(-\frac{2\pi k}{n}\right) \overset{n\to\infty}{\longrightarrow} 2\pi\int |k|\left|\hat{f}(2\pi k)\right|^2 dk = \frac{1}{2\pi}\int |k|\left|\hat{f}(k)\right|^2 dk.$$

The others appear as error terms and are Riemann sums converging to integrals on the hyperplane $\left\{k_1+\cdots+k_p=0\right\}\subset\mathbb{R}^p$.

$$\forall p\geq 2, \frac{1}{n^{p-1}}\sum_{\substack{k_1+\cdots+k_p=0\\ |k_1|+\cdots+|k_p|>n}}\frac{|k_1|}{n}\left|\hat{f}\left(\frac{2\pi k_1}{n}\right)\ldots\hat{f}\left(\frac{2\pi k_p}{n}\right)\right|$$
$$\overset{n\to\infty}{\longrightarrow}\int_{k_1+\cdots+k_p=0}\mathbb{1}_{|k_1|+\cdots+|k_p|>1}|k_1|\left|\hat{f}(2\pi k_1)\ldots\hat{f}(2\pi k_p)\right| dk.$$

Therefore, for every $p\geq 1$, the p-th cumulant of $X_{n,f}$ is bounded independently of n and the sequence $|X_{n,f}|^p$ is uniformly integrable. Thus, the convergence of $X_{n,f}$ to X_f is not only almost sure but also in every $L^p(\Omega)$, Ω being the underlying probability space.

Now since

$$\forall p\geq 1, C_p(f_n) \overset{n\to\infty}{\longrightarrow} C_p(f),$$

we have

$$\left|C_2(f) - \frac{1}{2\pi}\int \left|\hat{f}(k)\right|^2 |k| dk\right| \ll \int |k| \left|f(2\pi k)\right|^2 \mathbb{1}_{|k|\geq \frac{1}{2}} dk$$

$$\forall p \geq 3, \left|C_p(f)\right| \ll_p \int_{k_1+\dots+k_p=0} \mathbb{1}_{|k_1|+\dots+|k_p|>1} |k_1| \left|\hat{f}\left(2\pi k_1\right)\dots\hat{f}\left(2\pi k_p\right)\right| dk$$

After an obvious change of variables, we recover the claimed estimates. □

Corollary 4.3. *The map*

$$f \mapsto X_f$$

from $L^1(\mathbb{R}, \mathbb{C}) \cap H^{1/2}$ to $L^2(\Omega)$ admits a linear extension to $H^{1/2}$, which satisfies the following property of continuity:

$$\mathbb{E}\left(\left|X_f\right|^2\right)^{\frac{1}{2}} \ll \|f\|_{H^{\frac{1}{2}}},$$

uniformly, for all $f \in H^{1/2}$. This extension is unique up to almost sure equality.

Proof. The estimate on the second cumulant, given by Proposition 4.2, implies

$$\mathbb{E}\left(\left|X_f\right|^2\right)^{\frac{1}{2}} \ll \|f\|_{H^{\frac{1}{2}}}$$

for every smooth, real-valued function f with compact support. By linearity, this esimate remains true without the assumption that f is real-valued. We deduce the existence of a family $(Y_f)_{f\in H^{1/2}}$ of random variables such that $Y_f = X_f$ a.s. if f is smooth with compact support, and

$$\mathbb{E}\left(\left|Y_f\right|^2\right)^{\frac{1}{2}} \ll \|f\|_{H^{\frac{1}{2}}}$$

This family is unique up to almost sure equality. Then, we are done if we show that $X_f = Y_f$ almost surely as soon as $f \in L^1 \cap H^{1/2}$. Now, the map $f \mapsto X_f - Y_f$ from $f \in L^1 \cap H^{1/2}$ to $L^2(\Omega)$ is a.s. equal to zero on $\mathcal{C}_c^\infty(\mathbb{R}, \mathbb{C})$. Moreover, we have seen above, by using the two first correlation functions of the sine kernel process, that

$$\mathbb{E}\left(\left|X_f\right|^2\right)^{\frac{1}{2}} \leq ||f||_{L^1} + ||f||_{L^2}$$

which implies:

$$\mathbb{E}\left(\left|X_f - Y_f\right|^2\right)^{\frac{1}{2}} \ll ||f||_{L^1} + ||f||_{L^2} + ||f||_{H^{1/2}} \ll ||f||_{L^1} + ||f||_{H^{1/2}}.$$

Hence, the map $f \mapsto X_f - Y_f$ from $f \in L^1 \cap H^{1/2}$ is continuous, and since it vanishes on $\mathcal{C}_c^\infty$, which is dense in $L^1 \cap H^{1/2}$, it vanishes everywhere.

□

Proof of Theorem 4.1. It is sufficient to prove convergence in law of the one-dimensional marginals, for real-valued functions f. Indeed, if we have this convergence, if $f_1, \dots, f_m$

are real-valued functions in $H^{1/2}$, and if $\lambda_1, \dots, \lambda_m \in \mathbb{R}$, then we have the convergence in law

$$X_{f\left(\frac{\cdot}{L}\right)} = \sum_{j=1}^{m} \lambda_j X_{f_j\left(\frac{\cdot}{L}\right)} \overset{L\to\infty}{\longrightarrow} \mathcal{B}(f) = \sum_{j=1}^{m} \lambda_j \mathcal{B}(f_j),$$

for

$$f = \lambda_1 f_1 + \lambda_2 f_2 + \dots + \lambda_m f_m.$$

Applying the bounded, continuous function $x \mapsto e^{ix}$ gives the convergence of the Fourier transform of $(X_{f_j\left(\frac{\cdot}{L}\right)})_{1\le j\le m}$ towards the Fourier transform of $(\mathcal{B}(f_j))_{1\le j\le m}$, and then the convergence of the finite-dimensional marginals claimed in Theorem 4.1, for real-valued functions. The case of complex-valued functions is then deduced by linearity.

If remains to prove that for all $f \in H^{1/2}$, real-valued,

$$X_{f\left(\frac{\cdot}{L}\right)} \overset{L\to\infty}{\longrightarrow} \mathcal{B}(f).$$

Let us first assume that f is smooth function with compact support. If $C_p^{(L)}(f)$ is the p-th cumulant of $X_{f\left(\frac{\cdot}{L}\right)}$, then by rescaling the space variable:

$$\forall k \in \mathbb{R}, \widehat{f\left(\frac{\cdot}{L}\right)}(k) = L\hat{f}(Lk)$$

and

$$C_1^{(L)}(f) = 0$$

$$\left| C_2^{(L)}(f) - \frac{1}{2\pi} \int \left|\hat{f}(k)\right|^2 |k| dk \right| \ll \int |k| \left|\hat{f}(k)\right|^2 \mathbb{1}_{\{|k|\ge L\pi\}} dk$$

$$\forall p \ge 3, \left| C_p^{(L)}(f) \right| \ll_p \int_{k_1+\dots+k_p=0} \mathbb{1}_{\{|k_1|+\dots+|k_p|>2\pi L\}} |k_1| \left| \hat{f}(k_1) \dots \hat{f}(k_p) \right| dk$$

Therefore, as $L \to \infty$, $X_{f\left(\frac{\cdot}{L}\right)}$ converges in law to a centered Gaussian with variance $\frac{1}{2\pi} \int |k| \, |f(k)|^2 \, dk$, i.e. to $\mathcal{B}(f)$.

Now, if f is only supposed to be in $H^{\frac{1}{2}}$, let us consider a sequence of smooth compactly supported functions $(f_n)_{n\in\mathbb{N}}$ such that:

$$\|f - f_n\|_{H^{\frac{1}{2}}} \overset{n\to\infty}{\longrightarrow} 0$$

We will be done after proving that for any t in a compact set:

$$\mathbb{E}\left(e^{itX_{f\left(\frac{\cdot}{L}\right)}}\right) \overset{n\to\infty}{\longrightarrow} \exp\left(-\frac{t^2}{4\pi} \int |k| |\hat{f}(k)|^2 dk\right)$$

We have because of the triangular inequality, for fixed n:

$$\left| \mathbb{E}\left(e^{itX_{f\left(\frac{\cdot}{L}\right)}}\right) - \exp\left(-\frac{t^2}{4\pi} \int |k| |\hat{f}(k)|^2 dk\right) \right|$$

$$
\begin{aligned}
\leq & \left|\mathbb{E}\left(e^{itX_{f\left(\frac{\cdot}{L}\right)}}\right)-\mathbb{E}\left(e^{itX_{f_n\left(\frac{\cdot}{L}\right)}}\right)\right| \\
& +\left|\mathbb{E}\left(e^{itX_{f_n\left(\frac{\cdot}{L}\right)}}\right)-\exp\left(-\frac{t^2}{4\pi}\int|k||\hat{f}_n(k)|^2dk\right)\right| \\
& +\left|\exp\left(-\frac{t^2}{4\pi}\int|k||\hat{f}(k)|^2dk\right)-\exp\left(-\frac{t^2}{4\pi}\int|k||\hat{f}_n(k)|^2dk\right)\right|
\end{aligned}
$$

The third term is a $\mathcal{O}\left(t^2\|f-f_n\|^2_{H^{\frac{1}{2}}}\right)$. The second disappears when we take the $\limsup_{L\to\infty}$. As for the first term, we have for any $\varepsilon>0$:

$$
\begin{aligned}
& \left|\mathbb{E}\left(e^{itX_{f\left(\frac{\cdot}{L}\right)}}\right)-\mathbb{E}\left(e^{itX_{f_n\left(\frac{\cdot}{L}\right)}}\right)\right| \\
\leq & \mathbb{E}\left(\left|e^{itX_{f\left(\frac{\cdot}{L}\right)}-itX_{f_n\left(\frac{\cdot}{L}\right)}}-1\right|\right) \\
\leq & 2\mathbb{P}\left(\left|X_{f\left(\frac{\cdot}{L}\right)}-X_{f_n\left(\frac{\cdot}{L}\right)}\right|\geq\varepsilon\right)+\varepsilon|t| \\
\leq & 2\frac{\mathbb{E}\left(\left|X_{f\left(\frac{\cdot}{L}\right)}-X_{f_n\left(\frac{\cdot}{L}\right)}\right|^2\right)}{\varepsilon^2}+\varepsilon|t|
\end{aligned}
$$

By linearity and the second cumulant estimate:

$$
\mathbb{E}\left(\left|X_{f\left(\frac{\cdot}{L}\right)}-X_{f_n\left(\frac{\cdot}{L}\right)}\right|^2\right)^{\frac{1}{2}}=\mathbb{E}\left(\left|X_{(f-f_n)\left(\frac{\cdot}{L}\right)}\right|^2\right)^{\frac{1}{2}}\ll\|f-f_n\|_{H^{\frac{1}{2}}}
$$

Hence for any fixed n and $\varepsilon>0$:

$$
\begin{aligned}
& \limsup_{L\to\infty}\left|\mathbb{E}\left(e^{itX_{f\left(\frac{\cdot}{L}\right)}}\right)-\exp\left(-\frac{t^2}{2}\int|k||\hat{f}(k)|^2dk\right)\right| \\
\ll & \|f-f_n\|^2_{H^{\frac{1}{2}}}\left(\frac{1}{\varepsilon^2}+t^2\right)+\varepsilon|t|
\end{aligned}
$$

Taking $n\to\infty$, then $\varepsilon\to 0$ concludes the proof. □

4.2 Application to the Stieltjes transform of the sine kernel

For $z\in\mathbb{C}\backslash\mathbb{R}$, $f_z: t\mapsto 1/(z-t)$ is in $H^{1/2}$. Indeed, one can check (by using the inverse Fourier transform for example) that

$$
\hat{f}_z(k)=-i\sqrt{2\pi}\,\operatorname{sgn}\Im(z)e^{-izk}\mathbb{1}_{k\Im(z)<0},
$$

and then $\hat{f}_z$ decays exponentially at infinity. Moreover, X_{f_z} can be related to the logarithmic derivative of ξ_∞:

Proposition 4.4. *For all $z\notin\mathbb{R}$, we have almost surely,*

$$
X_{f_z}=\frac{\xi'_\infty(z)}{\xi_\infty(z)}-2i\pi\mathbb{1}_{\Im z<0}=i\pi\operatorname{sgn}\Im z+\frac{1}{z-y_0}+\sum_{k=1}^{\infty}\left(\frac{1}{z-y_k}+\frac{1}{z-y_{-k}}\right).
$$

Proof. Let φ be a smooth function from $\mathbb{R}$ to $[0,1]$, nonincreasing on $\mathbb{R}_+$, equal to 1 on $[-1,1]$ and to 0 on $\mathbb{R}\backslash[-2,2]$. If for $A>0$, $f_z^{(A)}(t) = f_z(t)\varphi(t/A)$, we have

$$|f_z^{(A)}(t) - f_z(t)| \leq |f_z(t)| \mathbb{1}_{|t|\geq A}$$

and

$$|(f_z^{(A)})'(t) - f_z'(t)| = \left| f_z'(t)\varphi(t/A) + \frac{1}{A}\varphi'(t/A) f_z(t) - f_z'(t) \right| \ll |f_z'(t)| \mathbb{1}_{|t|\geq A} + \frac{|f_z(t)|}{A}.$$

For z fixed, $|f_z(t)|$ is dominated by $1/(1+|t|)$, $|f_z'(t)|$ is dominated by $1/(1+|t|)^2$, and then

$$|f_z^{(A)}(t) - f_z(t)|^2 + |(f_z^{(A)})'(t) - f_z'(t)|^2 \ll \frac{\mathbb{1}_{|t|\geq A}}{(1+|t|)^2} + \frac{1}{A^2(1+|t|)^2}.$$

We deduce that $f_z^{(A)}$ converges to f_z in H^1, and a fortiori in $H^{1/2}$. Hence, in $L^2(\Omega)$,

$$\begin{aligned} X_{f_z} &= \lim_{A\to\infty} X_{f_z^{(A)}} = \lim_{A\to\infty} \left(\sum_{k\in\mathbb{Z}} \frac{\varphi(y_k/A)}{z-y_k} - \int_{\mathbb{R}} \frac{\varphi(y/A)}{z-y} dy \right) \\ &= \lim_{A\to\infty} \int_1^2 (-\varphi'(u)) \left(\sum_{k\in\mathbb{Z}} \frac{\mathbb{1}_{|y_k|\leq Au}}{z-y_k} - \int_{-Au}^{Au} \frac{dy}{z-y} \right) du \end{aligned}$$

From Proposition 3.5, one easily deduces that

$$\sum_{k\in\mathbb{Z}} \frac{\mathbb{1}_{|y_k|\leq B}}{z-y_k} - \int_{-B}^{B} \frac{dy}{z-y} \underset{B\to\infty}{\longrightarrow} \frac{\xi_\infty'(z)}{\xi_\infty(z)} - 2i\pi \mathbb{1}_{\Im z<0}$$

in $L^p(\Omega)$ for all $p\geq 1$, and in particular in $L^2(\Omega)$. Now, since $-\varphi'$ is nonnegative in $[1,2]$ and has integral 1, one has

$$\begin{aligned} &\left\| \int_1^2 (-\varphi'(u)) \left(\sum_{k\in\mathbb{Z}} \frac{\mathbb{1}_{|y_k|\leq Au}}{z-y_k} - \int_{-Au}^{Au} \frac{dy}{z-y} \right) du - \frac{\xi_\infty'(z)}{\xi_\infty(z)} + 2i\pi \mathbb{1}_{\Im z<0} \right\|_{L^2(\Omega)} \\ &\leq \int_1^2 (-\varphi'(u)) du \left\| \sum_{k\in\mathbb{Z}} \frac{\mathbb{1}_{|y_k|\leq Au}}{z-y_k} - \int_{-Au}^{Au} \frac{dy}{z-y} - \frac{\xi_\infty'(z)}{\xi_\infty(z)} + 2i\pi \mathbb{1}_{\Im z<0} \right\|_{L^2(\Omega)} \\ &\leq \sup_{B\in[A,2A]} \left\| \sum_{k\in\mathbb{Z}} \frac{\mathbb{1}_{|y_k|\leq B}}{z-y_k} - \int_{-B}^{B} \frac{dy}{z-y} - \frac{\xi_\infty'(z)}{\xi_\infty(z)} + 2i\pi \mathbb{1}_{\Im z<0} \right\|_{L^2(\Omega)}, \end{aligned}$$

which tends to zero when A goes to infinity. Hence, in $L^2(\Omega)$,

$$X_{f_z} = \lim_{A\to\infty} \int_1^2 (-\varphi'(u)) \left(\sum_{k\in\mathbb{Z}} \frac{\mathbb{1}_{|y_k|\leq Au}}{z-y_k} - \int_{-Au}^{Au} \frac{dy}{z-y} \right) du = \frac{\xi_\infty'(z)}{\xi_\infty(z)} - 2i\pi \mathbb{1}_{\Im z<0}.$$

□

A consequence of the previous proposition is the following result:

Proposition 4.5. *For $z \in \mathbb{C}\backslash\mathbb{R}$, let*

$$F(z) := X_{f_z} = \frac{\xi'_\infty(z)}{\xi_\infty(z)} - 2i\pi \mathbb{1}_{\Im z<0}.$$

Then, one has the convergence in law:

$$(LF(Lz))_{z\in\mathbb{C}\backslash\mathbb{R}} \underset{L\to\infty}{\longrightarrow} (G(z))_{z\in\mathbb{C}\backslash\mathbb{R}},$$

where $G(z) = \mathcal{B}(f_z)$ for all $z \in \mathbb{C}\backslash\mathbb{R}$. The centered gaussian process $(G(z))_{z\in\mathbb{C}\backslash\mathbb{R}}$ has the covariance structure given, for all $z_1, z_2 \notin \mathbb{R}$, by

$$\mathbb{E}[G(z_1)G(z_2)] = -\frac{\mathbb{1}_{\Im(z_1)\Im(z_2)<0}}{(z_2 - z_1)^2},$$

$$\mathbb{E}[G(z_1)\overline{G(z_2)}] = -\frac{\mathbb{1}_{\Im(z_1)\Im(z_2)>0}}{(\overline{z_2} - z_1)^2},$$

and in particular

$$\mathbb{E}[|G(z_1)|^2] = \frac{1}{4\Im^2(z_1)}.$$

Proof. We have, for $L > 0$,

$$f_z(t/L) = \frac{1}{z-(t/L)} = \frac{L}{Lz - t} = Lf_{Lz}(t),$$

and then

$$X_{f_z(\cdot/L)} = LX_{f_{Lz}} = LF(Lz).$$

The convergence in law given in this proposition is then a consequence of Theorem 4.1. It remains to compute the covariance structure. For $z_1, z_2 \in \mathbb{C}\backslash\mathbb{R}$,

$$\begin{aligned}&\mathbb{E}[\mathscr{B}(f_{z_1})\mathscr{B}(f_{z_2})] \\ &= \frac{1}{2\pi}\int_{\mathbb{R}} |k|(-i\sqrt{2\pi}\,\mathrm{sgn}\,\Im(z_1)e^{-iz_1k}\mathbb{1}_{k\Im(z_1)<0})(-i\sqrt{2\pi}\,\mathrm{sgn}\,\Im(z_2)e^{iz_2k}\mathbb{1}_{-k\Im(z_2)<0})dk.\end{aligned}$$

If $\Im(z_1)$ and $\Im(z_2)$ have the same sign, the product of the indicator functions vanishes for all $k \in \mathbb{R}$, so

$$\mathbb{E}[\mathscr{B}(f_{z_1})\mathscr{B}(f_{z_2})] = 0.$$

If $\Im(z_1)$ and $\Im(z_2)$ have not the same sign, we get

$$\mathbb{E}[\mathscr{B}(f_{z_1})\mathscr{B}(f_{z_2})] = \int_{\mathbb{R}} |k|e^{ik(z_2-z_1)}\mathbb{1}_{k\Im(z_2)>0}.$$

By doing the change of variable $k' = k\,\mathrm{sgn}\,\Im(z_2)$, we get

$$\mathbb{E}[\mathscr{B}(f_{z_1})\mathscr{B}(f_{z_2})] = \int_0^\infty ke^{ik(z_2-z_1)\,\mathrm{sgn}\,\Im(z_2)}dk$$

Now, for all $y > 0$,

$$\int_0^\infty k e^{-yk} dk = \int_0^\infty (u/y) e^{-u} d(u/y) = 1/y^2,$$

and by analytic continuation, this formula is true for all y with strictly positive real part. Applying this to $y = -i(z_2 - z_1) \operatorname{sgn} \Im(z_2)$, we have

$$\mathbb{E}[\mathscr{B}(f_{z_1})\mathscr{B}(f_{z_2})] = -1/(z_2 - z_1)^2$$

for $\Im(z_1)\Im(z_2) < 0$, and then in any case,

$$\mathbb{E}[\mathscr{B}(f_{z_1})\mathscr{B}(f_{z_2})] = -\frac{\mathbb{1}_{\Im(z_1)\Im(z_2)<0}}{(z_2 - z_1)^2}.$$

Since the blue noise here is real-valued for real functions, $\mathscr{B}(f_{\overline{z_2}}) = \overline{\mathscr{B}(f_{z_2})}$, and then

$$\mathbb{E}[\mathscr{B}(f_{z_1})\overline{\mathscr{B}(f_{z_2})}] = -\frac{\mathbb{1}_{\Im(z_1)\Im(z_2)>0}}{(\overline{z_2} - z_1)^2}.$$

□

Remark 4.6. *The covariance structure of F has been computed above in this paper. We have*

$$\mathbb{E}[F(z_1)F(z_2)] = -\frac{1 - e^{2i\pi(z_1 - z_2)\operatorname{sgn}\Im(z_1 - z_2)}}{(z_1 - z_2)^2}\mathbb{1}_{\Im(z_1)\Im(z_2)>0},$$

and then

$$\mathbb{E}[(LF(Lz_1))(LF(Lz_2))] \underset{L\to\infty}{\longrightarrow} -\frac{\mathbb{1}_{\Im(z_1)\Im(z_2)<0}}{(z_2 - z_1)^2} = \mathbb{E}[G(z_1)G(z_2)].$$

Similarly,

$$\mathbb{E}[(LF(Lz_1))(\overline{LF(Lz_2)})] \underset{L\to\infty}{\longrightarrow} \mathbb{E}[G(z_1)\overline{G(z_2)}].$$

This convergence is naturally expected once the previous proposition is proven.

The stochastic process $z \mapsto X_{f_z}$ admits the version

$$z \mapsto F(z) = \frac{\xi'_\infty(z)}{\xi_\infty(z)} - 2i\pi\mathbb{1}_{\Im z<0},$$

which is holomorphic on $\mathbb{C}\backslash\mathbb{R}$. One can ask if the situation is similar for G. The answer is positive:

Proposition 4.7. *The random function G admits a version which is holomorphic on $\mathbb{C}\backslash\mathbb{R}$. Moreover, $z \mapsto LF(Lz)$ converges in law to an holomorphic version of G when L goes to infinity, in the sense of the uniform convergence on compact sets of $\mathbb{C}\backslash\mathbb{R}$.*

Proof. We first compute the L^2 norm of $G(z_1) - G(z_2)$ when $z_1, z_2 \notin \mathbb{R}$:

$$\mathbb{E}[|G(z_1) - G(z_2)|^2] = \mathbb{E}[|G(z_1)|^2] + \mathbb{E}[|G(z_2)|^2] - \mathbb{E}[G(z_1)\overline{G(z_2)}] - \mathbb{E}[G(z_2)\overline{G(z_1)}]$$

$$= -\frac{1}{(z_1 - \overline{z_1})^2} - \frac{1}{(z_2 - \overline{z_2})^2} + \mathbb{1}_{\Im(z_1)\Im(z_2)>0}\left(\frac{1}{(z_1 - \overline{z_2})^2} + \frac{1}{(z_2 - \overline{z_1})^2}\right).$$

Let us now assume that z_1 and z_2 are in a given compact set K of $\mathbb{C}\backslash\mathbb{R}$. Let us denote:

$$c_K := \inf\{|\Im(z)|, z \in K\} > 0.$$

If $z_1, z_2 \in K$ have imaginary parts of different signs, necessarily $|z_1 - z_2| \geq 2c_K$ and from the computations above,

$$\mathbb{E}[|G(z_1) - G(z_2)|^2] = \frac{1}{4\Im^2(z_1)} + \frac{1}{4\Im^2(z_2)} \leq \frac{1}{2c_K^2}.$$

One deduces

$$\mathbb{E}[|G(z_1) - G(z_2)|^2] \leq \frac{1}{8c_K^4}|z_1 - z_2|^2.$$

If $z_1, z_2 \in K$ have imaginary parts with the same sign,

$$\mathbb{E}[|G(z_1) - G(z_2)|^2] = A(z_1, \overline{z_1}) + A(z_2, \overline{z_2}) - A(z_1, \overline{z_2}) - A(z_2, \overline{z_1})),$$

where

$$A(u, v) := -\frac{1}{(u - v)^2}.$$

The function A of two variables is holomorphic in the open set of $(a, b) \in \mathbb{C}^2$ such that $\Im(a)\Im(z_1) > 0$ and $\Im(b)\Im(z_1) < 0$. Since the set $[z_1, z_2] \times [\overline{z_1}, \overline{z_2}]$ is included in this domain (recall that $\Im(z_1)$ and $\Im(z_2)$ have the same sign), we have

$$A(z_1, \overline{z_1}) + A(z_2, \overline{z_2}) - A(z_1, \overline{z_2}) - A(z_2, \overline{z_1})) = \int_{z_1}^{z_2} \int_{\overline{z_1}}^{\overline{z_2}} A''_{1,2}(u, v)dudv,$$

where $A''_{1,2}$ is the second derivative of A with respect to the two variables. Hence,

$$\mathbb{E}[|G(z_1) - G(z_2)|^2] = 6 \int_{z_1}^{z_2} \int_{\overline{z_1}}^{\overline{z_2}} \frac{dudv}{(u - v)^4}.$$

Now, for $u \in [z_1, z_2]$, $v \in [\overline{z_1}, \overline{z_2}]$, we have $|\Im(u) - \Im(v)| \geq 2c_K$, since $z_1, z_2 \in K$. Hence, $|u - v|^4 \geq 16c_K^4$, and

$$\mathbb{E}[|G(z_1) - G(z_2)|^2] \leq \frac{3}{8c_K^4} \int_{z_1}^{z_2} \int_{\overline{z_1}}^{\overline{z_2}} |du||dv|.$$

Hence, similarly as in the case $\Im(z_1)\Im(z_2) < 0$, we have

$$\mathbb{E}[|G(z_1) - G(z_2)|^2] \leq \frac{3}{8c_K^4}|z_1 - z_2|^2.$$

By Kolmogorov's criterion, G admits a continuous version on $\mathbb{C}\backslash\mathbb{R}$. We now assume that G itself is continuous.

Let $\Gamma : [0,1] \mapsto \mathbb{C}$ be a closed, piecewise smooth contour in $\mathbb{C}\backslash\mathbb{R}$. Since G is continuous, the integral of G along Γ is well-defined, and one has

$$\left|\int_\Gamma G(z)dz\right|^2 = \int_0^1 \int_0^1 G(\Gamma(t))\overline{G(\Gamma(u))}\Gamma'(t)\overline{\Gamma'(u)}dtdu.$$

If we denote $\overline{\Gamma}$ the contour given by $\overline{\Gamma}(t) = \overline{\Gamma(t)}$, we can write

$$\left|\int_\Gamma G(z)dz\right|^2 = \int_0^1 \int_0^1 G(\Gamma(t))\widetilde{G}(\overline{\Gamma}(u))\Gamma'(t)\overline{\Gamma}'(u)dtdu,$$

where $\widetilde{G}$ is the function from $\mathbb{C}\backslash\mathbb{R}$, given by

$$\widetilde{G}(z) = \overline{G(\overline{z})}.$$

Hence,

$$\left|\int_\Gamma G(z)dz\right|^2 = \int_\Gamma \int_{\overline{\Gamma}} G(z_1)\widetilde{G}(z_2)dz_1dz_2.$$

Now, for $z_1 \in \Gamma$, $z_2 \in \overline{\Gamma}$

$$\mathbb{E}[|G(z_1)||\widetilde{G}(z_2)|] \leq \left(\mathbb{E}[|G(z_1)|^2]\right)^{1/2}\left(\mathbb{E}[|\overline{G(z_2)}|^2]\right)^{1/2} = \frac{1}{4|\Im(z_1)||\Im(z_2)|},$$

which implies

$$\int_\Gamma \int_{\overline{\Gamma}} \mathbb{E}[|G(z_1)\widetilde{G}(z_2)|]|dz_1|\,|dz_2| \leq \frac{(\ell(\Gamma))^2}{4c_\Gamma^2} < \infty,$$

where $\ell(\Gamma)$ is the length of Γ and c_Γ the infimum of $|\Im(z)|$ for $z \in \Gamma$. This bound allows to write

$$\mathbb{E}\left[\left|\int_\Gamma G(z)dz\right|^2\right] = \int_\Gamma \int_{\overline{\Gamma}} \mathbb{E}[G(z_1)\widetilde{G}(z_2)]dz_1dz_2.$$

Now, for $z_1 \in \Gamma$ and $z_2 \in \overline{\Gamma}$, $\Im(z_1)$ and $\Im(\overline{z_2})$ have the same sign, which implies

$$\mathbb{E}[G(z_1)\widetilde{G}(z_2)] = \mathbb{E}[G(z_1)\overline{G(\overline{z_2})}] = -\frac{1}{(z_2 - z_1)^2},$$

and then

$$\mathbb{E}\left[\left|\int_\Gamma G(z)dz\right|^2\right] = -\int_\Gamma \int_{\overline{\Gamma}} \frac{dz_1dz_2}{(z_2 - z_1)^2},$$

which is equal to zero, since the function $(z_1, z_2) \mapsto 1/(z_2 - z_1)^2$ is holomorphic and the contours Γ and $\overline{\Gamma}$ are closed. Hence, for all closed, piecewise smooth contours Γ on $\mathbb{C}\backslash\mathbb{R}$, one has almost surely

$$\int_\Gamma G(z)dz = 0.$$

One deduces that almost surely, this equality holds simultaneously for all polygonal closed contours whose vertices have rational real and imaginary parts. Then, by continuity of G, one can remove the condition of rationality, and deduces that almost surely, G is holomorphic on $\mathbb{C}\backslash\mathbb{R}$.

We know $z \mapsto LF(Lz)$ converges in law to G in the sense of the finite-dimensional marginals: it remains to prove that this convergence occurs in the space of continuous functions, i.e. that the family of laws of $(LF(Lz))_{z\in\mathbb{C}}$ is tight in this space. For a compact set K of $\mathbb{C}\backslash\mathbb{R}$, and for $z_1, z_2 \in K$, one has

$$\mathbb{E}[|LF(Lz_1) - LF(Lz_2)|^2] = \frac{1-e^{-4L\pi|\Im(z_1)|}}{4\Im^2(z_1)} + \frac{1-e^{-4L\pi|\Im(z_2)|}}{4\Im^2(z_2)}$$

if $\Im(z_1)\Im(z_2) < 0$, and

$$\begin{aligned}\mathbb{E}[|LF(Lz_1) - LF(Lz_2)|^2] &= A_L(z_1,\overline{z_1}) + A_L(z_2,\overline{z_2}) - A_L(z_1,\overline{z_2}) - A_L(z_2,\overline{z_1}) \\ &= \int_{z_1}^{z_2}\int_{\overline{z_1}}^{\overline{z_2}} (A_L'')_{1,2}(u,v)dudv\end{aligned}$$

if $\Im(z_1)\Im(z_2) > 0$, for

$$A_L(u,v) = -\frac{1-e^{2i\pi L(u-v)\,\mathrm{sgn}\,\Im(u-v)}}{(u-v)^2}.$$

In the first case, we get

$$\mathbb{E}[|LF(Lz_1) - LF(Lz_2)|^2] \leq \frac{|z_1-z_2|^2}{8c_K^4}$$

and in the second case,

$$\mathbb{E}[|LF(Lz_1) - LF(Lz_2)|^2] \leq |z_2-z_1|^2 \sup_{|\Im(u)|,|\Im(v)|>c_K} |(A_L'')_{1,2}(u,v)|.$$

Note that A_L is holomorphic in $\{(u,v)\in\mathbb{C}^2, \Im(u)\Im(v) < 0\}$, since $\mathrm{sgn}\,\Im(u-v)$ is locally constant on this set. Now,

$$(A_L')_1(u,v) = \frac{2(1-e^{2i\pi L(u-v)\,\mathrm{sgn}\,\Im(u-v)})}{(u-v)^3} + \frac{2i\pi L\,\mathrm{sgn}\,\Im(u-v)e^{2i\pi L(u-v)\,\mathrm{sgn}\,\Im(u-v)}}{(u-v)^2},$$

$$\begin{aligned}(A_L'')_{1,2}(u,v) = &\frac{6(1-e^{2i\pi L(u-v)\,\mathrm{sgn}\,\Im(u-v)})}{(u-v)^4} + \frac{8i\pi L\,\mathrm{sgn}\,\Im(u-v)e^{2i\pi L(u-v)\,\mathrm{sgn}\,\Im(u-v)}}{(u-v)^3} \\ &+ \frac{4\pi^2L^2e^{2i\pi L(u-v)\,\mathrm{sgn}\,\Im(u-v)}}{(u-v)^2},\end{aligned}$$

$$\begin{aligned}|(A_L'')_{1,2}(u,v)| &\leq \frac{6(1+e^{-2\pi L|\Im(u-v)|})}{|u-v|^4} + \frac{8\pi L e^{-2\pi L|\Im(u-v)|}}{|u-v|^3} + \frac{4\pi^2L^2e^{-2\pi L|\Im(u-v)|}}{|u-v|^2}. \\ &\leq \frac{12}{|\Im(u-v)|^4} + \frac{8\pi L e^{-2\pi L|\Im(u-v)|}}{|\Im(u-v)|^3} + \frac{4\pi^2L^2e^{-2\pi L|\Im(u-v)|}}{|\Im(u-v)|^2} \\ &\leq \frac{4\pi^2}{|\Im(u-v)|^4}\left(1 + (L|\Im(u-v)| + L^2(\Im(u-v))^2)e^{-2\pi L|\Im(u-v)|}\right)\end{aligned}$$

$$\leq \frac{\pi^2}{4c_K^4}\left(1+\sup_{x\geq 0}(x+x^2)e^{-2\pi x}\right).$$

Hence,

$$\sup_{L>0}\mathbb{E}[|LF(Lz_1)-LF(Lz_2)|^2] \leq \tilde{c}_K|z_2-z_1|^2,$$

where $\tilde{c}_K > 0$ depends only on K. By Kolmogorov's criterion, the laws of $(LF(Lz))_{z\in\mathbb{C}\setminus\mathbb{R}}$ form a tight family for the uniform convergence on compact sets of $\mathbb{C}\setminus\mathbb{R}$.

□

References

[AW13] M. Aizenman and S. Warzel. On the ubiquity of the Cauchy distribution in spectral problems. Arxiv, http://arxiv.org/pdf/1312.7769.pdf, 2013.

[BG06] D. Bump and A. Gamburd. On the averages of characteristic polynomials from classical groups. *Comm. Math. Phys.*, 265(1):227–274, 2006.

[BHNN13] Y. Barhoumi, C.-P. Hughes, J. Najnudel, and A. Nikeghbali. On the number of zeros of linear combinations of indepepndent characteristic polynomials of random unitary matrices. Arxiv, http://arxiv.org/pdf/1301.5144.pdf, 2013.

[BHNY08] P. Bourgade, C.-P. Hughes, A. Nikeghbali, and M. Yor. The characteristic polynomial of a random unitary matrix: a probabilistic approach. *Duke Math. J.*, 145(1):45–69, 2008.

[BNN12] P. Bourgade, J. Najnudel, and A. Nikeghbali. A unitary extension of virtual permutations. *IMRN*, 2013(18):4101–4134, 2012.

[BS06] A. Borodin and E. Strahov. Averages of charactersitic polynomials in random matrix theory. *Comm. Pure. Appl. Math.*, 59(2):161–253, 2006.

[CFZ08] B. Conrey, D.-W. Farmer, and M.-R. Zirnbauer. Autocorrelation of ratios of L-functions. *Commun. Number Theory Phys.*, 2(3):593–636, 2008.

[CL95] O. Costin and J. Lebowitz. Gaussian fluctuations in random matrices. *Phys. Rev. Lett.*, 75(1):69–72, 1995.

[CS07] B. Conrey and N. Snaith. Applications of the L-functions ratios conjectures. *Proc. Lond. Math. Soc.*, 94(3):594–646, 2007.

[CS08] B. Conrey and N. Snaith. Correlation of eigenvalues and riemann zeros. *Commun. Number Theory Phys.*, 2(3):477–536, 2008.

[CS14] B. Conrey and N. Snaith. In support of n-correlation. *Comm. Math. Phys.*, 330(2):639–653, 2014.

[FGLL13] D.-W. Farmer, S. M. Gonek, Y. Lee, and S. J. Lester. Mean values of $\zeta'/\zeta(s)$, correlations of zeros and the distribution of almost primes. *Q. J. Math.*, 64(4):1057–1089, 2013.

[FK14] Y.-V Fyodorov and J.-P. Keating. Freezing transitions and extreme values: random matrix theory and disordered landscapes. *Philos. Trans. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.*, 372(2007), 2014.

[GGM01] D. A. Goldston, S. M. Gonek, and H. L. Montgomery. Mean values of the logarithmic derivative of the Riemann zeta-function with applications to primes in short intervals. *J. Reine Angew. Math.*, 537:105–126, 2001.

[HKO01] C.-P. Hughes, J.-P. Keating, and N. O'Connell. On the Characteristic Polynomial of a Random Unitary Matrix. *Commun. Math. Physics*, 220:429–451, 2001.

[Hug01] C.-P. Hughes. On the Characteristic Polynomial of a Random Unitary Matrix and the Riemann Zeta Function. *PhD Thesis*, 2001.

[KS00] J.-P. Keating and N. Snaith. Random Matrix Theory and $\zeta(1/2+it)$. *Commun. Math. Physics*, 214:57–89, 2000.

[MM13] E.-S. Meckes and M.-W. Meckes. Spectral measures of powers of random matrices. *Electron. Commun. Probab.*, 18:no. 78, 13, 2013.

[MNN13] K. Maples, J. Najnudel, and A. Nikeghbali. Limit operators for circular ensembles. *arxiv*, 2013, arXiv:1304.3757.

[Mon73] H. L. Montgomery. The pair correlation of zeros of the zeta function. In *Analytic number theory (Proc. Sympos. Pure Math., Vol. XXIV, St. Louis Univ., St. Louis, Mo., 1972)*, pages 181–193. Amer. Math. Soc., Providence, R.I., 1973.

[RS96] Zeév Rudnick and Peter Sarnak. Zeros of principal L-functions and random matrix theory. *Duke Math. J.*, 81(2):269–322, 1996. A celebration of John F. Nash, Jr.

[Sos00] Alexander Soshnikov. The central limit theorem for local linear statistics in classical compact groups and related combinatorial identities. *Ann. Probab.*, 28(3):1353–1370, 2000.

[Sos02] A. Soshnikov. Gaussian limit for determinantal random point fields. *Ann. Probab.*, 30(1):171–187, 2002.

[Tit86] E.-C. Titchmarsh. *The theory of the Riemann zeta-function*. The Clarendon Press, Oxford University Press, New York, second edition, 1986. Edited and with a preface by D. R. Heath-Brown.