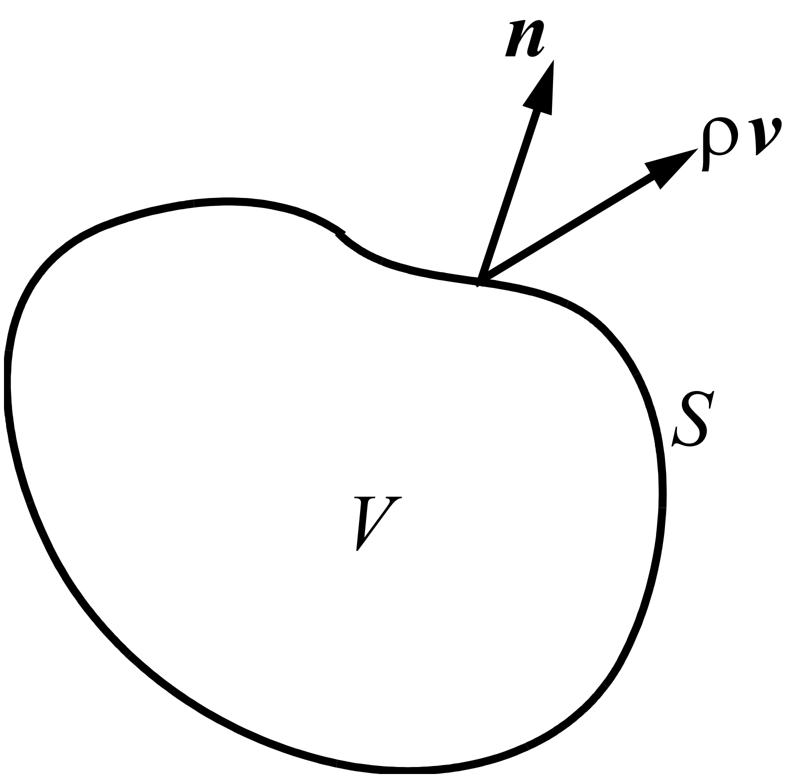
The Fundamental Equations of Gas Dynamics

1 References for astrophysical gas (fluid) dynamics

1. L. D. Landau and E. M. Lifshitz *Fluid mechanics*
2. F.H. Shu *The Physics of Astrophysics Volume II Gas Dynamics*
3. L. Mestel *Stellar Magnetism*

2 The equation of continuity

2.1 Derivation of the fundamental equation



Define

$$\begin{aligned} \rho &= \text{density} \\ v_i &= \text{velocity components} \end{aligned} \quad (1)$$

Mass within volume V is

$$M = \int_V \rho dV \quad (2)$$

Rate of flow of mass out of V is

$$\int_S \rho v_i n_i dS$$

Therefore, the mass balance within V is given by:

$$\frac{d}{dt}\int_V \rho dV = -\int_S \rho v_i n_i dS \tag{3}$$

Use the divergence theorem:

$$\int_S \rho v_i n_i dS = \int_V \frac{\partial}{\partial x_i}(\rho v_i) dV \tag{4}$$

and

$$\frac{d}{dt}\int_V \rho\, dV = \int_V \frac{\partial \rho}{\partial t} dV \tag{5}$$

to give

$$\int_V \left[\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_i}(\rho v_i)\right] dV = 0 \tag{6}$$

Since the volume V is arbitrary, then

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_i}(\rho v_i) = 0 \tag{7}$$

This is the equation of continuity.

Aside: Differentiation following the motion

Suppose we have a function $f(x_i, t)$ which is a function of both space and time. How does this function vary along the trajectory of a fluid element described by $x_i = x_i(t)$? We simply calculate

$$\frac{d}{dt}f(x_i(t), t)) = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x_i}\frac{dx_i}{dt} = \frac{\partial f}{\partial t} + v_i\frac{\partial f}{\partial x_i} \tag{8}$$

Useful result from the equation of continuity

Write the above equation in the form:

$$\frac{\partial \rho}{\partial t} + v_i \frac{\partial \rho}{\partial x_i} + \rho \frac{\partial v_i}{\partial x_i} = 0 \qquad (9)$$

Going back to the equation of continuity, then

$$\frac{d\rho}{dt} = -\rho \frac{\partial v_i}{\partial x_i} \Rightarrow \frac{\partial v_i}{\partial x_i} = -\frac{1}{\rho}\frac{d\rho}{dt} \qquad (10)$$

Putting it another way:

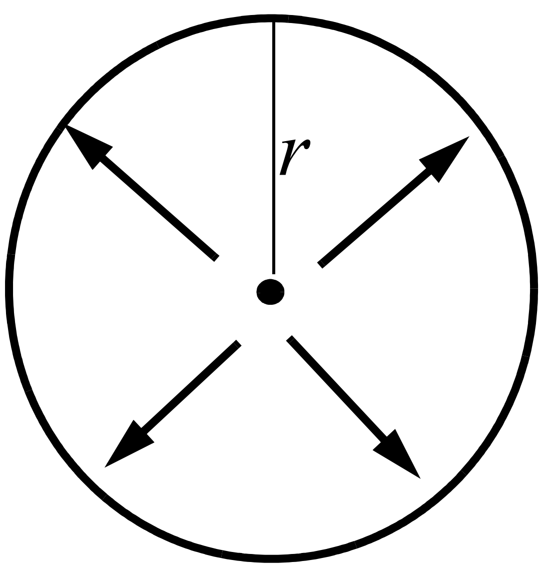
$$\frac{1}{\rho}\frac{d\rho}{dt} = -\frac{\partial v_i}{\partial x_i} \qquad (11)$$

This tells us that in a diverging velocity field $\partial v_i / \partial x_i > 0$, the density decreases and in a converging velocity field $\partial v_i / \partial x_i < 0$ the density increases.

2.2 Examples of mass flux

Wind from a massive star

Massive stars such as O and B stars produce winds with velocities of around $1,000\ \mathrm{km\ s^{-1}}$ and mass fluxes of around 10^{-6} solar masses per yr. We can use these facts to estimate the density in the wind as follows.



$$\begin{aligned}\text{Mass flux} &= \int_S \rho v_i n_i dS \\ &= 4\pi r^2 \rho(r) v(r)\end{aligned} \tag{12}$$

where the integral is over a sphere of radius r. We assume that the flow is steady so that the mass flux integrated over any 2 surfaces surrounding the star is the same. Hence,

$$\dot{M} = 4\pi r^2 \rho(r) v(r) = \text{constant} \qquad (13)$$

We shall show in later lectures that far from the star, the velocity is constant, i.e.

$$v(r) = v_\infty = \text{constant} \qquad (14)$$

Therefore the density is given by:

$$\rho(r) = \frac{\dot{M}}{4\pi r^2 v_\infty} \qquad (15)$$

For typical parameters:

$$\begin{aligned} \dot{M} &= 10^{-6} \text{ solar masses per yr} \\ r &= 0.1 \text{ pc} = 3.09\times10^{15} \text{ m} \\ v_\infty &= 1,000 \text{ km s}^{-1} = 10^{6} \text{ m s}^{-1} \end{aligned} \qquad (16)$$

Units

$$1 \text{ solar mass} = 2\times10^{30} \text{ kg} \qquad 1 \text{ pc} = 3.09\times10^{16} \text{ m}$$
$$1 \text{ year} = 3.16\times10^{7} \text{ seconds} \qquad (17)$$

The density of the wind at 0.1 pc from the star is:

$$\rho = 5.2\times10^{-22} \text{ kg m}^{-3} \qquad (18)$$

This is not particularly informative by itself. We usually express the density in terms of particles cm^{-3}. Assuming the gas is totally ionised, then

$$\rho = n_H m_p + n_{He} m_{He} + \ldots\ldots \tag{19}$$

Now for solar cosmic abundances,

$$n_{He} = 0.085 n_H \tag{20}$$

and

$$m_{He} \approx 4m \tag{21}$$

where m is an atomic mass unit

$$m = 1.66 \times 10^{-27}\ \mathrm{kg} \tag{22}$$

so that

$$\rho \approx n_H m + 4 \times 0.085 n_H m = 1.34 n_H m \tag{23}$$

Therefore, for the O-star wind:

$$1.34 n_H m \approx 5.2 \times 10^{-22} \text{ kg m}^{-3}$$

$$\Rightarrow n_H = \frac{5.2 \times 10^{-22}}{1.34 m} \text{m}^{-3} \tag{24}$$

$$= 2.3 \times 10^{5} \text{ Hydrogen atoms m}^{-3}$$

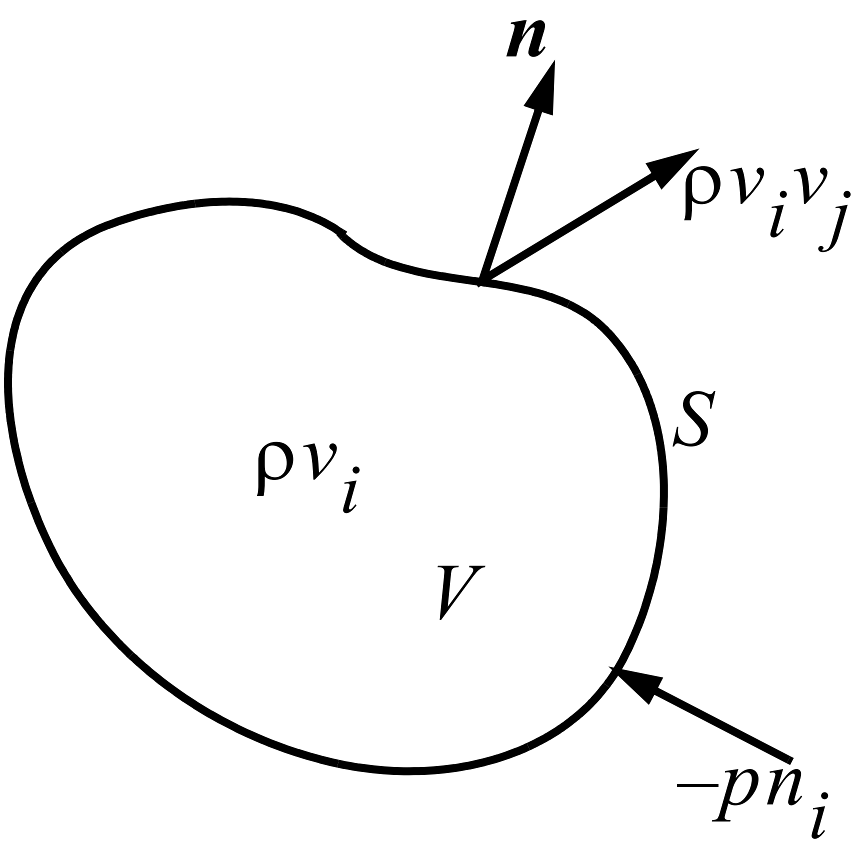
Often, instead of the Hydrogen density we use the total number density of ions plus electrons. In this case we put

$$\begin{aligned} \rho &= \mu n m \\ \mu &= \text{mean molecular weight} = 0.6156 \end{aligned} \qquad (25)$$

Hence,

$$n = \frac{5.2\times10^{-22}}{0.6156m} = 5.1\times10^{5} \text{ particles m}^{-3} \qquad (26)$$

3 Conservation of momentum



Consider first the rate of change of momentum within a volume as a result of the flux of momentum. Let Π_i = total momentum, then

$$\dot{\Pi}_i = \frac{d}{dt}\int_V \rho v_i dV \qquad (27)$$

is the rate of change of momentum in the volume, V.

The flux of momentum out of the surface S is

$$\int_S \rho v_i v_j n_j dS \tag{28}$$

3.1 Surface force

There are two complementary ways that we can look at the other aspects of the conservation of momentum.

In a perfect fluid there is a force per unit area perpendicular to the surface that exerts a force on the gas inside. This is the pessure p

$$\begin{array}{c}\text{Force on volume} \\ \text{within } S\end{array} = -\int_S p n_i dS \tag{29}$$

Therefore the total momentum balance for the volume is:

$$\frac{d}{dt}\int_V \rho v_i dV = -\int_S \rho v_i v_j n_j dS - \int_S p n_i dS \tag{30}$$

The surface integrals

Using the divergence we can write

$$\int_S \rho v_i v_j n_j dS = \int_V \frac{\partial}{\partial x_j}(\rho v_i v_j) dV \tag{31}$$

The surface integral of the pressure can be written

$$\int_S p n_i dS = \int_S p \delta_{ij} n_j dS = \int_V \frac{\partial}{\partial x_j}(p \delta_{ij}) dV \tag{32}$$

Entire momentum equation

Again, we take the time derivative inside and obtain:

$$\int_V \left[\frac{\partial}{\partial t}(\rho v_i) + \frac{\partial}{\partial x_j}(\rho v_i v_j) + \frac{\partial}{\partial x_j}(p\delta_{ij})\right] dV = 0 \qquad (33)$$

Since the volume is arbitrary, then

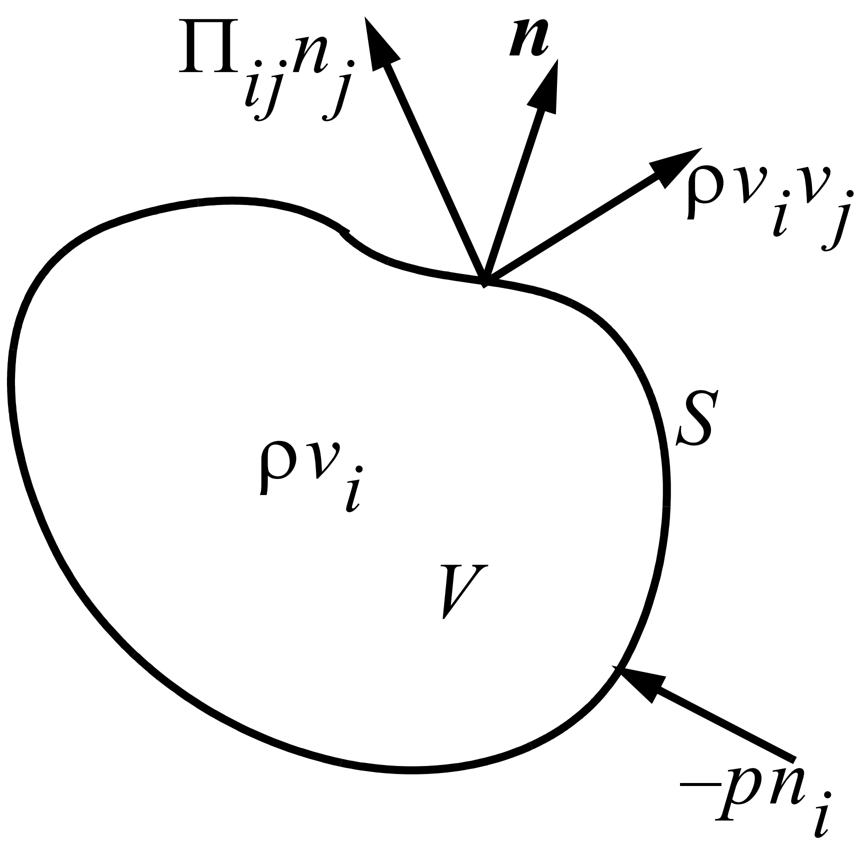
$$\frac{\partial}{\partial t}(\rho v_i) + \frac{\partial}{\partial x_j}(\rho v_i v_j + p\delta_{ij}) = 0 \qquad (34)$$

We often write this equation in the form:

$$\frac{\partial}{\partial t}(\rho v_i) + \frac{\partial}{\partial x_j}(\rho v_i v_j) = -\frac{\partial p}{\partial x_i} \qquad (35)$$

3.2 2nd way - momentum flux

Across any surface in a gas there is a flux of momentum due to the random motions of atoms in the gas.



We write the flux per unit area of momentum due to molecular motions as $\Pi_{ij}n_j$

The flux of momentum ***out*** of the volume due to molecular motions is then:

$$\int_V \Pi_{ij} n_j dS \qquad (36)$$

For a perfect fluid, the relation between Π_{ij} and p is

$$\Pi_{ij} = p\delta_{ij} \tag{37}$$

In viscous fluids (to be treated later) the tensor Π_{ij} is not diagonal.

Momentum balance

When we adopt this approach, the momentum balance of the volume of gas is:

$$\frac{d}{dt}\int_V \rho v_i dV = -\int_S \rho v_i v_j n_j dS - \int_S \Pi_{ij} n_j dS$$
$$\Rightarrow \int_V \left[\frac{\partial}{\partial t}(\rho v_i) + \frac{\partial}{\partial x_j}(\rho v_i v_j + \Pi_{ij})\right] dV = 0 \quad (38)$$

The corresponding partial differential equation is:

$$\frac{\partial}{\partial t}(\rho v_i) + \frac{\partial}{\partial x_j}(\rho v_i v_j + \Pi_{ij}) = 0 \quad (39)$$

The point to remember from this analysis is that pressure in a fluid is the result of a flux of momentum resulting from the microscopic motions of the particles.

Particles of mass m crossing a surface within the fluid with random atomic velocity u_i (relative to the bulk velocity) contribute an amount mu_iu_j to the flux of momentum. A particle with an equally opposite velocity contributes exactly the same amount to the momentum flux. Hence, atomic motions in both directions across the surface contribute equal amounts to the pressure.

3.3 Alternate forms of the momentum equations

Consider, the isotropic case where $\Pi_{ij} = p\delta_{ij}$. Then,

$$\frac{\partial}{\partial t}(\rho v_i) + \frac{\partial}{\partial x_j}(\rho v_i v_j) = -\frac{\partial p}{\partial x_i} \tag{40}$$

Expand the terms on the left:

$$v_i\frac{\partial \rho}{\partial t} + \rho\frac{\partial v_i}{\partial t} + v_i\frac{\partial}{\partial x_j}(\rho v_j) + \rho v_j\left(\frac{\partial v_i}{\partial x_j}\right) = -\frac{\partial p}{\partial x_i} \tag{41}$$

Using the continuity equation to eliminate the blue terms:

$$\rho\frac{\partial v_i}{\partial t} + \rho v_j\frac{\partial v_i}{\partial x_j} = -\frac{\partial p}{\partial x_i} \tag{42}$$

In terms of the derivative following the motion:

$$\rho \frac{dv_i}{dt} = -\frac{\partial p}{\partial x_i} \qquad (43)$$

3.4 Additional forces

We can in general have additional forces acting on the fluid. In particular, we can have a gravitational force derived from a gravitational potential. This is not a surface force like the pressure but a body force which is proportional to the volume of the region. We write:

$$\text{Gravitational force per unit mass} = -\frac{\partial \phi}{\partial x_i} \qquad (44)$$

where ϕ is the gravitational potential. Therefore the gravitational force acting on volume V is

$$F_i = -\int_V \rho \frac{\partial \phi}{\partial x_i} dV \qquad (45)$$

We add this term to the momentum equations to obtain:

$$\int_V \left[\frac{\partial}{\partial t}(\rho v_i) + \frac{\partial}{\partial x_j}(\rho v_i v_j + \Pi_{ij})\right] dV = -\int_V \rho \frac{\partial \phi}{\partial x_i} dV \qquad (46)$$

implying that:

$$\frac{\partial}{\partial t}(\rho v_i) + \frac{\partial}{\partial x_j}(\rho v_i v_j + \Pi_{ij}) = -\rho \frac{\partial \phi}{\partial x_i} \qquad (47)$$

or the alternative form:

$$\rho\frac{\partial v_i}{\partial t} + \rho v_j\frac{\partial v_i}{\partial x_j} = -\frac{\partial \Pi_{ij}}{\partial x_j} - \rho\frac{\partial \phi}{\partial x_i} \tag{48}$$

When the momentum flux is diagonal

$$\Pi_{ij} = p\delta_{ij} \tag{49}$$

$$\frac{\partial}{\partial t}(\rho v_i) + \frac{\partial}{\partial x_j}(\rho v_i v_j) = -\frac{\partial p}{\partial x_i} - \rho\frac{\partial \phi}{\partial x_i}$$

$$\rho\frac{\partial v_i}{\partial t} + \rho v_j\frac{\partial v_i}{\partial x_j} = -\frac{\partial p}{\partial x_i} - \rho\frac{\partial \phi}{\partial x_i} \tag{50}$$

3.5 Hydrostatic application of the momentum equations

X-ray emitting atmosphere in an elliptical galaxy

Elliptical galaxies have extended hot atmospheres extending for 10 – 100 kpc from the centre of the galaxy.

Our first approach to understanding the distribution of gas in such an atmosphere is to consider an hydrostatic model.

In this case $v_i = 0$ and the momentum equations reduce to

$$\frac{1}{\rho}\frac{\partial p}{\partial x_i} = -\frac{\partial \phi}{\partial x_i} \tag{51}$$

Let us restrict ourselves to the case of spherical symmetry:

$$\frac{1}{\rho}\frac{\partial p}{\partial r} = -\frac{\partial \phi}{\partial r} = -\frac{GM(r)}{r^2} \tag{52}$$

$$\begin{aligned} G &= \text{Newtons constant of gravitation} \\ &= 6.67\times 10^{-11}\,\text{SI units} \\ M(r) &= \text{Mass within } r \end{aligned} \tag{53}$$

This can be used to estimate the mass of the galaxy. Put

$$p = nkT = \frac{\rho kT}{\mu m} \tag{54}$$

then

$$\frac{1}{\rho}\frac{d}{dr}\left[\frac{\rho kT}{\mu m}\right] = -\frac{GM(r)}{r^2}$$

$$\frac{1}{\rho}\left[\frac{kT}{\mu m}\frac{d\rho}{dr} + \frac{\rho k}{\mu m}\frac{dT}{dr}\right] = -\frac{GM(r)}{r^2} \tag{55}$$

Rearrangement of terms gives:

$$M(r) = -\frac{r^2}{G}\frac{kT}{\mu m}\left[\frac{1}{\rho}\frac{d\rho}{dr} + \frac{1}{T}\frac{dT}{dr}\right] \tag{56}$$

Usually, at large radii, the temperature of the atmosphere is isothermal and the density of the X-ray emitting gas may be approximated by a power-law:

$$T \approx \text{constant} \qquad \rho \propto r^{-\alpha} \qquad \alpha \approx 0.7$$
$$\Rightarrow \frac{r}{\rho}\frac{d\rho}{dr} = -\alpha \tag{57}$$

Therefore,

$$\begin{aligned} M(r) &= -\frac{r^2}{G}\frac{kT}{\mu m_p}\left[\frac{1}{\rho}\frac{d\rho}{dr} + \frac{1}{T}\frac{dT}{dr}\right] = \frac{\alpha kT}{\mu m_p G} r \\ &= 3.1\times 10^{11}\alpha\left(\frac{T}{10^7}\right)\left(\frac{r}{10\ \text{kpc}}\right)\ \text{solar masses} \end{aligned} \tag{58}$$

The amount of matter implied by such observations is larger than can be accounted for by the stellar light from elliptical galaxies and implies that elliptical galaxies, like spiral galaxies, have large amounts of dark matter.

4 Entropy

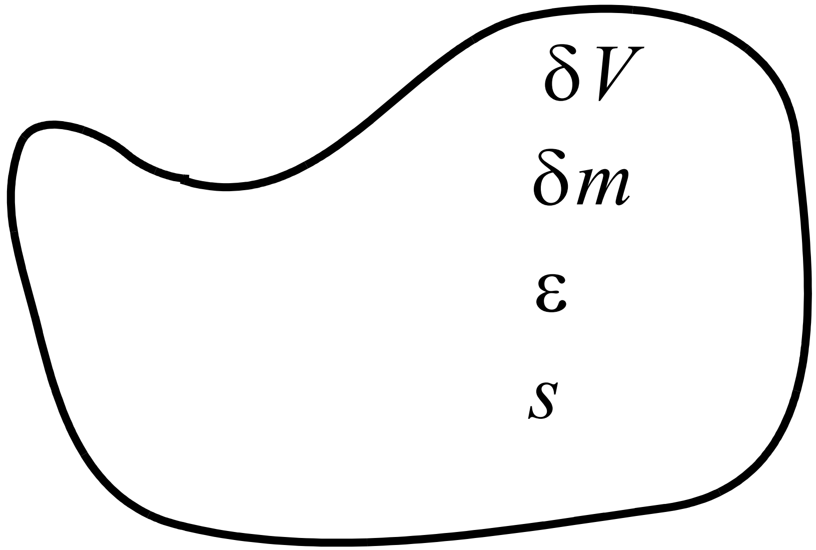
In any dynamical system, the conservation of mass, momentum and energy are the fundamental principles to consider.

However, before proceeding with the conservation of energy it is necessary to make an excursion into the domain of entropy.

4.1 Entropy of a fluid

Consider an element of fluid with:

$$\begin{aligned} \varepsilon &= \text{Internal energy density (per unit volume)} \\ p &= \text{pressure} \\ \rho &= \text{density} \\ s &= \text{entropy per unit mass} \\ \delta m &= \text{mass of the element} \end{aligned} \tag{59}$$



$$\begin{aligned} \text{Volume} &= \frac{\delta m}{\rho} \\ \text{Entropy} &= s\delta m \\ \text{Internal energy} &= \frac{\varepsilon \delta m}{\rho} \end{aligned} \qquad (60)$$

The relationship between internal energy (U), pressure, volume (V) and entropy (S) of a gas is

$$kTdS = dU + pdV \qquad (61)$$

For the infinitesimal volume, above

$$kTd(s\delta m) = d\left(\frac{\varepsilon\delta m}{\rho}\right) + pd\left(\frac{\delta m}{\rho}\right) \tag{62}$$

The mass δm is constant, therefore

$$kTds = d\left(\frac{\varepsilon}{\rho}\right) + pd\left(\frac{1}{\rho}\right) \tag{63}$$

Other ways of expressing the entropy relation

Expanding the differentials:

$$kTds = \frac{1}{\rho}d\varepsilon - \left(\frac{\varepsilon + p}{\rho^2}\right)d\rho$$

$$\rho kTds = d\varepsilon - \left(\frac{\varepsilon + p}{\rho}\right)d\rho \qquad (64)$$

Specific enthalpy

$$h = \frac{\varepsilon + p}{\rho} = \text{Enthalpy per unit mass} \qquad (65)$$

$$= \text{Specific enthalpy}$$

Expression using the enthalpy

$$
\begin{aligned}
kTds &= \frac{1}{\rho}d\varepsilon - \left(\frac{\varepsilon + p}{\rho^2}\right)d\rho \\
&= \frac{1}{\rho}d(\varepsilon + p) - \left(\frac{\varepsilon + p}{\rho^2}\right)d\rho - \frac{dp}{\rho} \\
&= dh - \frac{dp}{\rho}
\end{aligned}
\tag{66}
$$

To be symmetric with the previous expression

$$
\rho kTds = \rho dh - dp \tag{67}
$$

Relation between thermodynamic variables throughout the fluid

These relationships have been derived for a given fluid element. However, the equation of state of a gas can be expressed in the form

$$p = p(\rho, s) \tag{68}$$

so that any relationship derived between the thermodynamic variables is valid everywhere. Therefore the differential expressions

$$\begin{aligned} \rho k T ds &= d\varepsilon - \left(\frac{\varepsilon + p}{\rho}\right) d\rho = d\varepsilon - h d\rho \\ \rho k T ds &= \rho dh - dp \end{aligned} \tag{69}$$

are valid relationships between the differentials of s, ε, ρ, p throughout the fluid.

In particular when considering the energy equation, we look at the changes in these quantities resulting from temporal or spatial changes. We often use the first form for temporal or spatial changes and the second form for spatial changes.

Thus consider the change in thermodynamic variables due to temporal changes alone:

The first differential form tells us that:

$$\rho k T \frac{\partial s}{\partial t} = \frac{\partial \varepsilon}{\partial t} - \frac{(\varepsilon + p)}{\rho}\frac{\partial \rho}{\partial t} = \frac{\partial \varepsilon}{\partial t} - h\frac{\partial \rho}{\partial t} \qquad (70)$$

and for spatial changes

$$\rho kT\frac{\partial s}{\partial x_i} = \frac{\partial \varepsilon}{\partial x_i} - \frac{(\varepsilon + p)}{\rho}\frac{\partial \rho}{\partial x_i} \tag{71}$$

For spatial changes we often use:

$$\rho kT\frac{\partial s}{\partial x_i} = \rho\frac{\partial h}{\partial x_i} - \frac{\partial p}{\partial x_i} \tag{72}$$

Derivation along a fluid trajectory

Another use for the entropy equation is to consider the variation of the entropy along the trajectory of a fluid element. Take

$$\rho kTds = d\varepsilon - \left(\frac{\varepsilon + p}{\rho}\right)d\rho \tag{73}$$

then

$$\rho k T \frac{ds}{dt} = \frac{d\varepsilon}{dt} - \left(\frac{\varepsilon + p}{\rho}\right)\frac{d\rho}{dt} = \frac{d\varepsilon}{dt} - h\frac{d\rho}{dt} \tag{74}$$

4.2 Adiabatic gas

If the gas is adiabatic, then the change of entropy along the trajectory of an element of fluid is zero, i.e.

$$\frac{ds}{dt} = 0 \tag{75}$$

and

$$\frac{d\varepsilon}{dt} - h\frac{d\rho}{dt} = 0 \tag{76}$$

This is often re-expressed in a different form using:

$$\frac{1}{\rho}\frac{d\rho}{dt} = -\frac{\partial v_i}{\partial x_i} \qquad (77)$$

to give

$$\frac{d\varepsilon}{dt} = -(\varepsilon + p)\frac{\partial v_i}{\partial x_i} \qquad (78)$$

This equation describes the change in internal energy of the fluid resulting from expansion or compression.

4.3 Equation of state

The above equations can be used to derive the equation of state for a gas, given relationships between ε and p. One important case is the γ-law equation of state where the pressure and internal energy density are related by:

$$p = (\gamma - 1)\varepsilon \Rightarrow \varepsilon = \frac{1}{\gamma - 1}p \qquad (79)$$

where

$$\gamma = \frac{c_p}{c_v} = \text{Constant ratio of specific heats} \qquad (80)$$

We use this in conjunction with the perfect gas law

$$p = nkT = \frac{\rho kT}{\mu m} \qquad (81)$$

$$\Rightarrow \rho kT = \mu m p$$

Substitute this into:

$$\rho kT\frac{ds}{dt} = \frac{d\varepsilon}{dt} - \left(\frac{\varepsilon + p}{\rho}\right)\frac{d\rho}{dt}$$

$$\Rightarrow \mu m p\frac{ds}{dt} = \frac{1}{\gamma - 1}\frac{dp}{dt} - \frac{\gamma}{\gamma - 1}\frac{p}{\rho}\frac{d\rho}{dt} \qquad (82)$$

Multiply by $\dfrac{\gamma - 1}{p}$:

$$\frac{1}{p}\frac{dp}{dt} - \gamma\frac{1}{\rho}\frac{d\rho}{dt} = \mu m(\gamma - 1)\frac{ds}{dt} \tag{83}$$

so that

$$
\begin{aligned}
\frac{d}{dt}(\ln p) - \gamma\frac{d}{dt}\ln\rho &= \mu m(\gamma-1)\frac{ds}{dt} \\
\frac{d}{dt}\ln\left(\frac{p}{\rho^{\gamma}}\right) &= \mu m(\gamma-1)\frac{ds}{dt} \\
\ln\left(\frac{p}{\rho^{\gamma}}\right) &= \mu m(\gamma-1)(s-s_0) \\
\frac{p}{\rho^{\gamma}} &= \exp[\mu m(\gamma-1)(s-s_0)]
\end{aligned}
\tag{84}
$$

We write

(85)

$$K(s) = \exp[\mu m(\gamma - 1)(s - s_0)] = \text{Pseudo-entropy}$$

The equation of state is therefore written:

$$p = K(s)\rho^{\gamma} \tag{86}$$

Adiabatic flow:

In adiabatic flow

$$\frac{ds}{dt} = 0 \Rightarrow s = \text{constant along a streamline} \tag{87}$$

and $K(s)$ is a streamline constant, but may differ from streamline to streamline.

Special cases

$$\text{Monatomic gas (e.g. completely ionised gas)} \Rightarrow \gamma = \frac{5}{3}$$

$$\text{Diatomic gas} \Rightarrow \gamma = \frac{7}{5} = 1.4 \qquad (88)$$

4.4 Radiating gas

In astrophysics we often have to take account of the fact that the gas radiates energy at a sufficient rate, that we have to take into account the effect on the internal energy. Let the energy radiated per unit volume per unit time per steradian be j then the internal energy equation

$$\rho kT\frac{ds}{dt} = \frac{d\varepsilon}{dt}-\left(\frac{\varepsilon+p}{\rho}\right)\frac{d\rho}{dt} = \frac{d\varepsilon}{dt}-h\frac{d\rho}{dt} \qquad (89)$$

becomes

$$\frac{d\varepsilon}{dt}-\left(\frac{\varepsilon+p}{\rho}\right)\frac{d\rho}{dt} = -4\pi j \qquad (90)$$

The quantity j represents the total emissivity in units of energy per unit time per unit volume per steradian. The factor of 4π results from integrarting over 4π steradians.

Thermal gas

When we are dealing with a thermal gas, that is one in which the ions and electrons are more or less in thermal equilibrium, then the total emissivity may be expressed in the form:

$$4\pi j = n_e n_p \Lambda(T) \tag{91}$$

where

$$
\begin{aligned}
n_e &= \text{electron density} \\
n_p &= \text{proton density} \\
\Lambda(T) &= \text{cooling function}
\end{aligned}
\tag{92}
$$

This has been the time-honoured way of expressing thermal cooling. A more modern approach is to write

$$4\pi j = n^2 \Lambda(T) \tag{93}$$

where n is the ***total*** number density.

The cooling function is calculated using complex atomic physics calculations. Ralph Sutherland has published cooling functions for different plasma conditions.

5 The energy equations

5.1 Conservation of energy in classical mechanics

It's a good idea to look at how we derive the expression for energy in classical mechanics. Suppose we have a particle moving in a time invariant potential field, ϕ, with its equation of motion:

$$m\frac{dv_i}{dt} = -m\frac{\partial\phi}{\partial x_i} \tag{94}$$

Take the scalar product of this equation with v_i.

$$mv_i\frac{dv_i}{dt} = -mv_i\frac{\partial\phi}{\partial x_i} \tag{95}$$

Now

$$v_i \frac{dv_i}{dt} = \frac{1}{2} \frac{d}{dt}(v_i v_i) = \frac{d}{dt}\left(\frac{1}{2} v^2\right) \qquad (96)$$

Also, the differentiation following the particle motion of ϕ

$$\frac{d\phi}{dt} = \frac{\partial\phi}{\partial t} + v_i \frac{\partial\phi}{\partial x_i} = v_i \frac{\partial\phi}{\partial x_i} \qquad (97)$$

since $\frac{\partial\phi}{\partial t} = 0$. Hence

$$\frac{d}{dt}\left(\frac{1}{2} m v^2 + m\phi\right) = 0 \qquad (98)$$

As a consequence, the total energy,

$$E = \frac{1}{2}mv^2 + m\phi \qquad (99)$$

is a constant of the motion, i.e. it is conserved.

5.2 Conservation of energy in gas dynamics

The consideration of energy in gas dynamics follows a similar line – We start by taking the scalar product of the momentum equations with the velocity. The end result is not as simple but has some interesting and useful consequences.

We start with the momentum equations in the form

$$\rho\frac{\partial v_i}{\partial t} + \rho v_j\frac{\partial v_i}{\partial x_j} = -\frac{\partial p}{\partial x_i} - \rho\frac{\partial \phi}{\partial x_i} \tag{100}$$

and take the scalar product with the velocity:

$$\rho v_i\frac{\partial v_i}{\partial t} + \rho v_j v_i\frac{\partial v_i}{\partial x_j} = -v_i\frac{\partial p}{\partial x_i} - \rho v_i\frac{\partial \phi}{\partial x_i} \tag{101}$$

Now use:

$$v_i \frac{\partial v_i}{\partial t} = \frac{\partial}{\partial t}\left(\frac{1}{2} v_i v_i\right) = \frac{\partial}{\partial t}\left(\frac{1}{2} v^2\right)$$

$$v_i \frac{\partial v_i}{\partial x_j} = \frac{\partial}{\partial x_j}\left(\frac{1}{2} v_i v_i\right) = \frac{\partial}{\partial x_j}\left(\frac{1}{2} v^2\right) \qquad (102)$$

then:

$$\rho \frac{\partial}{\partial t}\left(\frac{1}{2} v^2\right) + \rho v_j \frac{\partial}{\partial x_j}\left(\frac{1}{2} v^2\right) = -v_i \frac{\partial p}{\partial x_i} - \rho v_i \frac{\partial \phi}{\partial x_i} \qquad (103)$$

We can now take the density and momentum inside the derivatives on the left hand side, using the continuity equation:

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho v^2\right) + \frac{\partial}{\partial x_j}\left(\frac{1}{2}\rho v^2 v_j\right) = -v_i\frac{\partial p}{\partial x_i} - \rho v_i\frac{\partial \phi}{\partial x_i} \qquad (104)$$

Bring in entropy equations

We introduce the entropy equations in order to eliminate the term $v_i\partial p/\partial x_i$. The first one to use is:

$$\rho kT\frac{\partial s}{\partial x_i} = \rho\frac{\partial h}{\partial x_i} - \frac{\partial p}{\partial x_i}$$

$$\Rightarrow -v_i\frac{\partial p}{\partial x_i} = \rho kTv_i\frac{\partial s}{\partial x_i} - \rho v_i\frac{\partial h}{\partial x_i} \qquad (105)$$

We then manipulate the enthalpy term as follows:

$$\rho v_i \frac{\partial h}{\partial x_i} = \frac{\partial}{\partial x_i}(\rho h v_i) - h\frac{\partial}{\partial x_i}(\rho v_i)$$
$$= \frac{\partial}{\partial x_i}(\rho h v_i) + h\frac{\partial \rho}{\partial t} \tag{106}$$

And now use the other form of the entropy equation

$$\rho k T\frac{\partial s}{\partial t} = \frac{\partial \varepsilon}{\partial t} - h\frac{\partial \rho}{\partial t}$$
$$\Rightarrow h\frac{\partial \rho}{\partial t} = \frac{\partial \varepsilon}{\partial t} - \rho k T\frac{\partial s}{\partial t} \tag{107}$$

Putting all of these equations together with appropriate signs:

$$
\begin{aligned}
-v_i\frac{\partial p}{\partial x_i} &= \rho k T v_i\frac{\partial s}{\partial x_i} - \rho v_i\frac{\partial h}{\partial x_i} \\
-\rho v_i\frac{\partial h}{\partial x_i} &= -\frac{\partial}{\partial x_i}(\rho h v_i) - h\frac{\partial \rho}{\partial t} \\
-h\frac{\partial \rho}{\partial t} &= -\frac{\partial \varepsilon}{\partial t} + \rho k T\frac{\partial s}{\partial t}
\end{aligned}
\qquad (108)
$$

Add all of these up:

$$-v_i \frac{\partial p}{\partial x_i} = \rho k T \left[\frac{\partial s}{\partial t} + v_i \frac{\partial s}{\partial x_i} \right] - \frac{\partial \varepsilon}{\partial t} - \frac{\partial}{\partial x_i}(\rho h v_i) \tag{109}$$

$$= \rho k T \frac{ds}{dt} - \frac{\partial \varepsilon}{\partial t} - \frac{\partial}{\partial x_i}(\rho h v_i)$$

Substitute in the intermediate form of the energy equation:

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho v^2\right) + \frac{\partial}{\partial x_j}\left(\frac{1}{2}\rho v^2 v_j\right) = -v_i \frac{\partial p}{\partial x_i} - \rho v_i \frac{\partial \phi}{\partial x_i} \tag{110}$$

gives:

$$\frac{\partial}{\partial t}\left[\frac{1}{2}\rho v^2 + \varepsilon\right] + \frac{\partial}{\partial x_j}\left[\frac{1}{2}\rho v^2 v_j + \rho h v_j\right] = -\rho v_i \frac{\partial \phi}{\partial x_i} + \rho k T \frac{ds}{dt} \qquad (111)$$

Gravitational potential term

Tha last term to deal with is $-\rho v_i \partial\phi / \partial x_i$:

$$
\begin{aligned}
-\rho v_i \frac{\partial\phi}{\partial x_i} &= -\frac{\partial}{\partial x_i}(\rho v_i \phi) + \phi \frac{\partial}{\partial x_i}(\rho v_i) \\
&= -\frac{\partial}{\partial x_i}(\rho v_i \phi) - \phi \frac{\partial \rho}{\partial t} \qquad (112) \\
&= -\frac{\partial}{\partial x_i}(\rho \phi v_i) - \frac{\partial}{\partial t}(\rho \phi)
\end{aligned}
$$

Final form of energy equation

Substitute the last expression into

$$\frac{\partial}{\partial t}\left[\frac{1}{2}\rho v^2 + \varepsilon\right] + \frac{\partial}{\partial x_j}\left[\frac{1}{2}\rho v^2 v_j + \rho h v_j\right] = -\rho v_i \frac{\partial \phi}{\partial x_i} + \rho k T \frac{ds}{dt} \qquad (113)$$

$$\frac{\partial}{\partial t}\left[\frac{1}{2}\rho v^2 + \varepsilon + \rho\phi\right] + \frac{\partial}{\partial x_j}\left[\frac{1}{2}\rho v^2 v_j + \rho h v_j + \rho\phi v_j\right] = \rho k T \frac{ds}{dt} \qquad (114)$$

and since j is a dummy repeated subscript, we can write

$$\frac{\partial}{\partial t}\left[\frac{1}{2}\rho v^2 + \varepsilon + \rho\phi\right] + \frac{\partial}{\partial x_i}\left[\left(\frac{1}{2}\rho v^2 + \rho h + \rho\phi\right)v_i\right] = \rho k T\frac{ds}{dt} \tag{115}$$

When the flow is adiabatic then $\rho k T\frac{ds}{dt} = 0$ and

$$\frac{\partial}{\partial t}\left[\frac{1}{2}\rho v^2 + \varepsilon + \rho\phi\right] + \frac{\partial}{\partial x_i}\left[\left(\frac{1}{2}\rho v^2 + \rho h + \rho\phi\right)v_i\right] = 0 \tag{116}$$

When there is radiation

$$\rho k T \frac{ds}{dt} = -4\pi j \tag{117}$$

and

$$\frac{\partial}{\partial t}\left[\frac{1}{2}\rho v^2 + \varepsilon + \rho\phi\right] + \frac{\partial}{\partial x_i}\left[\left(\frac{1}{2}\rho v^2 + \rho h + \rho\phi\right) v_i\right] = -4\pi j \tag{118}$$

Terms in the total energy

$$
\begin{aligned}
\frac{1}{2}\rho v^2 &= \text{Kinetic energy density} \\
\varepsilon &= \text{Internal energy density} \\
\rho\phi &= \text{Gravitational energy density} \\
\frac{1}{2}\rho v^2 + \varepsilon + \rho\phi &= E_{\mathrm{tot}} (\text{Total energy density})
\end{aligned}
\qquad (119)
$$

Note analogy with $E = \frac{1}{2}mv^2 + m\phi$ for a single particle.

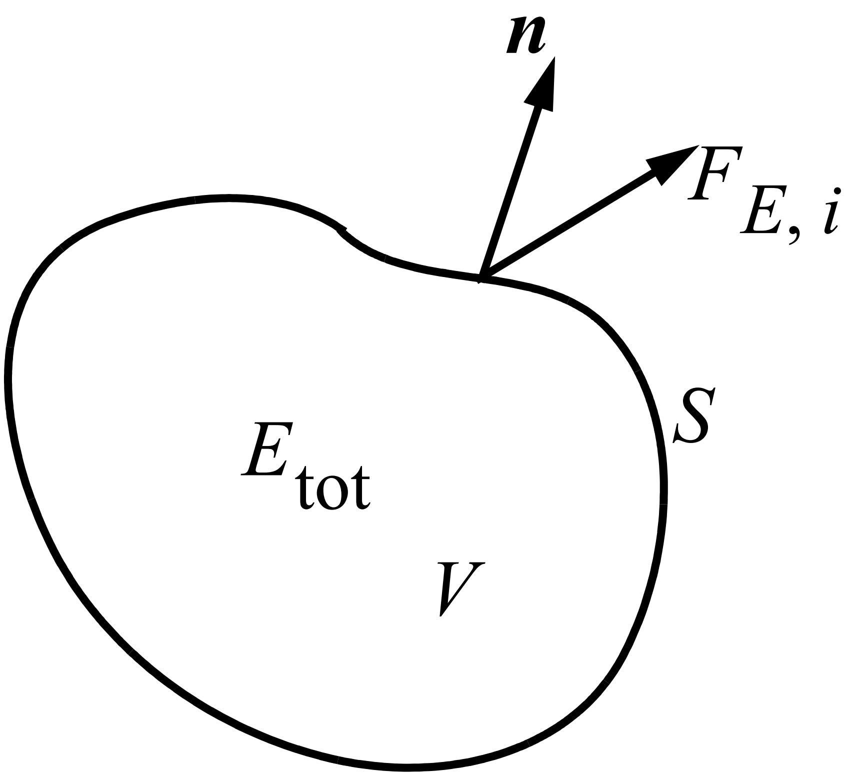
The energy flux

$$\text{Energy flux} = F_{Ei} = \left(\frac{1}{2}\rho v^2 + \rho h + \rho\phi\right) v_i$$

$$\begin{aligned} \frac{1}{2}\rho v^2 v_i &= \text{Flux of kinetic energy} \\ \rho h v_i &= \text{Enthalpy flux} \\ \rho\phi v_i &= \text{Flux of gravitational potential energy} \end{aligned} \tag{120}$$

An interesting point is that the flux associated with the internal energy is ***not*** εv_i as one might expect but the ***enthalpy flux*** $\rho h v_i = (\varepsilon + p) v_i$.

5.3 Integral form of the energy equation



We can integrate the energy equation over volume giving:

$$\int_V \left[\frac{\partial E_{tot}}{\partial t} + \frac{\partial F_{E,i}}{\partial x_i}\right] dV = 0$$

and then using the divergence theorem

$$\frac{\partial}{\partial t}\int E_{tot} dV + \int F_{E,i} n_i dS = -\int 4\pi j dV \qquad (121)$$

This says that the total energy within a volume changes as a result of the energy flux out of that volume and the radiative losses from the volume.

5.4 Examples of energy flux

Wind from O star

Suppose that the wind is spherically symmetric:

$$F_E = \int_{\text{sphere}} \left(\frac{1}{2}v^2 + \frac{\gamma}{\gamma - 1}\frac{p}{\rho}\right)\rho v_i n_i dS \qquad (122)$$

Since all variables are constant over the surface of the sphere, then

$$F_E = \left(\frac{1}{2}v^2 + \frac{\gamma}{\gamma - 1}\frac{p}{\rho}\right) \times \int_{\text{sphere}} \rho v_i n_i dS \tag{123}$$

$$= \dot{M}\left(\frac{1}{2}v^2 + \frac{\gamma}{\gamma - 1}\frac{p}{\rho}\right)$$

We neglect the enthalpy term in comparison to $v^2/2$ so that

$$F_E \approx \frac{1}{2}\dot{M}v^2 = \frac{1}{2} \times 10^{-6} \text{solar masses / yr} \times (10^3 \text{km s}^{-1})^2 \tag{124}$$

$$= 3.2 \times 10^{28} \text{ W}$$

Why do we neglect the enthalpy term?

Consider

$$\frac{1}{2}v^2 + \frac{\gamma}{\gamma-1}\frac{p}{\rho} = \frac{1}{2}v^2\left[1 + \frac{2\gamma}{\gamma-1}\frac{p}{\rho v^2}\right] = \frac{1}{2}v^2\left[1 + \frac{2}{\gamma-1}\frac{c_s^2}{v^2}\right] \quad (125)$$

$$= \frac{1}{2}v^2\left[1 + \frac{2}{\gamma-1}\frac{1}{M^2}\right]$$

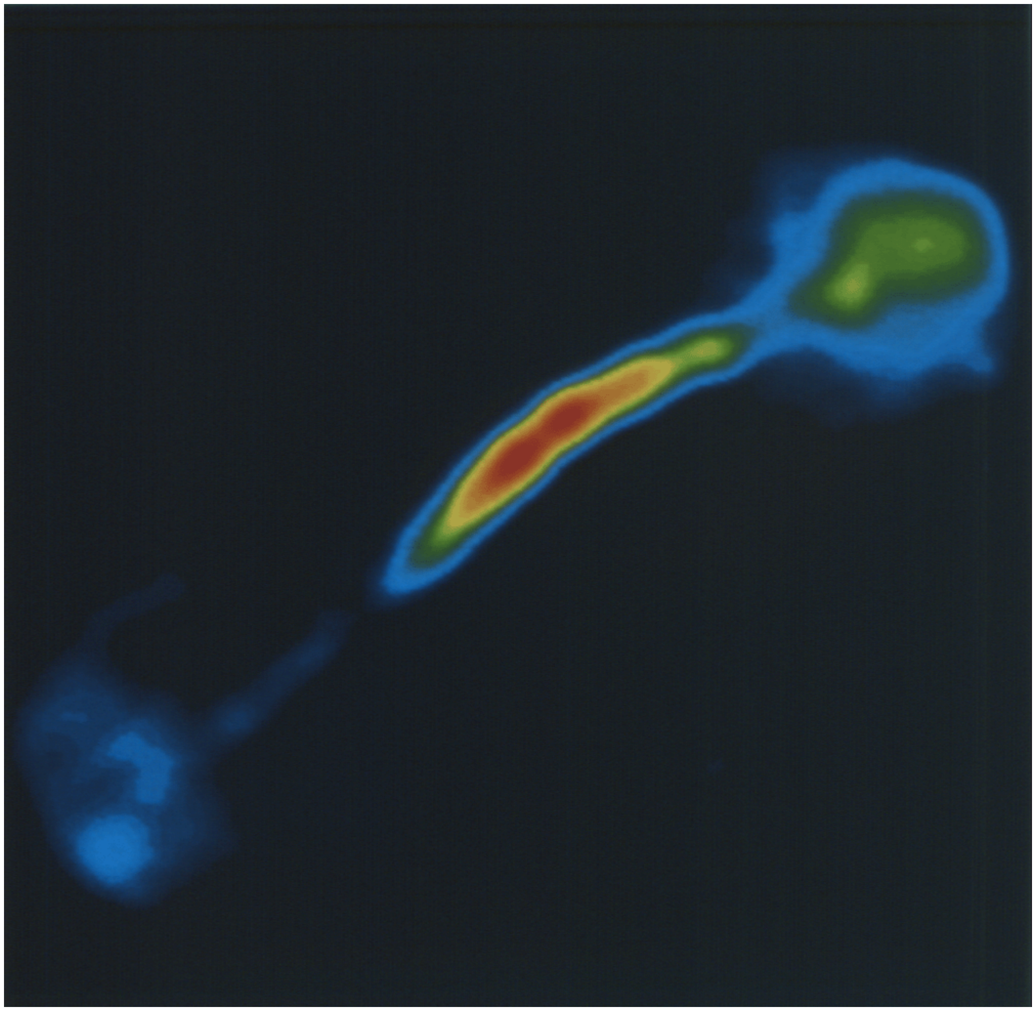
$$\text{where} \qquad c_s = \text{sound speed} \qquad M = \text{Mach number} \quad (126)$$

In the asymptotic zone the Mach number is high, so that

$$\frac{2}{\gamma-1}\frac{1}{M^2} \ll 1 \quad (127)$$

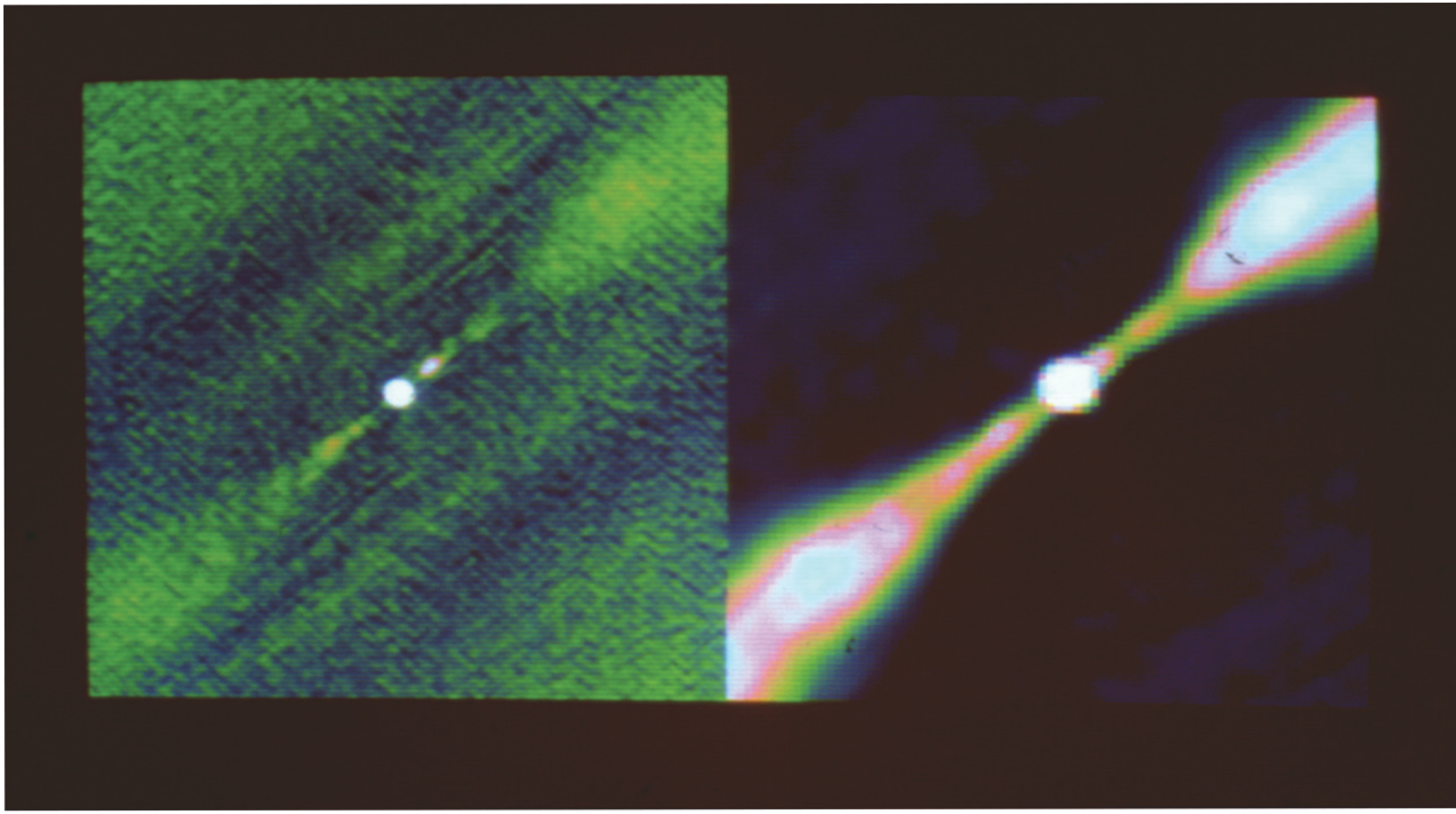
We shall justify these statements later.

Velocity of a jet



20 cm image of the radio galaxy, IC4296.

This image shows the jets (unresolved close to the core) and the lobes of the radio source.

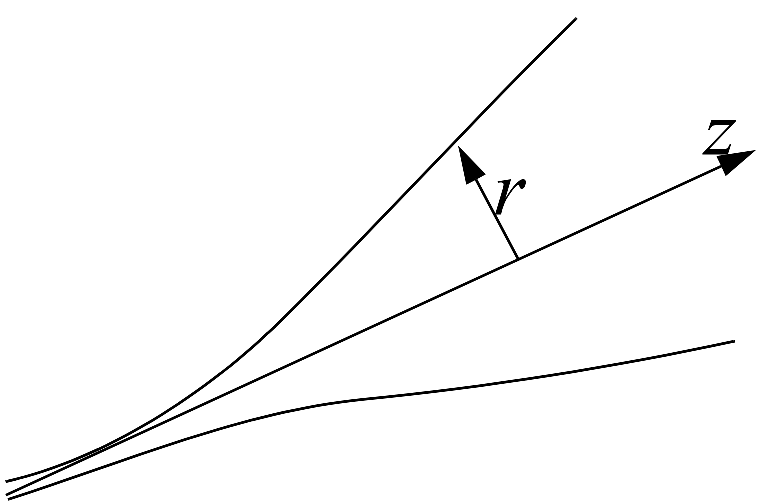


Images of the jets in the radio galaxy IC4296 close to the core. The resolutions are 1" and 3.2" on the left and right respectively.

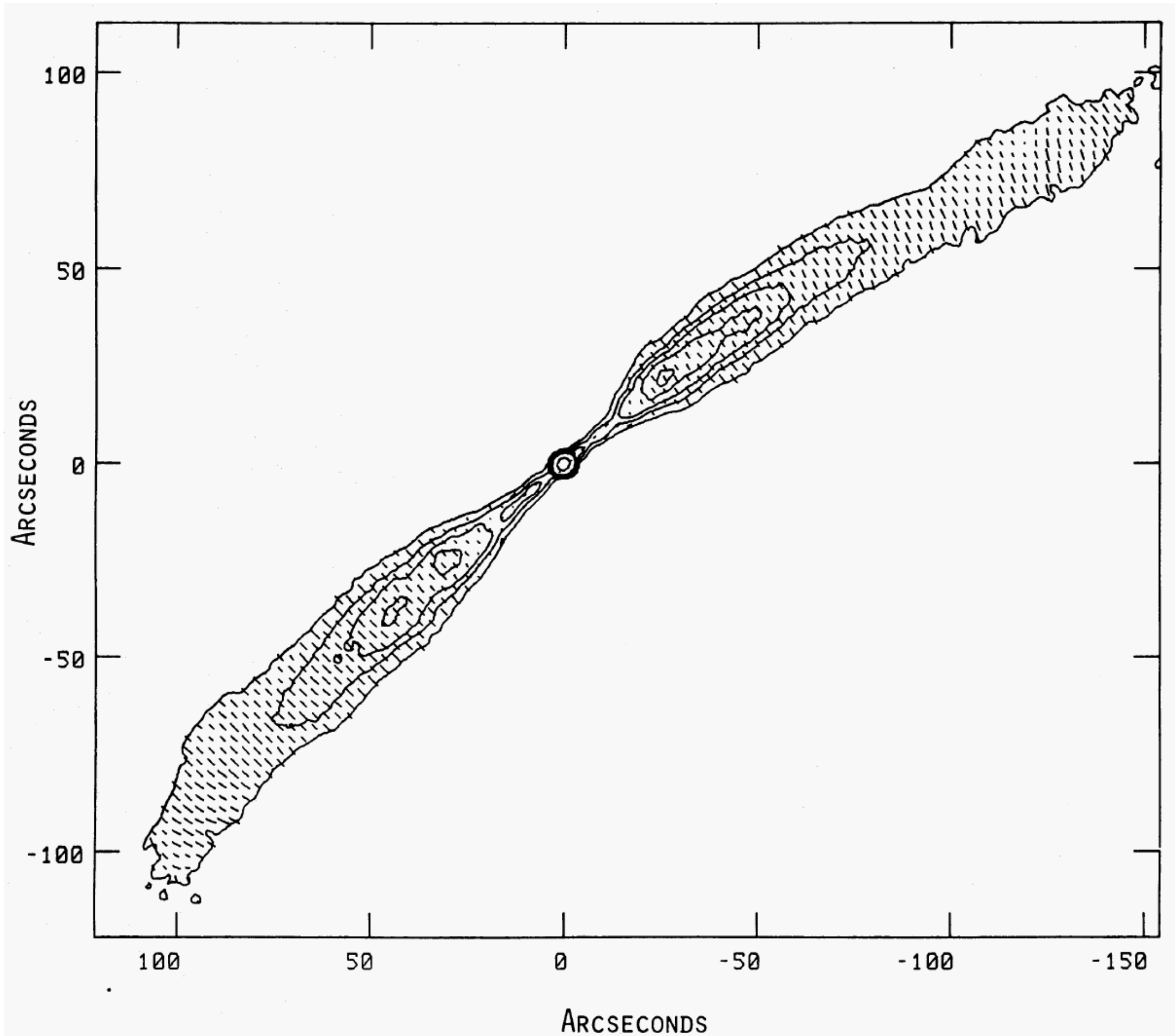
In this case we shall reverse the process to estimate the jet velocity. For a radio jet such as we observe in IC4296, once the jet has widened appreciably, then the Mach number is quite low probably of order unity. Hence the energy flux is dominated by the enthalpy flux.

This is the case when

$$\frac{2}{\gamma-1}\frac{1}{M^2} > 1 \Rightarrow M^2 < \frac{2}{\gamma-1} = 6 \text{ for } \gamma = \frac{4}{3} \qquad (128)$$



$$F_E \approx \int_{\text{jet}} \frac{\gamma}{\gamma-1} p v_z dS = 4 p v_z \pi R_{\text{jet}}^2 \qquad (129)$$



Contour image of the inner 150" of IC4296. The diameters of jets can be worked out from images such as this.

Using the radio data we can estimate the following parameters at a distance $100''$ from the core:

$$\begin{aligned} \text{Diameter} &= 15'' = 2.57\ \text{kpc} \\ \text{Minimum pressure} &= 5\times10^{-13}\ \text{N m}^{-2} \end{aligned} \qquad (130)$$

We also know from an analysis from the radio emission from the lobes of this radio galaxy that

$$F_E \sim 10^{36}\ \text{W} \qquad (131)$$

Since,

$$v_z = \frac{F_E}{4\pi p r^2} = \frac{10^{36}\ \mathrm{W}}{4\pi \times 5\times 10^{-13} \times \left(\frac{2.57}{2}\ \mathrm{kpc}\right)^2} \tag{132}$$

$$= 2.5\times 10^7\ \mathrm{m\ s^{-1}} = 25,000\ \mathrm{km\ s^{-1}} \approx 0.08c$$

This is actually an upper estimate of the velocity since we are using a minimum estimate of the pressure. However, it is unlikely that the pressure is ***too*** far from the minimum value, so that this is a reasonable estimate of the velocity in the western jet of IC4296 at this distance.

6 Summary of gas dynamics equations

6.1 Mass

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_i}(\rho v_i) = 0 \qquad (133)$$

6.2 Momentum

$$\frac{\partial}{\partial t}(\rho v_i) + \frac{\partial}{\partial x_j}(\rho v_i v_j) = -\frac{\partial p}{\partial x_i} - \rho\frac{\partial \phi}{\partial x_i}$$

$$\rho\frac{\partial v_i}{\partial t} + \rho v_j\frac{\partial v_i}{\partial x_j} = -\frac{\partial p}{\partial x_i} - \rho\frac{\partial \phi}{\partial x_i} \qquad (134)$$

6.3 Internal energy

$$\frac{d\varepsilon}{dt} = -(\varepsilon + p)\frac{\partial v_i}{\partial x_i} - 4\pi j \qquad (135)$$

6.4 Total energy

$$\frac{\partial}{\partial t}\left[\frac{1}{2}\rho v^2 + \varepsilon + \rho\phi\right] + \frac{\partial}{\partial x_i}\left[\left(\frac{1}{2}\rho v^2 + \rho h + \rho\phi\right)v_i\right] = -4\pi j \qquad (136)$$

6.5 Thermal cooling

$$4\pi j = n^2\Lambda(T)) \qquad (137)$$

6.6 Equation of state

$$p = (\gamma - 1)\varepsilon = K(s)\rho^{\gamma} \tag{138}$$