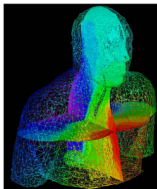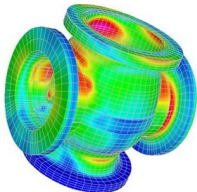# The Finite Element Method for the Analysis of Non-Linear and Dynamic Systems

Prof. Dr. Eleni Chatzi

Lecture 1 - 16 September, 2014

Course Information

Instructor
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Course Website
Lecture Notes and Homeworks will be posted at:
http://www.ibk.ethz.ch/ch/education

Suggested Reading

- Nonlinear Finite Elements for Continua and Structures by T. Belytschko, W. K. Liu, and B. Moran, John Wiley and Sons, 2000

- The Finite Element Method: Linear Static and Dynamic Finite Element Analysis by T. J. R. Hughes, Dover Publications, 2000

- The Finite Element Method Vol. 2 Solid Mechanics by O.C. Zienkiewicz and R.L. Taylor, Oxford : Butterworth Heinemann, 2000

Course Outline

- Review of the Finite Element method - Introduction to Non-Linear Analysis

- Non-Linear Finite Elements in solids and Structural Mechanics
  - Overview of Solution Methods
  - Continuum Mechanics & Finite Deformations
  - Lagrangian Formulation
  - Structural Elements

- Dynamic Finite Element Calculations
  - Integration Methods
  - Mode Superposition

- Eigenvalue Problems

- Special Topics
  - Boundary Element & Extended Finite Element methods

Performance Evaluation - Homeworks (100%)

Homework

- Homeworks are due in class 2-3 weeks after assignment
- Computer Assignments may be done using any coding language (MATLAB, Fortran, C, MAPLE) - example code will be provided in MATLAB
- Commercial software such as CUBUS, ABAQUS and SAP will also be used for certain Assignments

Homework Sessions will be pre-announced and it is advised to bring a laptop along for those sessions

Lecture #1: Structure

Review of the Finite Element Method

- Strong vs. Weak Formulation

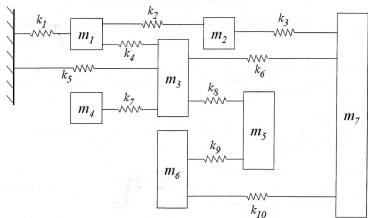- The Finite Element (FE) formulation

- The Iso-Parametric Mapping

Examples

- The Bar Element

- The Beam Element
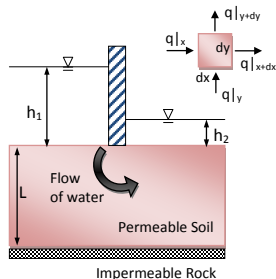
Classification of Engineering Systems

Discrete

$$F = KX$$

Direct Stiffness Method

Continuous

$$k \left(\frac{\partial^2 \phi}{\partial^2 x} + \frac{\partial^2 \phi}{\partial^2 y} \right) = 0$$

Laplace Equation

FEM: Numerical Technique for **approximating** the solution of **continuous systems**. We will use a displacement based formulation and a stiffness based solution (direct stiffness method).

Review of the Finite Element Method (FEM)

How is the Physical Problem formulated?

The formulation of the equations governing the response of a system under specific loads and constraints at its boundaries is usually provided in the form of a differential equation. The differential equation also known as the **strong form** of the problem is typically extracted using the following sets of equations:

① Equilibrium Equations
ex. $f(x) = R + \dfrac{aL + ax}{2}(L - x)$

② Constitutive Requirements Equations
ex. $\sigma = E\epsilon$

③ Kinematics Relationships
ex. $\epsilon = \dfrac{du}{dx}$

Axial bar Example

Review of the Finite Element Method (FEM)
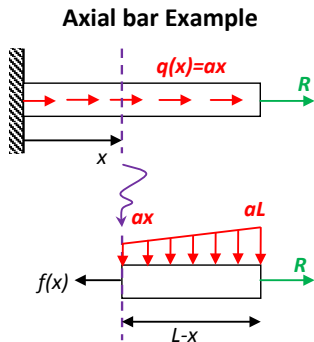
How is the Physical Problem formulated?

Differential Formulation (Strong Form) in 2 Dimensions

Quite commonly, in engineering systems, the governing equations are of a second order (derivatives up to u'' or $\frac{\partial^2 u}{\partial^2 x}$) and they are formulated in terms of variable u, i.e. displacement:

Governing Differential Equation ex: general 2nd order PDE

$$A(x,y)\frac{\partial^2 u}{\partial^2 x} + 2B(x,y)\frac{\partial^2 u}{\partial x \partial y} + C(x,y)\frac{\partial^2 u}{\partial^2 y} = \phi(x,y,u,\frac{\partial u}{\partial y},\frac{\partial u}{\partial y})$$

Problem Classification

- $B^2 - AC < 0 \Rightarrow$ elliptic
- $B^2 - AC = 0 \Rightarrow$ parabolic
- $B^2 - AC > 0 \Rightarrow$ hyperbolic

Boundary Condition Classification

- Essential (Dirichlet): $u(x_0, y_0) = u_0$
  order $m-1$ at most for C^{m-1}
- Natural (Neumann): $\frac{\partial u}{\partial y}(x_0, y_0) = \dot{u}_0$
  order m to $2m-1$ for C^{m-1}

Review of the Finite Element Method (FEM)

Differential Formulation (Strong Form) in 2 Dimensions

The previous classification corresponds to certain characteristics for each class of methods. More specifically,

- Elliptic equations are most commonly associated with a diffusive or dispersive process in which the state variable u is in an equilibrium condition.

- Parabolic equations most often arise in transient flow problems where the flow is down gradient of some state variable u. Often met in the heat flow context.

- Hyperbolic equations refer to a wide range of areas including elasticity, acoustics, atmospheric science and hydraulics.

Reference Problem

Consider the following 1 Dimensional (1D) strong form (parabolic)

$$\frac{d}{dx}\left(c(x)\frac{d\mathbf{u}}{dx}\right) + \mathbf{f}(x) = 0$$

$$-c(0)\frac{d}{dx}\mathbf{u}(0) = C_1 \qquad \text{(Neumann BC)}$$

$$u(L) = 0 \qquad\qquad \text{(Dirichlet BC)}$$

Physical Problem (1D)		Diff. Equation	Quantities	Constitutive Law
One dimensional Heat flow		$\frac{d}{dx}\left(Ak\frac{dT}{dx}\right) + Q = 0$	T=temperature A=area k=thermal conductivity Q=heat supply	Fourier $q = -k\,dT/dx$ q = heat flux
Axially Loaded Bar		$\frac{d}{dx}\left(AE\frac{du}{dx}\right) + b = 0$	u=displacement A=area E=Young's modulus B=axial loading	Hooke $\sigma = Edu/dx$ σ = stress

Approximating the Strong Form

The strong form requires strong continuity on the dependent field variables (usually displacements). Whatever functions define these variables have to be differentiable up to the order of the PDE that exist in the strong form of the system equations. Obtaining the exact solution for a strong form of the system equation is a quite difficult task for practical engineering problems.

The finite difference method can be used to solve the system equations of the strong form and obtain an approximate solution. However, this method usually works well for problems with simple and regular geometry and boundary conditions.

Alternatively we can use the **finite element method** on a **weak form** of the system. This weak form is usually obtained through energy principles which is why it is also known as variational form.

From Strong Form to Weak form

Three are the approaches commonly used to go from strong to weak form:

- Principle of Virtual Work

- Principle of Minimum Potential Energy

- Methods of weighted residuals (Galerkin, Collocation, Least Squares methods, etc)

*We will mainly focus on the third approach.
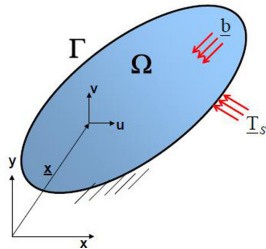
From Strong Form to Weak form - Approach #1

Principle of Virtual Work

For any set of compatible small virtual displacements imposed on the body in its state of equilibrium, the total internal virtual work is equal to the total external virtual work.

$$W_{int} = \int_{\Omega} \bar{\epsilon}^T \boldsymbol{\tau} \, d\Omega = W_{ext} = \int_{\Omega} \bar{\mathbf{u}}^T \mathbf{b} \, d\Omega + \int_{\Gamma} \bar{\mathbf{u}}^{ST} \mathbf{T_S} \, d\Gamma + \sum_i \bar{\mathbf{u}}^{iT} \mathbf{R_C}^i$$

where

- $\mathbf{T_S}$: surface traction (along boundary Γ)
- $\mathbf{b}$: body force per unit area
- $\mathbf{R_C}$: nodal loads
- $\bar{\mathbf{u}}$: virtual displacement
- $\bar{\epsilon}$: virtual strain
- $\boldsymbol{\tau}$: stresses

From Strong Form to Weak form - Approach #2

Principle of Minimum Potential Energy

Applies to elastic problems where the elasticity matrix is positive definite, hence the energy functional Π has a minimum (stable equilibrium).
Approach #1 applies in general.

The potential energy **Π** is defined as the strain energy **U** minus the work of the external loads **W**

$$\mathbf{\Pi} = \mathbf{U} - \mathbf{W}$$
$$\mathbf{U} = \frac{1}{2} \int_{\Omega} \boldsymbol{\epsilon}^T C \boldsymbol{\epsilon} d\Omega$$
$$\mathbf{W} = \int_{\Omega} \bar{\mathbf{u}}^T \mathbf{b} d\Omega + \int_{\Gamma_T} \bar{\mathbf{u}}^{ST} \mathbf{T_s} d\Gamma_T + \sum_i \bar{\mathbf{u}}_i^T \mathbf{R_C}^i$$

(**b T$_s$, R$_C$** as defined previously)

From Strong Form to Weak form - Approach #3

Galerkin's Method

Given an arbitrary weight function w, where

$$S = \{u|u \in \mathcal{C}^0, u(l) = 0\}, S^0 = \{w|w \in \mathcal{C}^0, w(l) = 0\}$$

$\mathcal{C}^0$ is the collection of all continuous functions.

Multiplying by w and integrating over Ω

$$\int_0^l w(x)[(c(x)u'(x))' + f(x)]dx = 0$$

$$[w(0)(c(0)u'(0) + C_1] = 0$$

Weak Form - 1D FEM

Using the divergence theorem (integration by parts) we reduce the order of the differential:

$$\int_0^l wg' dx = [wg]_0^l - \int_0^l gw' dx$$

The **weak form** is then reduced to the following problem. Also, in what follows we assume constant properties $c(x) = c = const$.

Find $u(x) \in \mathcal{S}$ such that:

$$\int_0^l w' c u' dx = \int_0^l wf dx + w(0)C_1$$
$$S = \{u | u \in \mathcal{C}^0, u(l) = 0\}$$
$$S^0 = \{w | w \in \mathcal{C}^0, w(l) = 0\}$$

Weak Form

Notes:

1. Natural (Neumann) boundary conditions, are imposed on the secondary variables like forces and tractions.
For example, $\frac{\partial u}{\partial y}(x_0, y_0) = \dot{u}_0$.

2. Essential (Dirichlet) or geometric boundary conditions, are imposed on the primary variables like displacements.
For example, $u(x_0, y_0) = u_0$.

3. A solution to the strong form will also satisfy the weak form, but not vice versa. Since the weak form uses a lower order of derivatives it can be satisfied by a larger set of functions.

4. For the derivation of the weak form we can choose any weighting function w, since it is arbitrary, so we usually choose one that satisfies homogeneous boundary conditions wherever the actual solution satisfies essential boundary conditions. Note that this does not hold for natural boundary conditions!

How to derive a solution to the weak form?

Step #1: Follow the **FE** approach:
Divide the body into finite elements, e, connected to each other through nodes.
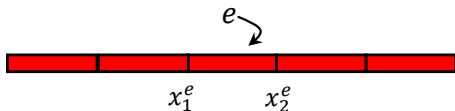
Then break the overall integral into a summation over the finite elements:

$$\sum_e \left[\int_{x_1^e}^{x_2^e} w'cu'dx - \int_{x_1^e}^{x_2^e} wfdx - w(0)C_1 \right] = 0$$

Step #2: Approximate the continuous displacement using a discrete equivalent:

Galerkin's method assumes that the approximate (or **trial**) solution, u, can be expressed as a linear combination of the nodal point displacements u_i, where i refers to the corresponding **node number**.

$$u(x) \approx u^h(x) = \sum_i N_i(x)u_i = \mathbf{N}(x)\mathbf{u}$$

where bold notation signifies a vector and $N_i(x)$ are the **shape functions**. In fact, the shape function can be any mathematical formula that helps us interpolate what happens at points that lie within the nodes of the mesh. In the **1-D case** that we are using as a reference, $N_i(x)$ are defined as 1st degree polynomials indicating a **linear interpolation**.

As will be shown in the application presented in the end of this lecture, for the case of a truss element the linear polynomials also satisfy the homogeneous equation related to the bar problem.
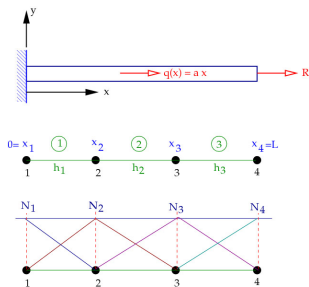
1D FE formulation: Galerkin's Method

Shape function Properties:

- Bounded and Continuous

- One for each node

- $N_i^e(x_j^e) = \delta_{ij}$, where
$$\delta_{ij} = \begin{cases} 1 & \text{if} \quad i = j \\ 0 & \text{if} \quad i \neq j \end{cases}$$

The shape functions can be written as piecewise functions of the x coordinate:

$$N_i(x) = \begin{cases} \dfrac{x - x_{i-1}}{x_i - x_{i-1}}, & x_{i-1} \leq x < x_i \\ \dfrac{xi + 1 - x}{x_{i+1} - x_i}, & x_i \leq x < x_{i+1} \\ 0, & \text{otherwise} \end{cases}$$

This is not a convenient notation. Instead of using the global coordinate x, things become simplified when using coordinate ξ referring to the local system of the element (see page 25).

1D FE formulation: Galerkin's Method

Step #2: Approximate $w(x)$ using a discrete equivalent:

The weighting function, w is usually (although not necessarily) chosen to be of the same form as u

$$w(x) \approx w^h(x) = \sum_i N_i(x)w_i = \mathbf{N}(x)\mathbf{w}$$

i.e. for 2 nodes:
$$\mathbf{N} = [N_1 \quad N_2] \qquad \mathbf{u} = [u_1 \quad u_2]^T \qquad \mathbf{w} = [w_1 \quad w_2]^T$$

Alternatively we could have a Petrov-Galerkin formulation, where $w(x)$ is obtained through the following relationships:

$$w(x) = \sum_i (N_i + \delta \frac{h^e}{\sigma} \frac{dN_i}{dx}) w_i$$

$$\delta = coth(\frac{Pe^e}{2}) - \frac{2}{Pe^e} \qquad coth = \frac{e^x + e^{-x}}{e^x - e^{-x}}$$

1D FE formulation: Galerkin's Method

Note: Matrix vs. Einstein's notation:

In the derivations that follow it is convenient to introduce the equivalence between the Matrix and Einstein's notation. So far we have approximated:

$$u(x) = \sum_i N_i(x)u_i \quad (\textit{Einstein's notation}) = \mathbf{N}(x)\mathbf{u} \quad (\textit{Matrix notation})$$

$$w(x) = \sum_i N_i(x)w_i = \mathbf{N}(x)\mathbf{w} \quad (\textit{similarly})$$

As an example, if we consider an element of 3 nodes:

$$u(x) = \sum_i^3 N_i(x)u_i = N_1 u_1 + N_2 u_2 + N_3 u_3 \Rightarrow$$

$$u(x) = [N_1 \ N_2 \ N_3]\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \mathbf{N}(x)\mathbf{u}$$

Step #3: Substituting into the weak formulation and rearranging terms we obtain the following in **matrix notation**:

$$\int_0^l w' c u' dx - \int_0^l w f dx - w(0) C_1 = 0 \Rightarrow$$

$$\int_0^l (\mathbf{w}^T \mathbf{N}^T)' c (\mathbf{N}\mathbf{u})' dx - \int_0^l \mathbf{w}^T \mathbf{N}^T f dx - \mathbf{w}^T \mathbf{N}(0)^T C_1 = 0$$

Since $\mathbf{w}$, $\mathbf{u}$ are vectors, each one containing a set of discrete values corresponding at the nodes i, it follows that the above set of equations can be rewritten in the following form, i.e. as a summation over the w_i, u_i components (**Einstein notation**):

$$\int_0^l \left(\sum_i u_i \frac{dN_i(x)}{dx} \right) c \left(\sum_j w_j \frac{dN_j(x)}{dx} \right) dx$$

$$- \int_0^l f \sum_j w_j N_j(x) dx - \sum_j w_j N_j(x) C_1 \Bigg|_{x=0} = 0$$

1D FE formulation: Galerkin's Method

This is rewritten as,

$$\sum_j w_j \left[\int_0^l \left(\sum_i c u_i \frac{dN_i(x)}{dx} \frac{dN_j(x)}{dx} \right) - f N_j(x) dx + \left. (N_j(x) C_1) \right|_{x=0} \right] = 0$$

The above equation has to hold $\forall w_j$ since the weighting function $w(x)$ is an arbitrary one. Therefore the following **system of equations** has to hold:

$$\int_0^l \left(\sum_i c u_i \frac{dN_i(x)}{dx} \frac{dN_j(x)}{dx} \right) - f N_j(x) dx + \left. (N_j(x) C_1) \right|_{x=0} = 0 \quad j = 1, ..., n$$

After reorganizing and moving the summation outside the integral, this becomes:

$$\sum_i \left[\int_0^l c \frac{dN_i(x)}{dx} \frac{dN_j(x)}{dx} \right] u_i = \int_0^l f N_j(x) dx + \left. (N_j(x) C_1) \right|_{x=0} = 0 \quad j = 1, ..., n$$

1D FE formulation: Galerkin's Method

We finally obtain the following discrete system in matrix notation:

$$\mathbf{K}\mathbf{u} = \mathbf{f}$$

where writing the integral from 0 to l as a summation over the subelements we obtain:

$$\mathbf{K} = \mathcal{A}_e \mathbf{K}^e \longrightarrow \mathbf{K}^e = \int_{x_1^e}^{x_2^e} \mathbf{N}_{,x}^T c \mathbf{N}_{,x} dx = \int_{x_1^e}^{x_2^e} \mathbf{B}^T c \mathbf{B} dx$$

$$\mathbf{f} = \mathcal{A}_e \mathbf{f}^e \longrightarrow \mathbf{f}^e = \int_{x_1^e}^{x_2^e} \mathbf{N}^T \mathbf{f} dx + \mathbf{N}^T h|_{x=0}$$

where $\mathcal{A}$ is not a sum but an assembly (see page and, x denotes differentiation with respect to x.

In addition, $\mathbf{B} = \mathbf{N}_{,x} = \dfrac{dN(x)}{dx}$ is known as the strain-displacement matrix.

Iso-Parametric Mapping

This is a way to move from the use of global coordinates (i.e.in (x, y)) into normalized coordinates (usually (ξ, η)) so that the finally derived stiffness expressions are uniform for elements of the same type.
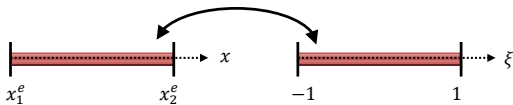
Shape Functions in Natural Coordinates

$$x(\xi) = \sum_{i=1,2} N_i(\xi) x_i^e = N_1(\xi) x_1^e + N_2(\xi) x_2^e$$

$$N_1(\xi) = \frac{1}{2}(1 - \xi), \qquad N_2(\xi) = \frac{1}{2}(1 + \xi)$$

1D FE formulation: Iso-Parametric Formulation

Map the integrals to the natural domain $\longrightarrow$ **element stiffness** matrix.
Using the chain rule of differentiation for $N(\xi(x))$ we obtain:

$$\mathbf{K}^e = \int_{x_1^e}^{x_2^e} \mathbf{N}_{,x}^T c \mathbf{N}_{,x} dx = \int_{-1}^{1} (\mathbf{N}_{,\xi} \xi_{,x})^T c (\mathbf{N}_{,\xi} \xi_{,x}) x_{,\xi} d\xi$$

$$\text{where } \mathbf{N}_{,\xi} = \frac{d}{d\xi} \left[\begin{array}{cc} \frac{1}{2}(1-\xi) & \frac{1}{2}(1+\xi) \end{array} \right] = \left[\begin{array}{cc} \frac{-1}{2} & \frac{1}{2} \end{array} \right]$$

and $x_{,\xi} = \dfrac{dx}{d\xi} = \dfrac{x_2^e - x_1^e}{2} = \dfrac{h}{2} = J$ (Jacobian) and h is the element length

$$\xi_{,x} = \frac{d\xi}{dx} = J^{-1} = 2/h$$

From all the above,

$$\mathbf{K}^e = \frac{c}{x_2^e - x_1^e} \left[\begin{array}{cc} 1 & -1 \\ -1 & 1 \end{array} \right]$$

Similary, we obtain the **element load vector**:

$$\mathbf{f}^e = \int_{x_1^e}^{x_2^e} \mathbf{N}^T \mathbf{f} dx + \mathbf{N}^T h|_{x=0} = \int_{-1}^{1} \mathbf{N}^T(\xi) \mathbf{f} x_{,\xi} d\xi + \mathbf{N}^T(\mathbf{x}) h|_{x=0}$$

Note: the iso-parametric mapping is only done for the integral.

So what is meant by assembly? ($\mathcal{A}_e$)

It implies adding the components of the stiffness matrix that correspond to the same degrees of freedom (dof).

In the case of a simple bar, it is trivial as the degrees of freedom (axial displacement) are as many as the nodes:

1:K^1 **2:K^2**

1 **2** **3**
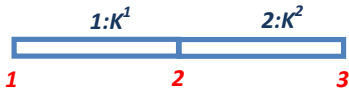
Red indicates the node each component corresponds to

Element Stiffness Matrices (2x2): $K^1 = \dfrac{c}{h}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{matrix} \mathbf{1} \\ \mathbf{2} \end{matrix}$, $K^2 = \dfrac{c}{h}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{matrix} \mathbf{2} \\ \mathbf{3} \end{matrix}$

with column labels $\mathbf{1}\ \mathbf{2}$ and $\mathbf{2}\ \mathbf{3}$

Total Stiffness Matrix (4x4): $K = \dfrac{c}{h}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 1+1 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{matrix} \mathbf{1} \\ \mathbf{2} \\ \mathbf{3} \end{matrix}$

with column labels $\mathbf{1}\ \mathbf{2}\ \mathbf{3}$

1D FE formulation: Galerkin's Method

In the case of a frame with beam elements, the stiffness matrix of the elements is typically of 4x4 size, corresponding to 2 dofs on each end (a displacement and a rotation):
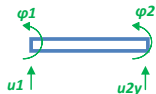
*The process will be shown explicitly during the HW sessions
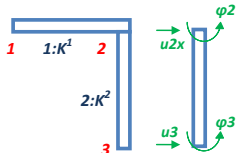
Green indicates the dof each component corresponds to

Element Stiffness Matrices (4x4):

$$K^1 = \begin{bmatrix} K_{11}^1 & K_{12}^1 & K_{13}^1 & K_{14}^1 \\ K_{12}^1 & K_{22}^1 & K_{23}^1 & K_{24}^1 \\ K_{13}^1 & K_{23}^1 & K_{33}^1 & K_{34}^1 \\ K_{14}^1 & K_{24}^1 & K_{34}^1 & K_{44}^1 \end{bmatrix} \begin{matrix} u1 \\ \varphi1 \\ u2y \\ \varphi2 \end{matrix} \quad K^2 = \begin{bmatrix} K_{11}^2 & K_{12}^2 & K_{13}^2 & K_{14}^2 \\ K_{12}^2 & K_{22}^2 & K_{23}^2 & K_{24}^2 \\ K_{13}^2 & K_{23}^2 & K_{33}^2 & K_{34}^2 \\ K_{14}^2 & K_{24}^2 & K_{34}^2 & K_{44}^2 \end{bmatrix} \begin{matrix} u2x \\ \varphi2 \\ u3 \\ \varphi3 \end{matrix}$$

Total Stiffness Matrix (2x2):

$$K = \begin{bmatrix} K_{44}^1 + K_{22}^2 & K_{34}^1 & 0 \\ K_{34}^1 & K_{33}^1 & K_{24}^2 \\ 0 & K_{24}^2 & K_{11}^2 \end{bmatrix} \begin{matrix} \varphi2 \\ u2y \\ u2x \end{matrix}$$

Fixed dofs are not included in the total stiffness matrix

Axially Loaded Bar Example

A. Constant End Load
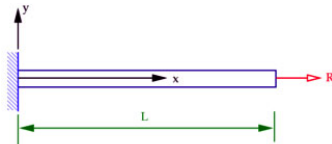
Given: Length L, Section Area A, Young's modulus E
Find: stresses and deformations.

Assumptions:

The cross-section of the bar does not change after loading.
The material is linear elastic, isotropic, and homogeneous.
The load is centric.
End-effects are not of interest to us.

Axially Loaded Bar Example

A. Constant End Load

Strength of Materials Approach

From the **equilibrium equation**, the axial force at a random point x along the bar is:

$$\mathbf{f}(x) = \mathbf{R}(= const) \Rightarrow \boldsymbol{\sigma}(x) = \frac{\mathbf{R}}{A}$$

From the **constitutive equation (Hooke's Law)**:

$$\epsilon(x) = \frac{\boldsymbol{\sigma}(x)}{E} = \frac{\mathbf{R}}{AE}$$

Hence, the deformation $\delta(x)$ is obtained from kinematics as:

$$\epsilon = \frac{\delta(x)}{x} \Rightarrow \delta(x) = \frac{\mathbf{R}x}{AE}$$

Note: The stress & strain is independent of x for this case of loading.

B. Linearly Distributed Axial + Constant End Load
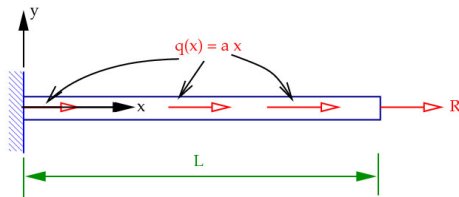
From the **equilibrium equation**, the axial force at random point x along the bar is:

$$\mathbf{f}(x) = \mathbf{R} + \frac{aL + ax}{2}(L - x) = \mathbf{R} + \frac{a(L^2 - x^2)}{2} (\text{ depends on x})$$

In order to now find stresses & deformations (which depend on x) we have to repeat the process for every point in the bar. This is computationally inefficient.

Axially Loaded Bar Example

From the **equilibrium equation**, for an infinitesimal element:

$$A\boldsymbol{\sigma} = \mathbf{q}(x)\Delta x + A(\boldsymbol{\sigma} + \Delta\sigma) \Rightarrow A \underbrace{lim}_{\Delta x \to 0} \frac{\Delta\boldsymbol{\sigma}}{\Delta x} + \mathbf{q}(x) = 0 \Rightarrow A\frac{d\boldsymbol{\sigma}}{dx} + \mathbf{q}(x) = 0$$

$$\text{Also, } \epsilon = \frac{d\mathbf{u}}{dx}, \boldsymbol{\sigma} = E\epsilon, \mathbf{q}(x) = ax \Rightarrow AE\frac{d^2\mathbf{u}}{dx^2} + ax = 0$$
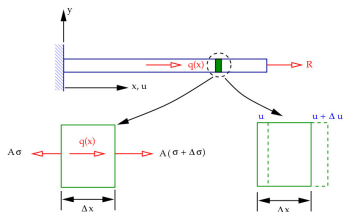
Strong Form

$$AE\frac{d^2\mathbf{u}}{dx^2} + ax = 0$$

$\mathbf{u}(0) = 0$ essential BC

$$\mathbf{f}(L) = \mathbf{R} \Rightarrow AE\left.\frac{d\mathbf{u}}{dx}\right|_{x=L} = \mathbf{R} \text{ natural BC}$$

Analytical Solution

$$\mathbf{u}(x) = \mathbf{u}_{hom} + \mathbf{u}_p \Rightarrow \mathbf{u}(x) = C_1 x + C_2 - \frac{ax^3}{6AE}$$

C_1, C_2 are determined from the BC

Axially Loaded Bar Example

An analytical solution cannot always be found

Approximate Solution - **The Galerkin Approach (#3)**: Multiply by the weight function **w** and integrate over the domain

$$\int_0^L AE \frac{d^2\mathbf{u}}{dx^2}\mathbf{w}dx + \int_0^L ax\mathbf{w}dx = 0$$

Apply integration by parts

$$\int_0^L AE \frac{d^2\mathbf{u}}{dx^2}\mathbf{w}dx = \left[AE \frac{d\mathbf{u}}{dx}\mathbf{w} \right]_0^l - \int_0^L AE \frac{d\mathbf{u}}{dx}\frac{d\mathbf{w}}{dx}dx \Rightarrow$$

$$\int_0^L AE \frac{d^2\mathbf{u}}{dx^2}\mathbf{w}dx = \left[AE \frac{d\mathbf{u}}{dx}(L)\mathbf{w}(L) - AE \frac{d\mathbf{u}}{dx}(0)\mathbf{w}(0) \right] - \int_0^L AE \frac{d\mathbf{u}}{dx}\frac{d\mathbf{w}}{dx}dx$$

But from BC we have $u(0) = 0$, $AE\frac{d\mathbf{u}}{dx}(L)\mathbf{w}(L) = \mathbf{R}\mathbf{w}(L)$, therefore the approximate weak form can be written as

$$\int_0^L AE \frac{d\mathbf{u}}{dx}\frac{d\mathbf{w}}{dx}dx = \mathbf{R}\mathbf{w}(L) + \int_0^L ax\mathbf{w}dx$$

Axially Loaded Bar Example

Variational Approach (#1)

Let us signify displacement by u and a small (variation of the) displacement by δu. Then the various works on this structure are listed below:

$$\delta W_{int} = A \int_0^L \sigma \delta\varepsilon \, dx$$

$$\delta W_{ext} = R\delta u|_{x=L}$$

$$\delta W_{body} = \int_0^L q\delta u \, dx$$

In addition, $\sigma = E\frac{du}{dx}$

Then, from equilibrium: $\delta W_{int} = \delta W_{ext} + \delta W_{body}$

$$\rightarrow \quad A \int_0^L E\frac{du}{dx}\frac{d(\delta u)}{dx} \, dx = \int_0^L q\delta u \, dx + R\delta u|_{x=L}$$

This is the same form as earlier via another path.

Axially Loaded Bar Example

In Galerkin's method we assume that the approximate solution, $\mathbf{u}$ can be expressed as

$$\mathbf{u}(x) = \sum_{j=1}^{n} u_j N_j(x)$$

$\mathbf{w}$ is chosen to be of the same form as the approximate solution (but with arbitrary coefficients w_i),

$$\mathbf{w}(x) = \sum_{i=1}^{n} w_i N_i(x)$$

Plug $\mathbf{u}(x), \mathbf{w}(x)$ into the approximate weak form:

$$\int_0^L AE \sum_{j=1}^n u_j \frac{dN_j(x)}{dx} \sum_{i=1}^n w_i \frac{dN_i(x)}{dx} dx = \mathbf{R} \sum_{i=1}^n w_i N_i(L) + \int_0^L ax \sum_{i=1}^n w_i N_i(x) dx$$

w_i is arbitrary, so the above has to hold $\forall\ w_i$:

$$\sum_{j=1}^n \left[\int_0^L \frac{dN_j(x)}{dx} AE \frac{dN_i(x)}{dx} dx \right] u_j = \mathbf{R} N_i(L) + \int_0^L ax N_i(x) dx \quad i = 1 \dots n$$

which is a system of n equations that can be solved for the unknown coefficients u_j.

Axially Loaded Bar Example

The matrix form of the previous system can be expressed as

$$\mathbf{K}_{ij}u_j = f_i \text{ where } K_{ij} = \int_0^L \frac{dN_j(x)}{dx} AE \frac{dN_i(x)}{dx} dx$$

$$\text{and } f_i = \mathbf{R}N_i(L) + \int_0^L ax N_i(x) dx$$

Finite Element Solution - using 2 discrete elements, of length h (3 nodes)
From the iso-parametric formulation we know the element stiffness matrix
$\mathbf{K}^e = \frac{AE}{h} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$. Assembling the element stiffness matrices we get:
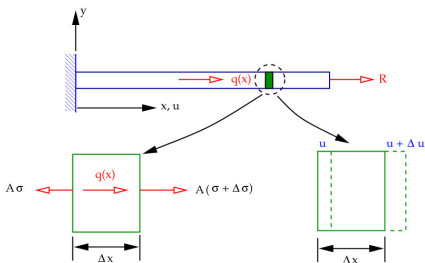
$$\mathbf{K}^{tot} = \begin{bmatrix} K_{11}^e & K_{12}^1 & 0 \\ K_{12}^1 & K_{22}^1 + K_{11}^2 & K_{12}^2 \\ 0 & K_{12}^2 & K_{22}^2 \end{bmatrix} \Rightarrow$$

$$\mathbf{K}^{tot} = \frac{AE}{h} \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

We also have that the element load vector is

$$f_i = \mathbf{R}N_i(L) + \int_0^L ax N_i(x)dx$$

Expressing the integral in iso-parametric coordinates $N_i(\xi)$ we have:

$$\frac{d\xi}{dx} = \frac{2}{h}, x = N_1(\xi)x_1^e + N_2(\xi)x_2^e, \Rightarrow$$

$$f_i = \mathbf{R}|_{i=4} + \int_0^L a(N_1(\xi)x_1^e + N_2(\xi)x_2^e)N_i(\xi)\frac{2}{h}d\xi$$

After the vectors are formulated we proceed with solving the main equation $\mathbf{Ku} = \mathbf{f} \Rightarrow \mathbf{u} = \mathbf{K}^{-1}\mathbf{f}$.

The results are plotted below using 3 elements:
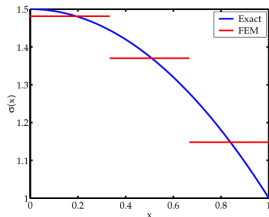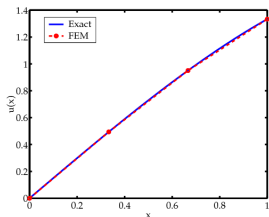
Notice how the approximation is able to track the displacement $u(x)$, despite the fact that in reality the solution is a cubic function of x (remember the analytical solution).

Since the **shape functions used, $N_i(x)$, are linear** the displacement is approximated as: $u(x) = \sum_i N_i(x)u_i$, where u_i corresponds to nodal displacements.

The **strain is then obtained as** $\epsilon = \dfrac{du}{dx} \Rightarrow \epsilon = \dfrac{dN}{dx}u_i$ where in slide 25 we have defined $B = \dfrac{dN}{dx}$ to be the so-called strain-displacement matrix.

*The section on the Beam Element is taken from **Prof. H. Waisman's notes** of the FEM II course - CEEM Department, Columbia University

F-16 Aeroelastic Structural Model
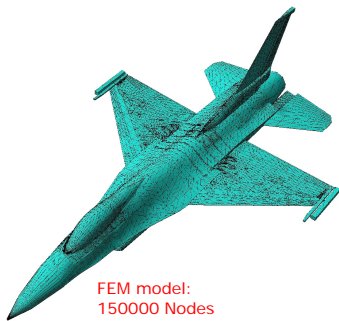
Exterior model 95% are shell elements

FEM model: 150000 Nodes
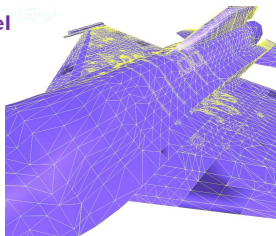
Internal structure zoom. Some Brick and tetrahedral elements

http://www.colorado.edu/engineering/CAS/Felippa.d/FelippaHome.d/Home.html
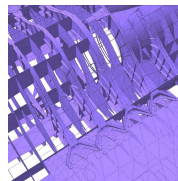
Beam Elements

Two main beam theories:

- Euler-Bernoulli theory (Engineering beam theory) -slender beams
- Timoshenko theory thick beams

Euler - Bernoulli Beam

Beam Elements
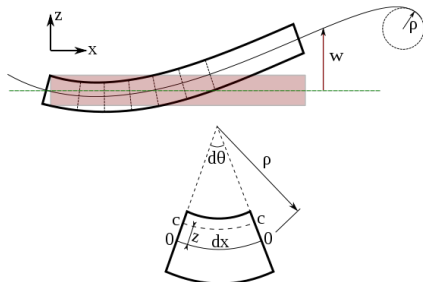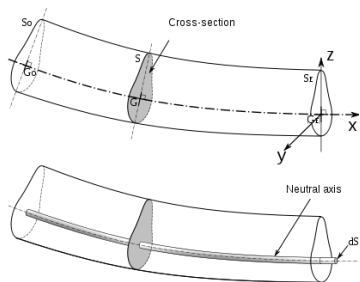
Euler Bernoulli Beam Assumptions - Kirchhoff Assumptions

- Normals remain straight (they do not bend)
- Normals remain unstretched (they keep the same length)
- Normals remain normal (they always make a right angle to the neutral plane)
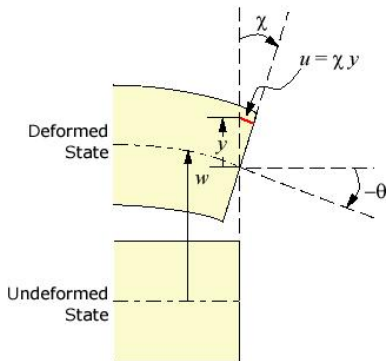
The Euler-Bernoulli Beam theory (small deformations)

$$\sigma = -\frac{M}{I}y$$

$$\epsilon = \frac{\sigma}{E}$$

$$\frac{d^2v}{dx^2} = \frac{M}{EI}$$

σ - normal stress

M - bending moment

ε - normal strain
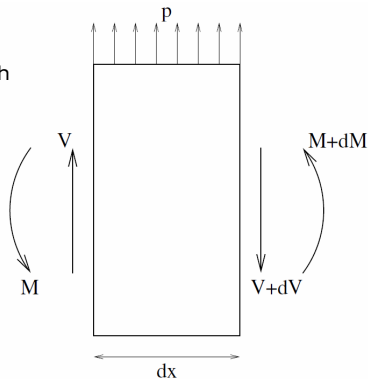
v - displacement of the centroid

EI - bending stiffness
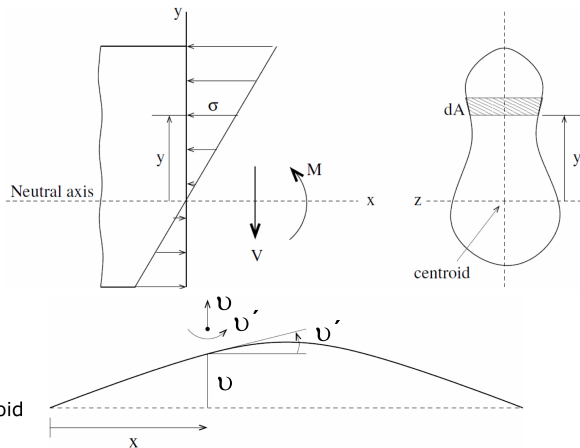
Equilibrium

$$\frac{dV}{dx} = p$$ – distributed load per unit length

$$\frac{dM}{dx} = V$$ – shear force

Combining the equations

$$\boxed{\frac{d^2}{dx^2}\left(EI\frac{d^2v}{dx^2}\right) - p = 0}$$

$$(S) \begin{cases} (1) \quad \dfrac{d^2}{dx^2}\left(EI\dfrac{d^2v}{dx^2}\right) - p(x) = 0 \quad \text{in } \Omega \\[2mm] (2) \quad v = \bar{v} \qquad \text{displacement} \qquad \text{on } \Gamma_u \\[2mm] (3) \quad \dfrac{dv}{dx} = -\bar{\theta} \qquad \text{angle} \qquad \text{on } \Gamma_\theta \\[2mm] (4) \quad EI\dfrac{d^2v}{dx^2} = \bar{M} \qquad \text{moment} \qquad \text{on } \Gamma_M \\[2mm] (5) \quad -EI\dfrac{d^3v}{dx^3} = \bar{S} \qquad \text{shear force} \qquad \text{on } \Gamma_S \end{cases}$$

$$\Gamma_u \cap \Gamma_S = \emptyset \qquad \Gamma_u \cup \Gamma_S = \Gamma$$

$$\Gamma_\theta \cap \Gamma_M = \emptyset \qquad \Gamma_\theta \cup \Gamma_M = \Gamma$$

Free end with applied load

$$EI\frac{d^2v}{dx^2} = \bar{M} \text{ on } \Gamma_M$$

$$-EI\frac{d^3v}{dx^3} = \bar{S} \text{ on } \Gamma_S$$

Simple support

$$EI\frac{d^2v}{dx^2} = 0 \text{ on } \Gamma_M$$

$$v = 0 \text{ on } \Gamma_u$$

Clamped support

$$\frac{dv}{dx} = 0 \text{ on } \Gamma_\theta$$

$$v = 0 \text{ on } \Gamma_u$$

Multiply Eqns. (1), (4) (5) by w and integrate over the domain

$$\int_0^L w\left((EIv'')'' - p(x)\right) dx = 0 \quad \text{in } \Omega$$

$$\left[w'\left(EIv'' - \bar{M}\right)\right]_{\Gamma_M} = 0$$

$$\left[w\left(-EIv''' - \bar{S}\right)\right]_{\Gamma_S} = 0$$

First integration by parts

$$\left[w\left(EIv''\right)'\right]_\Gamma - \int_0^L w'\left(EIv''\right)' dx - \int_0^L wp(x)dx = 0$$

Second integration by parts gives

$$\left[w\left(EIv''\right)'\right]_\Gamma - \left[w'\left(EIv''\right)\right]_\Gamma - \int_0^L w''EIv''dx - \int_0^L wp(x)dx = 0$$

Beam Elements - Strong Form to Weak Form

Arrive at the weak form

(W) $\begin{cases} \text{Find } v \in \mathcal{S} \text{ such that} \\[2mm] \displaystyle\int_0^L w'' EI v'' dx = \int_0^L wp(x)dx + \left[w'\bar{M}\right]_{\Gamma_M} + \left[w\bar{S}\right]_{\Gamma_S} \qquad \forall w \in \mathcal{S}^0 \\[3mm] \mathcal{S} = \{v | v \in \mathcal{C}^1, v = \bar{v} \text{ on } \Gamma_u, v' = \bar{\theta} \text{ on } \Gamma_\theta\} \\[3mm] \mathcal{S}^0 = \{w | w \in \mathcal{C}^1, w = 0 \text{ on } \Gamma_u, w' = 0 \text{ on } \Gamma_\theta\} \end{cases}$

Note:
1. The spaces are C^1 continuous, i.e. the derivative must also be continuous
2. The left side is symmetric in w and v (bi-linear form: $a(v,w)=a(w,v)$) this will lead to a symmetric Stiffness Matrix

Beam Elements - FE Formulation

Physical domain
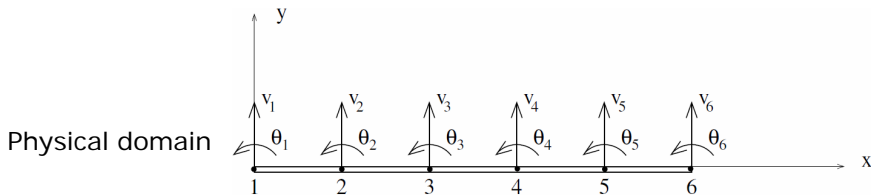
Natural domain
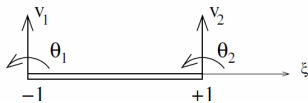
Element displacement vector
$$\boldsymbol{d}^e = \begin{bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{bmatrix}$$

Element force vector
$$\boldsymbol{f}^e = \begin{bmatrix} s_1 \\ m_1 \\ s_2 \\ m_2 \end{bmatrix}$$

Hermite Polynomials

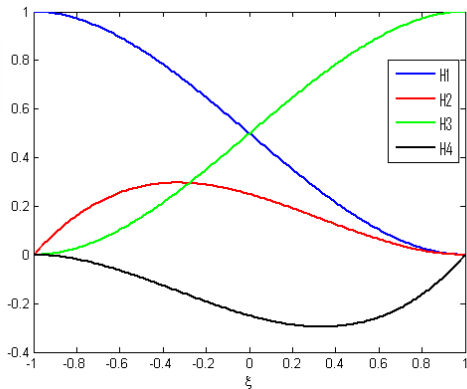$$H_I = a_I + b_I\xi + c_I\xi^2 + d_I\xi^3$$

$$H_{vI}(x_J) = \delta_{IJ} \qquad H_{\theta I,x}(x_J) = \delta_{IJ}$$

$$v_1 \rightarrow \qquad H_1 = \frac{1}{4}\left(1-\xi\right)^2\left(2+\xi\right)$$

$$\theta_1 \rightarrow \qquad H_2 = \frac{1}{4}\left(1-\xi\right)^2\left(\xi+1\right)$$

$$v_2 \rightarrow \qquad H_3 = \frac{1}{4}\left(1+\xi\right)^2\left(2-\xi\right)$$

$$\theta_2 \rightarrow \qquad H_4 = \frac{1}{4}\left(1+\xi\right)^2\left(\xi-1\right)$$

Note: The choice of a cubic polynomial is related to the homogeneous strong form of the problem $EIv'''' = 0$.

Finally, the weight functions and trial solutions are:

$$v(\xi) = H_1 v_1 + H_2 \left(\frac{dv}{d\xi}\right)_1 + H_3 v_2 + H_4 \left(\frac{dv}{d\xi}_1\right)_2$$

However note, that the rotation is actually the derivative of the (vertical) deflection: $\theta = \dfrac{dv}{dx}$

The connection between $\dfrac{dv}{dx}$ and $\dfrac{dv}{d\xi}$ is delivered via the **Jacobian**. This is calculated from the coordinate transformation relationship:

$$x = \frac{1-\xi}{2}x_1^e + \frac{1+\xi}{2}x_2^e \Rightarrow \frac{dv}{d\xi} = \theta = \frac{l^e}{2}\frac{dv}{dx}$$

$$J = \frac{dx}{d\xi} = \frac{l^e}{2}$$

where l^e is the length of the element.

Beam Elements - FE Matrices

From the weak form, we had

$$\int_0^L w'' E I v'' dx = \int_0^L w p(x) dx + \left[w' \bar{M}\right]_{\Gamma_M} + \left[w \bar{S}\right]_{\Gamma_S}$$

The second derivative of the shape functions of the element, $\mathbf{H}^e$, therefore needs to be calculated:

$$\frac{d\mathbf{H}^e}{dx^2} = \mathbf{B}^e = \frac{d\mathbf{H}^e}{d\xi^2} J^2 = \frac{1}{I^e} \left[\frac{6\xi}{I^e} \quad 3\xi - 1 \quad -\frac{6\xi}{I^e} \quad 3\xi + 1\right]$$

$$\text{where} \quad J = \frac{I^e}{2}$$

Matrix $\mathbf{B}^e$ now connects the curvature of the element, $\dfrac{d^2 v}{dx^2}$ to the nodal displacement vector $\mathbf{d}^e$:

$$\frac{d^2 v}{dx^2} = \mathbf{B}^e \mathbf{d}^e$$

Stiffness matrix

$$\boldsymbol{K}^e = \int_{\Omega^e} (\boldsymbol{B}^e)^T EI\boldsymbol{B}^e d\Omega^e = \frac{EI}{(l^e)^3} \begin{bmatrix} 12 & 6l^e & -12 & 6l^e \\ 6l^e & 4(l^e)^2 & -6l^e & 2(l^e)^2 \\ -12 & -6l^e & 12 & -6l^e \\ 6l^e & 2(l^e)^2 & -6l^e & 4(l^e)^2 \end{bmatrix}$$

Force vector

$$\boldsymbol{f}^e = \underbrace{\int_{\Omega} (\boldsymbol{H}^e)^T p(x)dx}_{\boldsymbol{f}^e_{\Omega}} + \underbrace{\left[\frac{d(\boldsymbol{H}^e)^T}{dx}\bar{M}\right]_{\Gamma_M} + \left[(\boldsymbol{H}^e)^T \bar{S}\right]_{\Gamma_S}}_{\boldsymbol{f}^e_{\Gamma}}$$

Assuming **constant** distributed force

$$\boldsymbol{f}^e_{\Omega} = \int_{\Omega} (\boldsymbol{H}^e)^T p dx = p \int_{\Omega} \begin{bmatrix} H_{v1} \\ H_{\theta 1} \\ H_{v2} \\ H_{\theta 2} \end{bmatrix} dx = \frac{pl^e}{2} \begin{bmatrix} 1 \\ \frac{l^e}{6} \\ 1 \\ -\frac{l^e}{6} \end{bmatrix}$$
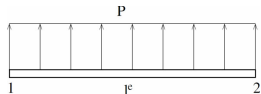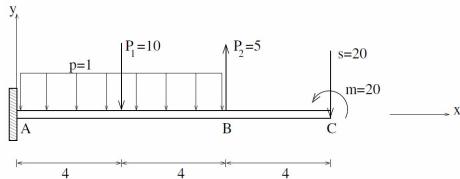
Consider a clamped-free beam
with $EI = 10^4 \, Nm^2$
$s = -20N$
$m = 20Nm$

Pre-processing

$$\boldsymbol{K}^e = \int_{\Omega^e} \left(\boldsymbol{B}^e\right)^T EI \boldsymbol{B}^e d\Omega^e = \frac{EI}{\left(l^e\right)^3} \begin{bmatrix} 12 & 6l^e & -12 & 6l^e \\ 6l^e & 4\left(l^e\right)^2 & -6l^e & 2\left(l^e\right)^2 \\ -12 & -6l^e & 12 & -6l^e \\ 6l^e & 2\left(l^e\right)^2 & -6l^e & 4\left(l^e\right)^2 \end{bmatrix}$$

For element (1)

$$K^1 = 10^3 \begin{bmatrix} 0.23 & 0.94 & -0.23 & 0.94 \\ 0.94 & 5.00 & -0.94 & 2.50 \\ -0.23 & -0.94 & 0.23 & -0.94 \\ 0.94 & 2.50 & -0.94 & 5.00 \end{bmatrix}$$

$$\quad\quad [1] \quad\; [2] \quad\;\; [3] \quad\;\; [4]$$

For element (2)

$$K^2 = 10^3 \begin{bmatrix} 1.88 & 3.75 & -1.88 & 3.75 \\ 3.75 & 10.00 & -3.75 & 5.00 \\ -1.88 & -3.75 & 1.88 & -3.75 \\ 3.75 & 5.00 & -3.75 & 10.00 \end{bmatrix}$$

$$\quad\quad [3] \quad\; [4] \quad\;\; [5] \quad\;\; [6]$$

Assembly into a *Global Stiffness Matrix*

$$K = 10^3 \begin{bmatrix} 0.23 & 0.94 & -0.23 & 0.94 & 0 & 0 \\ & 5.00 & -0.94 & 2.5 & 0 & 0 \\ & & 2.11 & 2.81 & -1.88 & 3.75 \\ & & & 15.00 & -3.75 & 5.00 \\ & SYM & & & 1.88 & -3.75 \\ & & & & & 10.00 \end{bmatrix}$$
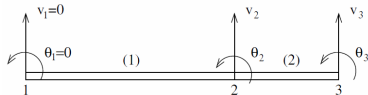
Boundary force matrix

$$\boldsymbol{f}_\Gamma^e = \left[\frac{d\left(\boldsymbol{H}^e\right)^T}{dx}\bar{M}\right]_{\Gamma_M} + \left[\left(\boldsymbol{H}^e\right)^T \bar{S}\right]_{\Gamma_S}$$

Element (1) has no boundary on Γ_S or Γ_M $\implies$ $\boldsymbol{f}_\Gamma^1 = \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}^T$

For element (2) we have
$$\boldsymbol{f}_\Gamma^2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}_{\Gamma_M} + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}_{\Gamma_S} = \begin{bmatrix} 0 \\ 0 \\ -20 \\ 20 \end{bmatrix} \begin{matrix} [3] \\ [4] \\ [5] \\ [6] \end{matrix}$$

Assembly to *global boundary force vector*

$$\boldsymbol{f}_\Gamma = \begin{bmatrix} 0 & 0 & 0 & 0 & -20 & 20 \end{bmatrix}^T$$

$$\boldsymbol{f}_\Omega^e = \int_\Omega (\boldsymbol{H}^e)^T \, p \, dx = p \int_\Omega \begin{bmatrix} H_{v1} \\ H_{\theta 1} \\ H_{v2} \\ H_{\theta 2} \end{bmatrix} dx = \frac{pl^e}{2} \begin{bmatrix} 1 \\ \frac{l^e}{6} \\ 1 \\ -\frac{l^e}{6} \end{bmatrix}$$ *(distributed loads)*

$$\boldsymbol{f}_\Omega^e = \boldsymbol{H}(\xi_A) P_A \quad \textit{(Point loads)}$$

For element (1) Given: $\quad P_1 = -10 \;\; \text{and} \;\; p(x) = -1$

$$\boldsymbol{f}_\Omega^1 = \int_\Omega (\boldsymbol{H}^e)^T \, p \, dx + \boldsymbol{H}(\xi_A) P_A = p \int_\Omega \begin{bmatrix} H_{v1} \\ H_{\theta 1} \\ H_{v2} \\ H_{\theta 2} \end{bmatrix} dx + P_1 \begin{bmatrix} H_{v1} \\ H_{\theta 1} \\ H_{v2} \\ H_{\theta 2} \end{bmatrix}_{\xi=0} = \begin{bmatrix} -9 \\ -15.3 \\ -9 \\ 15.3 \end{bmatrix}$$

For element (2) Given: $\quad P_2 = 5$

$$\boldsymbol{f}_\Omega^2 = P_2 \begin{bmatrix} H_{v1} \\ H_{\theta 1} \\ H_{v2} \\ H_{\theta 2} \end{bmatrix}_{\xi=-1} = \begin{bmatrix} 5 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$
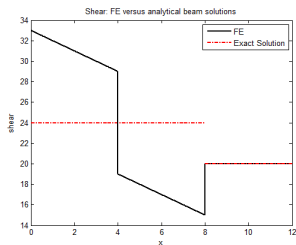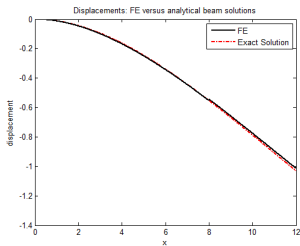
The global force vector

$$f = \begin{bmatrix} -9 \\ -15.3 \\ -4 \\ 15.3 \\ -20 \\ 20 \end{bmatrix}$$

⟹ $\boxed{\boldsymbol{K}\boldsymbol{d} = \boldsymbol{f}}$ ⟹

$$\begin{bmatrix} v_2 \\ \theta_2 \\ v_3 \\ \theta_3 \end{bmatrix} = \text{KNOWN}$$

Post-processing

$$m^1 = EI\frac{d^2 v^1}{dx^2} = EI\frac{d^2 \boldsymbol{H}}{dx^2}\boldsymbol{d}^1$$

$$s^1 = -EI\frac{d^3 v^1}{dx^3} = -EI\frac{d^3 \boldsymbol{H}}{dx^3}\boldsymbol{d}^1$$

$$m^2 = EI\frac{d^2 v^2}{dx^2} = EI\frac{d^2 \boldsymbol{H}}{dx^2}\boldsymbol{d}^2$$

$$s^2 = -EI\frac{d^3 v^2}{dx^3} = -EI\frac{d^3 \boldsymbol{H}}{dx^3}\boldsymbol{d}^2$$

The results are plotted below using 2 elements:

Displacements: FE versus analytical beam solutions

Moments: FE versus analytical beam solutions

Shear: FE versus analytical beam solutions
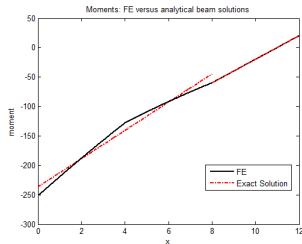
Notice how the approximation is able to accurately track the displacement $u(x)$

Since the **shape functions used, $H_i(x)$, are cubic** the moment is linear as a 2nd derivative and the shear is constant as a 3rd derivative.

If one wishes a better approximation, the use of 3 elements instead f two would be preferable in this case.

Strong Form - 2D Linear Elasticity FEM

Governing Equations

Equilibrium Eq: $\nabla_s \boldsymbol{\sigma} + \mathbf{b} = 0 \quad \in \Omega$

Kinematic Eq: $\boldsymbol{\epsilon} = \nabla_s u \quad \in \Omega$

Constitutive Eq: $\boldsymbol{\sigma} = D \cdot \boldsymbol{\epsilon} \quad \in \Omega$

Traction B.C.: $\boldsymbol{\tau} \cdot n = T_s \quad \in \Gamma_t$

Displacement B.C: $\mathbf{u} = \mathbf{u}_\Gamma \quad \in \Gamma_u$
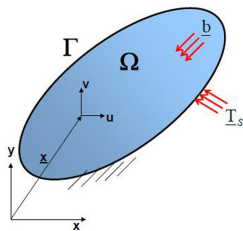
Hooke's Law - Constitutive Equation

Plane Stress
$\tau_{zz} = \tau_{xz} = \tau_{yz} = 0, \epsilon_{zz} \neq 0$

$$D = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}$$

Plane Strain
$\epsilon_{zz} = \gamma_{xz} = \gamma_{yz} = 0, \sigma_{zz} \neq 0$

$$D = \frac{E}{(1-\nu)(1+\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix}$$

Divide the body into finite elements connected to each other through nodes

$$\underline{d} = \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix}$$
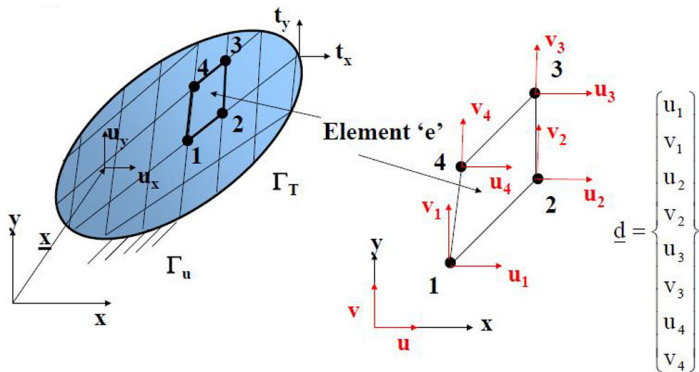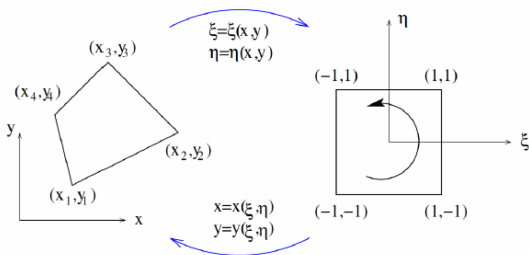
Shape Functions in Natural Coordinates

$$N_1(\xi, \eta) = \frac{1}{4}(1 - \xi)(1 - \eta), \qquad N_2(\xi, \eta) = \frac{1}{4}(1 + \xi)(1 - \eta)$$
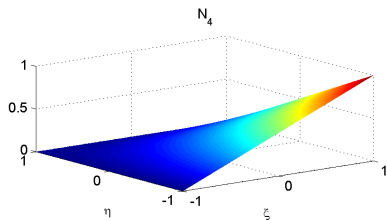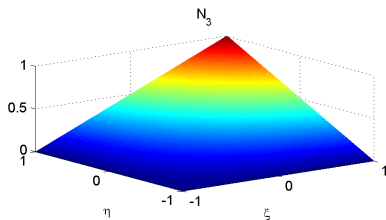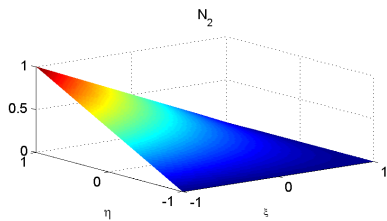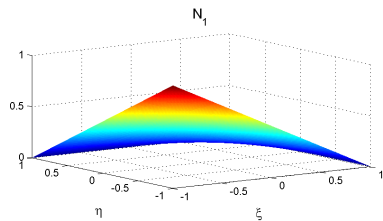
$$N_3(\xi, \eta) = \frac{1}{4}(1 + \xi)(1 + \eta), \qquad N_4(\xi, \eta) = \frac{1}{4}(1 - \xi)(1 + \eta)$$

Iso-parametric Mapping

$$x = \sum_{i=1}^{4} N_i(\xi, \eta) x_i^e$$

$$y = \sum_{i=1}^{4} N_i(\xi, \eta) y_i^e$$
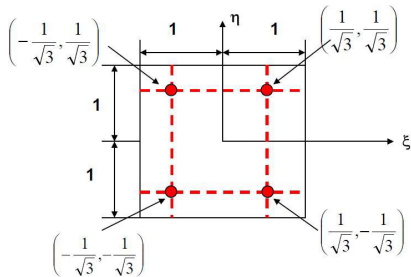
2D FE formulation: Matrices

from the Principle of Minimum Potential Energy (see slide #9)

$$\frac{\partial \Pi}{\partial d} = 0 \Rightarrow \mathbf{K} \cdot d = f$$

where

$$\mathbf{K^e} = \int_{\Omega^e} \mathbf{B}^T \mathbf{D} \mathbf{B} d\Omega, \quad f^e = \int_{\Omega^e} N^T B d\Omega + \int_{\Gamma_T^e} N^T t_s d\Gamma$$

Gauss Quadrature

$$I = \int_{-1}^{1} \int_{-1}^{1} f(\xi, \eta) d\xi d\eta$$

$$= \sum_{i=1}^{Ngp} \sum_{i=1}^{Ngp} W_i W_j f(\xi_i, \eta_j)$$

where W_i, W_j are the weights and (ξ_i, η_j) are the integration points.