CHAPTER 5

THE DIFFERENTIAL EQUATIONS OF FLOW

In Chapter 4, we used the Newton law of conservation of energy and the definition of viscosity to determine the velocity distribution in steady-state, uni-directional flow through a conduit. In this chapter, we shall examine the application of the same laws in the general case of three-dimensional, unsteady state flow. We will do so by developing and solving the *differential equations of flow*. These equations are very useful when detailed information on a flow system is required, such as the velocity, temperature and concentration profiles.

The differential equations of flow are derived by considering a differential volume element of fluid and describing mathematically

a) the conservation of mass of fluid entering and leaving the control volume; the resulting mass balance is called the **equation of continuity.**

b) the conservation of momentum entering and leaving the control volume; this energy balance is called the **equation of motion**.

In describing the momentum of a fluid, we should note that in the case of a solid body, its mass is readily defined and has the dimension, M; the same is true for its momentum which has the dimensions of M L t^{-1}. However, in the case of a fluid, we are dealing with a continuum and the only way to define mass at any given location is in terms of **mass flux**, i.e. mass transport rate per unit cross sectional area through which flow occurs. This quantity is equal to the product of density of the fluid times its velocity (ρu)

and has the dimensions of M t^{-1} L^{-2}. For the same reasons, the momentum of a fluid is expressed in terms of **momentum flux** ($\rho u\ u$), i.e. transport rate of momentum per unit cross sectional area (M t^{-2} L^{-1}).

In three-dimensional flow, the mass flux has three components (x,y,z) and the velocity also three (u_x, u_y, and u_z); therefore, in order to express momentum, we need to consider a total of nine terms.

5.1 THE EQUATION OF CONTINUITY

Let us consider an infinitesimal cubical element $dx.dy.dz$ (Fig. 5.1) in the three-dimensional flow of a fluid in a vessel. The mass conservation of fluid passing through this element is stated as follows:

rate of mass accumulation within element =

= transport rate of mass in - transport rate of mass out

(5.1.1)

The transport rate of mass into the element at location x and through the face $dy.dz$ of the element is equal to

mass flux of fluid in x-direction $\times$ surface area of face dy.dz , i.e.

$$\left(\rho\,u_x\right)_x (dy\ dz) \quad (5.1.2)$$

Similarly, the transport rate of mass out of the element at location $x+dx$ and through the face $dy.dz$ is expressed by

$$\left(\rho\,u_x\right)_{x+dx}(dy\ dz) \equiv \left[\left(\rho\,u_x\right)_x + \frac{\partial \rho\,u_x}{\partial x}dx\right](dy\ dz)\ \ (5.1.3)$$

The only way that mass can be accumulated (or depleted) within the control volume is by a corresponding change in the fluid density. Mathematically, the rate of accumulation is expressed as follows:

$$\frac{\partial \rho}{\partial t}(dx\ dy\ dz) \quad (5.1.4)$$

By substituting from (5.1.2)-(5.1.4) in the conservation equation (5.1.1) and eliminating redundant terms, we obtain

$$\frac{\partial \rho}{\partial t} = -\frac{\partial \rho\, u_x}{\partial x} \quad (5.1.5)$$

In the above derivation, we used partial differentials because we dealt only with the *x*-component of velocity. Of course, the same considerations apply to the *y*- and *z*-components. Therefore, the *general equation of continuity* in three-dimensional flow is expressed as follows:

$$\frac{\partial \rho}{\partial t} = -\left(\frac{\partial \rho\, u_x}{\partial x} + \frac{\partial \rho\, u_y}{\partial y} + \frac{\partial \rho\, u_z}{\partial z}\right) \quad (5.1.6)$$

or in abbreviated vector notation

$$\frac{\partial \rho}{\partial t} = -\Delta.\rho u \quad (5.1.7)$$

where $\boldsymbol{u}$ is the velocity vector and $\nabla.\rho\boldsymbol{u}$ is called the divergence of the quantity $\rho\boldsymbol{u}$.

By differentiating the product terms on the right hand side of (5.1.6) and then collecting all derivatives of density on the left hand side, we obtain:

$$\frac{\partial \rho}{\partial t} + u_x\frac{\partial \rho}{\partial x} + u_y\frac{\partial \rho}{\partial y} + u_z\frac{\partial \rho}{\partial z} = -\rho\left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z}\right) \quad (5.1.8)$$

The left side of (5.1.8) is called the **substantial time derivative** of density. In a physical sense, the substantial time derivative of a quantity designates its time derivative (i.e. rate of change) evaluated along a path that follows the motion of the fluid (*streamline*, Chapter 6). In general terms, the substantial time derivative of a variable w is defined by the following expression:

$$\frac{D(w)}{Dt} = \frac{\partial w}{\partial t} + u_x\frac{\partial w}{\partial x} + u_y\frac{\partial w}{\partial y} + u_z\frac{\partial w}{\partial z} \quad (5.1.9)$$

Therefore, (5.1.8) can be expressed in abbreviated vector notation as follows:

$$\frac{D\rho}{Dt} = -\rho\Delta.u \quad (5.1.10)$$

Equations (5.1.8) and (5.1.10) describe the rate of change of density as observed by someone who is moving along with the fluid. For steady state conditions, there is no mass accumulation and the equation of continuity becomes

$$\frac{\partial \rho u_x}{\partial x} + \frac{\partial \rho u_y}{\partial y} + \frac{\partial \rho u_z}{\partial z} = 0 \quad (5.1.11)$$

and for an incompressible fluid (i.e. for negligible variation in density of fluid)

$$\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z} = 0 \quad (5.1.12)$$

5.2 THE EQUATION OF MOTION

To develop the equation of motion, we start from the Newton law of conservation of energy:

rate of momentum accumulation =
= transport rate of momentum in - transport rate of momentum out +
+ sum of forces acting on element (5.2.1)

In order to express this equation mathematically, we must consider that momentum is transported in and out of the element in two ways:

a) *convective transport of momentum* is by means of the kinetic energy of the fluid mass moving in and out of the six faces of our cubical element. Thus, the convective transport of momentum in the *x*-direction (*x*-momentum) consists of three terms:

$$(\rho u_x)u_x, (\rho u_y)u_x, (\rho u_z)u_x$$

The **net** convective transport of x-momentum through the faces $dy.dz$ of the cubical element $dx.dy.dz$ (Fig. 5.2) is

$$[(\rho u_x)u_x - (\rho u_x)u_{x+dx}](dy\ dz) = -\frac{\partial \rho u_x u_x}{\partial x}(dx\ dy\ dz) \quad (5.2.2)$$

Similarly, the net convective transport of x-momentum through the faces $dx.dz$ and $dx.dy$ of the element is represented by the terms

$$-\frac{\partial \rho u_y u_x}{\partial y}(dx\ dy\ dz) \quad (5.2.3)$$

$$-\frac{\partial \rho u_z u_x}{\partial z}(dx\ dy\ dz) \quad (5.2.4)$$

Of course, there are six more symmetrical terms for convective transport of momentum in the y and z directions of flow.

b) *diffusive transport of momentum* also takes place at the six surfaces of the cubical element (Fig. 5.2) by means of the viscous shear stresses. Thus, the *net* diffusive transport of momentum in the x-direction through the faces $dy.dz$ of the element is

$$[(\tau_{x,x})_x - (\tau_{x,x})_{x+dx}](dy\ dz) = -\frac{\partial \tau_{x,x}}{\partial x}(dx\ dy\ dz) \quad (5.2.5)$$

Similarly, the net diffusive transport of momentum in the x-direction through the faces $dx.dz$ and $dx.dy$ is expressed by the terms

$$-\frac{\partial \tau_{y,x}}{\partial y}(dx\ dy\ dz) \quad (5.2.6)$$

$$-\frac{\partial \tau_{z,x}}{\partial z}(dx\ dy\ dz) \quad (5.2.7)$$

There are six more symmetrical terms for diffusive transport of momentum in the y and z directions of flow.

Let us now consider the forces acting on the control volume of Fig.

5.2. The *net* pressure force acting on the faces *dy.dz* of the element (i.e. the faces normal to the *x*-direction) is

$$\left(P_x - P_{x+dx}\right)(dy\ dz) = -\frac{\partial P}{\partial x}(dx\ dy\ dz) \quad (5.2.8)$$

There are two more symmetrical terms for the pressure on the faces *dx.dz* and *dx.dy* of the element.

The other force acting on the element is gravity; this is a *body force* and is equal to the density of the fluid times the volume of the element (i.e. its mass) times the gravitational acceleration. In the *x*-direction, the gravity force is expressed as follows:

$$\left(\rho\, g_x\right)(dx\ dy\ dz) \quad (5.2.9)$$

where g_x is the component of the gravitational acceleration in the direction *x*. There are two more symmetrical terms for the g_y and the g_z components of gravity.

In developing the equation of continuity (see (5.1.6)), we showed that the rate of accumulation of mass in the control element was equal to the time differential of density times the volume of the element. Similarly, the rate of accumulation of momentum in the *x*-direction is expressed by

$$\frac{\partial \rho\, u_x}{\partial t}(dx\ dy\ dz) \quad (5.2.10)$$

There are two more symmetrical terms for the *y* and *z* directions of momentum.

We now have mathematical expressions for all the terms of (5.2.1). By substitution and elimination of redundant terms, we obtain the following equation for the equation of motion in the *x*-direction of momentum:

$$\frac{\partial \rho\, u_x}{\partial t} = -\left(\frac{\partial \rho\, u_x u_x}{\partial x} + \frac{\partial \rho\, u_y u_x}{\partial y} + \frac{\partial \rho\, u_z u_x}{\partial z} \right) - \left(\frac{\partial \tau_{x,x}}{\partial x} + \frac{\partial \tau_{y,x}}{\partial y} + \frac{\partial \tau_{z,x}}{\partial z} \right) - \frac{\partial P}{\partial x} + \rho\, g$$

(5.2.11)

in the *y*-direction of momentum:

$$\frac{\partial \rho\, u_y}{\partial t} = -\left(\frac{\partial \rho\, u_x u_y}{\partial x} + \frac{\partial \rho\, u_y u_y}{\partial y} + \frac{\partial \rho\, u_z u_y}{\partial z} \right) - \left(\frac{\partial \tau_{x,y}}{\partial x} + \frac{\partial \tau_{y,y}}{\partial y} + \frac{\partial \tau_{z,y}}{\partial z} \right) - \frac{\partial P}{\partial y} + \rho\, g$$

(5.2.12)

and in the *z*-direction of momentum:

The above equations of motion can be expressed more conveniently in vector notation as

$$\frac{\partial \rho \mathbf{u}}{\partial t} = -\Delta.\rho \mathbf{u}\mathbf{u} - \Delta.\tau - \Delta P + \rho \mathbf{g} \qquad (5.2.14)$$

where the bold type indicates vector quantities. It should be noted that if, in addition to gravity, there were other "body" forces acting on the fluid (e.g. an electromagnetic force), their effect would be added to the gravitational components.

The vectorial shear stress in (5.2.14) represents six **shear stresses** (i.e. acting in the direction of flow) and three **normal stresses** (i.e.

compressive or tensile stresses normal to the direction of flow). Mathematically, these stresses are expressed as follows (3):

Shear stresses:

$$\tau_{x,y} = \tau_{y,x} = -\mu\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right), \_\tau_{y,z} = \tau_{z,y} = -\mu\left(\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}\right), \tau_{z,x} = \tau_{x,z} = -\mu\left(\frac{\partial \iota}{\partial z}\right)$$

Normal stresses:

$$\tau_{x,x} = -\mu\left(2\frac{\partial u_x}{\partial x} - \frac{2}{3}\Delta.u\right), \_\tau_{y,y} = -\mu\left(2\frac{\partial u_y}{\partial y} - \frac{2}{3}\Delta.u\right), \_\tau_{z,z} = -\mu\left(2\frac{\partial u_z}{\partial z} - \frac{2}{3}\Delta.u\right)$$

(5.2.15)

For *incompressible* flow (i.e. when changes in density are negligible), we can differentiate each component of convective momentum in the equation of motion ((5.2.11)-(5.2.13)) as follows:

$$\frac{\partial \rho u_x u_y}{\partial x} = \rho u_x \frac{\partial u_y}{\partial x} + \rho u_y \frac{\partial u_x}{\partial x} \qquad (5.2.16)$$

By eliminating the following terms (see (5.1.12))

$$\rho u_y\left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z}\right) = 0$$

and moving the remaining convective momentum terms to the left hand side of (5.2.11)-(5.2.13), we obtain the following simplified equation for the *x*-momentum balance

$$\rho\left(\frac{\partial u_x}{\partial t} + u_x\frac{\partial u_x}{\partial x} + u_y\frac{\partial u_x}{\partial y} + u_z\frac{\partial u_x}{\partial z}\right) = -\left(\frac{\partial \tau_{x,x}}{\partial x} + \frac{\partial \tau_{y,x}}{\partial y} + \frac{\partial \tau_{z,x}}{\partial z}\right) - \frac{\partial P}{\partial x} + \rho g_x \quad (5.$$

The left hand side of (5.2.17) will be recognized by the reader as the *substantial time derivative of velocity*, as defined earlier (see (5.1.8)). Therefore, (5.2.17 for all three dimensions is expressed as follows in vector notation:

$$\rho\frac{Du}{Dt} = -\Delta.\tau - \Delta P + \rho g \quad (5.2.18)$$

Table 5.1 shows the general equations of motion for incompressible flow in the three principal coordinate systems: rectangular, cylindrical and spherical. The angles shown in the last two systems are defined in Fig. 5.3. It can be seen that the complexity of these equations increases from rectangular to spherical coordinates. The reason is obvious: In rectangular coordinates, the cross-sectional area of flow does not change in all three dimensions; in cylindrical coordinates, this area does not change in the *z*-dimension; and in spherical, the cross-sectional area of flow changes in all three dimensions.

TABLE 5.1. General equations of motion

Rectangular coordinates, (x,y,z):

$$\rho\left(\frac{\partial u_x}{\partial t}+u_x\frac{\partial u_x}{\partial x}+u_y\frac{\partial u_x}{\partial y}+u_z\frac{\partial u_x}{\partial z}\right)=-\left(\frac{\partial \tau_{x,x}}{\partial x}+\frac{\partial \tau_{y,x}}{\partial y}+\frac{\partial \tau_{z,x}}{\partial z}\right)-\frac{\partial P}{\partial x}+\rho\, g_x$$

$$\rho\left(\frac{\partial u_y}{\partial t}+u_x\frac{\partial u_y}{\partial x}+u_y\frac{\partial u_y}{\partial y}+u_z\frac{\partial u_y}{\partial z}\right)=-\left(\frac{\partial \tau_{x,y}}{\partial x}+\frac{\partial \tau_{y,y}}{\partial y}+\frac{\partial \tau_{z,y}}{\partial z}\right)-\frac{\partial P}{\partial y}+\rho\, g_y$$

$$\rho\left(\frac{\partial u_z}{\partial t}+u_x\frac{\partial u_z}{\partial x}+u_y\frac{\partial u_z}{\partial y}+u_z\frac{\partial u_z}{\partial z}\right)=-\left(\frac{\partial \tau_{x,z}}{\partial x}+\frac{\partial \tau_{y,z}}{\partial y}+\frac{\partial \tau_{z,z}}{\partial z}\right)-\frac{\partial P}{\partial z}+\rho\, g_z$$

Cylindrical coordinates, $(r,\ \theta,\ z)$:

$$\rho\left(\frac{\partial u_r}{\partial t}+u_r\frac{\partial u_r}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_r}{\partial \theta}-\frac{u_\theta^2}{r}+u_z\frac{\partial u_r}{\partial z}\right)=-\left(\frac{1}{r}\right)-\frac{\partial P}{\partial r}+\rho\, g_r$$

$$\rho\left(\frac{\partial u_\theta}{\partial t}+u_r\frac{\partial u_\theta}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_\theta}{\partial \theta}+\frac{u_r u_\theta}{r}+u_z\frac{\partial u_\theta}{\partial z}\right)=-\left(\frac{1}{r^2}\frac{\partial(r^2\tau_{r,\theta})}{\partial r}+\frac{1}{r}\frac{\partial \tau_{\theta,\theta}}{\partial \theta}+\frac{\partial \tau_{\theta,z}}{\partial z}\right)-\frac{1}{r}\frac{\partial P}{\partial \theta}+\rho\, g_\theta$$

$$\rho\left(\frac{\partial u_z}{\partial t}+u_r\frac{\partial u_z}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_z}{\partial \theta}+u_z\frac{\partial u_z}{\partial z}\right)=-\left(\frac{1}{r}\frac{\partial(r\,\tau_{r,z})}{\partial r}+\frac{1}{r}\frac{\partial \tau_{\theta,z}}{\partial \theta}+\frac{\partial \tau_{z,z}}{\partial z}\right)-\frac{\partial P}{\partial z}+\rho\, g_z$$

Spherical coordinates, $(r,\ \theta,\ \varphi)$:

$$\rho\left(\frac{\partial u_\phi}{\partial t}+u_r\frac{\partial u_\phi}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_\phi}{\partial \theta}+\frac{u_\phi}{r\sin\theta}\frac{\partial u_\phi}{\partial \phi}+\frac{u_\phi u_r}{r}+\frac{u_\theta u_\phi\cot\theta}{r}\right)=$$

$$-\left(\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\tau_{r,r})+\frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\tau_{r\theta}\sin\theta)+\frac{1}{r\sin\theta}\frac{\partial \tau_{r,\phi}}{\partial \phi}-\frac{\tau_{\theta,\theta}+\tau_{\phi,\phi}}{r}\right)-\frac{\partial P}{\partial r}+\rho\, g_r$$

$$\rho\left(\frac{\partial u_\theta}{\partial t}+u_r\frac{\partial u_\theta}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_\theta}{\partial \theta}+\frac{u_\phi}{r\sin\theta}\frac{\partial u_\theta}{\partial \phi}+\frac{u_r u_\theta}{r}-\frac{u_\phi^2\cot\theta}{r}\right)=$$

$$-\left(\frac{1}{r^2}\right)-\frac{1}{r}\frac{\partial P}{\partial \theta}+\rho\, g_\theta$$

For incompressible, Newtonian fluids at constant viscosity, the general equations of motion can be simplified further by replacing the shear stress functions by the Newton law of viscosity. Thus, (5.2.16 becomes

$$\rho\left(\frac{\partial u_x}{\partial t}+u_x\frac{\partial u_x}{\partial x}+u_y\frac{\partial u_x}{\partial y}+u_z\frac{\partial u_x}{\partial z}\right)=\mu\left(\frac{\partial^2 u_x}{\partial x^2}+\frac{\partial^2 u_x}{\partial y^2}+\frac{\partial^2 u_x}{\partial z^2}\right)-\frac{\partial P}{\partial x}+\rho g_x \quad (.$$

This equation, and the symmetric equations for the y- and z-components of momentum, are called the *Navier-Stokes equations* of flow. In vector notation, they are expressed as:

$$\rho\frac{D\mathrm{u}}{Dt}=\mu\Delta^2\mathrm{u}-\Delta P+\rho\mathrm{g} \quad (5.2.19)$$

In cases where the viscous effects can be considered to be negligible, the Navier-Stokes equations may be written as follows:

$$\rho\frac{D\mathrm{u}}{Dt}=-\Delta P+\rho\mathrm{g} \quad (5.2.21)$$

This equation is known as the *Euler equation*. In the case of very slow motion, such as the flow of glass in a melting furnace, the inertia terms in the Navier-Stokes equations may be neglected to yield:

$$\Delta P=\mu\Delta^2\mathrm{u}+\rho\mathrm{g} \quad (5.2.22)$$

In the following chapter, we shall illustrate the manner in which the differential equations of flow can be solved to provide detailed information on the microstructure of flow systems.

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