5.5

The Greatest Common Factor and Factoring by Grouping

Greatest Common Factor

Greatest common factor – largest quantity that is a factor of all the integers or polynomials involved.

Finding the GCF of a List of Monomials

1) Find the GCF of the numerical coefficients.
2) Find the GCF of the variable factors.
3) The product of the factors found in Steps 1 and 2 is the GCF of the monomials.

Example

Find the GCF of each list of numbers.

12 and 8

$12 = \mathbf{2 \cdot 2} \cdot 3$

$8 = \mathbf{2 \cdot 2} \cdot 2$

So the GCF is $\mathbf{2 \cdot 2} = 4$.

7 and 20

$7 = 1 \cdot 7$

$20 = 2 \cdot 2 \cdot 5$

There are no common prime factors so the GCF is 1.

Example

Find the GCF of each list of terms.

a. x^3 and x^7

$x^3 = \mathbf{x} \cdot \mathbf{x} \cdot \mathbf{x}$

$x^7 = \mathbf{x} \cdot \mathbf{x} \cdot \mathbf{x} \cdot x \cdot x \cdot x \cdot x$

So the GCF is $\mathbf{x} \cdot \mathbf{x} \cdot \mathbf{x} = x^3$

b. $6x^5$ and $4x^3$

$6x^5 = \mathbf{2} \cdot 3 \cdot \mathbf{x} \cdot \mathbf{x} \cdot \mathbf{x}$

$4x^3 = \mathbf{2} \cdot 2 \cdot \mathbf{x} \cdot \mathbf{x} \cdot \mathbf{x}$

So the GCF is $\mathbf{2} \cdot \mathbf{x} \cdot \mathbf{x} \cdot \mathbf{x} = 2x^3$

Helpful Hint

Remember that the GCF of a list of terms contains the smallest exponent on each common variable.

smallest exponent on x

smallest exponent on y

The GCF of x^3y^5, x^6y^4, and x^4y^6 is x^3y^4.

Example

Factor out the GCF in each of the following polynomials.

a. $6x^3 - 9x^2 + 12x$

$= 3x \cdot 2x^2 - 3x \cdot 3x + 3x \cdot 4$

$= 3x(2x^2 - 3x + 4)$

b. $14x^3y + 7x^2y - 7xy$

$= 7xy \cdot 2x^2 + 7xy \cdot x - 7xy \cdot 1$

$= 7xy(2x^2 + x - 1)$

Example

Factor out the GCF in each of the following polynomials.

1) $6(x + 2) - y(x + 2) = 6(x + 2) - y(x + 2)$

$= (x + 2)(6 - y)$

2) $xy(y + 1) - (y + 1) = xy\ (y + 1) - 1(y + 1)$

$= (y + 1)(xy - 1)$

Factoring

Remember that factoring out the GCF from the terms of a polynomial should always be the first step in factoring a polynomial.

This will usually be followed by additional steps in the process.

Example

Factor by grouping. $15x^3 - 10x^2 + 6x - 4$

$$15x^3 - 10x^2 + 6x - 4 = (15x^3 - 10x^2) + (6x - 4)$$

$$= 5x^2(3x - 2) + 2(3x - 2)$$

$$= (3x - 2)(5x^2 + 2)$$

Example

Factor by grouping. $2a^2 + 5ab + 2a + 5b$

$$2a^2 + 5ab + 2a + 5b \quad = (2a^2 + 5ab) + (2a + 5b)$$

$$= a(2a + 5b) + 1(2a + 5b)$$

$$= (2a + 5b)(a + 1)$$

Example

Factor by grouping. $x^3 + 4x + x^2 + 4$

$$x^3 + 4x + x^2 + 4 = (x^3 + 4x) + (x^2 + 4)$$

$$= x(x^2 + 4) + 1(x^2 + 4)$$

$$= (x^2 + 4)(x + 1)$$

Example

Factor by grouping. $2x^3 - x^2 - 10x + 5$

$$2x^3 - x^2 - 10x + 5 = (2x^3 - x^2) - (10x + 5)$$

$$= x^2(2x - 1) - 5(2x - 1)$$

$$= (2x - 1)(x^2 - 5)$$

Example

Factor by grouping.

$$21x^3y^2 – 9x^2y + 14xy – 6$$

$$= (21x^3y^2 – 9x^2y) + (14xy – 6)$$

$$= 3x^2y(7xy – 3) + 2(7xy – 3)$$

$$= (7xy – 3)(3x^2 + 2)$$

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Factoring Trinomials

Factoring Trinomials of the Form $x^2 + bx + c$

Recall by using the FOIL method that

$$\begin{aligned} & \qquad\quad \textbf{\textit{F}} \qquad \textbf{\textit{O}} \qquad \textbf{\textit{I}} \qquad \textbf{\textit{L}} \\ (x + 2)(x + 4) &= x^2 + \mathbf{4}x + \mathbf{2}x + \mathbf{8} \\ &= x^2 + \mathbf{6}x + \mathbf{8} \end{aligned}$$

To factor $x^2 + \boldsymbol{b}x + \boldsymbol{c}$ into
(x + one #)(x + another #), note that ***b*** is the sum of the two numbers and ***c*** is the product of the two numbers.

So we'll be looking for 2 numbers whose product is ***c*** and whose sum is ***b***.

Example

Factor the polynomial $x^2 + 13x + 30$.

Since our two numbers must have a product of 30 and a sum of 13, the two numbers must both be positive.

Positive factors of 30	Sum of Factors
1, 30	31
2, 15	17
3, 10	13

There are other factors, but once we find a pair that works, we do not have to continue searching. $x^2 + 13x + 30 = (x + 3)(x + 10)$.

Example

Factor the polynomial $x^2 - 11x + 24$.

Since our two numbers must have a product of 24 and a sum of –11, the two numbers must both be negative.

Negative factors of 24	Sum of Factors
–1, –24	–25
–2, –12	–14
–3, –8	–11

So $x^2 - 11x + 24 = (x - 3)(x - 8)$.

Example

Factor the polynomial $x^2 - 2x - 35$.

Since our two numbers must have a product of –35 and a sum of – 2, the two numbers will have to have different signs.

Factors of –35	Sum of Factors
–1, 35	34
1, –35	–34
–5, 7	2
5, –7	–2

So $x^2 - 2x - 35 = (x + 5)(x - 7)$.

Example

Factor: $3m^2 - 24m - 60$

First factor out the greatest common factor, 3, from each term.

$$3m^2 - 24m - 60 = 3(m^2 - 8m - 20)$$

Now find two factors of –20 whose sum is –8.

$$3m^2 - 24m - 60 = 3(m + 2)(m - 10)$$

Example

Factor: $x^2 - 6x + 10$

We look for two numbers whose product is 10 and whose sum is –6. The two numbers will have to both be negative.

Negative factors of 10	Sum of Factors
–1, –10	–11
–2, –5	–7

Since there is not a factor pair whose sum is –6, $x^2 - 6x + 10$ is not factorable and we call it a **prime polynomial**.

Example

Factor: $25x^2 + 20x + 4$

Possible factors of $25x^2$: $25x^2 = x \cdot 25x$, $25x^2 = 5x \cdot 5x$.

Possible factors of 4: $4 = 1 \cdot 4$, $4 = 2 \cdot 2$.

We need to methodically try each pair of factors until we find a combination that works, or exhaust all of our possible pairs of factors.

Keep in mind that, because some of our pairs are not identical factors, we may have to exchange some pairs of factors and make 2 attempts before we can definitely decide a particular pair of factors will not work.

Continued

Example (cont)

We will be looking for a combination that gives the sum of the products of the outside terms and the inside terms equal to $20x$.

Factors of $25x^2$	Factors of 4	Resulting Binomials	Product of Outside Terms	Product of Inside Terms	Sum of Products
x, $25x$	1, 4	$(x + 1)(25x + 4)$	$4x$	$25x$	$29x$
		$(x + 4)(25x + 1)$	x	$100x$	$101x$
x, $25x$	2, 2	$(x + 2)(25x + 2)$	$2x$	$50x$	$52x$
5, $5x$	2, 2	$(5x + 2)(5x + 2)$	$10x$	$10x$	$\boxed{20x}$

Continued

Example (cont)

Check the resulting factorization using the FOIL method.

$$\begin{array}{lcccc} & \boldsymbol{F} & \boldsymbol{O} & \boldsymbol{I} & \boldsymbol{L} \\ (5x + 2)(5x + 2) = & 5x(5x) & + 5x(2) & + 2(5x) & + 2(2) \end{array}$$

$$= 25x^2 + 10x + 10x + 4$$

$$= 25x^2 + 20x + 4$$

Thus a factored form of $25x^2 + 20x + 4$ is
$(5x + 2)(5x + 2)$ or $(5x + 2)^2$.

Example

Factor: $21x^2 - 41x + 10$

Possible factors of $21x^2$: $21x^2 = x \cdot 21x$, $21x^2 = 3x \cdot 7x$.

Since the middle term is negative, possible factors of 10 must both be negative: $10 = -1 \cdot -10$, $10 = -2 \cdot -5$.

We need to methodically try each pair of factors until we find a combination that works, or exhaust all of our possible pairs of factors.

Continued

Example (cont)

We will be looking for a combination that gives the sum of the products of the outside terms and the inside terms equal to – 41x.

Factors of $21x^2$	Factors of 10	Resulting Binomials	Product of Outside Terms	Product of Inside Terms	Sum of Products
x, $21x$	1, 10	$(x - 1)(21x - 10)$	$-10x$	$-21x$	$-31x$
		$(x - 10)(21x - 1)$	$-x$	$-210x$	$-211x$
x, $21x$	2, 5	$(x - 2)(21x - 5)$	$-5x$	$-42x$	$-47x$
		$(x - 5)(21x - 2)$	$-2x$	$-105x$	$-107x$

Continued

Example (cont)

Factors of $21x^2$	Factors of 10	Resulting Binomials	Product of Outside Terms	Product of Inside Terms	Sum of Products
$3x$, $7x$	1, 10	$(3x - 1)(7x - 10)$	$-30x$	$-7x$	$-37x$
		$(3x - 10)(7x - 1)$	$-3x$	$-70x$	$-73x$
$3x$, $7x$	2, 5	$(3x - 2)(7x - 5)$	$-15x$	$-14x$	$-29x$
		$(3x - 5)(7x - 2)$	$-6x$	$-35x$	$\boxed{-41x}$

Continued

Example (cont)

Check the resulting factorization using the FOIL method.

$$\begin{array}{ccccc} & \textbf{\textit{F}} & \textbf{\textit{O}} & \textbf{\textit{I}} & \textbf{\textit{L}} \\ (3x - 5)(7x - 2) = & 3x(7x) & + 3x(-2) & - 5(7x) & - 5(-2) \end{array}$$

$$= 21x^2 - 6x - 35x + 10$$

$$= 21x^2 - 41x + 10$$

A factored form of $21x^2 - 41x + 10$ is $(3x - 5)(7x - 2)$.

Example

Factor: $6x^2y^2 - 2xy^2 - 60y^2$.

Remember that the larger the coefficient, the greater the probability of having multiple pairs of factors to check. So it is important that you attempt to factor out any common factors first.

$$6x^2y^2 - 2xy^2 - 60y^2 = 2y^2(3x^2 - x - 30)$$

The only possible factors for 3 are 1 and 3, so we know that, if we can factor the polynomial further, it will have to look like $2y^2(3x \qquad)(x \qquad)$ in factored form.

Continued

Example (cont)

Since the product of the last two terms of the binomials will have to be –30, we know that they must be different signs.
Factors of –30: –1 • 30, 1 • –30, –2 • 15, 2 • –15, –3 • 10, 3 • –10, –5 • 6, 5 • –6

We will be looking for a combination that gives the sum of the products of the outside terms and the inside terms equal to $-x$.

Continued

Example (cont)

Factors of -30	Resulting Binomials	Product of Outside Terms	Product of Inside Terms	Sum of Products
–1, 30	$(3x - 1)(x + 30)$	$90x$	$-x$	$89x$
	$(3x + 30)(x - 1)$ Common factor so no need to test.			
1, –30	$(3x + 1)(x - 30)$	$-90x$	x	$-89x$
	$(3x - 30)(x + 1)$ Common factor so no need to test.			
–2, 15	$(3x - 2)(x + 15)$	$45x$	$-2x$	$43x$
	$(3x + 15)(x - 2)$ Common factor so no need to test.			
2, –15	$(3x + 2)(x - 15)$	-45	$2x$	$-43x$
	$(3x - 15)(x + 2)$ Common factor so no need to test.			

Continued

Example (cont)

Factors of –30	Resulting Binomials	Product of Outside Terms	Product of Inside Terms	Sum of Products
{–3, 10}	$(3x - 3)(x + 10)$	Common factor so no need to test.		
	$(3x + 10)(x - 3)$	$-9x$	$10x$	x
{3, –10}	$(3x + 3)(x - 10)$	Common factor so no need to test.		
	$(3x - 10)(x + 3)$	$9x$	$-10x$	$\boxed{-x}$

Continued.

Example (cont)

Check the resulting factorization using the FOIL method.

$$\begin{array}{lcccc} & \textbf{\textit{F}} & \textbf{\textit{O}} & \textbf{\textit{I}} & \textbf{\textit{L}} \\ (3x-10)(x+3) = & 3x(x) & +\,3x(3) & -\,10(x) & -\,10(3) \end{array}$$

$$= 3x^2 + 9x - 10x - 30$$

$$= 3x^2 - x - 30$$

So our final answer when asked to factor the polynomial $6x^2y^2 - 2xy^2 - 60y^2$ will be $2y^2(3x - 10)(x + 3)$.

Factoring by Grouping

To Factor Trinomials by Grouping

Step 1: Find two numbers whose product is $a \cdot c$ and whose sum is b.

Step 2: Write the middle term, bx, using the factors found in Step 2.

Step 3: Factor by grouping.

Example

Factor $8x^2 - 14x + 5$

Step 1: Find two numbers whose product is *ac* or (8)(5), and whose sum if *b* or –14.

Factors of 40	Sum of Factors
–40, –1	–41
–20, –2	–22
–10, –4	–14

Step 2: Write –14 as –4*x* – 10*x* so that

$$8x^2 - 14x + 5 = 8x^2 - 4x - 10x + 5$$

Step 3: Factor by grouping.

$$8x^2 - 4x - 10x + 5 = 4x(2x-1) - 5(2x-1)$$

$$= (2x-1)(4x-5)$$

Example

Factor $6x^2 - 2x - 20$

Factor out the greatest common factor, 2.

$$6x^2 - 2x - 20 = 2(3x^2 - x - 10)$$

Find two numbers whose product is *ac* or (3)(–10) = –30 and whose sum is *b*, –1.

$$3x^2 - x - 10 = 3x^2 - 6x + 5x - 10$$

$$3x^2 - 6x + 5x - 10 = 3x(x - 2) + 5(x - 2)$$

$$= (x - 2)(3x + 5)$$

Factor by Substitution

- Sometimes complicated polynomials can be rewritten into a form that is easier to factor by using substitution.
- We replace a portion of the polynomial with a single variable, hopefully now creating a format that is familiar to us for factoring purposes.

Example

Factor $(4r + 1)^2 + 8(4r + 1) + 16$

Replace $4r + 1$ with the variable x.

Then our polynomial becomes

$$x^2 + 8x + 16$$

which factors into

$$(x + 4)^2$$

We then have to replace the original variable to get

$$(4r + 1 + 4)^2 = (4r + 5)^2$$