The Collateral Rule: Theory for the Credit Default Swap Market*

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Abstract

We develop a model of endogenous collateral requirements in the credit default swap (CDS) market. Our model provides an interpretation for the empirical findings of Capponi et al. (2020), according to which extreme tail risk measures have a higher explanatory power for observed collateral requirements than standard value at risk rules. The model predicts that this conservativeness of collateral levels can be explained through disagreement of market participants about the extreme states of the world, in which CDSs pay off and counterparties default.

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1 Introduction

The empirical study of Capponi et al. (2020) shows that collateral requirements in the cleared CDS market are set much more conservatively than the levels implied by standard Value-at-Risk (VaR) rules in over-the-counter (OTC) markets. Standard VaR rules (for example, those used by regulators and OTC market participants, and requiring collateral to cover 99% of 5-day losses) focus on moderate tail risk and are closely related to volatility. Empirically, they do not explain well the collateral levels or the time variation of collateralization rates in the CDS market. Instead, Capponi et al. (2020) show that extreme tail risk measures, such as maximum shortfall and aggregate short CDS notional, have substantially higher power in explaining observed collateral requirements. While in practice collateralization cannot fully eliminate *all* possible counterparty losses, collateral levels are set in the cleared CDS market to cover losses about 8 times larger than those experienced at the 99th percentile; and the variation of collateralization rates over time and in the cross-section is mostly driven by variation in the probability of extreme losses, much larger than the 99th percentile.

These findings broadly lend support to models where the collateral rule is determined endogenously, like in Fostel and Geanakoplos (2015): a key prediction of Fostel and Geanakoplos (2015) is that in a binomial economy (i.e., when there are two states of nature only), any collateral equilibrium is equivalent to one in which there is *no default* - that is, where collateral covers the most extreme losses. While these theories capture the general result that collateral is set based on extreme tail risks, linking the empirical results described above to their theoretical model is not straightforward for two main reasons: (1) the conclusions in Fostel and Geanakoplos (2015) only hold if there are two states of nature; and (2) counterparty defaults, and losses beyond the posted collateral, although rare, do arise in practice.

In this paper, we develop a new model of endogenous collateral that is specifically suited for the CDS market: it features a continuum of states (as opposed to just two), and non-zero default probability in equilibrium. This model can speak directly to the results of Capponi et al. (2020). Trade in this model occurs because of differences in beliefs. Our model builds on Simsek (2013), where belief disagreements are central to asset prices and endogenous margin requirements. Unlike Simsek (2013), who considers standard debt contracts and short selling, our model is specialized to an economy where the only contracts available for trading are state contingent promises (CDSs) backed by risk-free collateral (cash). As a result, the model presented here extends the framework of Simsek (2013) to the CDS market.

In our model, optimists naturally sell insurance (CDS protection) to pessimists, and pessimists require that the sellers post collateral in the form of cash.[1] The amount of cash required to collateralize the CDS contract arises endogenously in the model. We show that the main driver of the collateralization level is not the extent of disagreement between market participants per se, but rather the nature of their disagreement. In particular, when the optimists becomes more optimistic, the level of collateralization falls; but when the pessimists attach a larger weight to the negative tail events, the

[1] Given that CDS contracts have highly asymmetric payoffs, there is little need for collateral from a protection buyer – both in the theory developed here and in the data analysis of Capponi et al. (2020).

level of collateralization rises. In both cases, the level of disagreement between participants widens, but the change in the collateral requirement goes in opposite directions.

Our model is able to generate the high collateral requirements and low default probabilities observed in the data, when the clearinghouse, i.e., the pessimist in our model, places a large weight on extreme tail risks. Moreover, whilst the collateral requirements that arise in such an equilibrium may be viewed as onerous by the clearing members, the optimist in our model, they may nevertheless be insufficient to fully prevent defaults when viewed from the clearinghouse's perspective.[2]

2 A Review of the Empirical Findings of Capponi et al. (2020)

Capponi et al. (2020) provide an empirical analysis of collateral requirements in the cleared CDS market. Their analysis leverages a panel data set that contains, for each financial intermediary acting as a clearing member of the largest CDS clearinghouse, the daily time series of its portfolio positions and corresponding initial margin collateral posted.

The main finding of their analysis is that collateral is set much more conservatively, and, more importantly, in a qualitatively different way, than prescribed by standard Value-at-Risk rules. Specifically, they provide direct evidence that members' collateral exceeds by a large amount the levels implied by a 99% VaR rule with a 5-day margin period, the standard benchmark in OTC markets. In addition, even stricter VaR rules (that is, VaRs based on higher quantile levels) are also rejected by the cross-sectional evidence, because they fail to explain the different collateralization rates observed across participants.

Instead, Capponi et al. (2020) show that, in this market, collateral rules are set based on very extreme tail risks (like maximum shortfall and aggregate short notional) that determine the high level of collateral required, and drive its variation over time.

Their findings suggest that the mechanics of transmission of shocks through collateral may operate in a way that is qualitatively different from those implied by standard VaR. Specifically, they suggest that a special role is played by tail risks and worst-case events, and by participants' beliefs about them. By providing a lens through which to interpret the empirical patterns discussed above, the theoretical model in this paper provides guidance for designing models of the collateral feedback channel in over-the-counter markets with highly skewed payoffs, such as the CDS market.

3 Model Setup

Consider an economy with two periods $t = \{0, 1\}$ and two risk-neutral agents: one optimist and one pessimist. All agents trade in period $t = 0$ and consume in period $t = 1$. Uncertainty is captured by a continuum of states $s \in \mathcal{S} = \left[s^{min}, s^{max}\right]$ realized in period $t = 1$, with s^{min} normalized to zero for simplicity. The pessimist, denoted by $i = 0$, holds prior beliefs over $\mathcal{S}$ given by the distribution F_0

[2] Of course, there exist differences between the theoretical model of collateral presented here and the empirical setting in Capponi et al. (2020). Among them, it is worth noting that they analyze empirically a clearinghouse that determines collateral rules in an oligopolistic setting (given that ICC is the largest clearinghouse with a certain degree of market power), whereas the theoretical models of endogenous collateral assume a competitive market. That said, this theoretical model still provides important insights on the determinants of collateral.

with corresponding density f_0. The optimist, denoted by $i = 1$, has prior beliefs characterized by the distribution F_1 with density f_1. The optimist has a higher expectation than the pessimist on the state in period 1, i.e., $E_1[s] > E_0[s]$. These prior beliefs are common knowledge for all agents.

At the start of period $t = 0$, each agent $i = \{0, 1\}$ is endowed with n_i units of the numeraire consumption good which can be safely stored without depreciation for consumption at period $t = 1$. We assume that the only other asset available is cash, which also yields one unit of the consumption good in period $t = 1$, but - unlike the consumption good - cash can be used as collateral in CDS contracts. At $t = 0$, the entire endowment of cash (normalized to 1) is held by an un-modeled third party, who can sell cash in exchange for the numeraire consumption good at the equilibrium price p. The price of the consumption good is normalized to 1.

The optimist and pessimist have identical (linear) preferences over the consumption good, so trading between the two is driven purely by differences in beliefs. We assume that the only class of financial contracts available for trading is that of simple CDS contracts. Recall that the payoff of a CDS contract is zero if there is no default of the underlying (in our model, when s high), and $1 - R$ in the case of default, where R is the state-contingent recovery rate of the underlying bond. Since the recovery R worsens as the fundamentals of the underlying deteriorate, the payoff of the CDS becomes larger as the state s becomes worse. We assume that the promised payoff of a CDS is $s^{max} - s$. We can think of the case $s = s^{max}$ as the event in which the underlying bond does not default, so that the CDS does not pay anything; $0 < s < s^{max}$ as the intermediate case in which the underlying bond defaults, but there is positive recovery, so that the CDS pays off some amount; and $s = 0$ as the extreme case of zero recovery, where the payoff of the CDS is maximal (and equal to s^{max}).

To enforce payment of the promise, the CDS seller needs to post some amount of collateral. Following the endogenous collateral literature, consider the *family* of CDS contracts $\mathcal{B}^{CDS} = \left\{[s^{max} - s]_{s\in\mathcal{S}}, \gamma\right\}$, each composed of a promise of $(s^{max} - s)$ units of the consumption good in state s at $t = 1$, backed by γ units of cash as collateral. Denote by $q(\gamma)$ the $t = 0$ price of such a CDS contract with collateral level γ. In general, multiple CDS contracts may coexist in equilibrium: they have the same promised payment $(s^{max} - s)$, but different amounts of collateral posted γ, and – as a consequence – trade for different prices $q(\gamma)$. Thus we can index the different CDS contracts in $\mathcal{B}^{CDS}$ by γ.

Since the promised payment on a CDS contract is enforceable only through the potential of seizing the collateral, the *actual delivery* on each contract in state s is given by the minimum of the promised payment and the value of the collateral in that state: $\delta(s, \gamma) := \min\{s^{max} - s, \gamma\}$. In other words, in any state $\tilde{s}$ such that $s^{max} - \tilde{s} > \gamma$, the seller of the CDS contract would default on her promise, and the buyer would only receive the value of the collateral γ.

Denote by μ_i^+, μ_i^-, respectively, agent i's long and short positions on CDS contracts (where a long position means that the agent has purchased the corresponding CDS contract). Let $a_i \in \mathbb{R}_+$ denote agent i's holding of the numeraire consumption good; and $c_i \in \mathbb{R}^+$ be her cash holdings. Then agent i's budget constraint can be written as:

$$a_i + pc_i + \int_{\gamma\in\mathcal{B}^{CDS}} q(\gamma)\, d\mu_i^+ \leq n_i + \int_{\gamma\in\mathcal{B}^{CDS}} q(\gamma)\, d\mu_i^-, \tag{1}$$

where the left hand side represents the total value of the agent's portfolio, comprised of her holding of the numeraire good a_i, the value of her cash holding pc_i, and her long-position in CDS contracts $\int_{\gamma \in \mathcal{B}^{CDS}} q(\gamma)\, d\mu_i^+$. The right hand side represents the total value of the funding available to the agent, comprising of her endowment of the consumption good n_i and the amount she can raise by shorting CDS contracts, $\int_{\gamma \in \mathcal{B}^{CDS}} q(\gamma)\, d\mu_i^-$. If an agent i has a short position on the CDS contracts, then she is also subject to the collateral constraint:

$$\int_{\gamma \in \mathcal{B}^{CDS}} \gamma d\mu_i^- \leq c_i, \tag{2}$$

which means that agent i must have sufficient cash holdings to satisfy the collateral requirements for the CDS contracts sold. In contrast, the purchaser of the CDS contracts (the party who is long) is not subject to any collateral requirements.

The optimization problem for each agent i is given by:

$$\max_{(a_i, c_i, \mu_i^+, \mu_i^-) \in \mathbb{R}_+^4} a_i + c_i + E_i \left[\int_\gamma \min\{s^{max} - s, \gamma\}\, d\mu_i^+ \right] - E_i \left[\int_\gamma \min\{s^{max} - s, \gamma\}\, d\mu_i^- \right] \tag{3}$$

subject to the budget constraint (1) and the collateral constraint (2).

Definition 1. A *collateral equilibrium* is a set of portfolio choices $\left(\hat{a}_i, \hat{c}_i, \hat{\mu}_i^+, \hat{\mu}_i^-\right)_{i \in \{0,1\}}$ and a set of prices $(p \in \mathbb{R}_+, q : \gamma \to \mathbb{R}_+)$ such that the portfolio choices solve the optimization problem (3) of each agent $i \in \{0, 1\}$; and the prices are such that the market for cash clears, i.e., $\sum_{i \in \{0,1\}} \hat{c}_i = 1$, and the CDS markets clear, i.e., $\sum_{i \in \{0,1\}} \mu_i^+ = \sum_{i \in \{0,1\}} \mu_i^-$.

We show that although the entire family of CDS contracts is priced, it is only one contract that is traded in equilibrium. This result is line with the literature on collateral equilibrium (Fostel and Geanakoplos (2015); Simsek (2013)). Furthermore, the collateral level γ of the actively traded CDS contract, and the price of cash p, will be determined endogenously. The equilibrium level of collateral requirement γ will depend on the nature of belief differences between the optimist and the pessimist.

4 Existence and Uniqueness of the Collateral Equilibrium

Following the approach in Simsek (2013), it is possible to show that (under suitable assumptions over initial endowments and beliefs) the collateral equilibrium exists, is unique, and is equivalent to a principal-agent equilibrium where the optimist chooses her cash holdings and the optimal CDS contract to sell, subject to the pessimist's participation constraint. To this end, we impose the following assumptions on initial endowments and prior beliefs, that parallel similar assumptions in Simsek (2013):

Assumption A1: [Restriction on Initial Endowments]

$$n_1 < \frac{E_1[s]}{s^{max}} \tag{4}$$

$$\text{and } n_0 > \frac{E_0[s^{max} - s]}{E_1[s^{max} - s]} - n_1 \tag{5}$$

The first inequality ensures that the optimist's initial endowment is not large enough to purchase the entire supply of cash in the economy with her own resources alone (that is, she will need to raise some more of the numeraire consumption good by selling CDS contracts).[3] The second inequality ensures that the initial endowment of the pessimist (n_0) is large enough that the pessimist will always have some residual consumption after paying for the CDS.[4]

Since the pessimist is risk neutral, this implies that her expected return on any CDS contract purchased must also be equal to 1 in equilibrium. Thus the equilibrium price of a CDS contract with collateral γ must be given by:

$$q(\gamma) = E_0[\min(s^{max} - s, \gamma)]. \tag{6}$$

This pricing equation serves as a convenient characterization of the pessimist's participation constraint.

We can now formulate the optimist's problem as choosing the level of cash holdings c_1, and the CDS contract γ to sell, so as to maximize the expected payoffs, subject to the pessimist's participation constraint of achieving an expected return of one unit on the CDS contract sold:

$$\max_{(c_1,\gamma)\in\mathbb{R}_+^2} c_1 - \frac{c_1}{\gamma} E_1[\min\{s^{max} - s, \gamma\}] \tag{7}$$
$$\text{s.t. } pc_1 = n_1 + \frac{c_1}{\gamma} E_0[\min\{s^{max} - s, \gamma\}]$$

This leads us to define the *principal-agent equilibrium* as follows:

Definition 2. A *principal-agent equilibrium* is a pair of optimist's portfolio choices (c_1^*, γ^*) and price for cash (p^*) such that the optimist's portfolio solves her optimization problem (7), and the market for cash clears: $c_1^* = 1$.

In order to show equivalence between the principal-agent equilibrium outlined here and the collateral equilibrium defined previously, further restrictions on the nature of belief differences are required:

Assumption A2: [Restrictions on Prior Beliefs] The probability densities of the optimist's and the pessimist's beliefs satisfy the monotone likelihood ratio property:

$$\frac{f_1(s_1)}{f_0(s_1)} > \frac{f_1(s_0)}{f_0(s_0)} \quad \text{for every } s_1 > s_0 \tag{8}$$

[3] For a more detailed discussion of this assumption, see Appendix A.

[4] More specifically, this inequality implies that the sum of the endowments needs to be greater than the maximum price cash can take in equilibrium (which we will show is bounded from above by $\left(\frac{E_0[s^{max}-s]}{E_1[s^{max}-s]}\right)$).

Note that this assumption implies: (1) first-order stochastic dominance: $F_1(s) < F_0(s), \forall s \in \left(s^{min}, s^{max}\right)$; (2) monotone hazard rate: $\frac{f_1(s)}{1-F_1(s)} < \frac{f_0}{1-F_1(s)}, \forall s \in \left(s^{min}, s^{max}\right)$; and (3) $\frac{f_0(s)}{F_0(s)} < \frac{f_1(s)}{F_1(s)}$ $\forall s \in \left(s^{min}, s^{max}\right)$ (which in turn implies $\frac{d}{ds}\frac{F_0(s)}{F_1(s)} < 0$).

We can then prove the following Proposition:

Proposition 1. ***[Existence, Uniqueness, and Equivalence of Equilibria]*** *Under Assumptions A1 and A2:*

1. There exists a unique principal-agent equilibrium $(p^*, (c_1^*, \gamma^*))$ s.t. $p^* > 1$.

2. There exists a collateral general equilibrium, $\left(\hat{a}_i, \hat{c}_i, \hat{\mu}_i^+, \hat{\mu}_i^-\right)_{i \in \{0,1\}}$, whereby the optimist sells CDS to the pessimist (i.e. $\hat{\mu}_1^+ = \hat{\mu}_0^- = 0$), and only a single CDS contract is actively traded (i.e. $\hat{\mu}_0^+$ is a measure that puts weight only at one contract $\hat{\gamma} \in \mathcal{B}^{CDS}$). This collateral equilibrium is unique in the sense that the price of cash $\hat{p}$ and the price of the traded CDS contract $q(\hat{\gamma})$ are uniquely determined.

3. The collateral equilibrium and the principal-agent equilibrium are equivalent:

$$\hat{p} = p^*, \quad \hat{c}_1 = c_1^*, \quad \hat{\gamma} = \gamma^*$$
$$\text{and } q(\hat{\gamma}) = E_0\left[\min\left(s^{max} - s, \gamma^*\right)\right]$$

The detailed proof is reported in the Appendix. Intuitively, because under Assumption A1 the pessimist will hold a surplus of the consumption good in equilibrium (he has a larger endowment than what the optimist would want to borrow), he must be indifferent between holding the consumption good (with a sure return of 1) and holding the CDS sold by the optimist. Hence, the optimist effectively holds all the bargaining power when deciding which CDS they should trade, and will only trade in the contract that maximizes the optimist's expected return. Assumption A2 provides the sufficient conditions for there to exist a unique contract γ^* that solves the optimist's principal agent problem.

5 Characterizing the equilibrium

In this section, we show that in equilibrium the optimist will wish to sell CDS contracts to the pessimist (so as to bet on the events she thinks are more likely). But, to do so, the optimist must first obtain more units of the numeraire good from the pessimist by selling CDS contracts, in order to purchase the cash required to collateralize the CDS contracts.[5] Because cash is the only asset that can be used as collateral, its equilibrium price will exceed its fundamental value ($p > 1$), so cash is held exclusively by the optimist in equilibrium.

To formally characterize the equilibrium, first substitute $c_1 = \frac{n_1}{\left(p - \frac{1}{\gamma}E_0[\min\{s^{max} - s, \gamma\}]\right)}$ from the optimist's constraint into his objective function. This reduces the dimension of the problem by one,

[5] This mechanism is analogous to a mortgage contract, where the borrower is raising funds from the lender in order to purchase the collateral (i.e. the house) required to back the mortgage.

and allows us to restrict attention to choosing only the optimal contract γ. The resulting first order condition characterizes the optimal contract choice for a given p:

Proposition 2. *Under assumptions A1 and A2, and fixing a price for cash p, the optimal CDS contract, $\bar{s}$, with respect to the optimist's problem (7) is given by the unique solution to:*

$$p = F_0\left(s^{max} - \bar{s}\right) + \left(1 - F_0\left(s^{max} - \bar{s}\right)\right) \frac{E_0\left[s^{max} - s | s \geq s^{max} - \bar{s}\right]}{E_1\left[s^{max} - s | s \geq s^{max} - \bar{s}\right]} =: p^{opt}\left(\bar{s}\right) \tag{9}$$

Inverting $p^{opt}(\bar{s})$ gives the optimal CDS contract $\bar{s}$ for the optimist, for given price of cash p. This first order condition effectively determines how much collateralization the optimist (seller) chooses for the CDS contract, since the states $s \leq s^{max} - \bar{s}$ are the ones where the optimist is defaulting on the promise of delivering the CDS payment and is instead relinquishing the collateral. Once the equilibrium price of cash p^* is known (see Eq. (11) below), we can use Eq. (9) to derive the equilibrium level of collateral γ^* by setting $p^* = p^{opt}(\gamma^*)$.

Equation (9) also implies that the price of cash is composed of two parts: (a) the pessimist's assessment of the probability of default on the CDS: $(F_0(s^{max} - \bar{s}))$, multiplied by the pessimist's valuation of cash in the default state (1); plus (b) the pessimist's assessment of the probability of the no-default state, multiplied by the value of cash to the optimist in the non-default state (each unit of cash allowed the optimist to borrow $\frac{1}{\bar{s}} E_0\left[s^{max} - s | s \geq s^{max} - \bar{s}\right]$ from the pessimist, with an expected actual delivery of $\frac{1}{\bar{s}} E_1\left[s^{max} - s | s \geq s^{max} - \bar{s}\right]$). Given the differences in beliefs, the optimist expects to deliver less than what the pessimist envisages on the CDS contract: $\frac{E_0[s^{max} - s | s \geq s^{max} - \bar{s}]}{E_1[s^{max} - s | s \geq s^{max} - \bar{s}]} \geq 1$. Hence, $p \geq 1$, i.e., cash generates collateral value. Consistent with the "Asymmetric Disciplining of Optimism" result in Simsek (2013), the pessimist's beliefs are used in assigning weights to the default and non-default states. The belief of the optimist enters only in determining the value of collateral in the non-default state.

Further, we show in the proof of Proposition 2 that $p^{opt}(\bar{s})$ is an increasing function of $\bar{s}$. To see this intuitively, denote the optimist's perceived interest rate on (borrowing through selling) the CDS contract as

$$1 + r_1^{per}\left(\bar{s}\right) := \frac{E_1\left[\min\left\{s^{max} - s, \bar{s}\right\}\right]}{E_0\left[\min\left\{s^{max} - s, \bar{s}\right\}\right]} \tag{10}$$

Under assumption A2, it can be shown that the perceived interest is decreasing in $\bar{s}$, i.e., $\frac{d\left(1 + r_1^{per}(\bar{s})\right)}{d\bar{s}} < 0$. In other words, given the nature of belief differences, offering a CDS contract with higher margin requirements is more attractive for the optimist. Thus the higher the margins $\bar{s}$ posted, the greater the discrepancy between expected deliveries becomes, and the more attractive selling the CDS is to the optimist. Thus, increasing $\bar{s}$ increases the collateral value of cash.

To close the model, we impose the market clearing condition for cash. The budget constraint in equation (7) implies that the optimist's demand for cash is given by:

$$c_1 = \frac{n_1}{p - \frac{1}{\bar{s}} E_0\left[\min\left\{s^{max} - s, \bar{s}\right\}\right]}$$

Since the supply of cash is normalized to 1, we have:

$$p = n_1 + \frac{1}{\bar{s}} E_0 \left[\min\left\{s^{max} - s, \bar{s}\right\}\right] =: p^{mc}\left(\bar{s}\right) \tag{11}$$

We show that $p^{mc}(\bar{s})$ is a decreasing function of $\bar{s}$,[6] with boundary conditions $p^{mc}\left(s^{min}\right) > p^{opt}\left(s^{min}\right)$ and $p^{mc}\left(s^{max}\right) < p^{opt}\left(s^{max}\right)$. Hence, the unique intersection between $p^{opt}(\bar{s})$ and $p^{mc}(\bar{s})$ pins down the equilibrium price for cash p^* and margin requirement γ^*.

Proposition 3. *Under assumptions A1 and A2, there is a unique principal-agent equilibrium* (c_1^*, γ^*, p^*) *characterized by:*

$$\begin{aligned} c_1^* &= 1 \\ p^* &= p^{mc}\left(\gamma^*\right) = p^{opt}\left(\gamma^*\right) \end{aligned}$$

where $p^{opt}(\cdot)$ *and* $p^{mc}(\cdot)$ *are respectively defined in equations (9) and (11).*

6 Comparative statics and illustrative example

We present a simple numerical example to illustrate the collateral equilibrium and perform comparative statics, before presenting the more general results as propositions.

Suppose that the starting endowments of the numeraire consumption good are $n_0 = 3$ and $n_1 = 0.5$ for the pessimist and the optimist respectively. Let the set of states and the prior beliefs be given by: $\mathcal{S} = [0, 1]$ and

$$F_0(s) = s^{\frac{1}{2}} \quad \forall s \in \mathcal{S} \tag{12}$$

$$F_1(s) = s^3 \quad \forall s \in \mathcal{S} \tag{13}$$

Panel A of Figure 1 plots the upward sloping cash price schedule $\left\{p^{opt}(\bar{s})\right\}_{\bar{s}\in[0,s^{max}]}$ (equation (9)), and the downward sloping cash price schedule $\left\{p^{mc}(\bar{s})\right\}_{\bar{s}\in[0,s^{max}]}$ schedule (equation (11)), the intersection of which characterizes the equilibrium price of cash p^* and the equilibrium margin requirement γ^*.

Suppose instead that the optimist becomes less optimistic, such that:

$$\tilde{F}_1(s) = s^{\frac{3}{2}} \quad \forall s \in \mathcal{S}$$

Then, we can see from equation (9) that the decreased optimism of the optimist shifts the $p^{opt}(\bar{s})$ schedule downwards and leaves the $p^{mc}(\bar{s})$ schedule untouched. The resulting equilibrium, as illustrated

[6] $p^{mc}(\bar{s})$ is downward sloping because $\frac{1}{\bar{s}} E_0\left[\min\left\{s^{max} - s, \bar{s}\right\}\right]$ is decreasing in $\bar{s}$. Intuitively, even though a CDS contract with higher margin requirements demands a higher price (i.e. $E_0\left[\min\left\{s^{max} - s, \bar{s}\right\}\right]$ is increasing in $\bar{s}$), the need to post more margins reduces the number of such contracts the optimist can sell with a single unit of cash as collateral. The overall effect is that posting more margins reduces the total amount that the optimist can borrow, $\frac{1}{\bar{s}} E_0\left[\min\left\{s^{max} - s, \bar{s}\right\}\right]$, and this reduction in the purchasing power of optimists reduces the market clearing price of cash.

in panel B of Figure 1, is comprised of a lower price for cash $\tilde{p}^*$ and higher margin requirements $\tilde{\gamma}^*$. We generalize this intuition in the following general proposition:

Proposition 4. ***[Change in optimist's beliefs]*** *Suppose that the optimist becomes "less optimistic" in the sense that her beliefs change from F_1 to $\tilde{F}_1$, s.t. the monotone likelihood ratio condition is satisfied: $\frac{f_1(s_1)}{\tilde{f}_1(s_1)} > \frac{f_1(s_0)}{\tilde{f}_1(s_0)}$ for every $s_0, s_1 \in \left[s^{min}, s^{max}\right]$ and $s_1 > s_0$. Then the equilibrium level of collateral (γ^*) increases, and the equilibrium collateral value (p^*) falls. Conversely, when the optimist becomes "more optimistic", i.e., $\frac{f_1(s_1)}{\tilde{f}_1(s_1)} < \frac{f_1(s_0)}{\tilde{f}_1(s_0)}$ for every $s_0, s_1 \in \left[s^{min}, s^{max}\right]$ and $s_1 > s_0$, then the equilibrium level of collateral falls and the equilibrium collateral value rises.*

While the changes in the optimist's beliefs lead to unambiguous changes in the equilibrium, the same is not true for the pessimist's beliefs. Since the pessimist's beliefs feature in the cash price schedule of both the optimist (i.e, p^{opt}) and the pessimist (i.e., p^{mc}), a given change in the pessimist's beliefs can either increase or decrease equilibrium collateral levels depending on the initial equilibrium. Intuitively, when the pessimist becomes more pessimistic, she is willing to pay more for any given CDS contracts (i.e., $E_0\left[\min\left\{s^{max} - s, \bar{s}\right\}\right]$ increases). This means that the optimist is now both: a) able to raise more funding from selling CDS contracts; and b) willing to pay more for each unit of collateral. The former effect shifts up the market clearing schedule p^{mc}, and the latter effect can push up the p^{opt} schedule. Therefore, whether the equilibrium collateral level increases depends on the interaction of these two effects. Moreover, since the p^{opt} schedule also depends on the pessimist's evaluation of the default probability (i.e., $F_0\left(s^{max} - \bar{s}\right)$)[7], the effect of increased pessimism is not always monotone along $\mathcal{S} = [0, 1]$. For low levels of collateral $\bar{s}$, the protection offered may be deemed insufficient, so the new p^{opt} schedule may be lower than before; but for high levels of collateral, the new p^{opt} schedule might be higher. In short, the p^{opt} schedule may pivot as well as shift in response to a change in pessimist's beliefs, creating further ambiguity in the direction of the equilibrium collateral level γ^*.

Instead, we show that a sufficient (but not necessary) condition for the equilibrium collateral level to increase is for the pessimist to become more concerned about the tail risks (i.e., to place a larger weight on the default states $\{s \in S \ : \ \gamma < s^{max} - s\}$) such that the probability density she attaches to all the non-default states are consistently lower than before, i.e., $\tilde{f}_0(s) < f_0(s) \quad \forall s \in \left[s^{max} - \gamma^*, s^{max}\right]$.

For illustration purposes, consider a change in pessimist's beliefs from F_0 (eqn. 12) to $\tilde{F}_0$:

$$\tilde{F}_0(s) = s^{\frac{1}{4}} \quad \forall s \in \mathcal{S}$$

Then the equilibrium collateral level and collateral value both increase (Panel C of Figure 1). Note that under the specific beliefs outlined here, the equilibrium level of collateral in Panel C is 0.8251, and the optimist believes default will occur with probability 0.0053. This is consistent with the empirical findings of Capponi et al. (2020) that extreme tail risk measures are important in explaining the high collateral levels and low default rates observed in cleared CDS markets. We state this result more formally in the following proposition.

[7] Recall from equation (9) the p^{opt} is the weighted average of 1 and $\frac{E_0[s^{max}-s|s\geq s^{max}-\bar{s}]}{E_1[s^{max}-s|s\geq s^{max}-\bar{s}]}>1$, with weights $F_0\left(s^{max} - \bar{s}\right)$ and $\left(1 - F_0\left(s^{max} - \bar{s}\right)\right)$ respectively.

Proposition 5. ***[Pessimist's concern for tail events]:*** *For any given initial equilibrium (γ^*, p^*), if the pessimist becomes "more concerned about tail events" in the sense that her beliefs changes from F_0 to $\tilde{F}_0$, such that: $\tilde{f}_0(s) < f_0(s) \quad \forall s \in [s^{max} - \gamma^*, s^{max}]$, then the equilibrium collateral level (γ^*) increases.*

Finally, note that the key to high collateral requirements in the model is the nature – rather than the degree – of the belief difference between the two agents. The equilibrium level of collateral increases both when the optimist becomes more pessimistic (which reduces the extent of the disagreement), and when the pessimist becomes more concerned about the tail events (which increases the extent of the disagreement). What the market participants disagree about matters more than how much they disagree per se.

7 Conclusions

We have developed a general equilibrium model of collateralized trading in the CDS market, and shown that under suitable assumptions, a unique collateral equilibrium exists for trading of contracts between optimists and pessimists. In this equilibrium, the pessimist buys CDS contracts from the optimist. Since the seller of a CDS contract is required to post cash as collateral, the optimist buys cash in the market. In equilibrium, the price of cash will be higher than the non-collateralizable value of the numeraire consumption good, reflecting its value as collateral.

Our theoretical analysis shows that, in equilibrium, only one CDS contract with a particular level of collateral γ^* will be actively traded. Default on the CDS obligations will, in general, happen with positive probability. The level of collateral γ^*, the price of the CDS contract $q(\gamma^*)$, and the price of cash p^* all depend on the belief disagreement between the optimist and the pessimist. In particular, the higher the disagreement about the probability of states in which *counterparty defaults* may occur (i.e., the left tail of the distribution), the higher the margin requirements γ^* in equilibrium, and the higher the CDS price $q(\gamma^*)$ the buyer is willing to pay.

The methodological framework presented here can thus generalize the key intuition of Fostel and Geanakoplos (2015), which predicts that observed collateralization levels might be set extremely conservatively in order to cover extreme losses in tail events. But unlike Fostel and Geanakoplos (2015), which restricts attention to economies with two states and the collateral equilibrium without default, the model in this paper allows for a continuum of states, and a low but strictly positive probability of default. Furthermore, we show that as the optimist's beliefs become "less optimistic", or when the pessimist becomes even more concerned about tail events, the equilibrium level of collateralization rises, such that from the perspective of the market participants the probability of default on CDS contracts tend to zero.

The model can therefore explain the particularly high collateral levels observed in the data. Viewed through the lens of this endogenous collateral framework, the empirical results of Capponi et al. (2020) point to disagreement about the states of the world in which CDS sellers default on their obligations to the clearinghouse as the fundamental reason why collateral levels are set so high.

References

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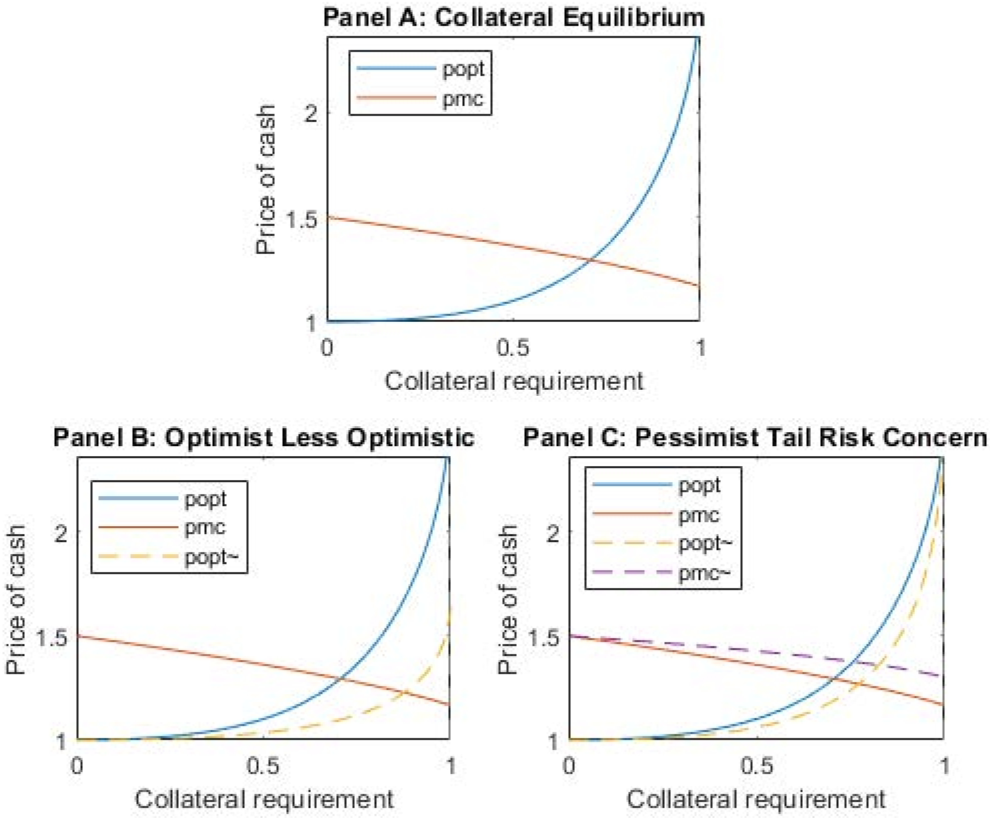
Figure 1: Collateral Equilibrium and Comparative Statics



Panel A illustrates the upward-sloping $p^{opt}(\bar{s})$ schedule (eqn 9, blue line) and the downward-sloping $p^{mc}(\bar{s})$ schedule (eqn 11, red line), the intersection of which gives the collateral equilibrium (γ^*, p^*). The $p^{opt}(\bar{s})$ schedule is increasing in the level of collateralization $(\bar{s})$ because the pessimist is willing to pay a higher price for a better collateralized CDS, which in turn increases the collateral value of cash (p) for the optimist. The $p^{mc}(\bar{s})$ schedule is decreasing in the level of collateralization $(\bar{s})$ because even though each unit of a more collateralized CDS contract is worth more, the scarcity of collateral implies that a fewer number of such contracts can be written and the total amount of funding the optimist can raise from the pessimist is lower. Therefore, in order for the market of collateral (i.e., cash) to clear, its equilibrium price (p) must fall as the level of collateralization increases.

Panel B illustrates the case where the optimist becomes "less optimistic", in the sense that her beliefs changes from $F_1(s)$ to $\tilde{F}_1(s)$ where $\frac{f_1(s_1)}{\tilde{f}_1(s_1)} > \frac{f_1(s_0)}{\tilde{f}_1(s_0)}$ for every $s_0, s_1 \in \left[s^{min}, s^{max}\right]$ and $s_1 > s_0$. When the optimist becomes more pessimistic, the $p^{opt}(\bar{s})$ schedule shifts down to $\tilde{p}^{opt}(\bar{s})$ (yellow dashed line). Intuitively, this is because trading in CDSs becomes less attractive for both as the optimist and pessimist's beliefs converge, thus the optimist's demand for cash (the required collateral) falls (weakly) for every level of collateralization. The resulting equilibrium is one where the price of cash (its collateral value) is lower and the level of collateralization is higher.

Panel C illustrates the case where the pessimist becomes more concerned about the tail risks associated with the default states $\{s \in S \, : \, s < s^{max} - \gamma^*\}$. Specifically, the pessimist's beliefs changes from $F_0(s)$ to $\tilde{F}_0(s)$, where $f_0(s) > \tilde{f}_0(s) \quad \forall s \in [s^{max} - \gamma^*, s^{max}]$. A greater concern for tail events increases the value of highly collateralized CDS contracts, and decreases the value of less collateralized contracts, so the collateral value of cash for the optimist is higher only when it is used to back highly collateralized CDS contracts (the p^{opt} schedule pivots to $\tilde{p}^{opt}$, yellow upward-sloping dashed line). The total amount of the numeraire good the optimist can raise through selling such CDS contracts also rises, pushing up the market clearing price for cash (the p^{mc} schedule shifts up to $\tilde{p}^{mc}$, purple downward-sloping dashed line). The resulting equilibrium is one where the equilibrium level of collateralization (γ^*) is higher (see Proposition 5). Note that under the specific beliefs outlined in the illustrative example section of the main text, the equilibrium level of collateral in Panel C is 0.8251, and the optimist believes default will occur with probability 0.0053.

Appendices

A Additional discussion of the assumptions

We discuss in great detail some of the model assumptions. The inequality (4) states that the optimist's initial endowment is not large enough to purchase the entire supply of cash by issuing only fully collateralized CDS contracts. To see this, note that the value of one unit of cash to the optimist when selling contract γ is given by:

$$1 + \frac{1}{\gamma}\left[E_0\left[\min\left\{s^{max} - s, \gamma\right\}\right] - E_1\left[\min\left\{s^{max} - s, \gamma\right\}\right]\right]$$

The total amount of funding the optimist can raise by selling riskless CDS contracts $\gamma = s^{max}$ (the contract with the maximum collateral) using 1 unit of collateral (i.e. cash) is:

$$\frac{1}{s^{max}} E_0\left[\min\left\{s^{max} - s, s^{max}\right\}\right]$$

We assume that the optimist's endowment is not large enough to finance the purchase of cash using riskless CDS contracts:

$$\begin{aligned}
n_1 + \frac{1}{s^{max}} E_0\left[\min\left\{s^{max} - s, s^{max}\right\}\right] &< 1 + \frac{1}{s^{max}}\left[E_0\left[\min\left\{s^{max} - s, s^{max}\right\}\right] - E_1\left[\min\left\{s^{max} - s, s^{max}\right\}\right]\right] \\
n_1 &< 1 - \frac{1}{s^{max}} E_1\left[\min\left\{s^{max} - s, s^{max}\right\}\right] \\
&= 1 - \frac{1}{s^{max}}\left[s^{max} - E_1\left[s\right]\right] = \frac{E_1\left[s\right]}{s^{max}}.
\end{aligned}$$

Since $n_1 < \frac{E_1[s]}{s^{max}} \leq 1$, this in turn ensures that the optimist cannot simply purchase the entire stock of cash (normalized to 1) without borrowing from the pessimist.

B. Proof of Proposition 2

Returning to the optimist's problem under the principal-agent formulation (7), we can substitute the pessimist's participation constraint into the objective function to reduce the dimension of the choice variable to one:

$$\begin{aligned}
&\max_{\gamma \in \mathbb{R}_+} \frac{n_1}{\left(p - \frac{1}{\gamma} E_0\left[\min\left\{s^{max} - s, \gamma\right\}\right]\right)}\left[1 - \frac{1}{\gamma} E_1\left[\min\left\{s^{max} - s, \gamma\right\}\right]\right] \qquad \text{(A.1)} \\
\Leftrightarrow\ &\max_{\gamma \in \mathbb{R}_+} n_1 R_1^{CDS}\left(\gamma\right),
\end{aligned}$$

where we define $R_1^{CDS}(\gamma)$ as the expected return to the optimist who buys one unit of cash and uses it to back the sale of the CDS contract:

$$R_1^{CDS}(\gamma) := \frac{1 - \frac{1}{\gamma} E_1 \left[\min\left\{s^{max} - s, \gamma\right\}\right]}{p - \frac{1}{\gamma} E_0 \left[\min\left\{s^{max} - s, \gamma\right\}\right]} = \frac{\gamma - E_1 \left[\min\left\{s^{max} - s, \gamma\right\}\right]}{p\gamma - E_0 \left[\min\left\{s^{max} - s, \gamma\right\}\right]} \tag{A.2}$$

(the numerator is the expected $t = 1$ payoff from purchasing the cash whilst simultaneously selling the CDS; the denominator is the down payment required to purchase the cash).

Note that since $E_i \left[\min\left\{s^{max} - s, \gamma\right\}\right] \equiv \gamma F_i\left(s^{max} - \gamma\right) + \int_{s^{max}-\gamma}^{s^{max}} \left(s^{max} - s\right) dF_i(s)$, we have

$$\begin{aligned}
\frac{dE_i \left[\min\left\{s^{max} - s, \gamma\right\}\right]}{d\gamma} &= F_i\left(s^{max} - \gamma\right) - \gamma f_i\left(s^{max} - \gamma\right) - (-1)\left(s^{max} - \left(s^{max} - \gamma\right)\right) f_i\left(s^{max} - \gamma\right) \\
&= F_i\left(s^{max} - \gamma\right).
\end{aligned}$$

Therefore, the derivative of $R_1^{CDS}(\gamma)$ with respect to λ can be expressed as:

$$\begin{aligned}
\frac{dR_1^{CDS}(\gamma)}{d\gamma} &= \frac{\left(1 - F_1\left(s^{max} - \gamma\right)\right)\left(p\gamma - E_0 \left[\min\left\{s^{max} - s, \gamma\right\}\right]\right)}{\left(p\gamma - E_0 \left[\min\left\{s^{max} - s, \gamma\right\}\right]\right)^2} \\
&\quad - \frac{\left(p - F_0\left(s^{max} - \gamma\right)\right)\left(\gamma - E_1 \left[\min\left\{s^{max} - s, \gamma\right\}\right]\right)}{\left(p\gamma - E_0 \left[\min\left\{s^{max} - s, \gamma\right\}\right]\right)^2} \\
&= \frac{1}{\left(p\gamma - E_0 \left[\min\left\{s^{max} - s, \gamma\right\}\right]\right)} \left[\left(1 - F_1\left(s^{max} - \gamma\right)\right) - \left(p - F_0\left(s^{max} - \gamma\right)\right) R_1^{CDS}(\gamma)\right].
\end{aligned}$$

The first order condition (FOC) for the optimist's problem simplifies to:

$$\left(1 - F_1\left(s^{max} - \bar{s}\right)\right) = \left(p - F_0\left(s^{max} - \bar{s}\right)\right) \frac{\bar{s} - E_1 \left[\min\left\{s^{max} - s, \bar{s}\right\}\right]}{p\bar{s} - E_0 \left[\min\left\{s^{max} - s, \bar{s}\right\}\right]}.$$

Re-arranging and simplifying notation (let $F_i := F_i\left(s^{max} - \bar{s}\right)$) gives:

$$\begin{aligned}
p &= \frac{\left(1 - F_1\right) E_0 \left[\min\left\{s^{max} - s, \bar{s}\right\}\right] + F_0 E_1 \left[\min\left\{s^{max} - s, \bar{s}\right\}\right] - \bar{s} F_0}{E_1 \left[\min\left\{s^{max} - s, \bar{s}\right\}\right] - \bar{s} F_1} \\
&= \frac{\left(1 - F_1\right)\left[\bar{s} F_0 + \int_{s^{max}-\bar{s}}^{s^{max}} \left(s^{max} - s\right) f_0(s)\, ds\right] + F_0 \left[\bar{s} F_1 + \int_{s^{max}-\bar{s}}^{s^{max}} \left(s^{max} - s\right) f_1(s)\, ds\right] - \bar{s} F_0}{\left[\bar{s} F_1 + \int_{s^{max}-\bar{s}}^{s^{max}} \left(s^{max} - s\right) f_1(s)\, ds\right] - \bar{s} F_1} \\
&= F_0 + \frac{\left(1 - F_1\right) \int_{s^{max}-\bar{s}}^{s^{max}} \left(s^{max} - s\right) f_0(s)\, ds}{\int_{s^{max}-\bar{s}}^{s^{max}} \left(s^{max} - s\right) f_1(s)\, ds} \\
&= F_0 + \frac{\left(1 - F_1\right)\left(1 - F_0\right) E_0 \left[s^{max} - s | s \geq s^{max} - \bar{s}\right]}{\left(1 - F_1\right) E_1 \left[s^{max} - s | s \geq s^{max} - \bar{s}\right]} \\
&= F_0\left(s^{max} - \bar{s}\right) + \left(1 - F_0\left(s^{max} - \bar{s}\right)\right) \frac{E_0 \left[s^{max} - s | s \geq s^{max} - \bar{s}\right]}{E_1 \left[s^{max} - s | s \geq s^{max} - \bar{s}\right]} \\
&=: p^{opt}(\bar{s}),
\end{aligned}$$

as required.

Lemma 1. *The price of cash (i.e., the collateral) is strictly increasing in the optimist's desired level of collateralization:* $\frac{dp^{opt}(\bar{s})}{d\bar{s}} > 0$ *for* $\bar{s} \in \left(s^{min}, s^{max}\right)$.

Proof. We first evaluate the following derivative

$$
\begin{aligned}
&\frac{d}{d\bar{s}} E_i\left[s^{max} - s | s \geq s^{max} - \bar{s}\right] \\
=&\frac{d}{d\bar{s}}\left(\frac{1}{1 - F_i\left(s^{max} - \bar{s}\right)} \int_{s^{max} - \bar{s}}^{s^{max}} \left(s^{max} - s\right) f_1(s)\, ds\right) \\
=&\frac{\left(\bar{s} f_i\left(s^{max} - \bar{s}\right)\right)\left(1 - F_i\left(s^{max} - \bar{s}\right)\right) - f_i\left(s^{max} - \bar{s}\right) \int_{s^{max} - \bar{s}}^{s^{max}} \left(s^{max} - s\right) f_1(s)\, ds}{\left(1 - F_i\left(s^{max} - \bar{s}\right)\right)^2} \\
=&\frac{\bar{s} f_i\left(s^{max} - \bar{s}\right) - f_i\left(s^{max} - \bar{s}\right) E_i\left[s^{max} - s | s \geq s^{max} - \bar{s}\right]}{1 - F_i\left(s^{max} - \bar{s}\right)} \\
=&\frac{f_i\left(s^{max} - \bar{s}\right)}{1 - F_i\left(s^{max} - \bar{s}\right)}\left(\bar{s} - E_i\left[s^{max} - s | s \geq s^{max} - \bar{s}\right]\right).
\end{aligned}
$$

Differentiating $p^{opt}(\bar{s})$ (and using shorthand notations: $f_i := f_i\left(s^{max} - \bar{s}\right)$, $F_i := F_i\left(s^{max} - \bar{s}\right)$, and $E_i := E_i\left[s^{max} - s | s \geq s^{max} - \bar{s}\right]$) yields:

$$
\begin{aligned}
\frac{dp^{opt}(\bar{s})}{d\bar{s}} &= -f_0 + \frac{\left[f_0 E_0 + (1 - F_0)\left(\frac{f_0}{1 - F_0}\left(\bar{s} - E_0\right)\right)\right]\left[E_1\right] - \left[\frac{f_1}{1 - F_1}\left(\bar{s} - E_1\right)\right]\left[(1 - F_0) E_0\right]}{\left(E_1\right)^2} \\
&= -f_0 + \frac{\bar{s} f_0}{E_1} - \bar{s} f_1 \frac{1 - F_0}{1 - F_1} \frac{E_0}{\left(E_1\right)^2} + f_1 \frac{1 - F_0}{1 - F_1} \frac{E_0}{E_1} \\
&= f_0\left(\frac{\bar{s}}{E_1} - 1\right) + f_1 \frac{1 - F_0}{1 - F_1} \frac{E_0}{E_1}\left(1 - \frac{\bar{s}}{E_1}\right) \\
&= \underbrace{\left(\frac{\bar{s}}{E_1} - 1\right)}_{>0} \underbrace{\left(f_0 - f_1 \frac{\int_{s^{max} - \bar{s}}^{s^{max}} \left(s^{max} - s\right) f_0(s)\, ds}{\int_{s^{max} - \bar{s}}^{s^{max}} \left(s^{max} - s\right) f_1(s)\, ds}\right)}_{>0},
\end{aligned}
$$

where the first inequality follows from $\bar{s} > E_1\left[s^{max} - s | s \geq s^{max} - \bar{s}\right] \quad \forall \bar{s} \in \left(s^{min}, s^{max}\right)$, and the second inequality holds by Assumption A2:

$$
\begin{aligned}
& f_0\left(s^{max} - \bar{s}\right) \int_{s^{max} - \bar{s}}^{s^{max}} \left(s^{max} - s\right) f_1(s)\, ds - f_1 \int_{s^{max} - \bar{s}}^{s^{max}} \left(s^{max} - s\right) f_0(s)\, ds \\
=& \int_{s^{max} - \bar{s}}^{s^{max}} \left(s^{max} - s\right)\left[f_0\left(s^{max} - \bar{s}\right) f_1(s) - f_1\left(s^{max} - \bar{s}\right) f_0(s)\right] ds \\
>& 0 \quad \text{since by A2: } \frac{f_1(s)}{f_0(s)} > \frac{f_1\left(s^{max} - \bar{s}\right)}{f_0\left(s^{max} - \bar{s}\right)} \quad \forall s > \left(s^{max} - s\right).
\end{aligned}
$$

□

Lemma 2. *The optimist's perceived interest rate on the CDS is strictly decreasing in the level of collateral* $\bar{s}$*:* $\frac{d\left(1 + r_1^{per}(\bar{s})\right)}{d\bar{s}} < 0$ *for* $\bar{s} \in \left(s^{min}, s^{max}\right)$.

Proof. Recall from equation (10) that the perceived interest rate is defined as $1+r_1^{per}(\bar{s}) := \frac{E_1[\min\{s^{max}-s,\bar{s}\}]}{E_0[\min\{s^{max}-s,\bar{s}\}]}$. Differentiating with respect to $\bar{s}$ (using the result $\frac{dE_i[\min\{s^{max}-s,\bar{s}\}]}{d\bar{s}} = F_i(s^{max}-\bar{s})$) yields:

$$\frac{d\left(1+r_1^{per}(\bar{s})\right)}{d\bar{s}} = \frac{F_1(s^{max}-\bar{s})\,E_0[\min\{s^{max}-s,\bar{s}\}] - F_0(s^{max}-\bar{s})\,E_1[\min\{s^{max}-s,\bar{s}\}]}{\left(E_0[\min\{s^{max}-s,\bar{s}\}]\right)^2}.$$

Because the denominator in the above expression is always positive, to prove that $\frac{d\left(1+r_1^{per}(\bar{s})\right)}{d\bar{s}} < 0$ is enough to show that for $\bar{s} \in \left(s^{min}, s^{max}\right)$, we have $\frac{E_1[\min\{s^{max}-s,\bar{s}\}]}{E_0[\min\{s^{max}-s,\bar{s}\}]} > \frac{F_1(s^{max}-\bar{s})}{F_0(s^{max}-\bar{s})}$. Proceed as follows:

$$\begin{aligned}
\frac{E_1[\min\{s^{max}-s,\bar{s}\}]}{E_0[\min\{s^{max}-s,\bar{s}\}]} &= \frac{\bar{s}F_1(s^{max}-\bar{s}) + \int_{s^{max}-\bar{s}}^{s^{max}}(s^{max}-s)\,f_1(s)\,ds}{\bar{s}F_0(s^{max}-\bar{s}) + \int_{s^{max}-\bar{s}}^{s^{max}}(s^{max}-s)\,f_0(s)\,ds} \\
&> \frac{\frac{F_1(s^{max}-\bar{s})}{F_0(s^{max}-\bar{s})}\bar{s}F_0(s^{max}-\bar{s}) + \int_{s^{max}-\bar{s}}^{s^{max}}(s^{max}-s)\,f_0(s)\,\frac{F_1(s)}{F_0(s)}ds}{\bar{s}F_0(s^{max}-\bar{s}) + \int_{s^{max}-\bar{s}}^{s^{max}}(s^{max}-s)\,dF_0(s)} \\
&> \frac{\frac{F_1(s^{max}-\bar{s})}{F_0(s^{max}-\bar{s})}\bar{s}F_0(s^{max}-\bar{s}) + \frac{F_1(s^{max}-\bar{s})}{F_0(s^{max}-\bar{s})}\int_{s^{max}-\bar{s}}^{s^{max}}(s^{max}-s)\,f_0(s)\,ds}{\bar{s}F_0(s^{max}-\bar{s}) + \int_{s^{max}-\bar{s}}^{s^{max}}(s^{max}-s)\,dF_0(s)} \\
&= \frac{F_1(s^{max}-\bar{s})}{F_0(s^{max}-\bar{s})},
\end{aligned}$$

where the first inequality follows from Assumption A2 ($f_0(s)\frac{F_1(s)}{F_0(s)} < f_1(s)$), and the second inequality above is implied from the previous inequality ($\frac{d}{ds}\frac{F_0(s)}{F_1(s)} < 0$) . □

C. Proof of Proposition 3

In the main body, we have argued that the principal-agent equilibrium is given by the intersection of the optimality condition $p^{opt}(\bar{s}) = F_0(s^{max}-\bar{s}) + (1-F_0(s^{max}-\bar{s}))\frac{E_0[s^{max}-s|s>s^{max}-\bar{s}]}{E_1[s^{max}-s|s>s^{max}-\bar{s}]}$ derived from the optimist's optimization problem, and the market clearing condition for cash: $p^{mc}(\bar{s}) = n_1 + \frac{1}{\bar{s}}E_0[\min\{s^{max}-s,\bar{s}\}]$. In the proof of Proposition 2, we have also established that $p^{opt}(\bar{s})$ is strictly increasing in $\bar{s}$ over the interval $\left(s^{min}, s^{max}\right)$. It remains to show that (i) $p^{mc}(\bar{s})$ is strictly decreasing in $\bar{s}$ over $\bar{s} \in \left(s^{min}, s^{max}\right)$; and (ii) the boundary conditions are such that an intersection exists: $p^{mc}\left(s^{min}\right) > p^{opt}(s^{max})$ and $p^{mc}(s^{max}) < p^{opt}(s^{max})$.

1. Show that $p^{mc}(\bar{s})$ is strictly decreasing in $\bar{s}$ over $\bar{s} \in \left(s^{min}, s^{max}\right)$.

   Using that $p^{mc}(\bar{s}) = n_1 + \frac{1}{\bar{s}}E_0[\min\{s^{max}-s,\bar{s}\}]$, differentiating with respect to $\bar{s}$ yields:

$$\begin{aligned}
\frac{dp^{mc}(\bar{s})}{d\bar{s}} &= \frac{F_0(s^{max}-\bar{s})\,\bar{s} - E_0[\min\{s^{max}-s,\bar{s}\}]}{\bar{s}^2} \\
&= \frac{\bar{s}F_0(s^{max}-\bar{s}) - \left(\bar{s}F_0(s^{max}-\bar{s}) + \int_{s^{max}-\bar{s}}^{s^{max}} s\,dF_0\right)}{\bar{s}^2} \\
&= -\frac{\int_{s^{max}-\bar{s}}^{s^{max}} s\,dF_0}{\bar{s}^2} < 0 \quad \forall \bar{s} \in \left(s^{min}, s^{max}\right)
\end{aligned}$$

2. Consider the boundary conditions for $p^{mc}(\bar{s})$ and $p^{opt}(\bar{s})$:

   (a) For $\bar{s} = s^{max}$, we have:

$$\begin{aligned}
p^{mc}(s^{max}) &= n_1 + \left(\frac{s^{max} - E_0[s]}{s^{max}}\right) \\
&< \frac{E_1[s]}{s^{max}} + \left(\frac{s^{max} - E_0[s]}{s^{max}}\right) \quad \text{by assumption A1} \\
&= 1 + \frac{E_0[s^{max} - s] - E_1[s^{max} - s]}{s^{max}} \\
&< 1 + \frac{E_0[s^{max} - s] - E_1[s^{max} - s]}{E_1[s^{max} - s]} \\
&= \frac{E_0[s^{max} - s]}{E_1[s^{max} - s]} = p^{opt}(s^{max}).
\end{aligned}$$

   (b) For $\bar{s} = s^{min} \equiv 0$, using L'Hospital's rule, we have:

$$\begin{aligned}
\lim_{\bar{s}\downarrow 0} p^{mc}(\bar{s}) &= n_1 + \lim_{\bar{s}\downarrow 0} \frac{\frac{d}{d\bar{s}}\left(E_0\left[\min\{s^{max} - s, \bar{s}\}\right]\right)}{\frac{d}{d\bar{s}}(\bar{s})} \\
&= n_1 + \lim_{\bar{s}\downarrow 0} \frac{F_0(s^{max} - \bar{s})}{1} = n_1 + 1,
\end{aligned}$$

   and

$$\begin{aligned}
\lim_{\bar{s}\downarrow 0} p^{opt}(\bar{s}) &= \lim_{\bar{s}\downarrow 0} \left[F_0(s^{max} - \bar{s}) + (1 - F_0(s^{max} - \bar{s})) \frac{E_0[s^{max} - s | s \geq s^{max} - \bar{s}]}{E_1[s^{max} - s | s \geq s^{max} - \bar{s}]}\right] \\
&= \lim_{\bar{s}\downarrow 0} F_0(s^{max} - \bar{s}) + \lim_{\bar{s}\downarrow 0} \frac{\int_{s^{max}-\bar{s}}^{s^{max}} (s^{max} - s) f_0(s)\, ds}{\frac{1}{1-F_1(s^{max}-\bar{s})} \int_{s^{max}-\bar{s}}^{s^{max}} (s^{max} - s) f_1(s)\, ds} \\
&= 1 + \lim_{\bar{s}\downarrow 0} (1 - F_1(s^{max} - \bar{s})) \lim_{\bar{s}\downarrow 0} \frac{\frac{d}{d\bar{s}} \int_{s^{max}-\bar{s}}^{s^{max}} (s^{max} - s) f_0(s)\, ds}{\frac{d}{d\bar{s}} \int_{s^{max}-\bar{s}}^{s^{max}} (s^{max} - s) f_1(s)\, ds} \\
&= 1 + 0 \times \lim_{\bar{s}\downarrow 0} \frac{\bar{s} f_0(s^{max} - \bar{s})}{\bar{s} f_1(s^{max} - \bar{s})} = 1 + 0 \times \frac{f_0(s^{max})}{f_1(s^{max})} \\
&= 1,
\end{aligned}$$

   so

$$p^{mc}\left(s^{min}\right) = n_1 + 1 > 1 = p^{opt}\left(s^{min}\right)$$

3. Because $p^{opt}(\gamma)$ and $p^{mc}(\gamma)$ are both continuous functions, by the intermediate value theorem they intersect at some interior point $\gamma^* \in \left(s^{min}, s^{max}\right)$ and $p^* \in [1, \frac{E_0[s^{max}-s]}{E_1[s^{max}-s]})$. Since $p^{mc}(\bar{s})$ is strictly decreasing, and $p^{opt}(\bar{s})$ is strictly increasing, the intersection is unique.

D. Proof of Proposition 1

The existence of a unique Principal-Agent equilibrium has been shown in the proof of propositions 2 and 3. In this section, we show that a collateral general equilibrium as defined in the main body exists and is equivalent to the principal-agent equilibrium.

1. Step 1: Simplifying observations for solving the collateral general equilibrium:

   (a) Without loss of generality, we can show that the equilibrium price of cash satisfies $\hat{p} \in \left[1, 1 + \frac{[E_1[s]-E_0[s]]}{s^{max}}\right)$. Since cash guarantees a safe return of 1, its equilibrium price will never fall below 1. The optimist attach a higher value to cash because, by using cash as collateral when selling CDS contracts indexed by the collateral requirement γ, the optimist also gains the difference in expected delivery:

   $$\frac{1}{\gamma}\left[E_0\left[\min\left\{s^{max}-s,\gamma\right\}\right]-E_1\left[\min\left\{s^{max}-s,\gamma\right\}\right]\right].$$[8]

   This difference in beliefs is fully exploited when the CDS is fully collateralized at $\gamma = s^{max}$. So the maximum price an optimist is willing to pay for cash is equal to:

   $$1+\frac{1}{s^{max}}\left[E_0\left[\min\left\{s^{max}-s,s^{max}\right\}\right]-E_1\left[\min\left\{s^{max}-s,s^{max}\right\}\right]\right]=1+\frac{\left[E_1\left[s\right]-E_0\left[s\right]\right]}{s^{max}},$$

   at which point the optimist would also weakly prefer to hold the illiquid asset instead. $\frac{[E_1[s]-E_0[s]]}{s^{max}}$ can be interpreted as the upper bound on the equilibrium collateral value for cash. For the rest of the proof, We restrict attention to the more interesting cases where $\hat{p}$ is strictly less than $1 + \frac{[E_1[s]-E_0[s]]}{s^{max}}$.

   (b) Note that each agent's optimization problem (3) is linear in the objective variables, thus their value functions will take the form $v_i n_i$, where v_i denotes the return on agent i's endowment n_i.

   (c) Since the agents can always just hold their endowment of the illiquid asset or buy cash in order to sell the fully collateralized CDS contract $\gamma = s^{max}$, we must have:

   $$v_i \geq \max\left\{1, \frac{1-\frac{1}{s^{max}}E_i\left[\min\left\{s^{max}-s,s^{max}\right\}\right]}{p-\frac{1}{s^{max}}E_0\left[\min\left\{s^{max}-s,s^{max}\right\}\right]}\right\} \quad \forall i=\{0,1\}. \tag{A.3}$$

   Above, 1 represents the rate of return on the illiquid asset; and $\left(\frac{1-\frac{1}{s^{max}}E_i[\min\{s^{max}-s,s^{max}\}]}{p-\frac{1}{s^{max}}E_0[\min\{s^{max}-s,s^{max}\}]}\right)$ represents the expected rate of return on buying cash to use as collateral in selling the CDS contract $\gamma = s^{max}$. For the latter, the expected payoff in the second period is 1 (from the

[8] Note the fact that $\frac{1}{\gamma}\left[E_0\left[\min\left\{s^{max}-s,\gamma\right\}\right]-E_1\left[\min\left\{s^{max}-s,\gamma\right\}\right]\right]$ is maximized at $\gamma = s^{max}$ does not imply that the optimist will always want to sell the fully collateralised CDS at any p. The $p^{opt}(\bar{s})$ curve plots the optimal collateral level $\bar{s}$ for the optimist at any given price p. (Derived from the interior solution to $\max_\gamma R$, where $R := \frac{1-\frac{1}{\gamma}E_1[\min\{s^{max}-s,\gamma\}]}{p-\frac{1}{\gamma}E_0[\min\{s^{max}-s,\gamma\}]}$).

cash) minus $\frac{1}{s^{max}} E_i \left[\min\left\{s^{max} - s, s^{max}\right\}\right]$ (the expected delivery on the CDS contract γ). The down-payment on this transaction is $\left(p - \frac{1}{s^{max}} E_0 \left[\min\left\{s^{max} - s, s^{max}\right\}\right]\right)$, where p is the price paid for the unit of cash, and $\frac{1}{s^{max}} E_0 \left[\min\left\{s^{max} - s, s^{max}\right\}\right]$ is the amount raised from selling the CDS to the pessimist who values it the most.

(d) Summing over the two agents' budget constraints (1), and imposing the market clearing conditions in equilibrium (holdings of CDS contracts must cancel out and the sum of total cash holdings is normalized to 1) yields:

$$a_0 + a_1 + \hat{p} \times 1 = n_0 + n_1$$

Recall that by Assumption A1 $n_0 + n_1 > \frac{E_0[s^{max} - s]}{E_1[s^{max} - s]} = 1 + \frac{E_1[s] - E_0[s]}{E_1[s^{max} - s]} > 1 + \frac{[E_1[s] - E_0[s]]}{s^{max}}$, so $\hat{p} \in [1, 1 + \frac{[E_1[s] - E_0[s]]}{s^{max}}]$ implies that $a_0 + a_1 > 0$ (i.e. one or more agents must hold the illiquid asset in equilibrium).

(e) Since $p < 1 + \frac{[E_1[s] - E_0[s]]}{s^{max}}$, we know from equation (A.3) that $v_1 > 1$. Therefore, the pessimist must be the one holding the illiquid asset in the collateral equilibrium, which gives us $v_0 = 1$.

(f) Lastly, without loss of generality, CDS contracts with $\gamma > s^{max}$ will not be used in equilibrium (such contracts tie down a larger amount of collateral without a compensating increase in price).

2. Agents' bid and ask prices for CDS contracts:

(a) An agent's bid price is the price that would make her indifferent between buying the CDS contract and simply receiving the equilibrium value per net worth v_i , so:

$$\begin{aligned} q_0^{bid}(\gamma) &= \frac{E_0\left[\min\left\{s^{max} - s, \gamma\right\}\right]}{v_0} = E_0\left[\min\left\{s^{max} - s, \gamma\right\}\right] \\ > q_1^{bid}(\gamma) &= \frac{E_1\left[\min\left\{s^{max} - s, \gamma\right\}\right]}{v_1} \end{aligned} \tag{A.4}$$

(b) An agent's ask price for the CDS contract γ is the price that would make the trader indifferent between taking a negative position in the CDS γ and simply receiving the equilibrium value v_i, so:

$$\begin{aligned} v_0 &= \frac{1 - \frac{1}{\gamma} E_0\left[\min\left\{s^{max} - s, \gamma\right\}\right]}{p - \frac{1}{\gamma} q_0^{ask}(\gamma)} = 1 \\ v_1 &= \frac{1 - \frac{1}{\gamma} E_1\left[\min\left\{s^{max} - s, \gamma\right\}\right]}{p - \frac{1}{\gamma} q_1^{ask}(\gamma)} > 1 \end{aligned} \tag{A.5}$$

(c) Market clearing for CDS contracts ($\hat{\mu}_1^+ + \hat{\mu}_0^+ = \hat{\mu}_1^- + \hat{\mu}_0^-$) implies:

$$\min_i q_i^{ask}(\gamma) \geq q(\gamma) \geq \max_i q_i^{bid}(\gamma) \quad \forall \gamma$$

(Suppose $\max_i q_i^{bid}(\gamma) > q(\gamma)$, then the buyer wants to buy an infinite amount, but the seller can only sell a finite amount due to the collateral constraint. Moreover, it cannot occur that $q(\gamma) > \max\left\{\min_i q_i^{ask}(\gamma), \max_i q_i^{bid}(\gamma)\right\}$, so we must have $\min_i q_i^{ask}(\gamma) \geq q(\gamma)$).

(d) A CDS contract is traded in positive quantities only if

$$q_i^{ask}(\hat{\gamma}) = q(\hat{\gamma}) = q_j^{bid}(\hat{\gamma}) \quad \text{for some } \{i, j\} = \{0, 1\}.$$

(e) Claim the pessimist's ask prices are always higher than optimist's bid prices:

$$\begin{aligned} q_0^{ask}(\gamma) &= \gamma(p-1) + E_0\left[\min\{s^{max} - s, \gamma\}\right] \quad \text{from eqn (A.5)} \\ &> E_0\left[\min\{s^{max} - s, \gamma\}\right] \quad \text{since } p > 1 \\ &= q_0^{bid}(\gamma) > q_1^{bid}(\gamma) \quad \text{from eqn , (A.4)} \end{aligned}$$

so there are no traded CDS contracts in which the optimist buy and the pessimist sell.

(f) The equilibrium prices of CDS contracts are therefore:

$$\begin{cases} q(\hat{\gamma}) = q_0^{bid}(\hat{\gamma}) = q_1^{ask}(\hat{\gamma}) & \text{for each } \hat{\gamma} \text{ with positive trade} \\ q(\gamma) \in \left[\max_i q^{bid}(\gamma), \min_i q^{ask}(\gamma)\right] & \text{for each } \gamma. \end{cases} \tag{A.6}$$

3. Characterize the equilibrium in CDS markets for a given price for cash $p \in [1, \frac{E_0[s^{max}-s]}{E_1[s^{max}-s]})$.

- The optimist faces quasi-equilibrium prices for all CDS contracts (even those that are not positively traded in equilibrium):

$$\tilde{q}(\gamma) = q_0^{bid}(\gamma) = E_0\left[\min\{s^{max} - s, \gamma\}\right] \tag{A.7}$$

- Given these quasi-equilibrium prices, optimists solve the following optimization problem:

$$v_1 n_1 = \max_{c_1 \geq 0, \mu_1^-} c_1 - \int_{\gamma \in B^{CDS}} \frac{1}{\gamma} E_1\left[\min\{s^{max} - s, \gamma\}\right] d\mu_1^- \tag{A.8}$$

$$\text{s.t. } pc_1 - \int_{\gamma \in B^{CDS}} \frac{1}{\gamma} E_0\left[\min\{s^{max} - s, \gamma\}\right] d\mu_1^- = n_1 \quad \text{[budget constraint]} \tag{A.9}$$

$$\int_{\gamma \in B^{CDS}} \frac{1}{\gamma} d\mu_1^- \leq c_1 \quad \text{[collateral constraint]} \tag{A.10}$$

- Since $v_1 > 1$, the collateral constraint binds in equilibrium. Let λ_1 denote the Lagrangian multiplier for the collateral constraint. v_1 will correspond to the multiplier for the budget constraint.

- The FOCs for c_1 and μ_1^- yields:

$$1+\lambda_1 = v_1 p$$
$$v_1 \frac{1}{\gamma} E_0\left[\min\left\{s^{max}-s,\gamma\right\}\right] \leq \frac{1}{\gamma} E_1\left[\min\left\{s^{max}-s,\gamma\right\}\right] + \lambda_1 \quad \text{with equality only if } \gamma \in supp\left(\mu_1^-\right).$$

- Combining the FOCs yield:

$$\begin{aligned} v_1 p &= 1+\lambda_1 \\ &\geq 1 + v_1 \frac{1}{\gamma} E_0\left[\min\left\{s^{max}-s,\gamma\right\}\right] - \frac{1}{\gamma} E_1\left[\min\left\{s^{max}-s,\gamma\right\}\right] \\ \Rightarrow v_1 &\geq \frac{1-\frac{1}{\gamma}E_1\left[\min\left\{s^{max}-s,\gamma\right\}\right]}{p-\frac{1}{\gamma}E_0\left[\min\left\{s^{max}-s,\gamma\right\}\right]} =: R_1^{CDS}(\gamma) \quad \text{with equality only in } \gamma \in \text{supp}\left(\mu_1^-\right). \end{aligned}$$

- As per the proof of Proposition 2, $R_1^{CDS}(\gamma)$ has a unique maximum characterized by $p^{opt}(\gamma=\tilde{s})$. So, again, the unique collateral-GE is pinned down by the intersection between $p^{opt}(\bar{s})$ and the market clearing condition for cash: $p^{mc}(\tilde{s}) = n_1 + \frac{1}{\bar{s}} E_0\left[\min\left\{s^{max}-s,\tilde{s}\right\}\right]$, s.t. the equilibrium collateral level $\hat{\gamma}$ and the price of cash $\hat{p}$ satisfies:

$$\hat{p} = p^{opt}(\hat{\gamma}) = p^{mc}(\hat{\gamma})$$

4. It follows that the unique general equilibrium is equivalent to the principal-agent equilibrium.

E. Proof of Proposition 4

We prove the case where the optimist becomes 'more pessimistic'. Let $g_1(s) := \frac{f_1(s)}{1-F_1(s^{max}-\bar{s})} \quad \forall s \in \left[s^{max}-\bar{s}, s^{max}\right], \ \forall \bar{s} \in \left[s^{min}, s^{max}\right)$ and $\tilde{g}_1(s) := \frac{\tilde{f}_1(s)}{1-\tilde{F}_1(s^{max}-\bar{s})} \quad \forall s \in \left[s^{max}-\bar{s}, s^{max}\right)$. Then, by Assumption A2, $g(\cdot)$ and $\tilde{g}(\cdot)$ must also satisfy the monotone likelihood ratio condition: $\frac{d}{ds}\left(\frac{f_1}{\tilde{f}_1}\right) > 0 \quad \forall s \in S \Rightarrow \frac{d}{ds}\left(\frac{g_1}{\tilde{g}_1}\right) > 0 \quad \forall s \geq \left(s^{max}-\bar{s}\right)$. This in turn implies $\tilde{E}_1\left[s|s > s^{max}-\bar{s}\right] < E_1\left[s|s \in s^{max}-\bar{s}\right]$ $\forall \bar{s} \in \left(s^{min}, s^{max}\right)$, so the upward sloping p^{opt} curve shifts down when the optimist becomes "more pessimistic". The converse of the proposition follows from the same logic.

F. Proof of Proposition 5

We show that a sufficient (but not necessary) condition for the equilibrium collateral level to increase is for the pessimist to attach sufficiently larger probability weights to the default states.

Let $F_0(s)$ and $\tilde{F}_0(s)$ denote the initial and the new beliefs of the pessimist respectively. For brevity, let $\tilde{E}_0[x] := E_{\tilde{F}_0}[x]$; $\tilde{p}^{opt}(\bar{s}) := \tilde{F}_0\left(s^{max}-\bar{s}\right) + \left(1-\tilde{F}_0\left(s^{max}-\bar{s}\right)\right)\frac{\tilde{E}_0\left[s^{max}-s|s>s^{max}-\bar{s}\right]}{E_1\left[s^{max}-s|s>s^{max}-\bar{s}\right]}$; and $\tilde{p}^{mc}(\bar{s}) := n_1 + \frac{1}{\bar{s}}\tilde{E}_0\left[\min\left\{s^{max}-s,\bar{s}\right\}\right]$. Define, respectively, the initial and the new equilibrium collateral levels

γ^* and γ^{**} implicitly as:

$$p^{opt}(\gamma^*) = p^{mc}(\gamma^*)$$
$$\tilde{p}^{opt}(\gamma^{**}) = \tilde{p}^{mc}(\gamma^{**}).$$

Then, given that $\tilde{p}^{mc}$ is strictly decreasing, p^{opt} is strictly increasing, and the two curves intersects within $\left(s^{min}, s^{max}\right)$ (see proofs for Propositions 2 and 3), we have that $\gamma^{**} > \gamma^*$ iff:

$$\tilde{p}^{mc}(\gamma^*) > \tilde{p}^{opt}(\gamma^*)$$
$$\Leftrightarrow \tilde{p}^{mc}(\gamma^*) - p^{mc}(\gamma^*) > \tilde{p}^{opt}(\gamma^*) - p^{opt}(\gamma^*)$$

With a little bit of algebra, we can show that:

$$\begin{aligned}
&\tilde{p}^{mc}(\gamma^*) - p^{mc}(\gamma^*) \\
=&\frac{1}{\gamma^*}\left[\tilde{E}_0\left[\min\left\{s^{max} - s, \gamma^*\right\}\right] - E_0\left[\min\left\{s^{max} - s, \gamma^*\right\}\right]\right] \\
=&\frac{1}{\gamma^*}\left[\gamma^*\tilde{F}_0 + \int_{s^{max}-\gamma^*}^{s^{max}}\left(s^{max} - s\right)\tilde{f}_0(s)\,ds - \gamma^* F_0 - \int_{s^{max}-\gamma^*}^{s^{max}}\left(s^{max} - s\right)f_0(s)\,ds\right] \\
=&\left(\tilde{F}_0 - F_0\right) + \frac{1}{\gamma^*}\left[\int_{s^{max}-\gamma^*}^{s^{max}}\left(s^{max} - s\right)\tilde{f}_0(s)\,ds - \int_{s^{max}-\gamma^*}^{s^{max}}\left(s^{max} - s\right)f_0(s)\,ds\right],
\end{aligned}$$

and

$$\begin{aligned}
&\tilde{p}^{opt}(\gamma^*) - p^{opt}(\gamma^*) \\
=&\tilde{F}_0 + \left(1 - \tilde{F}_0\right)\frac{\tilde{E}_0\left[s^{max} - s|s > s^{max} - \gamma^*\right]}{E_1\left[s^{max} - s|s > s^{max} - \gamma^*\right]} \\
&\cdots - \left[F_0 + \left(1 - F_0\right)\frac{E_0\left[s^{max} - s|s > s^{max} - \gamma^*\right]}{E_1\left[s^{max} - s|s > s^{max} - \gamma^*\right]}\right] \\
=&\left(\tilde{F}_0 - F_0\right) + \frac{\left[\int_{s^{max}-\gamma^*}^{s^{max}}\left(s^{max} - s\right)\tilde{f}_0(s)\,ds - \int_{s^{max}-\gamma^*}^{s^{max}}\left(s^{max} - s\right)f_0(s)\,ds\right]}{E_1\left[s^{max} - s|s > s^{max} - \gamma^*\right]}.
\end{aligned}$$

Taken together, we obtain:

$$\begin{aligned}
&\left[\tilde{p}^{mc}(\gamma^*) - p^{mc}(\gamma^*)\right] - \left[\tilde{p}^{opt}(\gamma^*) - p^{opt}(\gamma^*)\right] \\
=&\left[\frac{1}{\gamma^*} - \frac{1}{E_1\left[s^{max} - s|s > s^{max} - \gamma^*\right]}\right]\left[\int_{s^{max}-\gamma^*}^{s^{max}}\left(s^{max} - s\right)\tilde{f}_0(s)\,ds - \int_{s^{max}-\gamma^*}^{s^{max}}\left(s^{max} - s\right)f_0(s)\,ds\right] \\
=&\underbrace{\left[\frac{1}{E_1\left[s^{max} - s|s > s^{max} - \gamma^*\right]} - \frac{1}{\gamma^*}\right]}_{>0}\left[\int_{s^{max}-\gamma^*}^{s^{max}}\left(s^{max} - s\right)\left(f_0(s) - \tilde{f}_0(s)\right)ds\right] \qquad \text{(A.11)}
\end{aligned}$$

Therefore, whether the collateral requirement in the new equilibrium increases ($\gamma^{**} > \gamma^*$) depends on whether the second term $\int_{s^{max}-\hat{\gamma}}^{s^{max}}\left(s^{max} - s\right)\left(f_0(s) - \tilde{f}_0(s)\right)ds$ is positive, for which a sufficient condition is that $f_0\left(s\right) > \tilde{f}_0\left(s\right) \quad \forall s \in [s^{max} - \gamma^*, s^{max})$.