What you will learn today

- The Dot Product
- The Cross Product
- Equations of Lines and Planes

Definitions:
1. $\vec{a} : \langle a_1, a_2, a_3 \rangle$, $\vec{b} : \langle b_1, b_2, b_3 \rangle$, $\vec{a} \cdot \vec{b} := a_1 b_1 + a_2 b_2 + a_3 b_3$.
2. $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| cos\theta$.
The two definitions are equivalent.

Corollary

$$\theta = arccos \frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|}$$

Corollary

Two non zero vectors $\vec{a}$ and $\vec{b}$ are perpendicular iff $\vec{a} \cdot \vec{b} = 0$.

$0 \leq \theta < \frac{\pi}{2}$ iff $\vec{a}$ and $\vec{b}$ points in the same general direction iff $\vec{a} \cdot \vec{b} > 0$.

$\frac{\pi}{2} < \theta \leq \pi <$ iff $\vec{a}$ and $\vec{b}$ points in the opposite general direction iff $\vec{a} \cdot \vec{b} < 0$.

Direction angles of a non zero $\vec{a}$ are the angles α, β, γ that $\vec{a}$ makes with the positive x,y,z-axes.
The cosines of $cos\alpha, cos\beta, cos\gamma$ are the direction cosines.
For $\vec{a}: \langle a_1, a_2, a_3 \rangle$, $cos\alpha = \frac{\vec{a} \cdot \vec{i}}{|\vec{a}||\vec{i}|} = \frac{a_1}{|a|}$.
$cos\beta = \frac{a_2}{|\vec{a}|}$. $cos\gamma = \frac{a_3}{|\vec{a}|}$.
Therefore we have the convenient relation:

$$\vec{a} = |\vec{a}| \langle cos\alpha, cos\beta, cos\gamma \rangle$$

$\langle cos\alpha, cos\beta, cos\gamma \rangle$ is a unit vector in the same direction as $\vec{a}$.

Example

Find the direction angles of $\vec{a} = \langle 1, 2, 3 \rangle$.

$Proj_{\vec{a}}\vec{b}$: the vector projection of $\vec{b}$ along the direction of $\vec{a}$. (also called shadow of $\vec{b}$ along $\vec{a}$).
$Comp_{\vec{a}}\vec{b}$: the signed magnitude of $Proj_{\vec{a}}\vec{b}$. Called the scalar projection, or component of $\vec{b}$ along $\vec{a}$.

$$Comp_{\vec{a}}\vec{b} = |b|cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|}$$

$$Proj_{\vec{a}}\vec{b} = Comp_{\vec{a}}\vec{b}\frac{\vec{a}}{|\vec{a}|} = \frac{(\vec{a}\cdot\vec{b})\vec{a}}{|\vec{a}|^2}$$

Example

$\vec{a} = \langle -2, 3, 1 \rangle$, $\vec{b} = \langle 1, 1, 2 \rangle$, find $Comp_{\vec{a}}\vec{b}$, $Proj_{\vec{a}}\vec{b}$.

Application: Calculating work,

$$W = \vec{F} \cdot \vec{D} = |\vec{F}||\vec{D}|cos\theta$$

The cross product $\vec{a} \times \vec{b}$ is a vector,
Definition:
1, $\vec{a} \times \vec{b} = \langle a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1 \rangle$.
2, $\vec{a} \times \vec{b}$ is orthogonal to both $\vec{a}$ and $\vec{b}$, with direction determined by the right hand rule. The magnitude $|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}| sin\theta$.
The cross product is only defined for three-dimensional vectors.

Easier way to remember:

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Example

If $\vec{a} = \langle 1, 3, 4 \rangle$, $\vec{b} = \langle 2, 7, -5 \rangle$, find $\vec{a} \times \vec{b}$.

Two non zero vectors $\vec{a}$ and $\vec{b}$ are parallel to iff $\vec{a} \times \vec{b} = 0$.
The magnitude of the product $\vec{a} \times \vec{b}$ is equal to the area of the parallelogram determined by $\vec{a}$ and $\vec{b}$.

Find a vector perpendicular to the plane that passes through the points P(1,4,6), Q(-2,5,-1) and R(1,-1,1). Find the area of the triangle with vertices P, Q and R.

$\vec{i} \times \vec{j} = \vec{k}$, $\vec{j} \times \vec{k} = \vec{i}$, $\vec{k} \times \vec{i} = \vec{j}$.
If general, $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$.
$(\vec{a} \times \vec{b}) \times \vec{c} \neq \vec{a} \times (\vec{b} \times \vec{c})$.
$\vec{a} \cdot (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \cdot \vec{c}$.

$\vec{a} \cdot (\vec{b} \times \vec{c})$ is called the scalar triple product:

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Its geometric significance is that $|\vec{a} \cdot (\vec{b} \times \vec{c})|$ is the volume of the parallelepiped determined by $\vec{a}$, $\vec{b}$ and $\vec{c}$.
If however the volume is 0, then $\vec{a}$, $\vec{b}$ and $\vec{c}$ are coplanar.

Show the vectors $\vec{a} = \langle 1, 4, -7 \rangle$, $\vec{b} = \langle 2, -1, 4, \rangle$ and $\vec{c} = \langle 0, -9, 18 \rangle$ are coplanar.

Application: The torque (relative to the origin) is defined as the cross product of the position and force vectors:
$\vec{\tau} = \vec{r} \times \vec{F}$
measures the tendency of the body to rotate about the origin. The direction indicates the axis of rotation(its orientation).

A line L in three dimensional space is determined by a point on the line and its direction:

$$\vec{r} = \vec{r_0} + t\vec{v}$$

where t is a parameter. This is called the vector equation for L. As t varies, the line is traced out by the tip of the vector $\vec{r}$.
We can also write

$$\langle x, y, z\rangle = \langle x_0 + ta, y_0 + tb, z_0 + tc$$

or

$$x = x_0 + ta, y = y_0 + tb, z = z_0 + tc$$

which is called the parametric form.

Example

Find a vector equation and parametric equation for the line that passes through the point P(5,1,3) and is parallel to the vector $\langle 1, 4, -2 \rangle$.

Find two other points on the line.

If a vector $\vec{v} = \langle a, b, c \rangle$ is used to describe the direction of a line L, then the numbers a, b and c are called direction numbers of L. Any vector parallel to $\vec{v}$ can also be used.
If none of a,b or c is 0, we can solve each equation for t and equate them:

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

which is called the symmetric equations for L. The direction numbers just appear on the denominators.

Example

Find symmetric equations of the line that passes through the points A(2,4,-3) and B(3,-1,1).
At what point does this line intersect the xy-plane?
Hint: consider

$$\frac{x - x_0}{x_1 - x_0} = \frac{y - y_0}{y_1 - y_0} = \frac{z - z_0}{z_1 - z_0}$$

A line segment from $\vec{r_0}$ to $\vec{r_1}$ is given by the vector equation

$$\vec{r}(t) = (1-t)\vec{r_0} + t\vec{r_1}, 0 \leq t \leq 1$$

The lines not intersecting and not parallel are called skew lines.
Show $L_0 : x = 1+t, y = -2+3t, z = 4-t$ and
$L_1 : x = 2s, y = 3+s, z = -3+4s$ are skew lines.

A plane in three dimensional space is determined by a point in the plane and a vector $\vec{n}$ perpendicular to the plane. $\vec{n}$ is called the a normal vector.

Vector equation:

$$\vec{n} \cdot (\vec{r} - \vec{r_0}) = 0$$

or

$$\vec{n} \cdot \vec{r} = \vec{n} \cdot \vec{r_0}$$

Scalar equation:

$$\langle a, b, c \rangle \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0$$

Write $d = -(ax_0 + by_0 + cz_0)$, then we have linear equation $ax + by + cz + d = 0$.

1. Find an equation of the plane that passes through the points P(1,3,2), Q(3,-1,6) and R(5,2,0).
2. Find the point where the line $x = 2 + 3t, y = -4t, z = 5 + t$ intersects the plane $4x + 5y - 2z = 18$.

Two planes are parallel iff their normal directions are parallel. If they are no parallel, they intersect in a line. The angles between two planes is the acute angle between their normal vectors.

Find the angle between the planes $x + y + z = 1$ and $x - 2y + 3z = 1$.
Find symmetric equations for the line of intersection of the two planes.
Note: A pair of two linear equations represent a line, we can view the symmetric equations as a two linear equations.
Find the distance from a point (x_1, y_1, z_1) to the plane $ax + by + cz + d = 0$.

1. Find the distance between two planes $10x + 2y - 2z = 5$ and $5x + y - z = 1$.

2. Find the distance between the skew lines
$L_0 : x = 1 + t, y = -2 + 3t, z = 4 - t$ and
$L_1 : x = 2s, y = 3 + s, z = -3 + 4s$.

What you have learned today

- The Dot Product
- The Cross Product
- Equations of Lines and Planes