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The Fundamental Principles of Composite Material Stiffness Predictions

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bettertogether

Contents

- Description of example material for analysis
- Prediction of Stiffness using…
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- Overview of misconceptions in material property comparison between isotropic materials and composites

Lamina Axis Notation

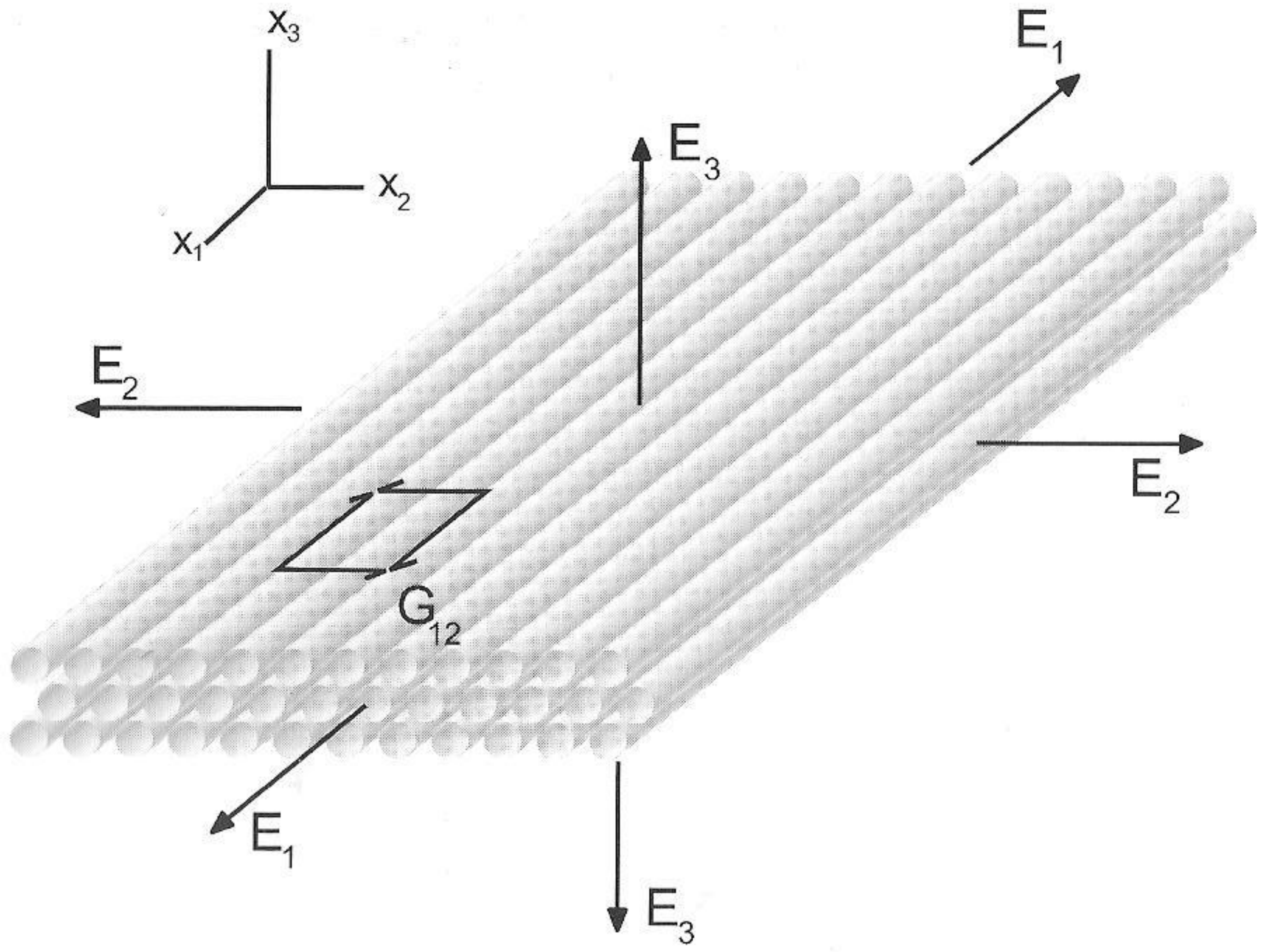


Diagram taken from Harris (1999)

Example Material for Analysis

- M21/35%/UD268/T700
  - A common Aerospace uni-directional pre-preg material called HexPly M21 from Hexcel
- E_f = 235 GPa E_m = 3.5 GPa
- ρ_f = 1.78 g/cm3 ρ_m = 1.28 g/cm3
- W_r = 35% (composite resin weight fraction)
- Layup = (0/0/0/+45/-45/0/0/0)

Stage 1

- Convert fibre weight fraction of composite to fibre volume fraction
  - Fibre weight fraction used by material suppliers
  - Fibre volume fraction needed for calculations

Fibre Volume Fraction

- Fibre mass fraction of M21 = 65% (0.65)
  - Data sheet says material is 35% resin by weight, therefore 65% fibre by weight
- Calculation of fibre volume fraction
- The resulting volume fraction is 57.2%

$$V_f = \frac{\frac{W_f}{\rho_f}}{\frac{W_f}{\rho_f} + \frac{W_m}{\rho_m}}$$

Methods of Stiffness Prediction

- Rule of Mixtures (with efficiency factor)
- Hart-Smith 10% Rule
  - Used in aerospace industry as a quick method of estimating stiffness
- Empirical Formulae
  - Based solely on test data
- Classical Laminate Analysis
  - LAP software

Rule of Mixtures

- A composite is a mixture or combination of two (or more) materials
- The Rule of Mixtures formula can be used to calculate / predict…
  - Young's Modulus (E)
  - Density
  - Poisson's ratio
  - Strength (UTS)
    - very optimistic prediction
    - 50% usually measured in test
    - Strength very difficult to predict – numerous reasons

Rule of Mixtures for Stiffness

- Rule of Mixtures for Young's Modulus
- Assumes uni-directional fibres
- Predicts Young's Modulus in fibre direction only
  - $E_c = E_f V_f + E_m V_m$
  - $Ec = 235 \times 0.572 + 3.5 \times 0.428$
  - $E_c = 136$ GPa

Rule of Mixtures: Efficiency Factor

- The Efficiency Factor or Krenchel factor can be used to predict the effect of fibre orientation on stiffness
- This is a term that is used to factor the Rule of Mixtures formula according to the fibre angle
  - See following slide

Reinforcing Efficiency

$$\eta_\theta = \sum a_n cos^4\theta$$

a_n= proportion of total fibre content
θ = angle of fibres
η_θ= composite efficiency factor (Krenchel)

$$E_c = \eta_\theta E_f V_f + E_m V_m$$

Efficiency (Krenchel) Factor

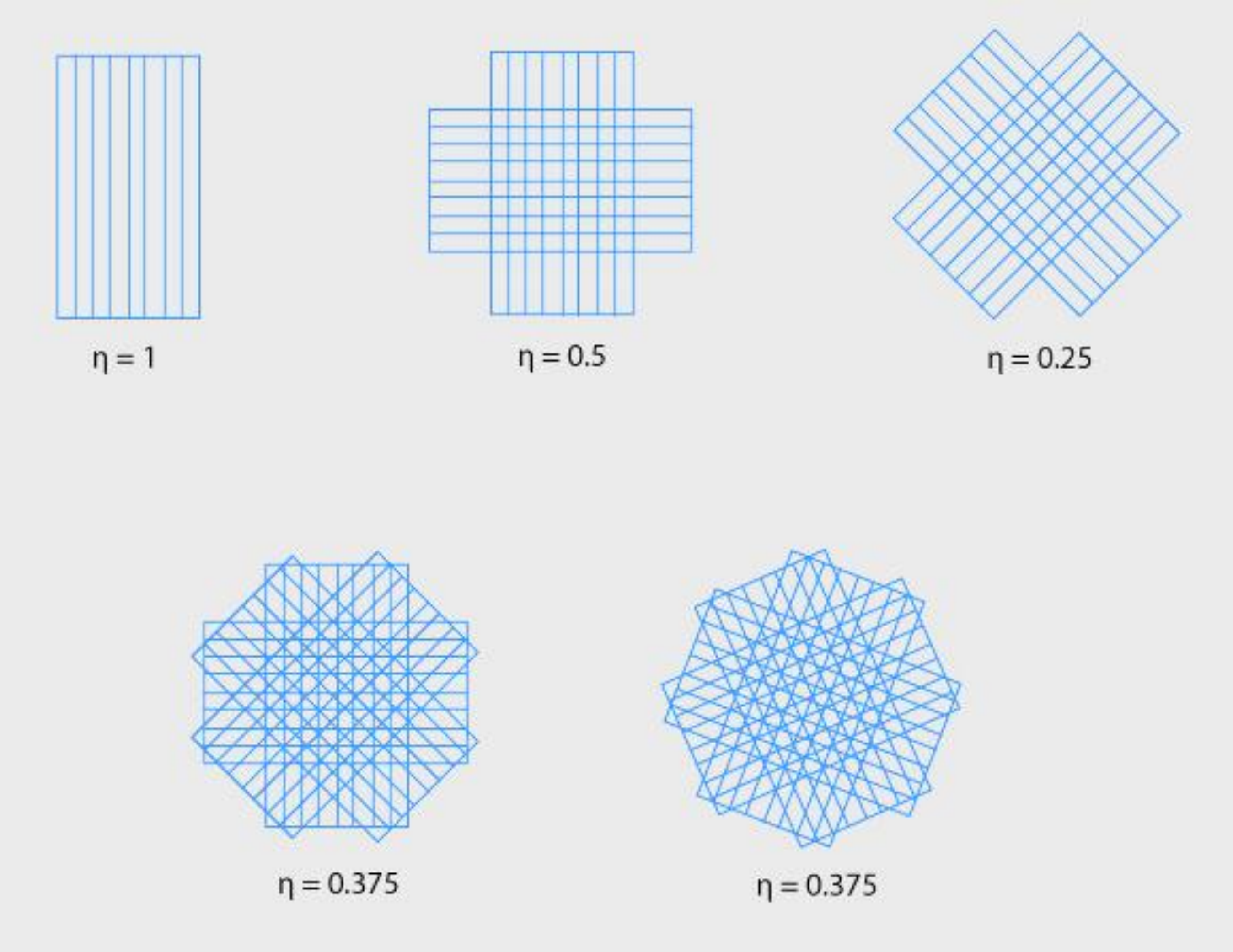


Diagram taken from Harris (1999)

Prediction of E for Example Ply

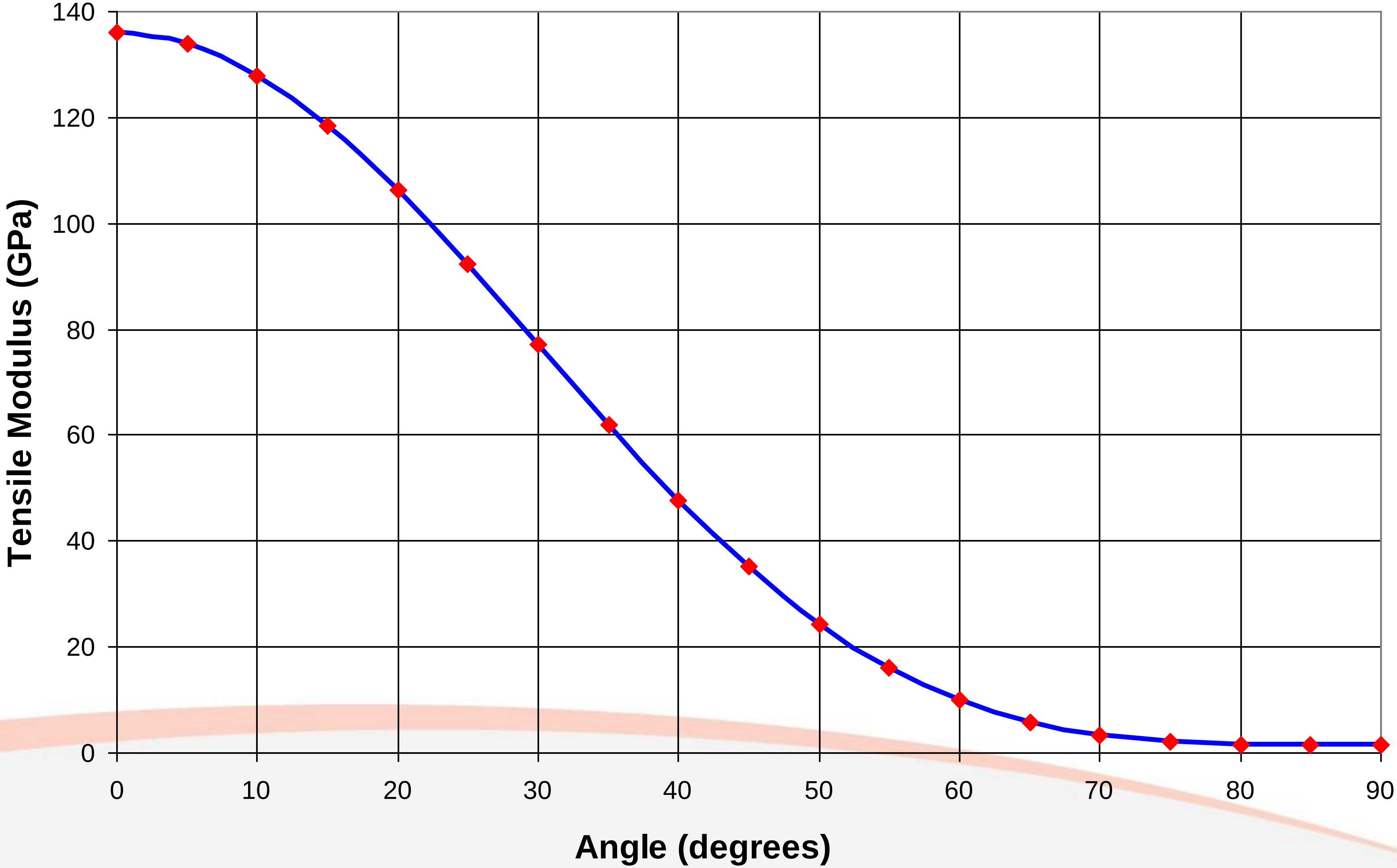
$$E_c = \eta_\theta E_f V_f + E_m V_m$$

E_f = 235 GPa
E_m= 3.5 GPa
V_f = 0.572

$$E(\theta) = (\text{Cos}^4\theta \times 235\times0.572) + (3.5\times0.428)$$

Predicted modulus versus angle plotted on following slide

Prediction of Tensile Modulus (Efficiency Factor)



Efficiency Factor for Laminate

- Layup = (0/0/0/+45/-45/0/0/0)
- $\eta = \cos^4\theta$
  - 0° =   η = 1
  - 45° = η = 0.25
  - 90° = η = 0
- Laminate in X-direction
  - (6/8 × 1) + (2/8 × 0.25)
  - (0.75 + 0.0625)
  - 0.8125
- Laminate in Y-direction
  - (6/8 × 0) + (2/8 × 0.25)
  - (0 + 0.0625)
  - 0.0625

Prediction of E for Example Ply

$$E_c = \eta_\theta E_f V_f + E_m V_m$$

E_f = 235 GPa E_m= 3.5 GPa V_f = 0.572

E_x = (0.8125 × 235×0.572) + (3.5×0.428)
E_x = 109 + 1.5 = 110.5 GPa

E_y = (0.0625 × 235×0.572) + (3.5×0.428)
E_y = 8.4 + 1.5 = 9.9 GPa

Ten-Percent Rule

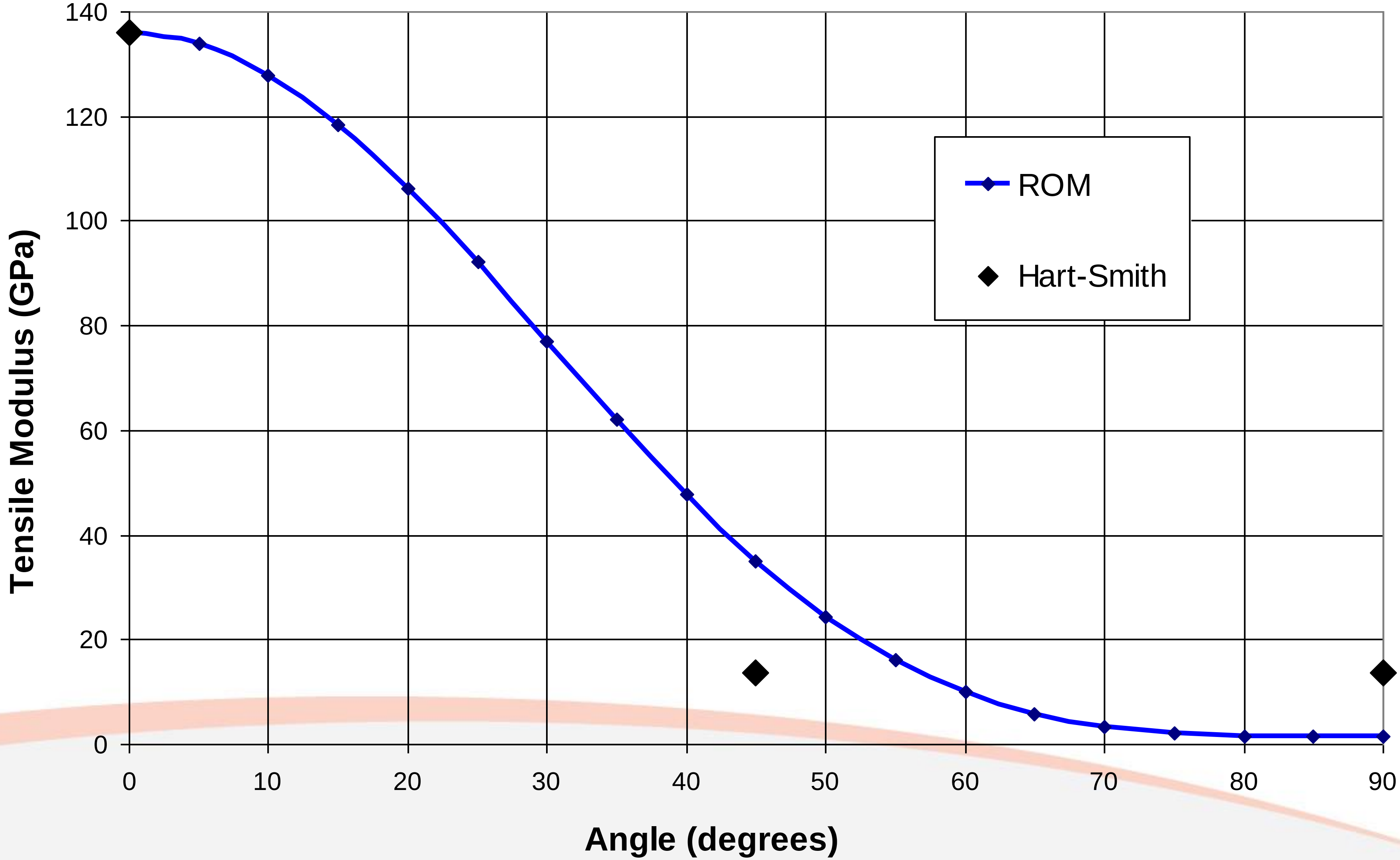
- Hart-Smith 1993
  - Each 45° or 90° ply is considered to contribute one tenth of the strength or stiffness of a 0° ply to the overall performance of the laminate
  - Rapid and reasonably accurate estimate
  - Used in Aerospace industry where standard layup [0/±45/90] is usually used

$$E_x = E_{11} \,.\, (0.1 + 0.9 \times \%\text{ plies at } 0°)$$

$$\sigma_x = \sigma_{11} \,.\, (0.1 + 0.9 \times \%\text{ plies at } 0°)$$

$$G_{xy} = E_{11} \,.\, (0.028 + 0.234 \times \%\text{ plies at } \pm 45°)$$

Prediction of Tensile Modulus



Calculation of E_{11} for Ply

- Using Rule of Mixtures
- $E_{11} = E_f V_f + E_m V_m$
- $E_{11} = 235 \times 0.572 + 3.5 \times 0.428$
- $E_{11} = 136$ GPa

- Layup = (0/0/0/+45/-45/0/0/0)
- 6/8 = 75% of plies in zero degree direction

Ten-Percent Rule

- $E_x = E_{11} \times (0.1 + 0.9 \times \%\ \text{plies at } 0°)$
- $E_x = 136 \times (0.1 + (0.9 \times 0.75))$
- $E_x = 136 \times (0.775)$
- $E_x = 105.4$ GPa

- $E_y = E_{11} \times (0.1 + 0.9 \times \%\ \text{plies at } 0°)$
- $E_y = 136 \times (0.1 + (0.9 \times 0))$
- $E_y = 136 \times (0.1)$
- $E_y = 13.6$ GPa

Classical Laminate Analysis

- 4 elastic constants are needed to characterise the in-plane macroscopic elastic properties of a ply
  - E_{11} = Longitudinal Stiffness
  - E_{22} = Transverse Stiffness
  - v_{12} = Major Poisson’s Ratio
  - G_{12}= In-Plane Shear Modulus

Elastic Constant Equations

- E_{11} = Longitudinal Stiffness (Rule of Mixtures Formulae)

$$E_c = E_f V_f + E_m(1 - V_f)$$

- E_{22} = Transverse Stiffness (Inverse Rule of Mixtures Formulae (Reuss Model))

$$\frac{1}{E_c} = \frac{V_f}{E_f} + \frac{V_m}{E_m}$$

- ν_{12} = Major Poisson's Ratio (Rule of Mixtures for Poisson's Ratio)

$$\nu_{12} = \nu_f V_f + \nu_m(1 - V_f)$$

- G_{12}= In-Plane Shear Modulus (Inverse Rule of Mixtures for Shear)

$$\frac{1}{G_{12}} = \frac{V_f}{G_f} + \frac{(1 - V_f)}{G_m}$$

Calculation of E_{11} for Ply

- Using Rule of Mixtures
- $E_{11} = E_f V_f + E_m V_m$
- $E_{11} = 235 \times 0.572 + 3.5 \times 0.428$
- $E_{11} = 136$ GPa

Calculation of E_{22} for Ply

- Using Inverse Rule of Mixtures Formulae (Reuss Model)
- E_{22} = 8 GPa

$$\frac{1}{E_c} = \frac{V_f}{E_f} + \frac{V_m}{E_m}$$

Calculate Poisson's Ratio (v_{12}) for Ply

- Using Rule of Mixtures formula
- However, we do not know
  - Poisson's ratio for carbon fibre
  - Poisson's ratio for epoxy matrix
  - We would need to find these for accurate prediction
  - We will assume a Poisson's Ratio (v) of 0.3

$$v_{12} = v_f V_f + v_m (1 - V_f)$$

Calculate Shear Modulus (G_{12}) of Ply

- Using Inverse Rule of Mixtures formula
- G for carbon fibre = 52 GPa (from test)
- G for epoxy = 2.26 GPa (from test)
  - Both calculated using standard shear modulus formula G = E/(2(1+v))
- G_{12} for composite = 5 GPa

$$\frac{1}{G_{12}} = \frac{V_f}{G_f} + \frac{(1 - V_f)}{G_m}$$

Resulting Properties of Ply

- E_{11} = 136 GPa
- E_{22} = 8 GPa
- v_{12} = 0.3
- G_{12} = 5 GPa

Matrix Representation

- 4 material elastic properties are needed to charaterise the in-plane behaviour of the linear elastic orthotropic ply
  - We conveniently define these in terms of measured engineering constants (as above)
  - These are usually expressed in matrix form
    - due to large equations produced
    - and subsequent manipulations required
- The stiffness matrix [Q]
- The compliance matrix [S] (inverse of stiffness)

Off-axis Orientation & Analysis

- The stiffness matrix is defined in terms of principal material directions, E_{11}, E_{22}
- However, we need to analyse or predict the material properties in other directions
  - As it is unlikely to be loaded only in principal direction
- We use stress transformation equations for this
  - Related to Mohr's stress circle
- The transformation equations are written in matrix form
  - They have nothing to do with the material properties, they are merely a rotation of stresses.

Single Ply

- [6 x 6] stiffness matrix [C] or
- [6 x 6] compliance matrix [S]
  - Often reduced stiffness matrix [Q] for orthotropic laminates [3 x 3]
  - Orthotropic = 3 mutually orthogonal planes of symetry
  - 4 elastic constants characterise the behaviour of the laminate
    - E_1, E_2, υ_{12}, G_{12}

Stiffness & Compliance Martricies

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{21} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}.$$

Stiffness Matrix [Q]

Calculates laminate stresses from laminate strains

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} \bar{S}_{11} & \bar{S}_{12} & \bar{S}_{16} \\ \bar{S}_{21} & \bar{S}_{22} & \bar{S}_{26} \\ \bar{S}_{16} & \bar{S}_{26} & \bar{S}_{66} \end{bmatrix} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}$$

Compliance Matrix [S]

Calculates laminate strains from laminate stresses

(inverse of compliance)

Transformation Matrix

The stress transformation equation that relates known stresses in the z, y coordinate system to stresses in the L, T coordinate system. These are related to the transformation performed using Mohr's stress circle.

$$\begin{Bmatrix} \sigma_L \\ \sigma_T \\ \tau_{LT} \end{Bmatrix} = \begin{bmatrix} \cos^2\theta & \sin^2\theta & 2\sin\theta\cos\theta \\ \sin^2\theta & \cos^2\theta & -2\sin\theta\cos\theta \\ -\sin\theta\cos\theta & \sin\theta\cos\theta & \cos^2\theta - \sin^2\theta \end{bmatrix} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}.$$

Transformed Stiffness Components

$$\overline{Q}_{11} = Q_{11}\cos^4\theta + Q_{22}\sin^4\theta + (2Q_{12} + 4\,Q_{66})\cos^2\theta\,\sin^2\theta$$

$$\overline{Q}_{12} = (Q_{11} + Q_{22} - 4Q_{66})\cos^2\theta\sin^2\theta + Q_{12}\,(\cos^4\theta + \sin^4\theta)$$

$$\overline{Q}_{22} = Q_{11}\sin^4\theta + Q_{22}\cos^4\theta + (2Q_{12} + 4\,Q_{66})\cos^2\theta\,\sin^2\theta$$

$$\overline{Q}_{66} = (Q_{11} + Q_{22} - 2Q_{12} - 2Q_{66})\cos^2\theta\,\sin^2\theta + Q_{66}\,(\cos^4\theta + \sin^4\theta)$$

$$\overline{Q}_{16} = (Q_{11} - 2Q_{66} - Q_{12})\cos^3\theta\,\sin\theta - (Q_{22} - Q_{12} - 2Q_{66})\cos\theta\sin^3\theta$$

$$\overline{Q}_{26} = (Q_{11} - 2Q_{66} - Q_{12})\cos\theta\,\sin^3\theta - (Q_{22} - Q_{12} - 2Q_{66})\cos^3\theta\,\sin\theta$$

Transformed Compliance Components

$$\overline{S}_{11} = S_{11}\cos^4\theta + S_{22}\sin^4\theta + (2S_{12} + S_{66})\cos^2\theta\sin^2\theta$$

$$\overline{S}_{12} = (S_{11} + S_{22} - S_{66})\cos^2\theta\sin^2\theta + S_{12}(\cos^4\theta + \sin^4\theta)$$

$$\overline{S_{22}} = S_{11}\sin^4\theta + S_{22}\cos^4\theta + (2S_{12} + S_{66})\cos^2\theta\sin^2\theta$$

$$\overline{S}_{66} = 4(S_{11} + S_{22} - 2S_{12})\cos^2\theta\sin^2\theta + S_{66}(\cos^2\theta - \sin^2\theta)^2$$

$$\overline{S}_{16} = (2S_{11} - S_{66} - 2S_{12})\cos^3\theta\sin\theta - (2S_{22} - 2S_{12} - S_{66})\cos\theta\sin^3\theta$$

$$\overline{S}_{26} = (2S_{11} - S_{66} - 2S_{12})\cos\theta\sin^3\theta - (2S_{22} - 2S_{12} - S_{66})\cos^3\theta\sin\theta$$

Individual Compliance & Stiffness terms

$$S_{11} = \frac{1}{E_L}, \quad S_{12} = -\frac{\mu_{LT}}{E_L} = -\frac{\mu_{TL}}{E_T}, \quad S_{22} = \frac{1}{E_T}, \quad S_{66} = \frac{1}{G_{LT}}.$$

$$Q_{11} = \frac{E_L}{(1-\mu_{LT}\mu_{TL})}, \quad Q_{12} = \frac{\mu_{LT}E_T}{(1-\mu_{LT}\mu_{TL})} = \frac{\mu_{TL}E_L}{(1-\mu_{LT}\mu_{TL})},$$

$$Q_{22} = \frac{E_T}{(1-\mu_{LT}\mu_{TL})}, \quad \text{and} \quad Q_{66} = G_{LT}.$$

CLA Derived Formula

- Formula from
  - Engineering Design with Polymers and Composites, J.C.Gerdeen et al.
- Suitable for calculating Young's Modulus of a uni-directional ply at different angles.

$$\frac{1}{E_x} = \frac{Cos^4\theta}{E_{11}} + \frac{Sin^4\theta}{E_{22}} + \left(\frac{1}{G_{12}} - \frac{2\nu_{12}}{E_{11}}\right) Cos^2\theta Sin^2\theta$$

Example Material ($\theta = 0°$)

- $E_{11} = 136$ GPa, $E_{22} = 8$ GPa
- $v_{12} = 0.3$, $G_{12} = 5$ GPa
- $1/E_x = 1/136 + 0 + 0$
- E_x for 0° fibres = 136 GPa

$$\frac{1}{E_x} = \frac{Cos^4\theta}{E_{11}} + \frac{Sin^4\theta}{E_{22}} + \left(\frac{1}{G_{12}} - \frac{2v_{12}}{E_{11}}\right)Cos^2\theta Sin^2\theta$$

Example Material (θ = 45°)

- E_{11} = 136 GPa, E_{22} = 8 GPa
- v_{12} = 0.3, G_{12} = 5 GPa
- $1/E_x = 0.25/136 + 0.25/8 + (1/5-0.6/136) \times 0.25$
- $1/E_x = 0.25/136 + 0.25/8 + (1/5-0.6/136) \times 0.25$
- E_x for 45° fibres = 12.2 GPa

$$\frac{1}{E_x} = \frac{Cos^4\theta}{E_{11}} + \frac{Sin^4\theta}{E_{22}} + \left(\frac{1}{G_{12}} - \frac{2v_{12}}{E_{11}}\right)Cos^2\theta Sin^2\theta$$

Example Material (θ = 90°)

- E_{11} = 136 GPa, E_{22} = 8 GPa
- v_{12} = 0.3, G_{12} = 5 GPa
- $1/E_x = 0 + 1/8 + 0$
- E_x for 90° fibres = 8 GPa

$$\frac{1}{E_x} = \frac{Cos^4\theta}{E_{11}} + \frac{Sin^4\theta}{E_{22}} + \left(\frac{1}{G_{12}} - \frac{2v_{12}}{E_{11}}\right)Cos^2\theta Sin^2\theta$$

Properties of Laminate

- CFRE (0/0/0/+45/-45/0/0/0)
  - 0° = 136 GPa
  - 45° = 12.2GPa
  - 90° = 8 GPa
- Laminate in X-direction
  - (6/8 × 136) + (2/8 × 12.2)
  - 102 + 3 = 105 GPa
- Laminate in Y-direction
  - (6/8 × 8) + (2/8 × 12.2)
  - 6 + 3 = 9 GPa
- *Note: This simplified calculation ignores the effect of coupling between extension and shear*

Classical Laminate Analysis

- The simplified laminate analysis approach taken ignores the effect of coupling between extension and shear
- Classical Laminate Analysis takes full account of this effect
  - However, this is too long and complex for hand calculations
  - Therefore build a spreadsheet or use software such as LAP (Laminate Analysis Programme)

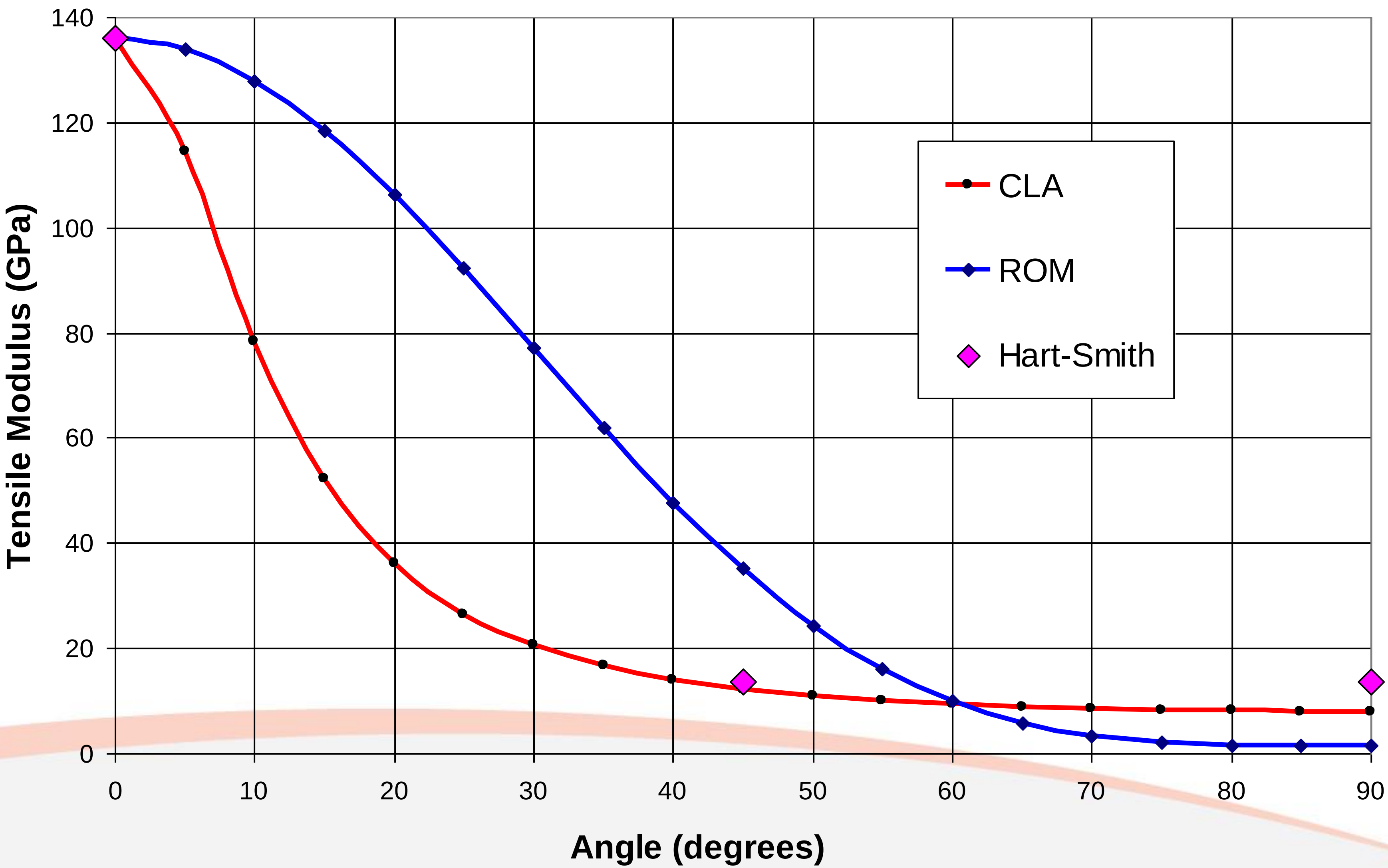
Summary of Results

For HexPly M21 Pre-preg

with (0/0/0/+45/-45/0/0/0) layup

	ROM & Efficiency factor	Hart-Smith 10% Rule	ROM of CLA rotation formula	Classical Laminate Analysis LAP
E_x	110.5	105.4	105.0	107.1
E_y	9.9	13.6	9.0	15.3

Prediction of Tensile Modulus



Procedure for Structural Analysis

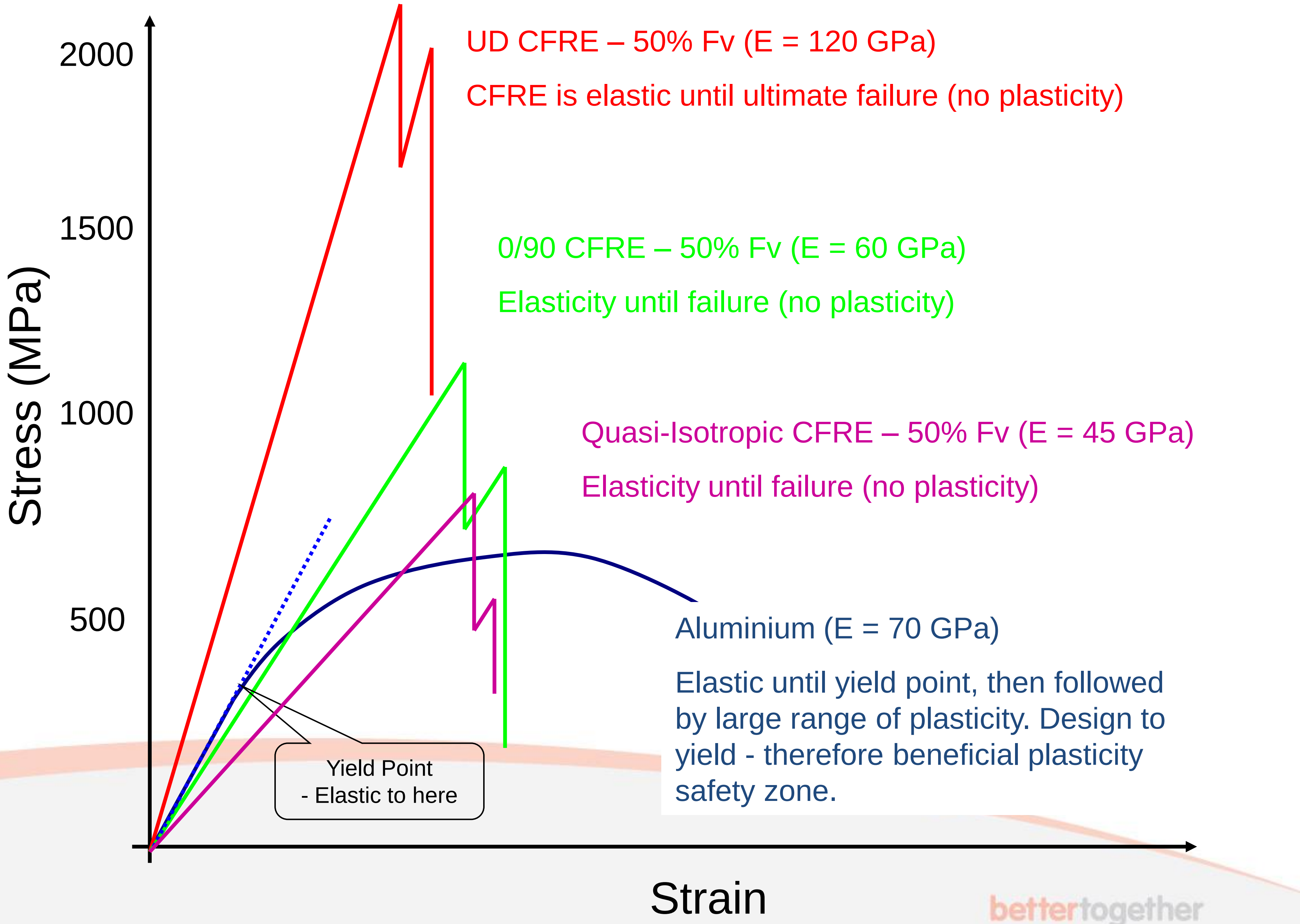
- Classical Laminate Analysis to provide
    - Relationship between in-plane load and strain
    - Relationship between out of plane bending moments and curvatures
  - Plate properties from the A, B and D matrices
- Then analysis is equivalent to classical analysis of anisotropic materials
  - FE analysis
  - Standard formulas to check stiffness

Material Property Comparison

- Young's Modulus of materials
  - Isotropic Materials
    - Aluminium = 70 GPa
    - Steel = 210 GPa
    - Polymers = 3 GPa
  - Fibres
    - Carbon = 240 GPa
      - common high strength T700 fibre
    - Glass = 70 GPa

Care with Property Comparison

- E for Aluminium = 70 GPa
- E for Glass fibre = 70 GPa
- However….
  - 50% fibre volume fraction
    - E now 35 GPa in x-direction
    - E in y-direction = matrix = 3 GPa
  - 0/90 woven fabric
    - 50% of material in each direction
    - E now 17.5 GPa in x and y direction
    - E at 45° = 9 GPa
- Glass reinforced polymer composite now has a low stiffness compared to Aluminium!


UD CFRE – 50% Fv (E = 120 GPa)
CFRE is elastic until ultimate failure (no plasticity)
0/90 CFRE – 50% Fv (E = 60 GPa)
Elasticity until failure (no plasticity)
Quasi-Isotropic CFRE – 50% Fv (E = 45 GPa)
Elasticity until failure (no plasticity)
Aluminium (E = 70 GPa)
Elastic until yield point, then followed by large range of plasticity. Design to yield - therefore beneficial plasticity safety zone.
Yield Point
- Elastic to here
Stress (MPa)
2000
1500
1000
500
Strain
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References & Bibliography

- Engineering Design with Polymers and Composites
  - J.C.Gerdeen, H.W.Lord & R.A.L.Rorrer
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