THE DIRAC EQUATION IN FIVE DIMENSIONS AND CONSEQUENCES

By

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Abstract

Recently a New-York based Nigerian, Professor Gabriel Oyibo, was reported to have developed a unified field theory along Einstein's program under the title "Generalized Mathematical Proof of Einstein's Theory Using A New Group Theory" which was published in a Russian and an American journal. He was reported to have shown that the mass conservation equation of physics is a conformal invariant of the energy conservation equation as well as the momentum conservation equations, and hence to have been led to the conclusion that mass can be transformed not only into energy (as Einstein states) but also into momentum (as "Oyibo states"). In this paper, we re-derive and explore the consequences of this conclusion in the classical relativity theory in which the set of local-time (*t*), position (*x,y,z*) and proper-time ($\tau \equiv s/c$) coordinates of a hypothetical massive particle of *variable* mass (*m*) in Minkowski space-time geometry is treated as the *five homogeneous coordinates* (ct, x, y, z, s) of the particle in a *projective space*, and the five canonically conjugate momenta of the particle, (p_t, p_x, p_y, p_z, mc), define the particle states in the corresponding projective 5-momentum space. The *geometric principle of duality* is applicable in the projective space and, as a consequence, one can define a "unified field" metric tensor $g_{\mu\nu} \equiv \frac{1}{2}\{(\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu) + (\tilde{\alpha}_\mu\tilde{\beta}\tilde{\alpha}_\nu + \tilde{\alpha}_\nu\tilde{\beta}\tilde{\alpha}_\mu)\} \equiv (1+\tilde{\beta})\eta_{\mu\nu}$, generated by Dirac's γ-matrices for spin-$\frac{1}{2}$ particles and Dirac's $(\tilde{\beta}, \tilde{\alpha})$-matrices for integral spin particles, with $\eta_{\mu\nu} = diag(+1,-1,-1,-1)$, such that a new light cone, $c^2(\tau \pm it)^2 - x^2 - y - z^2 = 0$, with *complex time*, is determined by the equation, $\det\left\|((r + ct\tilde{\beta})\eta_{\mu\nu} + s\delta_{\mu\nu})\right\| = 0$, where $r = \sqrt{x^2 + y^2 + z^2}$; and the Oyibo states are characterized by the corresponding mass-energy-momentum relation, $(mc \pm ip_t)^2 - p_x^2 - p_y^2 - p_z^2 = 0$, with *complex energy* where $p = \sqrt{p_x^2 + p_y^2 + p_z^2}$. The appropriate Dirac equation in five dimensions is set up and the consequences of the theory are discussed.

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0. Introduction

Recently, a New York-based Nigerian, Professor Gabriel Oyibo[1], published a paper in Russian and American journals under the title, *Generalized Mathematical Proof of Einstein's Theory Using A New Group Theory.* In the paper, Oyibo showed that the *mass* conservation equation of physics is a conformal invariant of the *energy* conservation equation as well as the 3-*momentum* conservation equations, from which he concluded that mass can be transformed not only into energy (as Einstein states) but also into momentum (as "Oyibo states"). One may compare the five relativistic conservation equations with the equations governing fluid mechanics, namely three for the momenta, one for the pressure or classical energy (corresponding with local-time momentum) and one for density (corresponding with the mass or proper-time momentum). Consequently, one may, as in the theory of variable mass particles developed by Greenberger[2] for describing some properties of classically decaying particles in which the rest mass (*m*) and the proper-time ($\tau = s/c$) are treated as canonically conjugate variables, approach Oyibo's relativistic conservation equations by treating the proper-time of an event as one of the *five homogeneous coordinates* $(ct, x, y, z, c\tau)$ representing the event in a four-dimensional projective space-time geometry. By being homogeneous is meant (see, p.3 of ref.[3]), that under scaling with $\rho \neq 0$, $ct : x : y : z : c\tau = \rho ct : \rho x : \rho y : \rho z : \rho c\tau$ This characteristics of the homogeneous coordinate system makes it possible to visualize the Euclidean geometry of an N+1-dimensional space as the projective geometry of an N-dimensional space, where N=1,2,3,... The main advantage lies, however, in the fact that the homogeneous coordinates system allows the *geometric principle of duality* to be introduced systematically in projective geometry; and we shall show that the new mass-energy-momentum relation envisaged by Oyibo arises precisely from the existence of such a dual principle in the projective 4-dimensional Minkowski space based on the five homogeneous coordinates $(ct, x, y, z, c\tau)$ of a particle and the corresponding momentum space based on the five canonically conjugate momenta of the particle, (p_t, p_x, p_y, p_z, mc).

I shall begin in Sec. 2 by describing (in Sec.2.1) the projective space-time geometry based on the five homogeneous coordinate system introduced in the preceding paragraph and show how the geometric principle of duality can be incorporated in terms of a "unified field" metric tensor, $g_{\mu\nu} \equiv (1+\widetilde{\beta})\eta_{\mu\nu}$, where $\eta_{\mu\nu} \equiv \frac{1}{2}(\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu)$ is the usual *symmetric* metric tensor of Minkowski space, i.e., $(\eta_{\mu\nu}) = diag(+1,-1,-1,-1)$, defined by the Dirac γ-matrices in the usual (1926) Dirac equation[4] for spin-$\frac{1}{2}$ particles and $\widetilde{\beta}$ is the *antisymmetric* matrix introduced by Dirac[5] in 1971 in his positive energy relativistic wave equation for integral spin particles. This unified field metric tensor will be used to construct a new equation for the light cone that will lead (in Sec. 2.2) to the mass-energy-momentum relation envisaged by Oyibo[1]. The dual of the usual Dirac equation in five dimensions will be set up in Sec. 3, and the consequences of the theory will be discussed and the attendant conclusions drawn in Sec.4.

2. Projective Space-Time and Mass-Energy-Momentum Relations

2.0: Introduction

In January 1964, I attended (as a post-graduate student) the lectures on advanced quantum theory delivered by Professor P.A.M. Dirac when he was at the Cavendish Laboratory, University of Cambridge, U.K. In the lectures, Dirac used the symbolism invented by van der Warden which involved unimodular transformations of two complex variables – a familiar concept in complex projective space - in the discussion of the theory of the two-component Pauli spinor (for massless particles) which he subsequently generalized to the four-component spinor (for massive particles) employed in his 1926 [Dirac] equation for an electron. Years later in 1974, following Dirac's statement recorded by his biographer[6] that although he used algebraic methods in presenting his discoveries, his thinking was based on projective geometry, I asked Dirac what he meant by the statement. He replied that projective geometry was convenient for visualizing four dimensional space, because the Euclidean geometry of a 4-dimensional space could be visualized as the projective geometry of a 3-dimensional projective space based on *four* homogeneous coordinates. This statement stuck to my mind since then because of its potential use for geometrizing the Dirac equation itself in line with Einstein's unified field program. Also while at Cambridge, I attended the lectures on Projective and Analytical Geometry by Professor J.A. Todd in the Department of Mathematics at the Cavendish Laboratory. The lectures were based on his book[3] which I had read in 1962 while an undergraduate student of mathematics at the University College, Ibadan, Nigeria. Professor Todd had emphasized at the lectures as well as in his book how homogeneous coordinates could be used to deal with the so-called "points at infinity" (represented by complex numbers) in Euclidean geometry, and how the use of homogeneous coordinates allows one to build into the theory, both the geometric principle of duality and conformal (including scale) invariance. Since then I had believed that, given the fact that the proper time τ (and hence the distance, $s = c\tau$) need not, like the speed of light (c), be a universal constant, one could avoid the singularities encountered in solving the usual Dirac equation by projecting the "points at infinity" onto the finite part of space-time – a process that could be achieved by introducing the *five* coordinates (ct, x, y, z, s) and the five canonically conjugate momenta $(p_t, p_x, p_y, p_z, p_m)$ with $p_m \equiv mc$ as homogeneous coordinates for a projective 4-dimensional space geometry.

Initially, in the late 1960's, I attempted[7] to achieve scale invariance (which demands the elimination of the scale-breaking mass term in the usual Dirac equation) by introducing higher-order derivative structure into the equation. This attempt was built on the Lorentz equation of motion [8] for an accelerated electron in an electromagnetic field which has the form of a linear ordinary differential equation of third order

$$m_e \frac{d^2\mathbf{x}}{dt^2} = e(\mathbf{E} + \frac{\mathbf{v}\times\mathbf{B}}{c}) + \left(\frac{2e^2}{3c^2}\right)\frac{d^3\mathbf{x}}{dt^3}$$

where $\mathbf{v} = d\mathbf{x}/dt$ is the velocity of the electron, and **E, B** are the electric and magnetic fields. Because the coefficient of the third order derivative term involves the ratio of two mass scales, $m_e/m_\mu = 2e^2/3hc$, m_e being the electron mass, m_μ the muon mass and h Planck's constant, such an equation can be used to unify the treatment of the leptons; and the geometric principle of duality comes into play in the study of the qualitative properties of the third order equation via Rene Thom's *catastrophe theory*[8]. The new perspective to be presented in this section is to go to higher space-time dimensions which will allow us to retain the first order (differential) forms of the Dirac equations for spin-$\frac{1}{2}$ and integral-spin particles and, at the same time, implement the geometric principle of duality in relativity physics. The required conceptual analytical tool comes from the 1986 Element Books-published book by John Evans[9] entitled, *Mind, Body and Electromagnetism*, in which Evans tried to explain the "timeless" state of human consciousness through what he called "space-time dualism" using projective geometry which we now proceed to discuss.

2.1 Projective 4-Dimensional Space-Time Geometry

For simplicity, let us begin by considering the representation of an event ("point") in a 2-dimensional (2D) projective space in which the time is treated as one of the three homogeneous coordinates (ct, x, y) by associating the event with the time evolution of a circle S (of radius ct) in the $(x, y)-$plane :

$$x^2 + y^2 = (ct)^2. \tag{2.1}$$

If the radius of the circle is reduced to zero, which defines the "timeless" state $(ct = 0)$, then Eq.(2.1) becomes the point-circle:

$$x^2 + y^2 = 0 \quad \text{or} \quad y = \pm ix. \tag{2.2a}$$

According to Evans[9], this is a natural starting point for introducing the *geometric principle of duality* in two-dimensional projective geometry. As shown in Fig.1a, the point-circle represented by Eq.(2.2a) is both a point (A) and the intersection of two non-physical straight lines, AC and AB. The points, B and C, are called "circular points at infinity" while BC is called the "line at infinity".

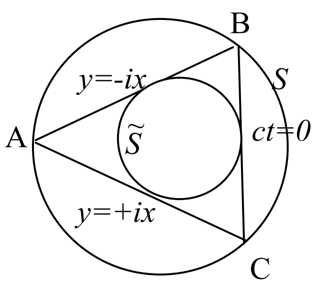

Fig.1a: Circle S and its dual $\tilde{S}$ in Projective 2D Space

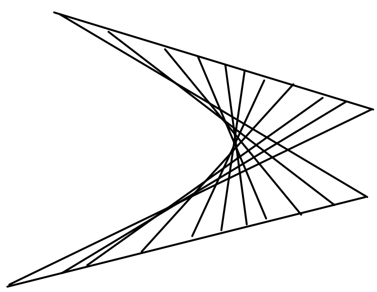

Fig.1b: Conic Locus as envelope of tangent lines

In like manner, the conic locus (Fig. 1b) may be regarded as the locus of the points of meet of corresponding lines of two related pencils or dually, as the envelope of the join of corresponding points of two related ranges on different bases. Note that in order to introduce "infinite" points, the standard analytical technique is to write the point (x, y) as $[(x/ct),(y/ct)]$ and then subsequently set c*t=0*. For this reason, we can talk about the point with *three homogeneous coordinates* (ct, x, y), where, for non-infinite points, *ct* can be regarded as 1, and the infinite points with *complex* coordinates $(0,1,i)$ and $(0,1,-i)$ always lie on the circle described by Eq.(2.1). Moreover, for $ct \neq 0$, we infer by rewriting the equation of the circle in the form, $(ct)^2 - (x-iy)(x+iy) = 0$, that the pair of imaginary lines defines by the linear homogeneous equations

$$\begin{pmatrix} ct & -(x-iy) \\ -(x+iy) & ct \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = 0\,, \tag{2.2b}$$

are associated with the circle, ABC in Fig.1a. In like manner, the rectangular hyperbola $(ct)^2 - (x-y)(x+y) = 0$ is associated with the pair of real (asymptotic) lines defined by the linear homogeneous equations

$$\begin{pmatrix} ct & -(x-y) \\ -(x+y) & ct \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = 0\,. \tag{2.2c}$$

In like manner, in 3D projective geometry based on four homogeneous coordinates, (ct, x, y, z), the point sphere is represented by a generalization of Eq.(2.2a) to the following equation:

$$x^2 + y^2 + z^2 = 0\,, \quad \text{or} \quad ix \pm jy \pm kz = 0\,. \tag{2.3a}$$

This involves the quaternion algebra ($i^2 = j^2 = k^2 = ijk = -1,\ ij = k$) and characterizes the three imaginary planes, ABC, ACD, ABD, as well as the "timeless" plane ($ct = 0$) given by the face BCD of the tetrahedron ABCD, as indicated in Fig. 2.

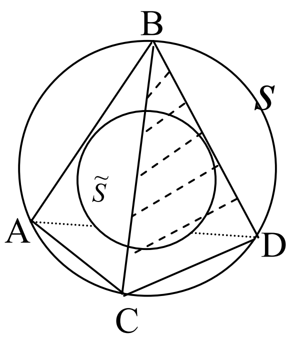

Fig. 2: Projective 3D Space

Moreover, for $ct \neq 0$, by writing the equation of the sphere (S) in the form $(ct-z)(ct+z) - (x-iy)(x+iy) = 0$, we infer that the two families of lines defined by the pairs of linear homogeneous equations

$$\begin{pmatrix} ct-z & -(x+iy) \\ -(x-iy) & ct+z \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = 0 \quad \text{and} \quad \begin{pmatrix} ct-z & -(x-iy) \\ -(x+iy) & ct+z \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0, \qquad (2.3b)$$

where u_s $(s = 1,2)$ and v_s $(s = 1,2)$ are homogeneous parameters, are associated with the sphere ABCD in Fig. 2.

It should be observed that (2.3b) may be rewritten in matrix notation as

$$(ct - \sigma_x x - \sigma_y y - \sigma_z z)u = 0 \text{ and } (ct - \sigma_x x + \sigma_y y - \sigma_z z)v = 0 \qquad (2.3c)$$

where $(\sigma_x, \sigma_y, \sigma_z)$ are the Pauli spin matrices:

$$\sigma_{\mathrm{x}} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_{\mathrm{y}} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_{\mathrm{z}} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

It should also be noted that the second equation in (2.3c) is the complex conjugate of the first equation, which is why the family of lines (called *regulus*) $\{v_s\}$ is said to be *conjugate* to the *regulus* $\{u_s\}$. *Conjugate reguli* have a number of interesting properties (p.273 of ref.[10]) which we shall simply state:

- Any two distinct lines of a regulus $\{u_s\}$ are non-intersecting (skew) lines;
- Any line of $\{u_s\}$ intersects any line of $\{v_s\}$ in one and only one point;
- Any plane defined by the two lines of the conjugate reguli and passing through the point of intersection of the lines on the sphere must be a tangent plane to the sphere at that point, as shown in Fig. 3a.

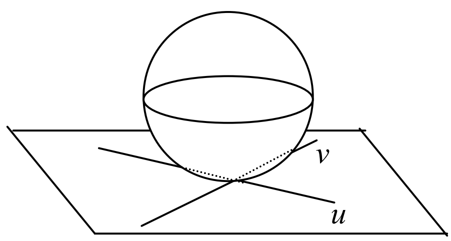

Fig.3a: Generating lines, *u* and *v*, of a sphere in 3D projective space

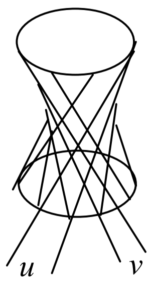

Fig.3b: Generating lines, *u* and *v*, of a ruled quadric in 3D projective space

We have also sketched in Fig. 3b the case of the ruled quadric (having real conjugate generating lines, *u* and *v*) which is represented by Eq.(2.8d) below.

Moreover, if the point of intersection of the two lines of the conjugate reguli on the sphere is (ct, x, y, z), then by solving Eqs.(2.3b) we find the results

$$ct + z = v_1 u_1, ct - z = v_2 u_2, x + iy = v_1 u_2, x - iy = v_2 u_1, \qquad (2.3d)$$

which yield

$$\begin{aligned} ct &= \tfrac{1}{2}(v_1 u_1 + v_2 u_2) \equiv \tfrac{1}{2} v \sigma_0 u; \quad x = \tfrac{1}{2}(v_1 u_2 + v_2 u_1) \equiv \tfrac{1}{2} v \sigma_1 u; \\ y &= -\tfrac{1}{2} i (v_1 u_2 - v_2 u_1) \equiv \tfrac{1}{2} v \sigma_2 u; \quad \mathrm{z} = \tfrac{1}{2}(v_1 u_1 - v_2 u_2) \equiv \tfrac{1}{2} v \sigma_3 u. \end{aligned} \qquad (2.3e)$$

These may be termed the parametric equations of the sphere. The lines $u = \text{constant}$ generate the regulus $\{u\}$ while the lines $v = \text{constant}$ generate the regulus $\{v\}$. In the sense of Eq.(2.3e), the "current", $j_\mu \propto \bar{u}\sigma_\mu u$, may be said to be admissible homogeneous coordinates of the points (x_μ) on the sphere.

Now, returning to the 2D space in Fig. 1a, the principle of duality also states that the lines AB, AC and BC envelope a conic $\widetilde{S}$ (the dual of S with respect to the triangle ABC) whose equation has the general form (p.119 of ref.[10]):

$$ax^2 + 2hxy + by^2 = 0 = ct, \tag{2.4a}$$

and is a hyperbola, a parabola, or an ellipse according as $h^2 - ab$ is greater than, equal to, or less than zero. It includes two pairs of straight lines corresponding to $a = b = 1 = \pm h$, given, when $ct = 0$, by

$$(x \pm y)^2 = 0 \text{ or } x = \pm y. \tag{2.4b}$$

These are the asymptotes of a rectangular hyperbola given (when $ct \neq 0$) by, $(x+y)(-x+y) - (ct)^2 = 0$, and leads to a generalization of (2.4b) to

$$\begin{pmatrix} x+y & -ct \\ -ct & -x+y \end{pmatrix}\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = 0, \text{ or } \{(x + \widetilde{\beta}_0 ct)(\sigma_3)_{\mu\nu} + y\delta_{\mu\nu}\}w_\nu = 0, \tag{2.4c}$$

where

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \widetilde{\beta}_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \widetilde{\beta}\sigma_{30} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}.$$

With the above preliminaries, we proceed to the 4D projective space based on the five homogenous coordinates (ct, x, y, z, s). We may represent an event, as done in special relativity theory, by a rectangular hyperbola in (ct, r)-space:

$$s^2 = (ct)^2 - r^2 \tag{2.5a}$$

where $r = \sqrt{x^2 + y^2 + z^2}$. The dual (as defined in "extended relativity" theory by Recami[11]) is given by

$$s^2 = -\{(ct)^2 - r^2\}. \tag{2.5b}$$

However, the novel ("Oyibo") states which we are after is obtained by generalizing (2.4c) in the form of the four linear homogeneous equations:

$$\begin{pmatrix} r+s & 0 & -ct & 0 \\ 0 & -r+s & 0 & -ct \\ -ct & 0 & -r+s & 0 \\ 0 & ct & 0 & -r+s \end{pmatrix}\begin{pmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{pmatrix} = 0, \tag{2.6a}$$

whose secular equation,

$$[s^2 - r^2 - (ct)^2][(s-r)^2 + (ct)^2] = 0, \tag{2.6b}$$

leads to the two possibilities:

$$s^2 - r^2 - (ct)^2 = 0, \text{ i.e., } s^2 = r^2 + (ct)^2; \tag{2.7a}$$

$$(s-r)^2 + (ct)^2 = 0, \text{ i.e., } s = r \pm ict \text{ or } |s| = \sqrt{r^2 + (ct)^2} \tag{2.7b}$$

Since $r = \sqrt{x^2 + y^2 + z^2}$ and $s = c\tau$, we may rewrite these in the forms:

$$s^2 = (ct)^2 + x^2 + y^2 + z^2, \tag{2.8a}$$

$$c^2(\tau \pm it)^2 - x^2 - y^2 - z^2 = 0 \tag{2.8b}$$

and observe that they are identical when $s = c\tau = 0$. However, by putting

$$c^2(\tau \pm it)^2 = c^2(\tau^2 + t^2)e^{\pm 2i\theta}, \theta = \tan^{-1}(t/\tau)$$

and re-defining $\hat{c}^2 = c^2 e^{\pm 2i\theta}$, Eq.(2.8b) can also be re-written so as to have the same form as Eq.(2.5b), i.e.,

$$(\hat{c}\tau)^2 = -\{(\hat{c}t)^2 - x^2 - y^2 - z^2\}. \tag{2.8c}$$

We recognize Eq.(2.8b) as the light cone with *complex time,* as proposed heuristically by Jannussis and co-workers[12] while Eq.(2.8c) is the dual of the Minkowski space defined in "extended relativity" by Recami and co-workers[11], (or *isodual* space in Santilli's terminology [13]) with complex speed of light ($\hat{c}$). Note that the projection of Eq.(2.8c) on the plane $z = 0$, takes the form

$$(\hat{c}\tau)^2 + (\hat{c}t)^2 - x^2 - y^2 = 0, \tag{2.8d}$$

which is the ruled quadric in $(\hat{c}\tau, \hat{c}t, x, y)$-space shown in Fig. 3b characterized by real generating lines.

At this juncture, two significant observations can be made. The first is that Eq.(2.6a) can be rewritten in the matrix form

$$[(r + ct\widetilde{\beta})\eta_{\mu\nu} + s\delta_{\mu\nu}]w_\nu = 0 \tag{2.9}$$

which involves the metric tensors,

$$(\eta_{\mu\nu}) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \quad \widetilde{\beta}\eta_{\mu\nu} = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}. \tag{2.10}$$

The sum of the two metric tensors defines a "unified field" metric tensor

$$g_{\mu\nu} \equiv \tfrac{1}{2}\{(\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu) + (\widetilde{\alpha}_\mu\widetilde{\beta}\widetilde{\alpha}_\nu + \widetilde{\alpha}_\nu\widetilde{\beta}\widetilde{\alpha}_\mu)\} \equiv (1 + \widetilde{\beta})\eta_{\mu\nu}. \tag{2.11}$$

which consists of the Dirac's γ-matrices for spin-$\frac{1}{2}$ particles [4]

$$\gamma_0 = \beta, \ \ \gamma_r = \beta\alpha_r (r = 1,2,3); \text{with} \ \ \gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2\eta_{\mu\nu}, \tag{2.12}$$

and the (dual) Dirac's $(\widetilde{\beta}, \widetilde{\alpha})-$matrices for integral spin particles [5]

$$(\widetilde{\alpha}_\mu\widetilde{\beta}\widetilde{\alpha}_\nu + \widetilde{\alpha}_\nu\widetilde{\beta}\widetilde{\alpha}_\mu) = 2\widetilde{\beta}\eta_{\mu\nu}; \ \text{with} \ \ \widetilde{\gamma}_\mu\widetilde{\gamma}_\nu + \widetilde{\gamma}_\nu\widetilde{\gamma}_\mu = -2\eta_{\mu\nu}, \ \widetilde{\gamma}_\mu = \widetilde{\beta}\widetilde{\alpha}_\mu \tag{2.13}$$

where,

$$\beta = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \alpha_1 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$$\alpha_2 = \begin{bmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{bmatrix}, \alpha_3 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}.$$

$$\tilde{\alpha}_0 = I, \tag{2.14}$$

$$\tilde{\beta} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}, \tilde{\alpha}_1 = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

$$\tilde{\alpha}_2 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \tilde{\alpha}_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}..$$

The second observation is that the equation of the conventional light cone given by

$$(ct)^2 - x^2 - y^2 - z^2 = 0\,, \tag{2.15a}$$

is a Minkowski space characterized by the O(3,1) Lorentz group while the equation of the *dual* light cone given by the $s \equiv c\tau = 0$ limit of Eq.(2.8a) or (2.8b)

$$(ct)^2 + x^2 + y^2 + z^2 = 0\,, \tag{2.15b}$$

is an Euclidean space characterized by an O(4) Lorentz group. Following Santilli[13], we may refer to Einstein's special relativity principle founded on (2.15a) as the "exterior" problem and that founded on (2.15b), as the "interior" problem. The significance of this dichotomy is that, whereas Einstein's principle of relativity for point-like objects is founded on the invariant theory of (2.15a) under the O(3,1) Lorentz group of transformations for the "exterior" problem, the appropriate principle of relativity for extended deformable objects which exhibit "exterior" and "interior" dichotomy ought to be founded on the invariant and covariant theories[3] of the *pair* of quadric surfaces represented by Eqs.(2.15a) and (2.15b). Explicitly, if we rewrite Eq.(2.15a)and (2.15b) in the respective forms:

$$\tilde{S} \equiv (ct)^2 - x^2 - y^2 - z^2 = \lambda_0 (ct)^2 + \lambda_1 x^2 + \lambda_2 y^2 + \lambda_3 z^2\,, \tag{2.16a}$$

$$S \equiv (ct)^2 + x^2 + y^2 + z^2, \tag{2.16b}$$

where

$$\lambda_0 = +1, \lambda_1 = \lambda_2 = \lambda_3 = -1\,, \tag{2.17}$$

then the two covariants of $\tilde{S}$ and S which are of physical interest are (from p. 249 of ref.[3]) given by:

$$\begin{aligned} F_1 &\equiv \lambda_0(\lambda_1 + \lambda_2 + \lambda_3)(ct)^2 + \lambda_1(\lambda_0 + \lambda_2 + \lambda_3)x^2 + \lambda_2(\lambda_0 + \lambda_1 + \lambda_3)y^2 \\ &\quad + \lambda_3(\lambda_0 + \lambda_1 + \lambda_2)z^2 \equiv -3(ct)^2 + x^2 + y^2 + z^2, \end{aligned} \tag{2.18a}$$

$$\begin{aligned} F_2 &\equiv \lambda_0(\lambda_2\lambda_3 + \lambda_3\lambda_1 + \lambda_1\lambda_2)x_0^2 + \lambda_1(\lambda_2\lambda_3 + \lambda_3\lambda_0 + \lambda_0\lambda_2)x_1^2 \\ &\quad + \lambda_2(\lambda_1\lambda_3 + \lambda_3\lambda_0 + \lambda_0\lambda_1)x_2^2 + \lambda_3(\lambda_1\lambda_2 + \lambda_2\lambda_0 + \lambda_0\lambda_1)x_3^2 \\ &\equiv 3(ct)^2 + x^2 + y^2 + z^2. \end{aligned} \tag{2.18b}$$

The physical interest of the covariants lies in the fact that the Nielson-Picek [14] modification of the Minkowski metric inside mesons as extended objects,

$$(\bar{g}_{\mu\nu}) = diag(1+\alpha, -1-\tfrac{\alpha}{3}, -1-\tfrac{\alpha}{3}, -1-\tfrac{\alpha}{3}) \qquad (2.19a)$$

is generated by the linear system of quadrics, $\widetilde{S}$ and F_1, given by:

$$\widetilde{S} - \tfrac{\alpha}{3}F_1 \equiv \bar{g}_{\mu\nu}x^{\mu}x^{\nu} \equiv (1+\alpha)(ct)^2 - (1+\tfrac{\alpha}{3})x^2 - (1+\tfrac{\alpha}{3})y^2 - (1+\tfrac{\alpha}{3})z^2. \qquad (2.19b)$$

Finally, we observe that the "unified field" metric $g_{\mu\nu} = (1+\widetilde{\beta})\eta_{\mu\nu}$ defined in Eq.(2.11) has the significance in 3D projective space that

$$x^{\mu}(1+\widetilde{\beta})\eta_{\mu\nu}x^{\nu} \equiv (ct)^2 - x^2 - y^2 - z^2 - 2cty = s^2 \qquad (2.20a)$$

$$x^{\mu}(1-\widetilde{\beta})\eta_{\mu\nu}x^{\nu} \equiv (ct)^2 - x^2 - y^2 - z^2 + 2cty = s^2 \qquad (2.20b)$$

are equivalent (when $ct = s$) to

$$[x^2 + (y-ct)^2 + z^2 - (ct)^2][x^2 + (y+ct)^2 + z^2 - (ct)^2] = 0 \qquad (2.20c)$$

which represents two spheres of radii ct in contact along the y-axis, as shown in Fig.4.

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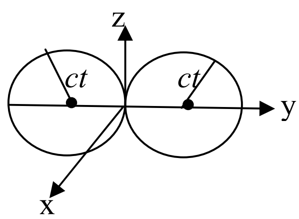

Fig. 4: Contact Interaction of two spheres in 3D space

This configuration represents the unset of contact interaction which ultimately leads to the penetration of one sphere into the other sphere, as depicted earlier in Fig.2. Such a visual image of the contact interaction of two electrons (of opposite spins) leading to the formation of Cooper pairs in superconductors was introduced recently by Animalu and Gill[15].

2.2. Mass-Energy-Momentum Relations

The corresponding 5-momentum space versions of the relations derived in Sec. 2.1 follow in terms of the *five* homogeneous momentum coordinates, $(p_t, p_x, p_y, p_z, p_m) \equiv (p_0, p_1, p_2, p_3, mc)$. In particular, we may represent the usual Einstein's mass-energy-momentum relation as a hyperbola in $(p_0, p)-$space

$$p_0^2 - p^2 = (mc)^2, \text{ where } p = \sqrt{p_1^2 + p_2^2 + p_3^2}\,. \qquad (2.21)$$

and the dual mass-energy-momentum relations by (cf, Eq.(2.6a)):

$$\det\left\|(p + \widetilde{\beta}p_0)\eta_{\mu\nu} + mc\delta_{\mu\nu}\right\| \equiv$$

$$\begin{vmatrix} p+mc & 0 & -p_0 & 0 \\ 0 & -p+mc & 0 & -p_0 \\ -p_0 & 0 & -p+mc & 0 \\ 0 & p_0 & 0 & -p+mc \end{vmatrix} \tag{2.22}$$
$$\equiv [(mc)^2 - p^2 - p_0^2][(mc-p)^2 + p_0^2] = 0.$$
which leads to the two possibilities,

$$(mc)^2 - p^2 - p_0^2 = 0, \text{ i.e., } mc = \pm\sqrt{p^2 + p_0^2}, \tag{2.23a}$$
$$(mc-p)^2 + p_0^2 = 0, \text{ i.e., } mc = p \pm ip_0 \text{ or } |mc| = \sqrt{p^2 + p_0^2}\ . \tag{2.23b}$$

Since $p = \sqrt{p_1^2 + p_2^2 + p_3^2}$, we can rewrite these in the respective standard forms

$$(mc)^2 - p_0^2 - p_1^2 - p_2^2 - p_3^2 = 0, \tag{2.24a}$$
$$(mc \pm ip_0)^2 - p_1^2 - p_2^2 - p_3^2 = 0, \tag{2.24b}$$

which are the new mass-energy-momentum relations arising from the existence of the geometric principle of duality in 5-momentum space.

On one hand, we can re-interpret $\hat{E} \equiv (mc \pm ip_0)c$ in Eq.(2.24b) as the complex energy of a massive particle traveling at the speed of light(c) in a conformal invariant space-time: this concurs with Oyibo's conclusion that mass conservation equation is a conformal invariant of energy conservation equation. On the other hand, by using

$$(mc \pm ip_0)^2 = [(mc)^2 + p_0^2]e^{\pm 2i\varphi} \equiv (m\hat{c})^2 + \hat{p}_0^2 \tag{2.25}$$

where $\varphi = \tan^{-1}(p_0 / mc)$, to rewrite Eq.(2.24b) in the form (cf Eq.(2.8c)),

$$(m\hat{c})^2 = -\{\hat{p}_0^2 - p_1^2 - p_2^2 - p_3^2\}, \tag{2.26}$$

we obtain mass-energy-momentum relation of "extended relativity" theory[11], in which $\hat{p}_0 = e^{\pm i\varphi} p_0 \equiv E / \hat{c}$, where $\hat{c} \equiv ce^{\pm i\varphi}$ is a complex velocity of light. Such a complex velocity of light implies that the underlying medium is dispersive.

It should be noted that the difference between Eqs.(2.21) and (2.23a) is mere *interchange* of mass (mc) and energy (p_0), which is possible because the two are treated on equal footing in 5-momentum space. There is also a striking analogy between Eq.(2.21) and the corresponding relation in superconductivity theory[15] in which mc is the analog of the superconducting energy gap, Δ.

3. Dirac Equation in Five Dimensions

It is apparent from the existence of the pair of distinct mass-energy-momentum relations defined by Eqs.(2.21) and (2.24b) or (2.26) which are related by the geometric principle of duality characterized by *interchange of mass and energy* in 5-momentum space that each of the two mass-energy-momentum relations would lead to a separate Dirac equation. We shall review the features of the two types of Dirac equation associated with Eq.(2.21) in Sec. 3.1 before turning to the general case of Dirac equation in projective 4D space (equivalent to Euclidean 5D space) in Sec. 3.2

3.1. Review of Dirac Equations for Fermions and Bosons

As is well-known[4], the usual Dirac equation for a spin-$\frac{1}{2}$ particle (fermion) has the form

$$(p_0 - \alpha_1 p_1 - \alpha_2 p_2 - \alpha_3 p_3 - \beta mc)\psi = 0\,, \tag{3.1a}$$

or

$$(\gamma_0 p_0 - \gamma_1 p_1 - \gamma_2 p_2 - \gamma_3 p_3 - mc)\psi = 0\,, \tag{3.1b}$$

where $p_0 = -ih\partial/c\partial t$, $p_j = -ih\partial/\partial x_j$ (j = 1,2,3), $\gamma_0 = \beta,\ \gamma_r = \beta\alpha_r (r = 1,2,3)$, and hence, by virtue of Eq.(2.14),

$$\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2\eta_{\mu\nu}\,. \tag{3.1c}$$

The main feature of the Dirac equation for fermions is that it is symmetrical between positive and negative energy solutions, with a finite transition probability from one type of solution to the other. The difficulty with the negative-energy solutions is that there is an infinite sea of such states. For this reason, anti-particles associated with "holes" in the negative energy sea of solutions are presumed to exist in the *same space-time* as the positive energy solutions. This was the situation up to 1971 when Dirac[5] proposed a relativistic wave equation, allowing only positive values for energy, but having integral values for the spin, which we next proceed to review.

In 1971, Dirac [5] proposed a new relativistic wave equation of the form

$$(\frac{\partial}{\partial x_0} + \tilde{\alpha}_1 \frac{\partial}{\partial x_1} + \tilde{\alpha}_2 \frac{\partial}{\partial x_2} + \tilde{\alpha}_3 \frac{\partial}{\partial x_3} + \tilde{\beta} mc)\tilde{\psi} = 0, \quad (\tilde{\psi} = q\psi) \tag{3.2a}$$

or, with $\partial^\mu \equiv \partial/\partial x_\mu$ and $(x_\mu) \equiv (ct, x, y, z)$,

$$(\tilde{\alpha}_\mu \partial^\mu + \tilde{\beta} mc)\tilde{\psi} = 0, \tag{3.2b}$$

where the $(\tilde{\beta}, \tilde{\alpha}_1, \tilde{\alpha}_2, \tilde{\alpha}_3)$-matrices are as defined at Eq.(2.14) above and $\tilde{\psi} (= q\psi)$ is a column matrix having four elements $(q_1\psi, q_2\psi, q_3\psi, q_4\psi)$, in which ψ has only one component and is a function of q_1 and q_2 as well as of the four x's. The q_a $(a = 1,2,3,4)$ characterize the dynamical variables (coordinates and conjugate momenta) of two harmonic oscillators, (q_1, p_1) for one oscillator and (q_2, p_2) for the other oscillator, where for economy of letters, one has put $p_1 = q_3$ and $p_2 = q_4$, so that their commutation relations (if one takes h =1) are related to the elements of the matrix $\tilde{\beta}$ as follows:

$$q_a q_b - q_b q_a = i\tilde{\beta}_{ab}\,. \tag{3.2c}$$

There are four equations in (3.2b) corresponding to the four values of a in

$$(\tilde{\alpha}_\mu \partial^\mu + \tilde{\beta} mc)_{ab} q_b \psi = 0, \tag{3.2d}$$

but, as pointed out by Dirac (see, eqs.(2.2)-2.5) of ref.5) only three of them are independent. Accordingly, by multiplying Eq.(3.2a) by $\tilde{\beta}$ on the left we get

$$(\gamma_\mu \partial^\mu - mc)\tilde{\psi} = 0 \qquad \text{with} \qquad \tilde{\gamma}_\mu \tilde{\gamma}_\nu + \tilde{\gamma}_\nu \tilde{\gamma}_\mu) = -\eta_{\mu\nu}\,. \tag{3.2e}$$

The transition from the usual Dirac equation for fermions into the Dirac equation for bosons may, therefore, be characterized by the dual transformation:

$$\eta_{\mu\nu} \equiv \tfrac{1}{2}(\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu) \rightarrow \widetilde{\eta}_{\mu\nu} \equiv \tfrac{1}{2}(\widetilde{\gamma}_\mu\widetilde{\gamma}_\nu + \widetilde{\gamma}_\nu\widetilde{\gamma}_\mu) = -\eta_{\mu\nu}. \quad (3.3)$$

3.2. Dirac Equation in Five Dimensions

An important feature of the transformation ($\eta_{\mu\nu} \rightarrow \widetilde{\beta}\eta_{\mu\nu}$) which is evident in Eq.(2.10) is that, even though it is non-singular, i.e. $\det(\widetilde{\beta}\eta_{\mu\nu}) \neq 0$, it reduces the number of dimensions of the Minkowski space from four to two and is presumably responsible for the transition from spin-$\frac{1}{2}$ of fermions to the zero (or integral) spin of the bosons. The principle of duality may be visualized in 3D projective space(as shown in Fig. 6) in terms of the quadric surfaces S and $\widetilde{S}$ circumscribing the tetrahedron of reference and inscribed in it.

3.2. Dirac Equation for Bosons

The important feature of Eq.(3.11)

4. Discussion and Conclusions

The principal new result of the geometric principle of duality is in Eqs.(2.8b) and (2.24a) which states that a massive particle can cross the light barrier in projective 4D space in which proper time and local time comprise the real and imaginery parts of one complex time, $\zeta = \tau + it$. Physically a pure-imaginery time can be interpreted 1/kT, where T is the temperature – unification of quantum mechanics and thermodynamics.

According to Santilli[12], the existence of two separate but complementary space-times (here identified as S and $\widetilde{S}$ connected by the dual transformation (3.1)) resolves the problem of the negative energy solutions without any need for the assumption of infinite sea of states and related “hole theory”, and before recourse to second quantization.

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2. The Oyibo-Einstein Relativities

Oyibo's unified field theory is described generically by the following equation:

$$(G_{0n})_t + (G_{1n})_x + (G_{2n})_y + (G_{3n})_z = 0 \quad (2.1)$$

whose solution is written in terms of η_n defined by the novel "metric" of 4-dimensional space-time geometry:

$$\eta_n = g_{n0}(ct)^{n+1} + g_{n1}x^{n+1} + g_{n2}y^{n+1} + g_{n3}z^{n+1} \quad (2.2)$$

It is clear that η_n is a function of the local-time and space coordinates (ct, x, y, z) and of the metric parameters $(g_{n0}, g_{n1}, g_{n2}, g_{n3})$ as well as *n*. By putting $\eta_n \equiv s^{n+1}$ and interpreting $\tau = s/c$ as the proper-time, Eq.(2.2) may be re-written in the form:

$$s^{n+1} = g_{n0}(ct)^{n+1} + g_{n1}x^{n+1} + g_{n2}y^{n+1} + g_{n3}z^{n+1}, \quad (2.3)$$

This reduces to (Einstein's) special relativity with a 2nd order algebraic structure when $n = 1$:

$$\eta_1 = g_{10}(ct)^2 + g_{11}x^2 + g_{12}y^2 + g_{13}z^2 \equiv (ct)^2 - x^2 - y^2 - z^2 \quad (2.4a)$$

and to a novel (Oyibo's) relativity with a 3rd order algebraic structure when $n = 2$

$$\eta_2 = g_{20}(ct)^3 + g_{21}x^3 + g_{22}y^3 + g_{23}z^3 . \quad (2.4b)$$

Since, in general relativity theory, the metric parameters $(g_{n0}, g_{n1}, g_{n2}, g_{n3})$ are functions of the four-coordinates (ct, x, y, z), it is pertinent to consider possible realizations of these (Einstein-Oyibo) relativities in the hierarchical form:

$$\eta_1 = (ct)^2 - r^2; \;\; \eta_2 = g_{20}(ct)^3 + g_{2r}r^3; \; ... \quad (2.5)$$

where $r = \sqrt{x^2 + y^2 + z^2}$. We observe that if g_{20}, g_{2r} were constants in this equation, we could rewrite the second term in the form

$$\eta_2 = g_{tt}(ct)^2 + g_{rr}r^2 \equiv g_{20}(ct)^3 + g_{2r}r^3 \quad (2.6)$$

and infer that $g_{tt} \propto ct$ and $g_{rr} \propto r$ .This suggests that these components of the metric tensor may be related to Einstein's "unified" field metric tensor,

$$\bar{g}_{\mu\nu} = f_{\mu\nu} + g_{\mu\nu}, \quad (2.7a)$$

in such a way that $(g_{\mu\nu}) = diag(s, -s, -s, -s)$ is proportional to the metric tensor of the conventional Minksowki space while $f_{\mu\nu}$ is an antisymmetric tensor of the 4th rank such that

$$(f_{01}, f_{02}, f_{03}) \equiv (x, y, z) = \mathbf{r} \propto (4\pi e / r^3)\mathbf{r} \equiv (F_{01}, F_{02}, F_{03}), \quad (2.7b)$$

i.e., the electric field part of $f_{\mu\nu}$ is proportional to the corresponding part of the electromagnetic field tensor $F_{\mu\nu}$ associated with the Coulomb (electric) force between two like electric point charges at a separation **r,** and the magnetic field part is proportional to a vector-time (**t**) in the form:

$$(f_{23}, f_{31}, f_{12}) \equiv (ct_1, ct_2, ct_3) = c\mathbf{t}, \quad (2.7c)$$

where $\sqrt{t_1^2 + t_2^2 + t_3^2} = t$ is the local-time. With the above characterization of Einstein's unified field, consider the proper distance (*s*) defined by the equation:

$$\det(\bar{g}_{\mu\nu}) \equiv \begin{vmatrix} s & x & y & z \\ -x & -s & t_3 & -t_2 \\ -y & -t_3 & -s & t_1 \\ -z & t_2 & -t_1 & -s \end{vmatrix}$$
$$\equiv s^4 - s^2[(c\mathbf{t})^2 - \mathbf{r}^2] - (c\mathbf{t}.\mathbf{r})^2 = 0. \qquad (2.8a)$$

This is a quadratic equation in s^2 and, accordingly, has two roots:

$$s_\pm^2 = \tfrac{1}{2}[(c\mathbf{t})^2 - \mathbf{r}^2] \pm \tfrac{1}{2}\sqrt{[(c\mathbf{t})^2 - \mathbf{r}^2]^2 - (2c\mathbf{t}.\mathbf{r})^2} \qquad (2.8b)$$

i.e. $s_\pm^2 = \tfrac{1}{2}[(c\mathbf{t})^2 - \mathbf{r}^2] \pm \tfrac{1}{2}\sqrt{[(c\mathbf{t})^2 - \mathbf{r}^2 + 2c\mathbf{t}.\mathbf{r}][(c\mathbf{t})^2 - \mathbf{r}^2 - 2c\mathbf{t}.\mathbf{r}]}$ (2.8c)

Accordingly, Einstein's special relativity is recovered in the form

$$s_+^2 = (c\mathbf{t})^2 - \mathbf{r}^2 = [ct, r]\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} ct \\ r \end{bmatrix}; \qquad s_-^2 = 0, \qquad (2.9)$$

in the limit when $\mathbf{t}.\mathbf{r} = 0$, i.e. when $\mathbf{t}$ is orthogonal to $\mathbf{r}$, (cf the transversality of the electric and magnetic fields, $\mathbf{E}.\mathbf{H} = 0$, in vacuum). Suppose that the medium between the point charges is dispersive and has a refractive index $n = v/c \equiv dr/cdt$ such that the bending of light in the medium is characterized by the transition:

$$s_+^2 = [ct, r]\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} ct \\ r \end{bmatrix} \to [ct, r]\begin{bmatrix} 1 & -dr/cdt \\ -dr/cdt & -1 \end{bmatrix}\begin{bmatrix} ct \\ r \end{bmatrix} = s_-^2 = 0\,. \qquad (2.10a)$$

which leads to the relation

$$[ct, r]\begin{bmatrix} 1 & -dr/cdt \\ -dr/cdt & -1 \end{bmatrix}\begin{bmatrix} ct \\ r \end{bmatrix} = (ct)^2 - r^2 - 2ctr(dr/cdt) = 0\,. \qquad (2.10b)$$

Then we get from Eq.(2.10b) the differential equation

$$\frac{dr}{cdt} = \frac{(ct)^2 - r^2}{2ctr} \qquad (2.11a)$$

whose solution is:

$$r^2 = \tfrac{1}{3}(ct)^2 + \tfrac{1}{3}\frac{r_0^3}{ct} \qquad \text{or} \quad 3ct\, r^2 = (ct)^3 + r_0^3 \qquad (2.11b)$$

where r_0 is a constant. Then, on the sphere $r^2 = r_0^2$ we have

$$3ct\, r_0^2 = (ct)^3 + r_0^3\,. \qquad (2.11c)$$

This is the desired solution of Eq.(2.6) with $(g_{20}, g_{2r}) = (1,1)$ and an explicit value of $\eta_2 = 3ctr_0^2$.

By the geometric principle of duality, Eq.(2.11c) may be rewritten as done in catastrophe theory[5], in the form

$$X + (ct)Y + (ct)^3 = 0\,. \qquad (2.12)$$

This means that for fixed *ct,* Eq.(2.12) represents a line in (X,Y)-space which is normal to a parabola parameterized by *ct* in such a way that any point on the parabola has coordinates $[(ct), (ct)^2]$ (p. 78 of ref.[5]) and the envelope of the normal (as *ct* varies) is a semi-cubical parabola (called *canonical cusp catastrophe*) defined by the discriminant equation:

$$27X^2 + Y^3 = 0. \qquad (2.13)$$

This represents the involute of the parabola (Fig. 2a). The corresponding involute of an ellipse is an hypocycloid shown in Fig. 2b and can be produced as "light caustic" formed by refraction of a laser beam in a periodically grated piece of glass (p. 271 of ref.[5])

Another feature of Eq.(2.11c) that follows from the geometric principle of duality is that, because of the invariance of Eq.(2.8a) under the interchange,

$$\mathbf{r} \to c\mathbf{t}, c\mathbf{t} \to -\mathbf{r}, \qquad (2.14)$$

we may rewrite Eq.(2.11b), under such an interchange, in the form

$$(ct)^2 = \tfrac{1}{3}r^2 + \frac{c_0^3 t_0^3}{3r} - (c_1 t_0)^2. \qquad (2.15)$$

where we have introduced an arbitrary constant (of integration) $(c_1 t_0)^2$. This means that in a static universe such that $t = t_0$, we have at small *r,*

$$c^2 \approx -c_1^2 + \frac{c_0^3 t_0^2}{3r}. \qquad (2.16)$$

This implies that if the metric in spherical polar (r, θ, φ) coordinates is

$$(ds)^2 = g_{tt}(dt)^2 + g_{rr}(dr)^2 + r^2[(d\theta)^2 + \sin^2\theta(d\varphi)^2], \qquad (2.17)$$

then we may identify g_{tt} with c^2. Accordingly, in this limit, Eq.(2.16) characterizes the usual Schwarzschild singularity of Einstein's equation of general relativity associated with the *black hole* [6] Moreover, in the special case of Eq.(2.15) corresponding to $t_0 = 0$,

$$(ct)^2 - \tfrac{1}{3}r^2 = 0, \quad \text{i.e.,} \quad 3(ct)^2 - r^2 \equiv 3(ct)^2 - x^2 - y^2 - z^2 = 0. (2.17)$$

This defines the F_1-quadric generated earlier in Eq.(1.6) as the covariant of the pair of quadrics defined by Eqs.(1.1a) and (1.1b).

Moreover, if the metric $g'_{\alpha\beta}$ of S' and $d_{\alpha\beta}$ of F_1 are normalized by rewriting Eq. (1.1b) and Eq.(1.6) in the forms,

$$S' \equiv 4g'_{\alpha\beta}x^{\alpha}x^{\beta}; F_1 \equiv -4d_{\alpha\beta}x^{\alpha}x^{\beta} \qquad (1.9)$$

then

$$(g'_{\alpha\beta}) = diag(\tfrac{1}{4},\tfrac{1}{4},\tfrac{1}{4},\tfrac{1}{4}); (d_{\alpha\beta}) = diag(\tfrac{3}{4},-\tfrac{1}{4},-\tfrac{1}{4},-\tfrac{1}{4}) \qquad (1.10)$$

and $(g'_{\alpha\beta} + d_{\alpha\beta}) = diag(1,0,0,0)$ is idempotent,; and there exists an orthogonal matrix (U) such that

$$U^{-1}(g'_{\alpha\beta} + d_{\alpha\beta})U = \begin{pmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{pmatrix} = (\hat{g}_{\alpha\beta}) \equiv \hat{g}, \qquad (1.11)$$

where,

$$U = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{pmatrix} = U^{+} = U^{-1}. \qquad (1.12)$$

This form of the idempotent matrix $\hat{g}$ is a generalization to 4×4 matrix of the result given for 3×3 matrix in Eq.(6) of ref.[4] and for 2×2 matrix in Eq.(17) of ref.[4]. The interest of $\hat{g}_{\alpha\beta} = g'_{\alpha\beta} + d_{\alpha\beta}$ is that it is an Einstein-like "unified field", except that here, $g'_{\alpha\beta}$ and $d_{\alpha\beta}$ are both symmetric metrics. This is a key to an appreciation of the unified field theory proposed by Oyibo[1].

1. NEW FRONTIERS OF DIRAC'S EQUATION

3.1 Dirac's Positive-Energy Relativistic Wave Equation

3.2 The n^2-representation, (n = 2,3,4,5) of fermion creation and annihilation operators.

3.3 O(4,2) algebra of Dirac matrices, and infinite-component wave equations.

However, while the three pi-meson $(\pi^{-},\pi^{0},\pi^{+})$ each has spin-0, the three (Konopinski-Mahmoud [4]) leptons (e^{-},ν,μ^{+}) each has spin-$\frac{1}{2}$, where the two massless electron-like neutrino (ν_e) and muon-like neutrino (ν_{μ}), have been put into a single four-component neutrino (ν) unifying ν_e and ν_{μ}^{c}. As I have pointed out in the application of projective geometry to particle physics described in ref.[3], these low-lying lepton and meson triplets and the low-lying spin-$\frac{1}{2}$ baryons $(p, n, \Lambda, \Sigma^0, \Sigma^\pm, \Xi^0, \Xi^-)$ comprise the “basic” representation of the octahedral symmetry group of the elementary particles (leptons, mesons and baryons) exhibiting the geometric principle of duality in LBQ-space, (L being lepton number, B baryon number and Q electric charge (in units of proton charge)), as shown in Figure 1.

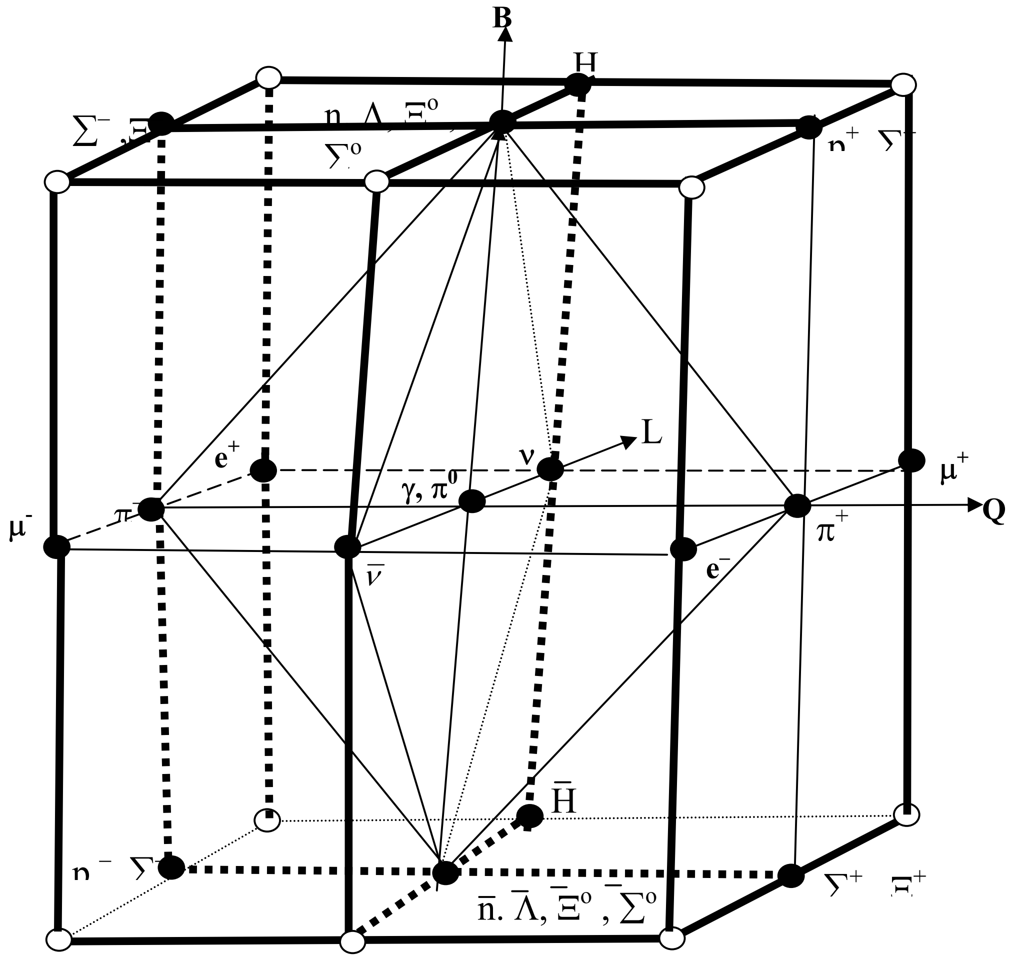

Fig. 1 : Representation of the elementary particles in 3-dimensional

(See, T. Poston and I. Stewart[5] *Catastrophe Theory and its Applications* (Pitman London, 1978, p.78 and 271).

[See, T. Frankel, *Gravitational Curvature: An Introduction to Einstein's Theory* (Freeman & Co, 1979) pp.46-50.

(See, J.A. Todd[7] *Projective and Analytical Geometry*, Sir Isaac Pitman & Sons, London, 1958 , pp. 246-250).