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The inverse scattering transform for the focusing nonlinear Schrödinger equation with asymmetric boundary conditions

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The inverse scattering transform (IST) as a tool to solve the initial-value problem for the focusing nonlinear Schrödinger (NLS) equation with non-zero boundary values $q_{l/r}(t) \equiv A_{l/r} e^{-2iA_{l/r}^2 t + i\theta_{l/r}}$ as $x \to \mp\infty$ is presented in the fully asymmetric case for both asymptotic amplitudes and phases, i.e., with $A_l \neq A_r$ and $\theta_l \neq \theta_r$. The direct problem is shown to be well-defined for NLS solutions $q(x,t)$ such that $\big(q(x,t) - q_{l/r}(t)\big) \in L^{1,1}(\mathbb{R}^{\mp})$ with respect to x for all $t \geq 0$, and the corresponding analyticity properties of eigenfunctions and scattering data are established. The inverse scattering problem is formulated both via (left and right) Marchenko integral equations, and as a Riemann-Hilbert problem on a single sheet of the scattering variables $\lambda_{l/r} = \sqrt{k^2 + A_{l/r}^2}$, where k is the usual complex scattering parameter in the IST. The time evolution of the scattering coefficients is then derived, showing that, unlike the case of solutions with equal amplitudes as $x \to \pm\infty$, here both reflection and transmission coefficients have a nontrivial (although explicit) time dependence. The results presented in this paper will be instrumental for the investigation of the long-time asymptotic behavior of fairly general NLS solutions with nontrivial boundary conditions via the nonlinear steepest descent method on the Riemann-Hilbert problem, or via matched asymptotic expansions on the Marchenko integral equations. *© 2014 AIP Publishing LLC*. [http://dx.doi.org/10.1063/1.4898768]

I. INTRODUCTION

Nonlinear Schrödinger (NLS) systems have attracted the attention of the physical community for almost 60 years, and equations of NLS type have been derived in such diverse fields as deep water waves,[4,37] plasma physics,[33] nonlinear fiber optics,[21] magnetic spin waves,[16,38] Bose-Einstein condensates,[29] and much more. As a matter of fact, most dispersive energy preserving systems give rise, in appropriate limits, to the scalar NLS, which explains the keen interest in it as a prototypical integrable system, and motivates the efforts devoted to advance our mathematical understanding of this equation. In this respect, the inverse scattering transform (IST) as a method to solve the initial-value problem for the scalar NLS equation,

$$iq_t = q_{xx} - 2\sigma\,|q|^2 q \tag{1.1}$$

(subscripts x and t denote partial differentiation throughout) has been extensively studied in the literature, both in the focusing ($\sigma = -1$) and in the defocusing ($\sigma = 1$) dispersion regimes; see, for instance, Refs. 2–4, 15, 26, and 37 for detailed accounts of the IST in the case of potentials $q(x,t)$ rapidly decaying as $x \to \pm\infty$. The situation is quite different when one is interested in potentials that do not decay at space infinity. In fact, even though the IST for the focusing NLS with rapidly decaying potentials was first proposed more than 40 years ago, and has been subsequently the subject of a vast amount of studies and applications, not as much is available in the literature in the case of nontrivial boundary conditions. The reason for this deficiency is twofold: on one hand, the technical difficulties resulting from the non-zero boundary conditions (NZBCs) significantly complicate the formulation of the IST; on the other hand, the onset of modulational instability, also known as the Benjamin-Feir instability[10,11] in the context of water waves, was believed to be an obstacle to the development of the IST, or at least to its validity. Nonetheless, a large number of exact solutions to the focusing NLS equation with NZBCs have been found over the years by the use of direct methods. Historically, the first such solution was found by Peregrine in 1983,[30] although a solitonic solution in the presence of a condensate was already reported in Refs. 23 and 24. In 1985, a second order Peregrine solution, periodic in space and homoclinic in time, was found.[6] "Multi-Peregrine" solutions and more general solitonic solutions were subsequently discovered in Refs. 7–9, 22, 25, 32, and 34. In recent years these solutions have been actively studied worldwide, and the renewed interest is due to the fact that the development of modulation instability in the governing equation has been recently suggested as a mechanism for the formation of "extreme" (also known as "rogue," or "freak") waves, where energy density exceeds the mean level by an order of magnitude.[28,35,36] General high-order rogue waves in the nonlinear Schrödinger equation have also been recently obtained by the bilinear method in Ref. 27, where it is shown that the general Nth order rogue wave contains $N-1$ free irreducible complex parameters, and that the specific rogue waves obtained by Akhmediev *et al.* in Ref. 8 correspond to special choices of these free parameters, yielding the highest peak amplitudes among all rogue waves of the same order. At the same time, the observation of rogue waves has been reported in an optical system, based on a microstructured optical fiber.[31] The generation of these rogue waves has been modelled using a generalized NLS equation, and shown to be an infrequent evolution from initially smooth pulses owing to power transfer seeded by a small noise perturbation.

In view of these recent developments, it is natural to wonder about the role that soliton solutions play in the nonlinear development of the modulation instability, which makes the study of the long-time asymptotics of NLS solutions of great practical importance, crucial for developing a consistent theory for rogue waves in the ocean, and for extreme events in optical fibers. In this respect, the investigation of the IST for the focusing case with NZBCs [Eq. (1.1) with $\sigma = -1$], i.e.,

$$i q_t = q_{xx} + 2|q|^2 q \,, \tag{1.2}$$

as a means to provide the time evolution of a fairly general initial one-dimensional pulse/wave profile over a nontrivial background, should receive a greater deal of attention, since it allows the study of the long-time asymptotic behavior via the nonlinear steepest descent method,[13,14,17] matched asymptotic expansions,[1,5] or other germane techniques. Until this year, the only general study of IST for the focusing NLS with NZBCs available in the literature was in Ref. 24, which only contains partial results as it is limited to the case of completely symmetric boundary conditions with $\lim_{x\to+\infty} q(x,t) = \lim_{x\to-\infty} q(x,t)$, i.e., only the case in which the potential exhibits no asymptotic phase difference and no amplitude difference is treated. Some interesting results can also be found in Ref. 20, where the authors propose a perturbation theory for the focusing NLS equation with symmetric boundary conditions based on the IST in order to analyze the effect of different dispersive, diffusive (damping), or nonlinear perturbations to the soliton propagation. Most recently, Biondini and Kovačič[12] have developed the IST for potentials with an arbitrary asymptotic phase difference, although assuming equal amplitudes at both space infinities. They also discuss the general behavior of the soliton solutions, as well as the reductions to all special solutions previously known in the literature and mentioned above.

In this work, we will develop the IST for the scalar focusing NLS (1.2) with fully asymmetric NZBCs,

$$q(x,t) \to q_{l/r}(t) = A_{l/r} e^{-2iA_{l/r}^2 t + i\theta_{l/r}} \quad \text{as } x \to \mp\infty, \tag{1.3}$$

where $A_r \geq A_l > 0$ and $0 \leq \theta_{l/r} < 2\pi$ are arbitrary constants. This is a highly nontrivial generalization of the work,[12] and it involves dealing with additional technical difficulties, the most important of which being the fact that when the amplitudes of the NLS solutions as $x \to \pm\infty$ are different, in the spectral domain one cannot introduce a uniformization variable that allows mapping the multiply sheeted Riemann surface for the scattering parameter to a single complex plane. From the point of view of physical applications, such a generalization would be particularly significant for the theoretical investigation of rogue waves and perturbed soliton solutions in microstructured fiber optical systems with different background amplitudes enforced at either end of the fiber. This work would also be relevant in clarifying the role that soliton solutions play in the nonlinear development of modulation instability in such systems.

The plan of the paper is outlined below. Section II is devoted to the study of the direct scattering problem. We will prove that the direct problem is well defined for potentials $q(x,t)$ such that $\big(q(x,t)-q_{l/r}(t)\big)\in L^{1,1}(\mathbb{R}^{\mp})$ with respect to x for all $t\ge 0$, $L^{1,s}(\mathbb{R})$ being the complex Banach space of all measurable functions $f(x)$ for which $(1+|x|)^s f(x)$ is integrable. We will then establish analyticity of eigenfunctions and scattering data, and obtain integral representations for the latter for potentials in this class. In Sec. III, we will formulate the inverse problem both in terms of (left and right) Marchenko integral equations (Sec. III B), and as a Riemann-Hilbert (RH) problem on a single sheet of the scattering variables $\lambda_{l/r}=\sqrt{k^2+A_{l/r}^2}$, where k is the usual complex scattering parameter in the IST (Sec. III C). Important differences with respect to the symmetric case also arise in the inverse problem, where, in addition to solitons (corresponding to the discrete eigenvalues of the scattering problem), and to radiation (corresponding to the continuous spectrum of the scattering operator, and represented in the inverse problem by the reflection coefficients for $k\in\mathbb{R}\cup[-iA_l,iA_l]$), one also has a nontrivial contribution from additional spectral data for $k\in(-iA_r,-iA_l)\cup(iA_l,iA_r)$, which appears in both formulations of the inverse problem. [Note that with a slight abuse of notation, in the paper we will denote the relevant segments on the imaginary k-axis by $\Sigma_{r/l}\equiv[-iA_{r/l},iA_{r/l}]$.] In particular, this implies that no pure soliton solutions exist, and solitons are always accompanied by a radiative contribution of some sort. As a consequence, unlike the equal-amplitude case dealt with in Ref. 12, here no explicit solution can be obtained by simply reducing the inverse problem to a set of algebraic equations. In view of this, the present study provides a very powerful tool for the asymptotic investigation of NLS solutions that cannot be obtained by direct methods. Specifically, the RH formulation of the inverse problem makes it amenable to the study of the long-time asymptotic behavior via the nonlinear steepest descent method, as was done, for instance, in Ref. 17 for the modified KdV equation, or in Refs. 14 and 13 for the focusing NLS with special step-like initial conditions. The Marchenko integral equations provide an alternative setup for the study of the long-time behavior of the solutions by means of matched asymptotics, as was recently done for KdV in Ref. 1. Section IV deals with the time evolution of eigenfunctions and scattering coefficients, and Sec. V is devoted to some concluding remarks. For a better readability of the paper, more technical proofs are collected in the Appendix.

II. DIRECT PROBLEM

It is well-known that the focusing NLS Eq. (1.2) can be associated with the following Lax pair:

$$\frac{\partial v}{\partial x}=(-ik\sigma_3+Q)\,v\,,\qquad \frac{\partial v}{\partial t}=\big[(2ik^2-i|q|^2+iQ_x)\sigma_3-2kQ\big]\,v\,,\tag{2.1}$$

where $v(x,k,t)$ is a two component vector, $k\in\mathbb{C}$ is the scattering parameter, and

$$\sigma_2=\begin{pmatrix}0&-i\\ i&0\end{pmatrix},\qquad \sigma_3=\begin{pmatrix}1&0\\0&-1\end{pmatrix},\qquad Q(x,t)=\begin{pmatrix}0&q(x,t)\\-q^*(x,t)&0\end{pmatrix}.\tag{2.2}$$

[Here and in the following the asterisk indicates complex conjugate; σ_2 is given for future reference.] The first equation in the Lax pair is the well-known Zakharov-Shabat (ZS) scattering problem,[2,37] and the matrix $Q(x,t)$ [or, equivalently, its entry $q(x,t)$] is referred to as the "potential" of the ZS scattering problem. Here we will consider potentials with nontrivial boundary conditions (BCs) as in (1.3), where we assume that $A_r\ge A_l>0$. Note that while the asymptotic amplitudes of the BCs $A_{l/r}$ can be assumed to be time independent, the asymptotic phases evolve as follows: $\theta_{l/r}(t)=-2A_{l/r}^2t+\theta_{l/r}$ (see Sec. IV for details). As a result, unlike the equal-amplitude BCs ($A_l=A_r$) considered in Ref. 12, here it is not possible to subtract out the background and make both BCs time-independent.

In the formulation of the direct problem we will omit to explicitly specify the time-dependence for brevity. It will be clear from the context whether one is considering $t = 0$ or an arbitrary $t > 0$. We also assume the integrability condition

$$(\boldsymbol{H}_s): \qquad \int_0^\infty dx\,(1+|x|)^s\,\{|q(-x)-q_l| + |q(x)-q_r|\} < +\infty, \tag{2.3}$$

where $s = 0, 1, 2$ depending on the situation [and, again, the relevant condition will be assumed to hold for all $t \geq 0$]. Note that the condition $(\boldsymbol{H}_s)$ is equivalent to assuming $(q(x) - q_{l/r}) \in L^{1,s}(\mathbb{R}^\mp)$.

For later convenience, we denote the limits of $Q(x, t)$ as $x \to +\infty$ and $x \to -\infty$ as $Q_r(t)$ and $Q_l(t)$, respectively. We also introduce the "free" potential matrix $Q_f(x, t)$ as follows:

$$Q_f(x,t) = Q_r(t)\,\theta(x) + Q_l(t)\,\theta(-x), \tag{2.4}$$

where $\theta(x)$ is the Heaviside function [as specified above, in the following the time dependence of Q_f and $Q_{l/r}$ will be omitted for brevity.]

It is convenient to introduce asymptotic scattering operators as $x \to \mp\infty$,

$$\Lambda_{l/r}(k) = -ik\sigma_3 + Q_{l/r}\,, \tag{2.5}$$

$$\Lambda(x,k) = -ik\sigma_3 + Q_f(x) = \Lambda_r(k)\theta(x) + \Lambda_l(k)\theta(-x)\,, \tag{2.6}$$

and define the *fundamental eigensolutions* $\tilde{\Psi}(x, k)$ and $\tilde{\Phi}(x, k)$ as those 2×2 matrix solutions to the first of (2.1) which satisfy the asymptotic conditions

$$\tilde{\Psi}(x,k) = e^{x\,\Lambda_r(k)}[I_2 + o(1)] \qquad x \to +\infty\,, \tag{2.7a}$$

$$\tilde{\Phi}(x,k) = e^{x\,\Lambda_l(k)}[I_2 + o(1)] \qquad x \to -\infty\,. \tag{2.7b}$$

[Here and in the following I_2 denotes the 2×2 identity matrix]. Note that $e^{x\,\Lambda_{l/r}(k)}$ is a bounded group for all $x \in \mathbb{R}$ iff $\Lambda_{l/r}(k)$ has only zero or purely imaginary eigenvalues and is diagonalizable. This occurs iff $k \in \mathbb{R} \cup (-iA_{l/r}, iA_{l/r})$, respectively. For $k = \pm iA_{l/r}$, the norm of the group $e^{x\,\Lambda_{l/r}(k)}$ grows linearly in x as $x \to \mp\infty$ (see the Appendix for the explicit computation of the relevant norm estimates for the groups $e^{x\,\Lambda_{l/r}(k)}$). Then the following result can be established similarly as to what was done for the defocusing NLS equation in a preceding paper.[18] For convenience we sketch the proof in the Appendix.

Proposition 2.1. Let the potential satisfy $(\boldsymbol{H}_0)$. *Then for* $k \in \mathbb{R} \cup (-iA_r, iA_r)$ *the fundamental eigensolution* $\tilde{\Psi}(x, k)$ *with asymptotic behavior* (2.7a) *can be obtained as the unique solution to the integral equation*

$$\tilde{\Psi}(x,k) = e^{x\,\Lambda_r(k)} - \int_x^\infty dy\, e^{(x-y)\Lambda_r(k)}[Q(y) - Q_r]\tilde{\Psi}(y,k)\,. \tag{2.8a}$$

Moreover, $\tilde{\Psi}(x, k)$ *is continuous for* $x_0 \leq x$ *for any finite* x_0, *and, as a function of* k, *for all* $k \in \mathbb{R} \cup (-iA_r, iA_r)$. *Similarly, for* $k \in \mathbb{R} \cup (-iA_l, iA_l)$ *the fundamental eigensolution* $\tilde{\Phi}(x, k)$ *is given by the unique solution to the integral equation*

$$\tilde{\Phi}(x,k) = e^{x\,\Lambda_l(k)} + \int_{-\infty}^x dy\, e^{(x-y)\Lambda_l(k)}[Q(y) - Q_l]\tilde{\Phi}(y,k)\,, \tag{2.8b}$$

continuous for $x \leq x_0$ *for any finite* x_0, *and for all* $k \in \mathbb{R} \cup (-iA_l, iA_l)$. *In addition, under the hypothesis* $(\boldsymbol{H}_1)$, (2.8a) *has a unique, continuous solution for* $k \in [-iA_r, iA_r]$, *and* (2.8b) *has a unique, continuous solution for* $k \in [-iA_l, iA_l]$ *(i.e., the result can be extended up to the corresponding branch points).*

Assuming ($\boldsymbol{H}_1$), one can replace the integral equations (2.8a)–(2.8b) by different ones. Let us introduce the fundamental matrix $\mathcal{G}(x, y; k)$ as follows:

$$\mathcal{G}(x, y; k) = \theta(x)\theta(y)\, e^{(x-y)\Lambda_r(k)} + \theta(-x)\theta(-y)\, e^{(x-y)\Lambda_l(k)} + \theta(x)\theta(-y)\, e^{x\Lambda_r(k)} e^{-y\Lambda_l(k)} + \theta(-x)\theta(y)\, e^{x\Lambda_l(k)} e^{-y\Lambda_r(k)}. \tag{2.9}$$

$\mathcal{G}(x, y; k)$ is a continuous matrix function of $(x, y, k) \in \mathbb{R}^2 \times \mathbb{C}$ which satisfies the initial value problems

$$\frac{\partial \mathcal{G}(x, y; k)}{\partial x} = \Lambda(x, k)\mathcal{G}(x, y, k)\,, \qquad \mathcal{G}(y, y; k) = I_2\,, \tag{2.10a}$$

$$\frac{\partial \mathcal{G}(x, y; k)}{\partial y} = -\mathcal{G}(x, y, k)\Lambda(y, k)\,, \qquad \mathcal{G}(x, x; k) = I_2\,, \tag{2.10b}$$

where $\Lambda(x, k)$ is given by (2.6). For further details on the fundamental matrix we refer to Appendix A of Ref. 18, where their properties were investigated for the defocusing NLS equation. Then using (2.10a)–(2.10b), one can easily check that the fundamental eigenfunctions also satisfy the integral equations

$$\tilde{\Psi}(x, k) = \mathcal{G}(x, 0; k) - \int_x^\infty dy\, \mathcal{G}(x, y; k)[Q(y) - Q_f(y)]\tilde{\Psi}(y, k), \tag{2.11a}$$

$$\tilde{\Phi}(x, k) = \mathcal{G}(x, 0; k) + \int_{-\infty}^x dy\, \mathcal{G}(x, y; k)[Q(y) - Q_f(y)]\tilde{\Phi}(y, k), \tag{2.11b}$$

with $Q_f(x)$ as in (2.4), and $\mathcal{G}(x, 0; k) = \theta(x)\, e^{x\Lambda_r(k)} + \theta(-x)\, e^{x\Lambda_l(k)}$ according to (2.9). Note that (2.11a) coincides with (2.8a) for $x \geq 0$, and (2.11b) coincides with (2.8b) for $x \leq 0$. On the other hand, using (2.9) we get

$$\tilde{\Psi}(x, k) = e^{x\Lambda_l(k)} \left[I_2 - \int_x^\infty dy\, \mathcal{G}(0, y; k)[Q(y) - Q_f(y)]\tilde{\Psi}(y, k) \right] \quad x \leq 0, \tag{2.12a}$$

$$\tilde{\Phi}(x, k) = e^{x\Lambda_r(k)} \left[I_2 + \int_{-\infty}^x dy\, \mathcal{G}(0, y; k)[Q(y) - Q_f(y)]\tilde{\Phi}(y, k) \right] \quad x \geq 0, \tag{2.12b}$$

where, unlike (2.8a)–(2.8b), the integrals in the right-hand sides converge absolutely as $x \to \mp\infty$. Note, however, that since the fundamental matrix $\mathcal{G}(x, y; k)$ depends on both groups $e^{x\Lambda_{l/r}(k)}$, the integral equations (2.11) can only be used to define both fundamental eigenfunctions $\tilde{\Phi}(x, k)$ and $\tilde{\Psi}(x, k)$ for $k \in \mathbb{R} \cup [-iA_l, iA_l]$.

Finally, note that any matrix solution of the scattering problem (2.1) satisfies $\frac{\partial}{\partial x} \det M = \operatorname{trace}(-ik\sigma_3 + Q) \det M = 0$. Consequently, the fundamental eigensolutions have determinants which are independent of x, and from their asymptotic behavior (2.7a)–(2.7b) it follows that

$$\det \tilde{\Psi}(x, k) = \det \tilde{\Phi}(x, k) = 1\,. \tag{2.13}$$

A. Jost solutions

In this section, we define the Jost solutions and derive their continuity and analyticity properties. Since the asymptotic scattering operators $\Lambda_{l/r}(k)$ are traceless, and such that $\Lambda_{l/r}^2(k) = -(k^2 + A_{l/r}^2)I_2$, it is natural to consider the conformal mappings

$$\lambda_l = \sqrt{k^2 + A_l^2}, \qquad \lambda_r = \sqrt{k^2 + A_r^2}, \tag{2.14}$$

with branch cuts along the imaginary segments $\Sigma_{l/r} = [-iA_{l/r}, iA_{l/r}]$. Introducing appropriate local polar coordinates, we define

$$\lambda_r = \sqrt{r_1 r_2}\, e^{i(\theta_1+\theta_2)/2}, \quad \lambda_l = \sqrt{r_3 r_4}\, e^{i(\theta_3+\theta_4)/2}, \tag{2.15}$$

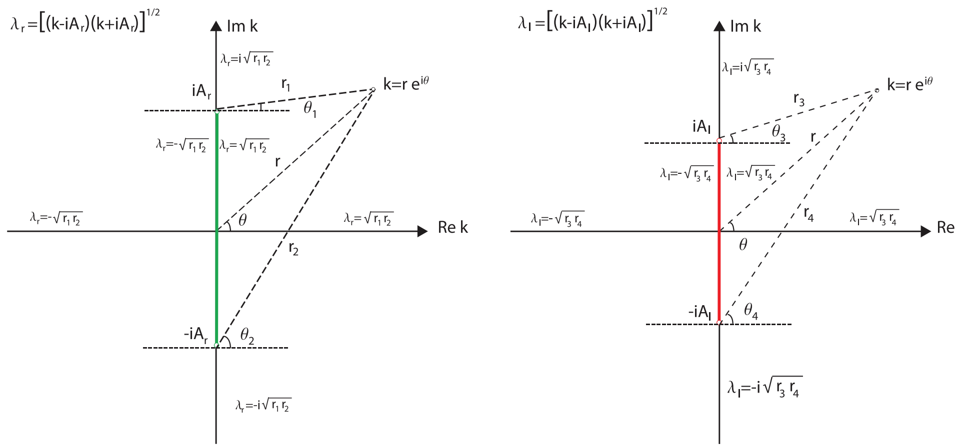

FIG. 1. The branch cuts for $\lambda_r = \sqrt{k^2 + A_r^2}$ and $\lambda_l = \sqrt{k^2 + A_l^2}$: we define $\lambda_r = \sqrt{r_1 r_2} e^{i\Theta}$ with $r_1 = |k - iA_r|$, $r_2 = |k + iA_r|$ and $\Theta = (\theta_1 + \theta_2)/2$ and angles $-\pi/2 < \theta_1\,, \theta_2 \le 3\pi/2$. Similarly, $\lambda_l = \sqrt{r_3 r_4} e^{i\Theta'}$ with $r_3 = |k - iA_l|$, $r_4 = |k + iA_l|$, and $\Theta' = (\theta_3 + \theta_4)/2$ and angles $-\pi/2 < \theta_3\,, \theta_4 \le 3\pi/2$.

with $r_j \ge 0$, and $-\pi/2 \le \theta_j < 3\pi/2$ for $j = 1, \ldots, 4$ as indicated in Fig. 1. We will consider a single sheet of the complex plane for k, and denote by $\mathbb{K}_{l/r}$ the plane cut along the segments $\Sigma_{l/r}$ on the imaginary axis. $\mathbb{C}^\pm$ will denote the open upper/lower complex half planes, and $\mathbb{K}^\pm_{r/l}$ the open upper/lower complex half-planes cut along $\Sigma_{r/l}$, respectively. Then the following results can easily be established.

λ_r provides one-to-one correspondences between the following sets:

- $k \in \mathbb{K}_r^+ \equiv \mathbb{C}^+ \setminus (0, iA_r]$ and $\lambda_r \in \mathbb{C}^+$
- $k \in \partial\mathbb{K}_r^+ \equiv \mathbb{R} \cup \{is - 0^+ : 0 < s < A_r\} \cup \{iA_r\} \cup \{is + 0^+ : 0 < s < A_r\}$ and $\lambda_r \in \mathbb{R}$
- $k \in \mathbb{K}_r^- \equiv \mathbb{C}^- \setminus [-iA_r, 0)$ and $\lambda_r \in \mathbb{C}^-$
- $k \in \partial\mathbb{K}_r^- \equiv \mathbb{R} \cup \{is - 0^+ : -A_r < s < 0\} \cup \{-iA_r\} \cup \{is + 0^+ : -A_r < s < 0\}$ and $\lambda_r \in \mathbb{R}$.

Similarly, λ_l provides one-to-one correspondences between the following sets:

- $k \in \mathbb{K}_l^+ \equiv \mathbb{C}^+ \setminus (0, iA_l]$ and $\lambda_l \in \mathbb{C}^+$
- $k \in \partial\mathbb{K}_l^+ \equiv \mathbb{R} \cup \{is - 0^+ : 0 < s < A_l\} \cup \{iA_l\} \cup \{is + 0^+ : 0 < s < A_l\}$ and $\lambda_l \in \mathbb{R}$
- $k \in \mathbb{K}_l^- \equiv \mathbb{C}^- \setminus [-iA_l, 0)$ and $\lambda_l \in \mathbb{C}^-$
- $k \in \partial\mathbb{K}_l^- \equiv \mathbb{R} \cup \{is - 0^+ : -A_l < s < 0\} \cup \{-iA_l\} \cup \{is + 0^+ : -A_l < s < 0\}$ and $\lambda_l \in \mathbb{R}$.

Note that with this choice for the branch cuts, one has $\lambda_r \sim \lambda_l \sim k$ as $k \to \infty$ in the entire plane (cf. (A11)). In the following, $\lambda_l^\pm(k)$ will denote the boundary values taken by $\lambda_l(k)$ for $k \in \Sigma_l$ from the right/left edge of the cut, with

$$\lambda_l^\pm(k) = \pm\sqrt{A_l^2 - |k|^2}\,, \qquad k = is \pm 0^+, \quad |s| \le A_l \tag{2.16a}$$

on the right/left edge, and $\lambda_r^\pm(k)$ will denote the boundary values taken by $\lambda_r(k)$ for $k \in \Sigma_r$ from the right/left edge of the cut, with

$$\lambda_r^\pm(k) = \pm\sqrt{A_r^2 - |k|^2}\,, \qquad k = is \pm 0^+, \quad |s| \le A_r \tag{2.16b}$$

on the right/left edge (cf. Fig. 1).

Clearly, $\pm i\lambda_{l/r}$ are the eigenvalues of $\Lambda_{l/r}(k)$, and the eigenvector matrices $W_{l/r}(k)$, such that

$$\Lambda_{l/r}(k) W_{l/r}(k) = -i\lambda_{l/r} W_{l/r}(k)\sigma_3\,, \tag{2.17}$$

can be conveniently chosen as follows:

$$W_{l/r}(k) = I_2 - \frac{i}{\lambda_{l/r} + k}\sigma_3 Q_{l/r}\,. \tag{2.18}$$

We can then define the Jost solutions in terms of the fundamental eigensolutions (2.7a)–(2.7b) as follows:

$$\Phi(x,k) = \big(\phi(x,k)\,\bar\phi(x,k)\big) := \tilde\Phi(x,k)W_l(k)\,, \tag{2.19a}$$

$$\Psi(x,k) = \big(\bar\psi(x,k)\,\psi(x,k)\big) := \tilde\Psi(x,k)W_r(k)\,, \tag{2.19b}$$

or, equivalently, as those solutions to the scattering problem in (2.1) with the following asymptotic behavior:

$$\Phi(x,k) \sim W_l(k)\,e^{-i\lambda_l x\sigma_3} \qquad x \to -\infty\,, \tag{2.20a}$$

$$\Psi(x,k) \sim W_r(k)\,e^{-i\lambda_r x\sigma_3} \qquad x \to +\infty\,. \tag{2.20b}$$

The Jost solutions $\bar\psi(x,k)$, $\psi(x,k)$ are then defined for $\lambda_r \in \mathbb{R}$, corresponding to $k \in \partial\mathbb{K}_r^+ \cup \partial\mathbb{K}_r^-$, and when $k = is \in [-iA_r, iA_r]$ we will label with a superscript $^\pm$ the values on the right/left edge of the cut in both half-planes, i.e.,

$$\Psi^\pm(x,is) \equiv \big(\bar\psi^\pm(x,is)\;\;\psi^\pm(x,is)\big) := \tilde\Psi(x,is)\,W_r(is\pm 0^+) \qquad |s| \le A_r\,, \tag{2.21a}$$

since $\tilde\Psi(x,k)$ is single-valued across the cut, and $W_r(k)$ has right/left limits defined via (2.16b).

Analogously, the Jost solutions $\phi(x,k)$, $\bar\phi(x,k)$ are defined for $\lambda_l \in \mathbb{R}$, corresponding to $k \in \partial\mathbb{K}_l^+ \cup \partial\mathbb{K}_l^-$, and when $k = is \in [-iA_l, iA_l]$ a superscript $^\pm$ will be used to denote the values on the right/left edge of the cut in both half-planes, i.e.,

$$\Phi^\pm(x,is) \equiv \big(\phi^\pm(x,is)\,\bar\phi^\pm(x,is)\big) := \tilde\Phi(x,is)\,W_l(is\pm 0^+) \qquad |s| \le A_l\,, \tag{2.21b}$$

since $\tilde\Phi(x,k)$ is single-valued across the cut, and $W_l(k)$ has right/left limits defined via (2.16a).

The following theorems establish the analyticity properties of the Jost solutions as functions of the scattering parameter k. The standard over-bar notation is used over the symbol of a set to denote its closure.

Proposition 2.2. Suppose ($\boldsymbol{H}_1$) *holds. Then, for every* $x \in \mathbb{R}$, *the Jost solution* $\psi(x,k)$ *[resp.* $\bar\psi(x,k)$*] extends to a function that is continuous for* $k \in \overline{\mathbb{K}_r^+} \cup \partial\mathbb{K}_r^-$ *[resp.* $k \in \overline{\mathbb{K}_r^-} \cup \partial\mathbb{K}_r^+$*] and analytic for* $k \in \mathbb{K}_r^+$ *[resp.* $k \in \mathbb{K}_r^-$*]. Similarly, the Jost solution* $\phi(x,k)$ *[resp.* $\bar\phi(x,k)$*] extends to a function that is continuous for* $k \in \overline{\mathbb{K}_l^+} \cup \partial\mathbb{K}_l^-$ *[resp.* $k \in \overline{\mathbb{K}_l^-} \cup \partial\mathbb{K}_l^+$*] and analytic for* $k \in \mathbb{K}_l^+$ *[resp.* $k \in \mathbb{K}_l^-$*].*

Proposition 2.3. Suppose ($\boldsymbol{H}_2$) *holds. Then, for every* $x \in \mathbb{R}$, *the derivatives* $\partial_k\psi(x,k)$, $\partial_k\bar\psi(x,k)$, $\partial_k\phi(x,k)$, *and* $\partial_k\bar\phi(x,k)$ *extend to functions that are continuous for:* $k \in \mathbb{K}_r^+ \cup \partial\mathbb{K}_r^- \setminus \{-iA_r\}$, $k \in \mathbb{K}_r^- \cup \partial\mathbb{K}_r^+ \setminus \{iA_r\}$, $k \in \mathbb{K}_l^+ \cup \partial\mathbb{K}_l^- \setminus \{-iA_l\}$, *and* $k \in \mathbb{K}_l^- \cup \partial\mathbb{K}_l^+ \setminus \{iA_l\}$, *respectively; and analytic for:* $k \in \mathbb{K}_r^-$, $k \in \mathbb{K}_r^+$, $k \in \mathbb{K}_l^-$, *and* $k \in \mathbb{K}_l^+$, *respectively.*

Note that $\mathbb{K}_{r/l}^\pm$ are intended as analytic manifolds, and continuity of the Jost solutions across the cuts is intended as the existence of right/left continuous limits only in the domains that have the branch cut as part of their boundary as an analytic manifold. In the half-planes where locally there is no analytic continuation off the branch cut, the functions $\psi^\pm(x,k)$, $\bar\psi^\pm(x,k)$ [resp. $\phi^\pm(x,k)$ and $\bar\phi^\pm(x,k)$] are as given in (2.21a) [resp. (2.21b)] with the two choices of $\lambda_r^\pm$ [resp. $\lambda_l^\pm$], and can be obtained as the unique solutions of the corresponding Volterra integral equations.

B. Scattering coefficients

From the integral equations for the fundamental matrices (2.12a)–(2.12b), one can easily find

$$\tilde\Psi(x,k) = e^{x\Lambda_l(k)}[\boldsymbol{B}_r(k) + o(1)] \qquad x \to -\infty, \tag{2.22a}$$

$$\tilde\Phi(x,k) = e^{x\Lambda_r(k)}[\boldsymbol{B}_l(k) + o(1)] \qquad x \to +\infty, \tag{2.22b}$$

where the coupling matrices

$$\boldsymbol{B}_r(k) = I_2 - \int_{-\infty}^{\infty} dy\, \mathcal{G}(0, y; k)[Q(y) - Q_f(y)]\tilde{\Psi}(y, k), \tag{2.23a}$$

$$\boldsymbol{B}_l(k) = I_2 + \int_{-\infty}^{\infty} dy\, \mathcal{G}(0, y; k)[Q(y) - Q_f(y)]\tilde{\Phi}(y, k), \tag{2.23b}$$

are each other's inverses. Under the assumptions of Proposition 2.1, in Eqs. (2.22a)–(2.22b) and (2.23a)–(2.23b), one needs to take $k \in \mathbb{R} \cup [-iA_l, iA_l]$, where (2.11a)–(2.11b) are defined. Furthermore, Eqs. (2.13), (2.22a)–(2.22b), and (A1) imply that $\det \boldsymbol{B}_r(k) = \det \boldsymbol{B}_l(k) = 1$ for $k \in \mathbb{R} \cup [-iA_l, iA_l]$.

Using (2.19a)–(2.19b) and (2.22a)–(2.22b) to obtain the asymptotic behavior of the Jost solutions as $x \to \pm\infty$, we can then express each set of Jost solutions as a linear combination of the other set, i.e.,

$$\begin{pmatrix} \phi(x,k) & \bar{\phi}(x,k) \end{pmatrix} = \begin{pmatrix} \bar{\psi}(x,k) & \psi(x,k) \end{pmatrix} S(k), \tag{2.24a}$$

$$\begin{pmatrix} \bar{\psi}(x,k) & \psi(x,k) \end{pmatrix} = \begin{pmatrix} \phi(x,k) & \bar{\phi}(x,k) \end{pmatrix} \bar{S}(k), \tag{2.24b}$$

where the *scattering matrices* $S(k)$ and $\bar{S}(k)$ are obviously each other's inverse, and they are given by

$$S(k) = W_r^{-1}(k)\boldsymbol{B}_l(k)W_l(k), \qquad \bar{S}(k) = W_l^{-1}(k)\boldsymbol{B}_r(k)W_r(k). \tag{2.25}$$

For later convenience, we write

$$S(k) = \begin{pmatrix} a(k) & \bar{b}(k) \\ b(k) & \bar{a}(k) \end{pmatrix}, \qquad \bar{S}(k) = \begin{pmatrix} \bar{c}(k) & d(k) \\ \bar{d}(k) & c(k) \end{pmatrix}, \tag{2.26}$$

where the entries of the scattering matrices are usually referred to as *scattering coefficients*. It is clear from (2.24a)–(2.24b) that $S(k)$ [resp. $\bar{S}(k)$] is in general defined where all four Jost solutions are, i.e., for $k \in \partial\mathbb{K}_l^- \cup \partial\mathbb{K}_l^+$ [resp. $k \in \partial\mathbb{K}_l^- \cup \partial\mathbb{K}_l^+ \setminus \{\pm iA_l\}$, the branch points being excluded because of the second of (2.25)]. When $k \in [-iA_l, iA_l]$, the above scattering matrices and their entries are defined by means of the corresponding values on the right/left edge of the cut, and labeled with superscripts $^\pm$ as clarified below. Since $\det \Phi(x,k)$ and $\det \Psi(x,k)$ are independent of x [and hence these determinants can be computed from their limits as $x \to \mp\infty$], one can easily verify that

$$\det S(k) = \frac{\det W_l(k)}{\det W_r(k)} = \frac{\lambda_l(\lambda_r + k)}{\lambda_r(\lambda_l + k)}, \qquad \det \bar{S}(k) = \frac{\det W_r(k)}{\det W_l(k)} = \frac{\lambda_r(\lambda_l + k)}{\lambda_l(\lambda_r + k)}, \qquad k \in \mathbb{R}, \tag{2.27a}$$

and

$$\det S^\pm(k) = \frac{\lambda_l^+(k)(\lambda_r^\pm(k) + k)}{\lambda_r^+(k)(\lambda_l^\pm(k) + k)}, \quad \det \bar{S}^\pm(k) = \frac{\lambda_r^+(k)(\lambda_l^\pm(k) + k)}{\lambda_l^+(k)(\lambda_r^\pm(k) + k)}, \quad k \in (-iA_l, iA_l), \tag{2.27b}$$

where we have taken into account (2.16a) and (2.16b).

If we now denote by $\mathrm{Wr}(v_1, v_2) \stackrel{\text{def}}{=} \det \begin{pmatrix} v_1 & v_2 \end{pmatrix}$, the Wronskian of any two vector solutions v_1, v_2 of the first of (2.1), then Eqs. (2.24a)–(2.24b) yield the following "Wronskian" representations for the scattering coefficients:

$$a(k) = \frac{\mathrm{Wr}(\phi, \psi)}{\mathrm{Wr}(\bar{\psi}, \psi)} = \frac{\lambda_r + k}{2\lambda_r}\mathrm{Wr}(\phi, \psi), \qquad \bar{a}(k) = \frac{\mathrm{Wr}(\bar{\psi}, \bar{\phi})}{\mathrm{Wr}(\bar{\psi}, \psi)} = \frac{\lambda_r + k}{2\lambda_r}\mathrm{Wr}(\bar{\psi}, \bar{\phi}), \tag{2.28a}$$

$$b(k) = \frac{\mathrm{Wr}(\bar{\psi}, \phi)}{\mathrm{Wr}(\bar{\psi}, \psi)} = \frac{\lambda_r + k}{2\lambda_r}\mathrm{Wr}(\bar{\psi}, \phi), \qquad \bar{b}(k) = \frac{\mathrm{Wr}(\bar{\phi}, \psi)}{\mathrm{Wr}(\bar{\psi}, \psi)} = \frac{\lambda_r + k}{2\lambda_r}\mathrm{Wr}(\bar{\phi}, \psi), \tag{2.28b}$$

and

$$c(k) = \frac{\lambda_l + k}{2\lambda_l}\mathrm{Wr}(\phi, \psi) = \frac{\lambda_r(\lambda_l + k)}{\lambda_l(\lambda_r + k)}a(k), \quad \bar{c}(k) = \frac{\lambda_l + k}{2\lambda_l}\mathrm{Wr}(\bar{\psi}, \bar{\phi}) = \frac{\lambda_r(\lambda_l + k)}{\lambda_l(\lambda_r + k)}\bar{a}(k)\,, \tag{2.28c}$$

$$d(k) = \frac{\lambda_l + k}{2\lambda_l}\mathrm{Wr}(\psi, \bar{\phi}) = -\frac{\lambda_r(\lambda_l + k)}{\lambda_l(\lambda_r + k)}\bar{b}(k), \quad \bar{d}(k) = \frac{\lambda_l + k}{2\lambda_l}\mathrm{Wr}(\phi, \bar{\psi}) = -\frac{\lambda_r(\lambda_l + k)}{\lambda_l(\lambda_r + k)}b(k)\,, \tag{2.28d}$$

where the arguments (x, k) of the Jost solutions have been omitted for brevity, and the second set of identities in each of (2.28c) and (2.28d) are obtained from $\bar{S}(k) = S^{-1}(k)$. Note that the above Wronskian representations can be used to define the values of the scattering coefficients from the right/left edge of the cuts $\Sigma_{l/r}$, consistently with (2.21a)–(2.21b) and (2.24a)–(2.24b). Explicitly, one has

$$\begin{aligned}
a^{\pm}(k) &= \frac{\lambda_r^{\pm}(k) + k}{2\lambda_r^{\pm}(k)}\mathrm{Wr}(\phi^{\pm}(x,k), \psi^{\pm}(x,k)) \qquad k \in (-iA_l, iA_r)\,,\\
\bar{a}^{\pm}(k) &= \frac{\lambda_r^{\pm}(k) + k}{2\lambda_r^{\pm}(k)}\mathrm{Wr}(\bar{\psi}^{\pm}(x,k), \bar{\phi}^{\pm}(x,k)) \qquad k \in (-iA_r, iA_l)\,,\\
b^{\pm}(k) &= \frac{\lambda_r^{\pm}(k) + k}{2\lambda_r^{\pm}(k)}\mathrm{Wr}(\bar{\psi}^{\pm}(x,k), \phi^{\pm}(x,k)) \qquad k \in (-iA_l, iA_r)\,,\\
\bar{b}^{\pm}(k) &= \frac{\lambda_r^{\pm}(k) + k}{2\lambda_r^{\pm}(k)}\mathrm{Wr}(\bar{\phi}^{\pm}(x,k), \psi^{\pm}(x,k)) \qquad k \in (-iA_r, iA_l)\,,
\end{aligned}$$

and similarly for the scattering coefficients from the left.

Now Eqs. (2.28a)–(2.28d) allow one to extend some of the scattering coefficients under the hypothesis ($\boldsymbol{H}_1$). In fact, (2.28a)–(2.28d) and Proposition 2.2 imply:

- $a(k)$ is continuous for $k \in \overline{\mathbb{K}_r^+} \cup \partial\mathbb{K}_l^- \setminus \{iA_r\}$ [with values across the cut denoted as $a^{\pm}(k)$], and analytic in $k \in \mathbb{K}_r^+$, while

$$a(k) \sim \frac{iA_r}{2\lambda_r}\mathrm{Wr}(\phi(x, iA_r), \psi(x, iA_r)), \quad k \to iA_r. \tag{2.29a}$$

- $\bar{a}(k)$ is continuous in $k \in \overline{\mathbb{K}_r^-} \cup \partial\mathbb{K}_l^+ \setminus \{-iA_r\}$ [with values across the cut denoted as $\bar{a}^{\pm}(k)$] and analytic in $k \in \mathbb{K}_r^-$, while

$$\bar{a}(k) \sim \frac{-iA_r}{2\lambda_r}\mathrm{Wr}(\bar{\psi}(x, -iA_r), \bar{\phi}(x, -iA_r)), \quad k \to -iA_r. \tag{2.29b}$$

- $b(k)$ is continuous for $\partial\mathbb{K}_r^+ \cup \partial\mathbb{K}_l^- \setminus \{iA_r\}$ [with values across the cut denoted as $b^{\pm}(k)$], and $\bar{b}(k)$ is continuous for $k \in \partial\mathbb{K}_r^- \cup \partial\mathbb{K}_l^+ \setminus \{-iA_r\}$ [with values across the cut denoted as $\bar{b}^{\pm}(k)$]; at the branch points where they are defined

$$b^{\pm}(k) \sim \frac{iA_r}{2\lambda_r}\mathrm{Wr}(\bar{\psi}(x, iA_r), \phi(x, iA_r)), \quad k \to iA_r\,, \tag{2.29c}$$

$$\bar{b}^{\pm}(k) \sim \frac{-iA_r}{2\lambda_r}\mathrm{Wr}(\bar{\phi}(x, -iA_r), \psi(x, -iA_r)), \quad k \to -iA_r\,. \tag{2.29d}$$

Similar results can be derived for the other four scattering coefficients, although the corresponding properties can also be obtained from those above using (2.28c) and (2.28d). Note Eqs. (2.29a)–(2.29d) show that the scattering coefficients generically have singularities at the branch points $k = \pm iA_r$, i.e., for $\lambda_r = 0$. The behavior of the scattering coefficients at the branch points will be further discussed in Sec. II D.

For future convenience, we also define the *reflection coefficients* from the right as follows:

$$\rho(k)=\frac{b(k)}{a(k)}\qquad \text{for } k\in\mathbb{R},\qquad \rho^{\pm}(k)=\frac{b^{\pm}(k)}{a^{\pm}(k)}\quad \text{for } k\in[-iA_l,iA_r)\,, \tag{2.30a}$$

$$\bar\rho(k)=\frac{\bar b(k)}{\bar a(k)}\qquad \text{for } k\in\mathbb{R},\qquad \bar\rho^{\pm}(k)=\frac{\bar b^{\pm}(k)}{\bar a^{\pm}(k)}\quad \text{for } k\in(-iA_r,iA_l]\,, \tag{2.30b}$$

and the *reflection coefficients* from the left as

$$r(k)=\frac{d(k)}{c(k)}\equiv-\frac{\bar b(k)}{a(k)}\quad \text{for } k\in\mathbb{R},\qquad r^{\pm}(k)=\frac{d^{\pm}(k)}{c^{\pm}(k)}\equiv\frac{\bar b^{\pm}(k)}{a^{\pm}(k)}\quad \text{for } k\in[-iA_l,iA_l]\,, \tag{2.30c}$$

$$\bar r(k)=\frac{\bar d(k)}{\bar c(k)}\equiv-\frac{b(k)}{\bar a(k)}\quad \text{for } k\in\mathbb{R},\qquad \bar r^{\pm}(k)=\frac{\bar d^{\pm}(k)}{\bar c^{\pm}(k)}\equiv-\frac{b^{\pm}(k)}{\bar a^{\pm}(k)}\quad \text{for } k\in[-iA_l,iA_l]\,. \tag{2.30d}$$

Note that in order to express $r(k)$ and $\bar r(k)$ in terms of the entries of $S(k)$, we have used that $S(k)\bar S(k)=I_2$. The coefficients $1/a(k)$ [resp. $1/\bar a(k)$] for $k\in\mathbb{K}_r^+$ [resp. for $k\in\mathbb{K}_r^-$], and $1/a^{\pm}(k)$ [resp. $1/\bar a^{\pm}(k)$] for $k\in[-iA_l,A_r)$ [resp. $k\in(-iA_r,iA_l]$ are usually referred to as (right) *transmission coefficients*. Similarly definitions can obviously be introduced for the transmission coefficients from the left, $1/c(k)$ and $1/\bar c(k)$.

As in the defocusing case,[18] from (2.25) and (2.23a)–(2.23b), using (2.17) and (2.19a)–(2.19b), we can also obtain the following integral representation for the scattering matrix:

$$\begin{aligned}S(k)=&\int_0^{\infty}dy\,e^{i\lambda_r y\sigma_3}W_r^{-1}(k)[Q(y)-Q_r]\Phi(y,k)\\&+W_r^{-1}(k)W_l(k)\left[I_2+\int_{-\infty}^{0}dy\,e^{i\lambda_l y\sigma_3}W_l^{-1}(k)[Q(y)-Q_l]\Phi(y,k)\right],\end{aligned} \tag{2.31}$$

which could serve as an alternative to the Wronskian representations to establish the analytic continuation in the appropriate half planes of the scattering coefficients $a(k)$ and $\bar a(k)$.

C. Symmetries of eigenfunctions and scattering data

The scattering problem (2.1) admits two involutions: $(k,\lambda_{l/r})\to(k^*,\lambda_{l/r}^*)$ and $(k,\lambda_{l/r})\to(k,-\lambda_{l/r})$. Correspondingly, eigenfunctions and scattering data satisfy two sets of symmetry relations. In the asymmetric case treated here, where four branch points and two separate branch cuts have to be considered, care must be taken in distinguishing between the case when both λ_r and λ_l are discontinuous, which happens for $k\in[-iA_l,iA_l]$, and when only one is [here λ_r, because of the choice $A_l\le A_r$], corresponding to $k\in[-iA_r,-iA_l]\cup[iA_l,iA_r]$.

First symmetry. On the single sheet for k we are considering, the involution $k\to k^*$ implies $\lambda_{r/l}\to\lambda_{r/l}^*$. One can easily check that if $v(x,k)=(v_1(x,k)\,v_2(x,k))^T$ [superscript ^{T} denotes matrix transpose] is a solution of the ZS system (2.1), then $\hat v^*(x,k)=i\sigma_2 v(x,k^*)$, σ_2 being the second Pauli matrix as given in (2.2), is a solution of the ZS system (2.1) as well. Taking into account the boundary conditions (2.20a)–(2.20b), the symmetries for the Jost solutions are:

$$\bar\psi^*(x,k^*)=i\sigma_2\psi(x,k)\text{ for }k\in\mathbb{K}_r^+\cup\mathbb{R}\,,\qquad \psi^*(x,k^*)=-i\sigma_2\bar\psi(x,k)\text{ for }k\in\mathbb{K}_r^-\cup\mathbb{R}, \tag{2.32a}$$

$$(\bar\psi^{\pm}(x,k^*))^*=i\sigma_2\psi^{\pm}(x,k)\text{ for }k\in[0,iA_r],\quad (\psi^*\pm(x,k^*))^*=-i\sigma_2\bar\psi^{\pm}(x,k)\text{ for }k\in[-iA_r,0], \tag{2.32b}$$

$$\phi^*(x,k^*)=i\sigma_2\bar\phi(x,k)\text{ for }k\in\mathbb{K}_l^-\cup\mathbb{R},\qquad \bar\phi^*(x,k^*)=-i\sigma_2\phi(x,k)\text{ for }k\in\mathbb{K}_l^+\cup\mathbb{R}, \tag{2.32c}$$

$$(\phi^{\pm}(x,k^*))^*=i\sigma_2\bar\phi^{\pm}(x,k)\ \text{ for }k\in[-iA_l,0]\,,\quad (\bar\phi^{\pm}(x,k^*))^*=-i\sigma_2\phi^{\pm}(x,k)\ \text{ for }k\in[0,iA_l]\,. \tag{2.32d}$$

From (2.24a)–(2.24b) we then obtain $S^*(k^*) = \sigma_2 S(k)\sigma_2$ wherever all entries in the scattering matrix are simultaneously defined. In particular, under the assumption $(\boldsymbol{H}_1)$ for the potential, the symmetry relations for the scattering coefficients can be written as

$$\bar{a}^*(k^*) = a(k) \quad \text{for } k \in \mathbb{K}_l^+ \cup \mathbb{R}\,, \qquad \left(\bar{a}^{\pm}(k^*)\right)^* = a^{\pm}(k) \quad \text{for } k \in [-iA_l, iA_r)\,, \tag{2.33a}$$

$$\bar{b}^*(k) = -\,b(k) \quad \text{for } k \in \mathbb{R}\,, \qquad \left(\bar{b}^{\pm}(k^*)\right)^* = -b^{\pm}(k) \quad \text{for } k \in [-iA_l, iA_r)\,. \tag{2.33b}$$

It is worth noticing that the above symmetries relate the values of the scattering coefficients in the upper/lower half plane of k, and from the same side of the cuts. Taking into account (2.30a)–(2.30d), one can easily establish the symmetry relations satisfied by the reflection coefficients:

$$\bar{\rho}^*(k) = -\rho(k) \quad \text{for } k \in \mathbb{R}\,, \qquad \left(\bar{\rho}^{\pm}(k^*)\right)^* = -\rho^{\pm}(k) \quad \text{for } k \in [-iA_l, iA_r)\,, \tag{2.33c}$$

$$\bar{r}^*(k) = -r(k) \quad \text{for } k \in \mathbb{R}\,, \qquad \left(\bar{r}^{\pm}(k^*)\right)^* = -r^{\pm}(k) \quad \text{for } k \in [-iA_l, iA_l]\,. \tag{2.33d}$$

Second symmetry. When using a single sheet for the Riemann surface of the functions $\lambda_{l/r}^2 = k^2 + A_{l/r}^2$, the involution $(k, \lambda_{l/r}) \to (k, -\lambda_{l/r})$ can only be considered across the cuts. So this second involution relates values of eigenfunctions and scattering coefficients for the same value of k from either side of the cut. On the innermost cut, where both λ_l and λ_r are discontinuous, i.e., for $k \in \Sigma_l$, one has

$$\bar{\psi}^{\mp}(x,k) = \frac{\lambda_r^{\pm} + k}{-iq_r}\psi^{\pm}(x,k)\,, \qquad \bar{\phi}^{\mp}(x,k) = \frac{\lambda_l^{\pm} + k}{-iq_l^*}\phi^{\pm}(x,k) \qquad \text{for } k \in [-iA_l, iA_l]\,. \tag{2.34}$$

For $k \in \Sigma_r \setminus \Sigma_l$, the symmetries for $\psi, \bar{\psi}$ remain as above, while $\phi^+(x,k) = \phi^-(x,k)$ for $k \in [iA_l, iA_r]$ and $\bar{\phi}^+(x,k) = \bar{\phi}^-(x,k)$ for $k \in [-iA_r, -iA_l]$. In addition, taking into account that $q_{r/l} = A_{r/l}e^{i\theta_{r/l}}$, the above symmetry relations also yield

$$\bar{\psi}(x, \pm iA_r) = \mp e^{-i\theta_r}\psi(x, \pm iA_r)\,, \qquad \bar{\phi}(x, \pm iA_l) = \mp e^{i\theta_l}\phi(x, \pm iA_l)\,. \tag{2.35}$$

Using the symmetries in the Wronskian representations for the scattering coefficients (2.26) one obtains for $k \in \Sigma_l$,

$$a^{\pm}(k) = \frac{q_l^* q_r}{(\lambda_l^{\pm}(k) + k)(\lambda_r^{\pm}(k) - k)}\bar{a}^{\mp}(k) \equiv \frac{q_r(\lambda_l^{\pm}(k) + k)}{q_l(\lambda_r^{\pm}(k) - k)}\bar{a}^{\mp}(k)\,, \tag{2.36a}$$

$$b^{\pm}(k) = \frac{q_l^*(\lambda_r^{\pm}(k) + k)}{q_r(\lambda_l^{\pm}(k) + k)}\bar{b}^{\mp}(k) \equiv \frac{q_r^*(\lambda_l^{\pm}(k) - k)}{q_l(\lambda_r^{\pm}(k) - k)}\bar{b}^{\mp}(k)\,. \tag{2.36b}$$

On the other hand, for $k \in \Sigma_r \setminus \Sigma_l$ the symmetry relations become

$$a^{\pm}(k) = \frac{\lambda_r^{\mp}(k) - k}{iq_r^*}b^{\mp}(k) \qquad \text{for } k \in [iA_l, iA_r]\,, \tag{2.36c}$$

$$\bar{a}^{\mp}(k) = \frac{\lambda_r^{\mp}(k) + k}{-iq_r}\bar{b}^{\pm}(k) \qquad \text{for } k \in [-iA_r, -iA_l]\,. \tag{2.36d}$$

Note that the above symmetries indeed hold also as $k \to iA_r$ for $a(k)$, $b(k)$ and as $k \to -iA_r$ for $\bar{a}(k)$, $\bar{b}(k)$, even when the scattering coefficients have singularities at these points (cf. (2.29a)–(2.29d) and Sec. II D below for more details). Moreover, for $k \in [iA_l, iA_r]$ using (2.36c) we have

$$\rho^{\pm}(k) = \frac{iq_r^*}{\lambda_r^{\pm}(k) - k}\frac{a^{\mp}(k)}{a^{\pm}(k)} \quad \Rightarrow \quad \frac{a^-(k)}{a^+(k)} = \frac{\lambda_r^+(k) - k}{iq_r^*}\rho^+(k)\,. \tag{2.37a}$$

Note that the above relationships imply that $\rho^+(k)$ and $\rho^-(k)$ are related to each other by

$$\rho^+(k)\rho^-(k) = q_r^*/q_r \qquad \text{for } k \in [iA_l, iA_r]\,. \tag{2.37b}$$

Similarly, for $k \in [-iA_r, -iA_l]$ (2.36d) yields

$$\bar\rho^{\pm}(k) = \frac{-iq_r}{\lambda_r^{\mp}(k)+k}\frac{\bar a^{\mp}(k)}{\bar a^{\pm}(k)} \quad \Rightarrow \quad \frac{\bar a^{+}(k)}{\bar a^{-}(k)} = \frac{\lambda_r^{+}(k)+k}{-iq_r}\bar\rho^{-}(k) \tag{2.37c}$$

also implying

$$\bar\rho^+(k)\bar\rho^-(k) = q_r/q_r^* \qquad \text{for } k \in [-iA_r, -iA_l]\,. \tag{2.37d}$$

For $k \in \Sigma_r \setminus \Sigma_l$, the symmetry relations for the scattering coefficients from the left are given by

$$c^{\pm}(k) = \frac{-iq_r}{\lambda_r^{\pm}(k)+k}\bar d^{\mp}(k) \qquad \text{for } k \in [iA_l, iA_r]\,, \tag{2.38a}$$

$$\bar c^{\pm}(k) = \frac{\lambda_r^{\mp}(k)+k}{-iq_r}d^{\mp}(k) \qquad \text{for } k \in [-iA_r, -iA_l]\,. \tag{2.38b}$$

D. Discrete eigenvalues, spectral singularities, and virtual levels

A *discrete eigenvalue* is a value of $k \in \mathbb{K}_r^+ \cup \mathbb{K}_r^-$ (corresponding to $\lambda_r, \lambda_l \in \mathbb{C} \setminus \mathbb{R}$) for which there exists a nontrivial solution v to (2.1) with entries in $L^2(\mathbb{R})$. These eigenvalues occur for $k \in \mathbb{K}_r^+$ iff the functions $\phi(x,k)$ and $\psi(x,k)$ are linearly dependent (i.e., iff $a(k) = 0$), and for $k \in \mathbb{K}_r^-$ iff the functions $\bar\psi(x,k)$ and $\bar\phi(x,k)$ are linearly dependent (i.e., iff $\bar a(k) = 0$). Equations (2.20a)–(2.20b), together with the linear dependence requirement, imply that the corresponding eigenfunctions are exponentially decaying as $x \to \pm\infty$. The conjugation symmetry (2.33a) then ensures that the discrete eigenvalues occur in complex conjugate pairs. The algebraic multiplicity of each discrete eigenvalue coincides with the multiplicity of the corresponding zero of $a(k)$ [for $k \in \mathbb{K}_r^+$] or $\bar a(k)$ [for $k \in \mathbb{K}_r^-$]. In the following, we will assume discrete eigenvalues are simple, and finite in number.

At each discrete eigenvalue $k_n \in K_r^+$, $n = 1, \ldots, N$, the eigenfunctions $\phi(x,k_n)$ and $\psi(x,k_n)$ are proportional, i.e., there exists a complex constant b_n such that $\phi(x,k_n) = b_n\psi(x,k_n)$. Then, denoting by τ_n the residue of $1/a(k)$ at the (simple) pole $\lambda_r = \lambda_r(k_n)$, we can write

$$\lim_{k\to k_n}(\lambda_r(k) - \lambda_r(k_n))\frac{\phi(x,k)}{a(k)} = C_n\,\psi(x,k_n)\,, \qquad C_n = b_n\tau_n\,, \tag{2.39a}$$

and C_n is referred to as the *norming constant* associated with the discrete eigenvalue k_n.

Similarly, at the (simple) discrete eigenvalues $k_1^*, \ldots, k_N^*$ in $\mathbb{K}_r^-$ (necessarily also finite in number, and the complex conjugates of the zeros of $a(k)$ in $\mathbb{K}_r^+$) the eigenfunctions $\bar\phi(x,k_n^*)$ and $\bar\psi(x,k_n^*)$ are proportional to each other, i.e., there exist complex constants $\bar b_n$ such that $\bar\phi(x,k_n^*) = \bar b_n\bar\psi(x,k_n^*)$. Then, denoting by $\bar\tau_n$ the residue of $1/\bar a(k)$ at the pole $\lambda_r = \lambda_r(k_n^*)$, we can write

$$\lim_{k\to k_n^*}(\lambda_r - \lambda_r(k_n^*))\frac{\bar\phi(x,k)}{\bar a(k)} = \bar C_n\bar\psi(x,k_n^*)\,, \qquad \bar C_n = \bar b_n\bar\tau_n\,, \tag{2.39b}$$

and $\bar C_n$ is referred to as the *norming constant* associated with the discrete eigenvalue k_n^*. Using the symmetry relations (2.32a)–(2.32d) and (2.33a)–(2.33d), and the definitions (2.39a) and (2.39b), we get

$$\bar\tau_n = \tau_n^*\,, \qquad \bar b_n = -b_n^*\,, \qquad \bar C_n = -C_n^*\,. \tag{2.40}$$

A *spectral singularity* is a value of $k \in \mathbb{R}$ for which $a(k) = 0$, or a value $k \in [-A_l, iA_l]$ for which $a^{\pm}(k) = 0$ [similarly for $\bar a(k) = 0$, according to (2.33a) and (2.36a)–(2.36d)]. In this work, we assume no spectral singularities exist; also, we assume that $a^{\pm}(k) \neq 0$ for $k \in \Sigma_r^+ \setminus \Sigma_l^+$ (and, correspondingly, $\bar{a}^{\pm}(k) \neq 0$ for $k \in \Sigma_r^- \setminus \Sigma_l^-$). Unlike discrete eigenvalues, zeros of $a^{\pm}(k)$ and $\bar{a}^{\pm}(k)$ is $\Sigma_r \setminus \Sigma_l$ would not correspond to bound states for the eigenfunctions, since $\psi(x,k)$ and $\bar{\psi}(x,k)$ would not be exponentially decaying as $x \to +\infty$ for any such value of k. Note that according to (2.37a) and (2.37c), requiring absence of spectral singularities implies that $\rho^{\pm}(k) \neq 0$ [resp. $\bar{\rho}^{\pm}(k) \neq 0$] for all $k \in \Sigma_r^+ \setminus \Sigma_l^+$ [resp. $k \in \Sigma_r^- \setminus \Sigma_l^-$]. Under these assumptions the transmission coefficients $1/a^{\pm}(k)$ [resp. $1/\bar{a}^{\pm}(k)$] and the reflection coefficients $\rho^{\pm}(k)$ [resp. $\bar{\rho}^{\pm}(k)$] are continuous in $[-iA_l, iA_r]$ [resp. $[-iA_r, iA_l]$. Similar statements hold for the transmission coefficients $1/c^{\pm}(k)$, $1/\bar{c}^{\pm}(k)$, and reflection coefficients $r^{\pm}(k)$, $\bar{r}^{\pm}(k)$.

Establishing conditions on the asymptotic amplitudes and phases that guarantee absence of spectral singularities, as well as of eigenvalues that lie in $\Sigma_r \setminus \Sigma_l$, is an interesting problem, but is beyond the scope of this paper and will be the subject of future investigation. In any event, we mention that spectral singularities can be incorporated in the inverse problem with slight modifications of the approach presented in Sec. III.

Let us now investigate the behavior of the scattering coefficients at $k = \pm iA_r$. While all eigenfunctions are continuous at these branch points (cf. Proposition 2.2), from the definition (2.28a)–(2.28b), we expect the scattering coefficients to have singularities at the branch points of λ_r. We say that the *generic case* holds if $\phi(x, iA_r)$ and $\psi(x, iA_r)$ are linearly independent, or equivalently [see (2.32a)–(2.32d)] iff $\bar{\psi}(x, -iA_r)$ and $\bar{\phi}(x, -iA_r)$ are linearly independent. The *exceptional case* holds if the above pairs of functions are linearly dependent, in which case $k = \pm iA_r$ are called *virtual levels* (cf. Ref. 19, where this terminology is introduced for the defocusing NLS with equal-amplitude BCs). In the generic case, the transmission coefficient $1/a(k)$ is continuous at $k = iA_r$ [equivalently, $1/\bar{a}(k)$ is continuous at $k = -iA_r$]. In the exceptional case and under the hypothesis $(\boldsymbol{H}_2)$ [cf. Theorem 2.3], we write (2.28a) in the form

$$a(k) = \frac{\lambda_r + k}{2} \frac{\mathrm{Wr}(\phi,\psi) - \mathrm{Wr}(\phi,\psi)|_{k=iA_r}}{\lambda_r} \to \frac{iA_r}{2}\, \frac{\partial}{\partial \lambda_r} W(\phi,\psi)\bigg|_{k=iA_r}, \tag{2.41}$$

as $k \to iA_r$. Thus $a(k)$ is nonzero [hence, $1/a(k)$ is continuous] as $k \to iA_r$ iff the Wronskian $\mathrm{Wr}(\phi,\psi)$ has a simple zero at $k = iA_r$.

Note that the reflection coefficients $\rho^{\pm}(k)$ and $\bar{\rho}^{\pm}(k)$ are bounded as $k \to iA_r$ and $k \to -iA_r$ respectively, both in the generic and in the exceptional case. In fact, using (2.35), from (2.29a) and (2.29c) it follows that $\rho^{\pm}(k) = b^{\pm}(k)/a^{\pm}(k)$, is finite at $k = iA_r$; in particular, in the generic case, one has

$$\rho^{\pm}(k)|_{k=iA_r} = e^{-i\theta_r}\,. \tag{2.42}$$

In the exceptional case, as long as the Wronskian $\mathrm{Wr}(\phi,\psi)$ has a simple zero at $k = iA_r$, $a(k)$ is finite, and so $\rho^{\pm}(k)$ are bounded at $k = iA_r$. Similarly, one can show that $\bar{\rho}^{\pm}(k)$ is bounded at $k = -iA_r$, and in the generic case $\bar{\rho}^{\pm}(k)|_{k=-iA_r} = e^{i\theta_r}$. In particular, we note that one has $|\rho^{\pm}(iA_r)| = |\bar{\rho}^{\pm}(-iA_r)| = 1$.

Obviously, similar considerations hold for the transmission coefficients from the left, $1/c(k)$, $1/\bar{c}(k)$, and corresponding reflection coefficients $r(k)$, $\bar{r}(k)$. More specifically, from (2.28c) it follows that $c(k)$ and $\bar{c}(k)$ have the same zeros as $a(k)$ and $\bar{a}(k)$, respectively, and therefore the set of discrete eigenvalues and (possibly) spectral singularities is completely determined by the latter.

The scattering coefficients from the left generically have singularities at the branch points $k = \pm iA_l$ (cf. (2.28c) and (2.28d)), unless the eigenfunctions become linearly dependent at these points, which again will be referred to as the exceptional case. In both the generic and the exceptional case (with a simple zero for the Wronskian of the eigenfunctions), the transmission and reflection coefficients from the left are bounded at the branch points $\pm iA_l$.

E. Large *k* behavior of eigenfunctions and scattering data

We will derive the large k asymptotic expansion for the eigenfunctions from the Volterra integral equations (A9a)–(A9d). Let us introduce for convenience the modified eigenfunction $N(x,k) = \psi(x,k)\,e^{-i\lambda_r x}$, and denote with a subscript $j = 1, 2$ its jth component. The integral equation (A9b) can be written explicitly in terms of the components of $N(x,k)$ as follows:

$$N_1(x,k) = -\frac{iq_r}{\lambda_r + k} - \frac{\lambda_r + k}{2\lambda_r}\int_x^\infty dy\ (q(y) - q_r)\, e^{2i\lambda_r(y-x)} N_2(y,k)$$

$$- \frac{\lambda_r - k}{2\lambda_r}\int_x^\infty dy\ (q(y) - q_r)\, N_2(y,k)$$

$$+ \frac{iq_r}{2\lambda_r}\int_x^\infty dy\ \left(q^*(y) - q_r^*\right) e^{2i\lambda_r(y-x)} N_1(y,k) - \frac{iq_r}{2\lambda_r}\int_x^\infty dy\ \left(q^*(y) - q_r^*\right) N_1(y,k)\,,$$

$$N_2(x,k) = 1 + \frac{\lambda_r + k}{2\lambda_r}\int_x^\infty dy\ \left(q^*(y) - q_r^*\right) N_1(y,k)$$

$$+ \frac{\lambda_r - k}{2\lambda_r}\int_x^\infty dy\ \left(q^*(y) - q_r^*\right) e^{2i\lambda_r(y-x)} N_1(y,k)$$

$$- \frac{iq_r^*}{2\lambda_r}\int_x^\infty dy\ (q(y) - q_r)\, N_2(y,k) + \frac{iq_r^*}{2\lambda_r}\int_x^\infty dy\ (q(y) - q_r)\, e^{2i\lambda_r(y-x)} N_2(y,k)\,.$$

As a result of Proposition 2.2, in the above integral equations $N_1(x,k)$ and $N_2(x,k) - 1$ are uniformly bounded for $(x,k) \in [x_0, +\infty) \times [\mathbb{K}_r^+ \cup \partial\mathbb{K}_r^+ \cup \partial\mathbb{K}_r^-]$. Also, the iteration of the integral equations converges uniformly for (x,k) belonging to the same set. Assuming the potential is such that its distributional derivative $\partial_x q \in L^1(\mathbb{R})$, we can write

$$\int_x^\infty dy\, (q(y) - q_r) e^{2i\lambda_r(y-x)} = \frac{i}{2\lambda_r}\left[q(x) - q_r + \int_x^\infty dy\, \partial_y q(y) e^{2i\lambda_r(y-x)}\right]$$
$$= \frac{i}{2}\frac{q(x) - q_r}{k} + o(1/k)\,. \tag{2.43}$$

Iterating once in the integral equation for $N_1(x,k)$ [resp. $N_2(x,k)$] for the other unknown $N_2(x,k)$ [resp. $N_1(x,k)$], and using (2.43), yields for the inhomogeneous terms the following expansions in powers of $1/k$:

$$N_1^{\text{inh}}(x,k) = -\frac{iq_r}{2k} - \frac{i[q(x) - q_r]}{2k} + o(1/k) = -\frac{iq(x)}{2k} + o(1/k)\,,$$

$$N_2^{\text{inh}}(x,k) = 1 - \frac{i}{2k}\int_x^\infty dy\, (q^*(y) - q_r^*) q(y) + o(1/k)\,.$$

Substituting these expressions into the integral equations and computing their first iterates, it is clear that only the last but one term in the right-hand side of the second integral equation leads to an additional contribution of order $1/k$, namely,

$$-\frac{iq_r^*}{2k}\int_x^\infty dy\, (q(y) - q_r).$$

The iterations do not lead to any other terms of $O(1)$ or $O(1/k)$ [in particular, the last term in the second integral equation is of higher order, because of Riemann-Lebesgue Lemma]. As a result, we obtain

$$N_1(x,k) = -\frac{iq(x)}{2k} + o(1/k)\,, \tag{2.44a}$$

$$N_2(x,k) = 1 - \frac{iq_r^*}{2k}\int_x^\infty dy\, (q(y) - q_r) - \frac{i}{2k} q(y) \int_x^\infty dy\, (q^*(y) - q_r^*) + o(1/k)$$
$$\equiv 1 - \frac{i}{2k}\int_x^\infty dy\, (|q(y)|^2 - A_r^2) + o(1/k)\,. \tag{2.44b}$$

Note that in the above proof we have not assumed *a priori* the existence of an asymptotic expansion for $N(x,k)$ at large k. Instead, under the hypotheses $(\boldsymbol{H}_1)$ and $\partial_x q \in L^1(\mathbb{R})$, the expansion follows automatically. Higher order asymptotic expansions [up to $O(1/k^n)$] require assuming $(\boldsymbol{H}_1)$ and $\partial_x^j q \in L^1(\mathbb{R})$ for $j = 1, 2, \ldots, n$. Note also that $\partial_x q \in L^1(\mathbb{R})$ implies that q is absolutely continuous as a function of x, which is consistent with (2.44a). Moreover, the absolute continuity of the potential and the assumption $(\boldsymbol{H}_0)$ ensure that the integral in (2.44b) is convergent as well. In a similar way, one can determine the large k asymptotic behavior for the other eigenfunctions.

Summarizing, the Volterra integral equations (A9a)–(A9d) yield the following asymptotic behaviors for the eigenfunctions as $|k| \to \infty$ in the appropriate half planes:

$$\Psi(x,k)e^{i\lambda_r\sigma_3 x} = \left(I_2 + \frac{iQ(x)\sigma_3}{2k}\right)[1+o(1)], \tag{2.45a}$$

$$\Phi(x,k)e^{i\lambda_l\sigma_3 x} = \left(I_2 + \frac{iQ(x)\sigma_3}{2k}\right)[1+o(1)]. \tag{2.45b}$$

For later convenience, we also observe that

$$\partial_x\left[\Psi(x,k)e^{i\lambda_r\sigma_3 x}\right] = \frac{i\partial_x Q(x)\sigma_3}{2k}[1+o(1)]. \tag{2.46}$$

From the Wronskian representations (2.28a)–(2.28b) for the scattering coefficients, and taking again into account that $\lambda_r \sim \lambda_l \sim k$ as $k \to \infty$, we then obtain the asymptotic behavior of the scattering coefficients

$$a(k) = \frac{\lambda_r + k}{2\lambda_r}\mathrm{Wr}\big(\phi(x,k) \quad \psi(x,k)\big) \sim 1 \quad \text{as} \quad |k| \to \infty,\ k \in \mathbb{K}_r^+ \cup \mathbb{R}, \tag{2.47a}$$

$$\bar{a}(k) = -\frac{\lambda_r + k}{2\lambda_r}\mathrm{Wr}\big(\bar{\phi}(x,k) \quad \bar{\psi}(x,k)\big) \sim 1 \quad \text{as} \quad |k| \to \infty,\ k \in \mathbb{K}_r^- \cup \mathbb{R}, \tag{2.47b}$$

while

$$b(k) = O(1/k^2), \qquad \bar{b}(k) = O(1/k^2) \quad \text{as} \quad |k| \to \infty,\ k \in \mathbb{R}.$$

Taking into account (2.30a)–(2.30d), the above also imply that

$$\rho(k) = O(1/k^2), \qquad \bar{\rho}(k) = O(1/k^2) \quad \text{as} \quad |k| \to \infty,\ k \in \mathbb{R}, \tag{2.47c}$$

$$r(k) = O(1/k^2), \qquad \bar{r}(k) = O(1/k^2) \quad \text{as} \quad |k| \to \infty,\ k \in \mathbb{R}. \tag{2.47d}$$

F. Trace formula

In order to derive a representation for the scattering coefficient $a(k)$ in terms of discrete eigenvalues and reflection coefficients, which is usually referred to as *trace formula*, the starting point is the quasi-unitarity of the scattering matrix $S(k)$. Taking into account the symmetries (2.33a)–(2.33d), Eq. (2.27a)–(2.27b) for $\det S(k)$ becomes

$$|a(k)|^2 + |b(k)|^2 = \frac{\lambda_l(\lambda_r + k)}{\lambda_r(\lambda_l + k)}, \qquad k \in \mathbb{R},$$

$$a^\pm(k)(a^\pm(k^*))^* + b^\pm(k)(b^\pm(k^*))^* = \frac{\lambda_l^+(\lambda_r^\pm + k)}{\lambda_r^+(\lambda_l^\pm + k)}, \qquad k \in \Sigma_l.$$

In turn, the above equations can be written in terms of the reflection coefficients as follows:

$$|a(k)|^2 = \frac{\lambda_l(\lambda_r + k)}{\lambda_r(\lambda_l + k)}\left[1 + |\rho(k)|^2\right]^{-1}, \qquad k \in \mathbb{R}, \tag{2.48a}$$

$$a^\pm(k)(a^\pm(k^*))^* = \frac{\lambda_l^+(\lambda_r^\pm + k)}{\lambda_r^+(\lambda_l^\pm + k)}\left[1 + \rho^\pm(k)(\rho^\pm(k^*))^*\right]^{-1}, \qquad k \in \Sigma_l. \tag{2.48b}$$

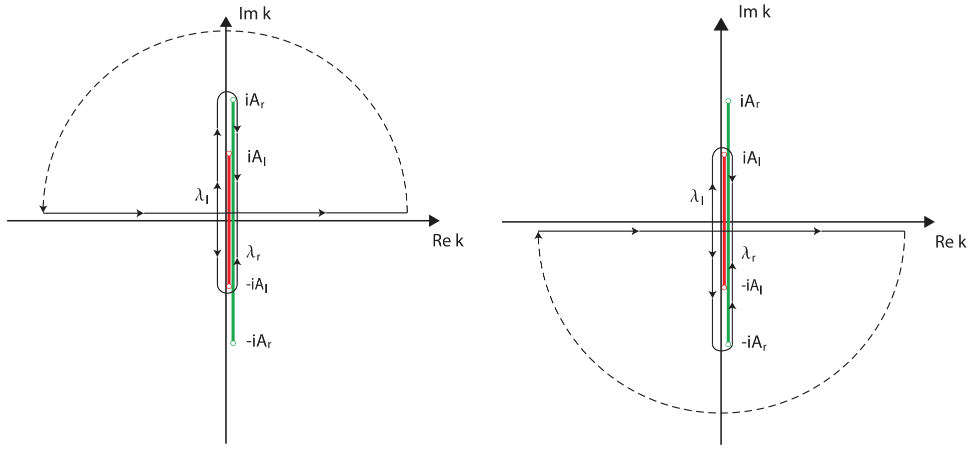

FIG. 2. Oriented contours C_+ (left) and C_- (right).

Since $a(k)$ [resp. $\bar{a}(k)$] is analytic in $\mathbb{K}_r^+$ [resp. $\mathbb{K}_r^-$] and continuous in $\overline{\mathbb{K}_r^+}$ [resp. $\overline{\mathbb{K}_r^-}$], approaches 1 as $k \to \infty$, and has (simple) zeros at $k = k_n$, $n = 1, \cdots, N$, [resp. $k = k_n^*$, $n = 1, \cdots, N$], we introduce

$$\alpha(k) = a(k) \prod_{n=1}^{N} \frac{k - k_n^*}{k - k_n}, \qquad \bar{\alpha}(k) = \bar{a}(k) \prod_{n=1}^{N} \frac{k - k_n}{k - k_n^*} . \tag{2.49}$$

Because of the analyticity properties of $\alpha(k)$ and $\bar{\alpha}(k)$, Cauchy's integral formula for $k \in \mathbb{K}_r^+$ yields

$$\begin{aligned}\log \alpha(k) =& \frac{1}{2\pi i} \int_{C_+} \frac{\log \alpha(\zeta)}{\zeta - k} d\zeta \\ =& \frac{1}{2\pi i} \left[\int_{\mathbb{R}} \frac{\log \alpha(\zeta)}{\zeta - k} d\zeta - \int_{-iA_l}^{0} \frac{\log \alpha^-(\zeta)}{\zeta - k} d\zeta + \int_{-iA_l}^{0} \frac{\log \alpha^+(\zeta)}{\zeta - k} d\zeta + \int_{0}^{iA_l} \frac{\log \alpha^-(\zeta)}{\zeta - k} d\zeta \right. \\ & \left. + \int_{iA_l}^{iA_r} \frac{\log \alpha^-(\zeta)}{\zeta - k} d\zeta - \int_{iA_l}^{iA_r} \frac{\log \alpha^+(\zeta)}{\zeta - k} d\zeta - \int_{0}^{iA_l} \frac{\log \alpha^+(\zeta)}{\zeta - k} d\zeta \right],\end{aligned} \tag{2.50a}$$

$$\begin{aligned}0 =& \frac{1}{2\pi i} \int_{C_-} \frac{\log \bar{\alpha}(\zeta)}{\zeta - k} d\zeta \\ =& \frac{1}{2\pi i} \left[\int_{\mathbb{R}} \frac{\log \bar{\alpha}(\zeta)}{\zeta - k} d\zeta - \int_{-iA_l}^{0} \frac{\log \bar{\alpha}^-(\zeta)}{\zeta - k} d\zeta - \int_{-iA_r}^{-iA_l} \frac{\log \bar{\alpha}^-(\zeta)}{\zeta - k} d\zeta + \int_{-iA_r}^{-iA_l} \frac{\log \bar{\alpha}^+(\zeta)}{\zeta - k} d\zeta \right. \\ & \left. + \int_{-iA_l}^{0} \frac{\log \bar{\alpha}^+(\zeta)}{\zeta - k} d\zeta + \int_{0}^{iA_l} \frac{\log \bar{\alpha}^-(\zeta)}{\zeta - k} d\zeta - \int_{0}^{iA_l} \frac{\log \bar{\alpha}^+(\zeta)}{\zeta - k} d\zeta \right],\end{aligned} \tag{2.50b}$$

where $C_\pm$ are the oriented contours illustrated in Fig. 2, and the superscripts $^\pm$ in $\alpha(k)$, $\bar{\alpha}(k)$ are chosen depending on whether the integration is performed on the right or the left edge of each cut.

Adding the two equations and using the definitions of $\alpha(k)$ and $\bar{\alpha}(k)$, we get

$$\begin{aligned}\log a(k) =& \sum_{n=1}^{N} \log \left(\frac{k - k_n}{k - k_n^*} \right) + \frac{1}{2\pi i} \left\{ \int_{\mathbb{R}} \frac{\log[a(\zeta)\bar{a}(\zeta)]}{\zeta - k} d\zeta + \int_{0}^{iA_l} \frac{\log \frac{a^-(\zeta)\bar{a}^-(\zeta)}{a^+(\zeta)\bar{a}^+(\zeta)}}{\zeta - k} d\zeta \right. \\ & \left. + \int_{-iA_l}^{0} \frac{\log \frac{a^+(\zeta)\bar{a}^+(\zeta)}{a^-(\zeta)\bar{a}^-(\zeta)}}{\zeta - k} d\zeta + \int_{iA_l}^{iA_r} \frac{\log \frac{a^-(\zeta)}{a^+(\zeta)}}{\zeta - k} d\zeta + \int_{-iA_r}^{-iA_l} \frac{\log \frac{\bar{a}^+(\zeta)}{\bar{a}^-(\zeta)}}{\zeta - k} d\zeta \right\} .\end{aligned} \tag{2.51}$$

Using (2.48a)–(2.48b) and (2.37a), (2.37c), we arrive at the so-called trace formula

$$a(k) = \prod_{n=1}^{N} \left(\frac{k - k_n}{k - k_n^*} \right) \exp \left\{ -\frac{1}{2\pi i} \int_{\Sigma} \frac{\log \{\gamma(\zeta)\,[1 + \rho(\zeta)\rho^*(\zeta^*)]\}}{\zeta - k} d\zeta \right.$$
$$+ \frac{1}{2\pi i} \int_{iA_l}^{iA_r} \frac{\log \left[\frac{|\lambda_r(\zeta)| - \zeta}{iq_r^*}\, \rho^+(\zeta) \right]}{\zeta - k} d\zeta$$
$$\left. + \frac{1}{2\pi i} \int_{-iA_r}^{-iA_l} \frac{\log \left[\frac{|\lambda_r(\zeta)| + \zeta}{-iq_r}\, \bar{\rho}^-(\zeta) \right]}{\zeta - k} d\zeta \right\}, \tag{2.52}$$

where $\gamma(k) = [\lambda_r(\lambda_l + k)]/[\lambda_l(\lambda_r + k)]$ is a short-hand notation for $1/\det S(k)$ (cf. (2.27a)–(2.27b)), and $\Sigma = \mathbb{R} \cup [-iA_l, 0]_{\text{left}} \cup [-iA_l, 0]_{\text{right}} \cup [0, iA_l]_{\text{left}} \cup [0, iA_l]_{\text{right}}$, oriented as in Fig. 2. Note in particular that the term in square bracket in the numerator in the first integral of (2.52) becomes $[1 + |\rho(\zeta)|^2]$ for $\zeta \in \mathbb{R}$, and $[1 + \rho^{\pm}(\zeta)(\rho^{\pm}(\zeta^*))^*]$ for $\zeta \in [-iA_l, iA_l]$. Equation (2.52) shows that $a(k)$ is completely determined for $k \in \mathbb{K}_r^+$ in terms of: (i) its zeros (discrete eigenvalues) $k_n \in \mathbb{K}_r^+$; (ii) the reflection coefficient $\rho(k)$ for $k \in \mathbb{R}$, and $\rho^{\pm}(k)$ for $k \in \Sigma_l$; (iii) $\rho^+(k)$ for $k \in [iA_l, iA_r]$, and $\bar{\rho}^-(k)$ for $k \in [-iA_r, -iA_l]$.

III. INVERSE SCATTERING PROBLEM

In the IST machinery, the inverse scattering problem consists of first reconstructing the eigenfunctions in terms of scattering data, and then obtaining the potential (i.e., the NLS solution) in terms of the eigenfunctions. For instance, in formulating the inverse problem from the right, the following set of scattering data is required: (i) the reflection coefficient $\rho(k)$ for $k \in \mathbb{R}$, and its values $\rho^{\pm}(k)$ for $k \in [-iA_l, iA_l]$ on either side of the cut [this accounts for the continuous spectrum of the scattering operator, and it plays the role of the direct Fourier transform of the initial datum in the solution of the initial-value problem for a linear PDE via Fourier transform; note that the reflection coefficient $\bar{\rho}(k)$ is related to $\rho(k)$ by the symmetry (2.33c)]; (ii) discrete eigenvalues $k_n \in \mathbb{K}_r^+$, and associated norming constants C_n, $n = 1, \cdots, N$, as in (2.39a) [note that discrete eigenvalues in $\mathbb{K}_r^-$ and associated norming constants are not independent data, as they can be obtained from the above by conjugation symmetry]; (iii) additional scattering data $\rho^+(k)$ for $k \in [iA_l, iA_r]$ and $\bar{\rho}^-(k)$ for $k \in [-iA_r, iA_l]$ [note $\rho^-(k)$ and $\bar{\rho}^+(k)$ are related to the latter by symmetries (2.37b) and (2.37d); also note that according to the trace formula in Sec. II F, the values of the transmission coefficient $1/a(k)$ for all $k \in \mathbb{K}_r^+$, and $1/a^{\pm}(k)$ for $k \in \Sigma_r^+$ can be obtained from the above scattering data]. In this section, we will formulate the inverse problem, i.e., the problem of reconstructing the eigenfunctions from the scattering data, both in terms of (left and right) Marchenko equations, and as a Riemann-Hilbert problems. As explained below, simple algorithms then allow one to obtain the potential from either the Marchenko kernels, or the solution of the Riemann-Hilbert problem.

A. Triangular representations for the eigenfunctions

In this section, we establish the following two triangular representations for the fundamental eigenfunctions:

$$\tilde{\Psi}(x, k)e^{-x\Lambda_r(k)} = I_2 + \int_x^{\infty} ds\, \boldsymbol{K}(x, s)e^{(s-x)\Lambda_r(k)}, \tag{3.1a}$$

$$\tilde{\Phi}(x, k)e^{-x\Lambda_l(k)} = I_2 + \int_{-\infty}^{x} ds\, \boldsymbol{J}(x, s)e^{(s-x)\Lambda_l(k)}, \tag{3.1b}$$

where the kernels $\boldsymbol{K}(x, s) = \left[\boldsymbol{K}_{ij}(x, s)\right]_{i,j=1,2}$ and $\boldsymbol{J}(x, s) = \left[\boldsymbol{J}_{ij}(x, s)\right]_{i,j=1,2}$ are "triangular" kernels, i.e., such that $\boldsymbol{K}(x, y) \equiv 0$ for $x > y$, and $\boldsymbol{J}(x, y) \equiv 0$ for $x < y$. We note that (3.1a)–(3.1b) yield the corresponding triangular representations for the Jost solutions (2.19a)–(2.19b),

$$\Psi(x,k)=W_r(k)e^{-i\lambda_r\sigma_3 x}+\int_x^\infty ds\,\boldsymbol{K}(x,s)W_r(k)\,e^{-i\lambda_r\sigma_3 s}\,, \tag{3.2a}$$

$$\Phi(x,k)=W_l(k)e^{-i\lambda_l\sigma_3 x}+\int_{-\infty}^x ds\,\boldsymbol{J}(x,s)W_l(k)\,e^{-i\lambda_l\sigma_3 s}\,. \tag{3.2b}$$

From the explicit expression of the groups $e^{x\Lambda_{r/l}(k)}$ in (A1), and taking into account that $k\to -k$ corresponds to $\lambda_{r/l}\to -\lambda_{r/l}$ (cf. (2.15)), one can obtain from (3.1a), for $x\in\mathbb{R}$ and $k\in\mathbb{R}\cup[-iA_r,iA_r]$,

$$\boldsymbol{K}(x,y)=\frac{1}{4\pi}\int_{-\infty}^\infty d\lambda_r e^{-i\lambda_r[y-x]}\left\{\left[\tilde\Psi(x,k)e^{-x\Lambda_r(k)}-I_2\right]\left[I_2-\frac{i}{k}\sigma_3 Q_r-\frac{\lambda_r}{k}\sigma_3\right]\right. \tag{3.3a}$$
$$\left.+\left[\tilde\Psi(x,-k)e^{-x\Lambda_r(-k)}-I_2\right]\left[I_2+\frac{i}{k}\sigma_3 Q_r+\frac{\lambda_r}{k}\sigma_3\right]\right\}.$$

In a similar way, from the second of (3.1b), one obtains

$$\boldsymbol{J}(x,y)=\frac{1}{4\pi}\int_{-\infty}^\infty d\lambda_l e^{-i\lambda_l[y-x]}\left\{\left[\tilde\Phi(x,k)e^{-x\Lambda_l(k)}-I_2\right]\left[I_2-\frac{i}{k}\sigma_3 Q_r-\frac{\lambda_l}{k}\sigma_3\right]\right. \tag{3.3b}$$
$$\left.+\left[\tilde\Phi(x,-k)e^{-x\Lambda_l(-k)}-I_2\right]\left[I_2+\frac{i}{k}\sigma_3 Q_r+\frac{\lambda_l}{k}\sigma_3\right]\right\}.$$

The above inversion formulas for the Marchenko kernels will be derived in the Appendix. We can now prove the existence of the kernels $\boldsymbol{K}(x,y)$ and $\boldsymbol{J}(x,y)$ as the Fourier transforms given by (3.3a)–(3.3b). To show this, we need to assume that the potential $q(x)$ satisfies ($\boldsymbol{H}_2$) and $\partial_x q\in L^1(\mathbb{R})$ (which implies that $q(x)$ is continuous in $x\in\mathbb{R}$, and $q(x)\to q_{r/l}$ as $x\to\pm\infty$). If (3.3a) defines, as a function of x, $\boldsymbol{K}(x,y)$ as the Fourier transform of a matrix function with entries in $L^2(\mathbb{R},d\lambda_r)$, we can state the existence of the kernel $\boldsymbol{K}(x,y)$. First of all, we observe that assuming ($\boldsymbol{H}_1$) and $\partial_x q\in L^1(\mathbb{R})$, we can write

$$\tilde\Psi(x,k)e^{-x\Lambda_r(k)}-I_2=\frac{i}{2\lambda_r}\sigma_3 Q_r-\frac{\lambda_r-k}{2\lambda_r}I_2+\frac{1}{k}\left[\boldsymbol{N}^{(1)}(x)+o(1)\right]W_r^{-1}(k)$$
$$=\frac{i}{2\lambda_r}\sigma_3 Q_r+\frac{1}{k}\left[\boldsymbol{N}^{(1)}(x)+o(1)\right]+O(1/k^2),$$

where $\boldsymbol{N}(x,k)=\Psi(x,k)\,e^{i\lambda_r x\sigma_3}$, and $\boldsymbol{N}^{(1)}(x)=\begin{pmatrix}\bar N^{(1)}(x) & N^{(1)}(x)\end{pmatrix}$ is the $O(1/k)$ term of the large k asymptotic expansion of $\boldsymbol{N}(x,k)$ (cf. (2.19b) and Sec. II E). Similarly,

$$\tilde\Psi(x,-k)e^{-x\Lambda_r(-k)}-I_2=-\frac{i}{2\lambda_r}\sigma_3 Q_r-\frac{\lambda_r-k}{2\lambda_r}I_2-\frac{1}{k}\left[\boldsymbol{N}^{(1)}(x)+o(1)\right]W_r^{-1}(-k)$$
$$=-\frac{i}{2\lambda_r}\sigma_3 Q_r-\frac{1}{k}\left[\boldsymbol{N}^{(1)}(x)+o(1)\right]+O(1/k^2).$$

We can now show that $\boldsymbol{K}(x,y)$ belongs to $L^2(x,+\infty)$ in y. In order to do so, we need to prove that

$$\int_{-\infty}^\infty d\lambda_r\left\|\left[\tilde\Psi(x,k)e^{-x\Lambda_r(k)}-I_2\right]\left[I_2-\frac{i}{k}\sigma_3 Q_r-\frac{\lambda_r}{k}\sigma_3\right]\right.$$
$$\left.+\left[\tilde\Psi(x,-k)e^{-x\lambda_r(-k)}-I_2\right]\left[I_2+\frac{i}{k}\sigma_3 Q_r+\frac{\lambda_r}{k}\sigma_3\right]\right\|^2$$

is finite for each $x\in\mathbb{R}$. Let us split up the integral into two parts to avoid the potentially bad points $k=0$ [i.e., $\lambda_r=\pm A_r$] and $k=\pm iA_r$ [i.e., $\lambda_r=0$]. Under the hypothesis ($\boldsymbol{H}_1$), the integral with respect to λ_r over any neighborhood $\lambda_r\in[-\delta,\delta]$ with $0<\delta<A_r$ avoids the point $k=0$, and the corresponding integrand function is continuous in λ_r, which leads to a well-behaved (and

hence finite) integral. The integral with respect to λ_r over the remainder of the real λ_r-line involves integration over a domain which does not bypass $\lambda_r = \pm A_r$, i.e., $k = 0$. However, if we replace this integral by an integral with respect to k, use that $d\lambda_r = (k/\lambda_r)dk$, assuming ($\boldsymbol{H}_2$), the existence of the limit

$$\lim_{k\to 0} \frac{\tilde{\Psi}(x,k)e^{-x\Lambda_r(k)} - \tilde{\Psi}(x,-k)e^{-x\lambda_r(-k)}}{2k} = \left[\frac{\partial}{\partial k}\tilde{\Psi}(x,k)e^{-x\Lambda_r(k)}\right]_{k=0}$$

[see Proposition 2.3] makes the integral well-behaved around $k = 0$. Thus the only thing left to consider is the behavior as $k \to \pm\infty$ [i.e., as $\lambda_r \to \pm\infty$]. Specifically, it remains to be proved that, say,

$$\left(\int_{-\infty}^{-1} + \int_{1}^{\infty}\right) dk \left\| \tilde{\Psi}(x,k)e^{-x\Lambda_r(k)} - I_2 \right\|^2$$

is finite for each $x \in \mathbb{R}$. This indeed follows from the large k expansion in Sec. II E, since under the assumptions ($\boldsymbol{H}_1$) and $\partial_x q \in L^1(\mathbb{R})$, the worst one can have as $k \to \pm\infty$ is a term of order $O(1/k)$. We have therefore proved the existence of $\boldsymbol{K}(x, y)$ as a measurable function in y (for each $x \in \mathbb{R}$), satisfying

$$\int_x^\infty dy\, \|\boldsymbol{K}(x,y)\|^2 < +\infty.$$

Consequently, as a function of y, $\boldsymbol{K}(x, y)$ is the Fourier transform of an L^2 matrix function and hence an L^2 matrix function itself, uniformly in $x \geq x_0$ for each $x_0 \in \mathbb{R}$. A similar argument can be used for $\boldsymbol{J}(x, y)$. Finally, the following result allows one to recover the potential in terms of the kernels $\boldsymbol{K}(x, y)$ and $\boldsymbol{J}(x, y)$.

Proposition 3.1. Suppose ($\boldsymbol{H}_2$) *holds, and in addition* $\partial_x q \in L^1(\mathbb{R})$. *Then one has*

$$Q(x) - Q_r = 2\,\sigma_3 \boldsymbol{K}(x,x)\sigma_3\,, \qquad Q(x) - Q_l = 2\,\sigma_3 \boldsymbol{J}(x,x)\sigma_3\,. \tag{3.4}$$

In particular, (3.4) *shows that the diagonal entries of both Marchenko kernels* $\boldsymbol{K}(x, x)$ *and* $\boldsymbol{J}(x, x)$ *are zero, while the off-diagonal entries satisfy the following relation:*

$$q(x) = q_r - 2\boldsymbol{K}_{12}(x,x) = q_l + 2\boldsymbol{J}_{12}(x,x)\,, \qquad q^*(x) = q_r^* - 2\boldsymbol{K}_{21}(x,x) = q_l^* + 2\boldsymbol{J}_{21}(x,x)\,. \tag{3.5}$$

Assuming in addition $q \in C^1(\mathbb{R})$ ensures that (3.4) and hence (3.5) are defined everywhere, and not merely almost everywhere.

Note that here we have omitted the time dependence for brevity. If all the above assumptions on the potential hold for all $t \geq 0$, then inserting the time dependence in the Jost and fundamental eigenfunctions (see Sec. IV for details) yields a parametric t-dependence for the Marchenko kernels, and the reconstruction formulas (3.5) for the potential for all $t \geq 0$ read

$$q(x,t) = q_r(t) - 2\boldsymbol{K}_{12}(x,x;t) = q_l(t) + 2\boldsymbol{J}_{12}(x,x;t)\,. \tag{3.6}$$

It is worth pointing out that the existence of the Marchenko kernels can be related to the following Goursat problem

$$(\partial_x + \partial_y)\begin{pmatrix} \boldsymbol{K}_{11}(x,y) \\ \boldsymbol{K}_{22}(x,y) \end{pmatrix} = \begin{pmatrix} -q_r^* & q(x) \\ -q^*(x) & q_r \end{pmatrix} \begin{pmatrix} \boldsymbol{K}_{12}(x,y) \\ \boldsymbol{K}_{21}(x,y) \end{pmatrix}, \tag{3.7a}$$

$$(\partial_x - \partial_y)\begin{pmatrix} \boldsymbol{K}_{12}(x,y) \\ \boldsymbol{K}_{21}(x,y) \end{pmatrix} = \begin{pmatrix} -q_r & q(x) \\ -q^*(x) & q_r^* \end{pmatrix} \begin{pmatrix} \boldsymbol{K}_{11}(x,y) \\ \boldsymbol{K}_{22}(x,y) \end{pmatrix}, \tag{3.7b}$$

with boundary conditions

$$q(x) = q_r - 2\boldsymbol{K}_{12}(x,x) = q_r + 2\boldsymbol{K}^*_{21}(x,x), \tag{3.7c}$$

$$\lim_{s\to\infty} \boldsymbol{K}_{jl}(x,s) = 0, \qquad j,l = 1,2\,, \tag{3.7d}$$

which can be derived (under additional assumptions on the potential) from the scattering problem. Here, however, we have provided an explicit inversion formula for the kernels that will appear in the inverse problem in terms of the eigenfunctions of the direct scattering problem, and used Plancherel's theorem to find an explicit expression for $\boldsymbol{K}(x,y)$.

B. Marchenko equations

In this section, we derive (left and right) Marchenko integral equations as a means to solve the inverse problem, i.e., reconstructing the eigenfunctions, and from them the potential, in terms of the scattering data.

1. Right Marchenko equations

Let us write (2.24a) explicitly as

$$\frac{\phi(x,k)}{a(k)} = \bar{\psi}(x,k) + \rho(k)\psi(x,k) \qquad k\in\mathbb{R}\,, \tag{3.8a}$$

$$\frac{\phi^{\pm}(x,k)}{a^{\pm}(k)} = \bar{\psi}^{\pm}(x,k) + \rho^{\pm}(k)\psi^{\pm}(x,k) \qquad k\in[-iA_l, iA_r]\,, \tag{3.8b}$$

$$\frac{\bar{\phi}(x,k)}{\bar{a}(k)} = \psi(x,k) + \bar{\rho}(k)\bar{\psi}(x,k) \qquad k\in\mathbb{R}\,, \tag{3.8c}$$

$$\frac{\bar{\phi}^{\pm}(x,k)}{\bar{a}^{\pm}(k)} = \psi^{\pm}(x,k) + \bar{\rho}^{\pm}(k)\bar{\psi}^{\pm}(x,k) \qquad k\in(-iA_r, iA_l]\,, \tag{3.8d}$$

where $\rho(k)$, $\rho^{\pm}(k)$ and $\bar{\rho}(k)$, $\bar{\rho}^{\pm}(k)$ are given by (2.30a) and (2.30b), respectively. It is important to point out that even though Eq. (2.24a) is only defined for $k\in\mathbb{R}\cup[-iA_l, iA_l]$, Eq. (3.8b) shows the first column of (2.24a) can be extended to $k\in[iA_l, iA_r]$, and (3.8d) extends the second column of (2.24a) to $k\in[-iA_r, -iA_l]$. In the following we will assume, in accordance with the discussion in Sec. II D, that: (i) there are no spectral singularities; (ii) all discrete eigenvalues are simple; (iii) at $k = iA_r$ the Wronskian $W(\phi,\psi)$ does not have multiple zeros [recall the symmetries imply the same holds at $k=-iA_r$ for $W(\bar{\phi},\bar{\psi})$]; (iv) $(\boldsymbol{H}_1)$ holds in the generic case and $(\boldsymbol{H}_2)$ in the exceptional case.

Multiply (3.8a) by $e^{i\lambda_r y}$ for $y > x$, and substituting the triangular representations (3.2a), we obtain

$$\left[\frac{e^{i\lambda_r x}\phi(x,k)}{a(k)} - W_{r,1}(k)\right] e^{i\lambda_r(y-x)} = \int_x^\infty ds\, \boldsymbol{K}(x,s)W_{r,1}(k)\, e^{i\lambda_r(y-s)} \tag{3.9}$$
$$+\,\rho(k)\left[e^{i\lambda_r(x+y)}W_{r,2}(k) + \int_x^\infty ds\, \boldsymbol{K}(x,s)W_{r,2}(k)\, e^{i\lambda_r(s+y)}\right],$$

where $W_{r,j}(k)$ denotes the jth column of the eigenvector matrix $W_r(k)$ in (2.18). We recall that $\lambda_r \sim \lambda_l$ as $|k|\to\infty$, so that the term in the left-hand side decays as $|k|\to\infty$ in $\mathbb{K}_r^+\cup\mathbb{R}$. For the purpose of this section, it will be convenient to consider the eigenfunctions as functions of λ_r, i.e.,

$$k = k(\lambda_r) \equiv \sqrt{\lambda_r^2 - A_r^2}\,.$$

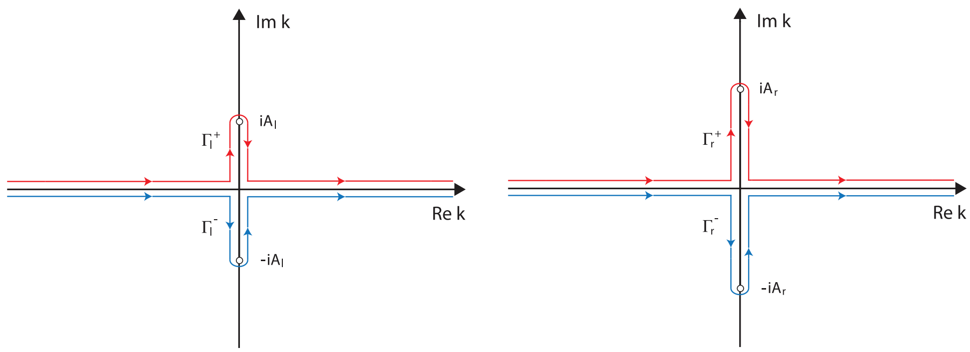

FIG. 3. The oriented contours $\Gamma_l^{\pm}$ (left) and $\Gamma_r^{\pm}$ (right).

Note that $\lambda_r \in \mathbb{R}$ is in one-to-one correspondence with either $k \in \Gamma_r^+$ or $k \in \Gamma_r^-$ (cf. Fig. 3). In the following we will assume $k \in \Gamma_r^+$ for the eigenfunction $\psi(x,k)$ [analytic for $k \in \mathbb{K}_r^+$], and $k \in \Gamma_r^-$ for $\bar{\psi}(x,k)$ [analytic for $k \in \mathbb{K}_r^-$]. We then formally integrate (3.9) with respect to λ_r, exchange the order of integration, and evaluate

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} d\lambda_r \begin{pmatrix} 1 \\ -iq_r^*/(\lambda_r + k) \end{pmatrix} e^{i\lambda_r(y-s)} = \begin{pmatrix} \delta(y-s) \\ 0 \end{pmatrix}$$

to obtain

$$\mathcal{I} = \boldsymbol{K}(x,y)\begin{pmatrix} 1 \\ 0 \end{pmatrix} + F(x+y) + \int_x^{\infty} ds\, \boldsymbol{K}(x,s)F(s+y), \tag{3.10}$$

where

$$\mathcal{I} \equiv \frac{1}{2\pi}\int_{-\infty}^{\infty} d\lambda_r \left[\frac{e^{i\lambda_r x}\phi(x,k)}{a(k)} - W_{r,1}(k)\right] e^{i\lambda_r(y-x)},$$

$$F(z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\lambda_r\, \rho(k) W_{r,2}(k)\, e^{i\lambda_r z}.$$

As explained above, in the above integrals $k = k(\lambda_r)$ with $k \in \Gamma_r^+$. The next task is to express $\mathcal{I}$ in terms of the Marchenko kernel $\boldsymbol{K}(x,y)$. Recall that we assumed that the discrete eigenvalues $k_1, \ldots, k_N$, corresponding to the zeros of $a(k)$ in $\mathbb{K}_r^+$, are simple. Then, if we consider the function $g(\lambda_r)$ obtained by subtracting to the integrand in $\mathcal{I}$ its poles, taking into account (2.39a), we obtain

$$g(\lambda_r) = e^{i\lambda_r(y-x)}\left[\frac{e^{i\lambda_r x}\phi(x,k)}{a(k)} - W_{r,1}(k)\right] - \sum_{n=1}^{N} \frac{e^{i\lambda_r(k_n)y}C_n\psi(x,k_n)}{\lambda_r - \lambda_r(k_n)}.$$

Since $g(\lambda_r)$ is analytic for $\lambda_r \in \mathbb{C}^+$, Residue Theorem and Jordan's Lemma yield

$$\mathcal{I} = i\sum_{n=1}^{N} e^{i\lambda_r(k_n)y} C_n \psi(x,k_n).$$

Taking into account the triangular representation (3.2a), we get

$$\mathcal{I} = F_d(x+y) + \int_x^{\infty} ds\, \boldsymbol{K}(x,s)F_d(s+y),$$

where

$$F_d(z) = i\sum_{n=1}^{N} e^{i\lambda_r(k_n)z} C_n W_{r,2}(k_n).$$

Substituting the above expression for $\mathcal{I}$ into (3.10), we then arrive at the *right* Marchenko integral equation

$$\boldsymbol{K}(x,y)\begin{pmatrix}1\\0\end{pmatrix}+\Omega_r(x+y)+\int_x^\infty ds\,\boldsymbol{K}(x,s)\Omega_r(s+y)=\begin{pmatrix}0\\0\end{pmatrix}, \tag{3.11}$$

where

$$\Omega_r(z)=F(z)-F_d(z)=\frac{1}{2\pi}\int_{-\infty}^{\infty}d\lambda_r\,\rho(k)W_{r,2}(k)\,e^{i\lambda_r z}-i\sum_{n=1}^{N}e^{i\lambda_r(k_n)z}C_nW_{r,2}(k_n)\,. \tag{3.12}$$

Again, note in the integral in (3.12) $\lambda_r\in\mathbb{R}$, and correspondingly $k=k(\lambda_r)\in\Gamma_r^+$.

Next, let us multiply (3.8c) by $e^{-i\lambda_r y}$ for $y>x$ and substitute (3.2a)–(3.2b). We then obtain

$$\left[\frac{e^{-i\lambda_r x}\bar{\phi}(x,k)}{\bar{a}(k)}-W_{r,2}(k)\right]e^{i\lambda_r(x-y)}=\int_x^\infty ds\,\boldsymbol{K}(x,s)W_{r,2}(k)\,e^{i\lambda_r(s-y)}$$
$$+\bar{\rho}(k)e^{-i\lambda_r y}\left[e^{-i\lambda_r x}W_{r,1}(k)+\int_x^\infty ds\,\boldsymbol{K}(x,s)W_{r,1}(k)\,e^{-i\lambda_r s}\right].$$

Formally integrating with respect to λ_r and proceeding as before, we obtain

$$\bar{\mathcal{I}}=\boldsymbol{K}(x,y)\begin{pmatrix}0\\1\end{pmatrix}+\int_x^\infty ds\,\boldsymbol{K}(x,s)\bar{F}(s+y)\,,$$

where

$$\bar{\mathcal{I}}\equiv\frac{1}{2\pi}\int_{-\infty}^{\infty}d\lambda_r\left[\frac{e^{-i\lambda_r x}\bar{\phi}(x,k)}{\bar{a}(k)}-W_{r,2}(k)\right]e^{-i\lambda_r(y-x)},$$
$$\bar{F}(z)=\frac{1}{2\pi}\int_{-\infty}^{\infty}d\lambda_r\,\bar{\rho}(k)W_{r,1}(k)\,e^{-i\lambda_r z}\,.$$

In the above integrals, we assume $k=k(\lambda_r)\in\Gamma_r^-$ (see Fig. 3). Taking into account (2.39b), exactly as before, we can express $\bar{\mathcal{I}}$ in terms of the Marchenko kernel

$$\bar{\mathcal{I}}=\bar{F}_d(x+y)+\int_x^\infty ds\,\boldsymbol{K}(x,s)\bar{F}_d(s+y)\,,$$

with

$$\bar{F}_d(z)=-i\sum_{n=1}^{N}e^{-i\lambda_r(k_n^*)z}\bar{C}_nW_{r,1}(k_n^*)\,,$$

thus arriving at the Marchenko integral equation

$$\boldsymbol{K}(x,y)\begin{pmatrix}0\\1\end{pmatrix}+\bar{\Omega}_r(x+y)+\int_x^\infty ds\,\boldsymbol{K}(x,s)\bar{\Omega}_r(s+y)=\begin{pmatrix}0\\0\end{pmatrix}, \tag{3.13}$$

where

$$\bar{\Omega}_r(z)=\bar{F}(z)-\bar{F}_d(z)=\frac{1}{2\pi}\int_{-\infty}^{\infty}d\lambda_r\,e^{-i\lambda_r z}\bar{\rho}(k)W_{r,1}(k)+i\sum_{n=1}^{N}e^{-i\lambda_r(k_n^*)z}\bar{C}_nW_{r,1}(k_n^*)\,. \tag{3.14}$$

Note that in the integral in (3.14) $\lambda_r\in\mathbb{R}$, and $k=k(\lambda_r)\in\Gamma_r^-$.

As a result of the symmetries of the scattering data, one has

$$F^*(z)=i\sigma_2\bar{F}(z),\quad F_d^*(z)=i\sigma_2\bar{F}_d(z),\quad \Omega_r^*(z)=i\sigma_2\bar{\Omega}_r(z). \tag{3.15}$$

In conclusion, we can write the Marchenko equations (3.11) and (3.13) as a single 2×2 Marchenko equation with a 2×2 Marchenko kernel as follows:

$$\boldsymbol{K}(x,y)+\boldsymbol{\Omega}_r(x+y)+\int_x^\infty ds\,\boldsymbol{K}(x,s)\boldsymbol{\Omega}_r(s+y)=0_{2\times 2}\,, \tag{3.16}$$

where

$$\boldsymbol{\Omega}_r(z) = \begin{pmatrix} \Omega_r(z) & \bar{\Omega}_r(z) \end{pmatrix}. \tag{3.17}$$

Ω_r , $\bar{\Omega}_r$ are given by (3.12) and (3.14), and satisfy $\bar{\Omega}_r(z) = i\sigma_2\Omega_r^*(z)$. Note that the 2×2 kernel $\boldsymbol{\Omega}_r(z)$ anticommutes with the Pauli matrix σ_3, and satisfies the conjugation symmetry relation

$$\boldsymbol{\Omega}_r^*(z) = \sigma_2\boldsymbol{\Omega}_r(z)\sigma_2, \tag{3.18}$$

in compliance with $\boldsymbol{K}^*(x, s) = \sigma_2\boldsymbol{K}(x, s)\sigma_2$.

2. Left Marchenko equations

In order to derive the left Marchenko equations, let us write (2.24b) explicitly as

$$\frac{\bar{\psi}(x,k)}{\bar{c}(k)} = \phi(x,k) + \bar{r}(k)\bar{\phi}(x,k) \qquad k \in \mathbb{R}\,, \tag{3.19a}$$

$$\frac{\bar{\psi}^{\pm}(x,k)}{\bar{c}^{\pm}(k)} = \phi^{\pm}(x,k) + \bar{r}^{\pm}(k)\bar{\phi}^{\pm}(x,k) \qquad k \in (-iA_l, iA_l)\,, \tag{3.19b}$$

$$\frac{\psi(x,k)}{c(k)} = \bar{\phi}(x,k) + r(k)\phi(x,k) \qquad k \in \mathbb{R}\,, \tag{3.19c}$$

$$\frac{\psi^{\pm}(x,k)}{c^{\pm}(k)} = \bar{\phi}^{\pm}(x,k) + r(k)^{\pm}\phi^{\pm}(x,k) \qquad k \in (-iA_l, iA_l)\,, \tag{3.19d}$$

where $r(k)$, $r^{\pm}(k)$ and $\bar{r}(k)$, $\bar{r}^{\pm}(k)$ are given by (2.30c) and (2.30d), respectively. Under the same assumptions as in Sec. III B 1 regarding the potential and the discrete spectrum, and considering in this case the eigenfunctions as functions of λ_l, with

$$k = k(\lambda_l) = \sqrt{\lambda_l^2 - A_l^2}\,,$$

we multiply (3.19c) by $e^{-i\lambda_l y}$ for $y < x$ and substitute the triangular representations (3.2b). Formally integrating with respect to λ_l and exchanging the order of integration, we have

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} d\lambda_l \left[\frac{e^{-i\lambda_l x}\psi(x,k)}{c(k)} - W_{l,2}(k)\right] e^{i\lambda_l(x-y)} = \frac{1}{2\pi}\int_{-\infty}^{x} ds\, \boldsymbol{J}(x,s) \int_{-\infty}^{\infty} d\lambda_l\, W_{l,2}(k)\, e^{i\lambda_l(s-y)} \tag{3.20}$$

$$+ \frac{1}{2\pi}\int_{-\infty}^{\infty} d\lambda_l\, r(k) W_{l,1}(k)\, e^{-i\lambda_l(x+y)} + \frac{1}{2\pi}\int_{-\infty}^{x} ds\, \boldsymbol{J}(x,s)\int_{-\infty}^{\infty} d\lambda_l\, r(k) W_{l,1}(k)\, e^{-i\lambda_l(s+y)}\,,$$

where $W_{l,j}(k)$ denotes the jth column of the matrix of asymptotic eigenvectors $W_l(k)$ (cf. (2.18)). As previously noted, $\lambda_l \in \mathbb{R}$ is in one-to-one correspondence with either $k \in \Gamma_l^+$, or $k \in \Gamma_l^-$ (cf. Fig. 3). We will consider $k \in \Gamma_l^+$ for the eigenfunction $\phi(x,k)$ [analytic for $k \in \mathbb{K}_l^+$], and $k \in \Gamma_l^-$ for $\bar{\phi}(x,k)$ [analytic for $k \in \mathbb{K}_l^-$]. As before, we can reduce the identity (3.20) to

$$\tilde{\mathcal{I}} = \boldsymbol{J}(x,y)\begin{pmatrix} 0 \\ 1 \end{pmatrix} + G(x+y) + \int_{-\infty}^{x} ds\, \boldsymbol{J}(x,s) G(s+y)\,, \tag{3.21}$$

where

$$\tilde{\mathcal{I}} \equiv \frac{1}{2\pi}\int_{-\infty}^{\infty} d\lambda_l \left[\frac{e^{-i\lambda_l x}\psi(x,k)}{c(k)} - W_{l,2}(k)\right] e^{i\lambda_l(x-y)}\,,$$

$$G(z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\lambda_l\, r(k) W_{l,1}(k)\, e^{-i\lambda_l z}\,.$$

In order to compute $\tilde{\mathcal{I}}$ so as to express it in terms of the Marchenko kernel $\boldsymbol{J}(x,y)$, one needs to be able to close the contour at infinity in the upper half-plane of λ_l. Unlike what happens for the

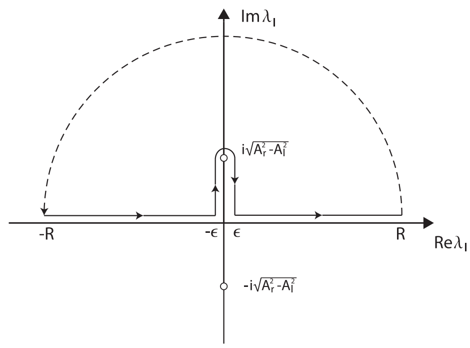

FIG. 4. The contour $\Gamma(R, \varepsilon)$.

Marchenko equations from the right, in this case closing the contour at infinity requires including the contribution of the additional branch cut that in the k-plane corresponds to $\Sigma_r \setminus \Sigma_l$. To this end, let us consider, for $0 < \varepsilon < R < +\infty$, the closed contour $\Gamma(R, \varepsilon)$ consisting of the following pieces, with the orientation specified in Fig. 4: (i) $[-R, -\varepsilon]$; (ii) the segment $\left[-\varepsilon + i0, -\varepsilon + i\sqrt{A_r^2 - A_l^2}\right]$ along the imaginary λ_l axis; (iii) the semicircle $\left\{i\sqrt{A_r^2 - A_l^2} + \varepsilon e^{i[\pi-\theta]} : 0 \le \theta \le \pi\right\}$ clockwise oriented; (iv) the segment $\left[\varepsilon + i0, \varepsilon + i\sqrt{A_r^2 - A_l^2}\right]$ along the imaginary λ_l axis; (v) $[\varepsilon, R]$; (vi) $\{Re^{i\theta} : 0 \le \theta \le \pi\}$ counterclockwise oriented. R is assumed large enough, and ε is small enough so that for all of the finitely many discrete eigenvalues $k_n \in \mathbb{K}_l^+$, $n = 1, 2, \ldots, N$, $\lambda_l(k_n)$ belong to the interior region of the contour. Since $\psi(x, k)$ and $1/c(k)$ have finite limits as $k \to iA_r$, the integral defining $\tilde{\mathcal{I}}$ with the integration confined to the semicircle around the branch point does not contribute as $\varepsilon \to 0^+$. Because of Jordan's lemma, the integral defining $\tilde{\mathcal{I}}$ when confined to the large semicircle (vi) does not contribute either as $R \to +\infty$ [note $\lambda_l \sim \lambda_r$ as $k \to \infty$, ensuring the integrand vanishes as $k \to \infty$]. Thus there are two nontrivial contributions to the integral, $\tilde{\mathcal{I}} = \tilde{\mathcal{I}}_1 + \tilde{\mathcal{I}}_2$: the contribution $\tilde{\mathcal{I}}_1$ pertaining to the residues of the function under the integral sign at the poles $k \in \mathbb{K}_r^+$; and the contribution $\tilde{\mathcal{I}}_2$ pertaining to the integral around $\lambda_l \in \left[0, i\sqrt{A_r^2 - A_l^2}\right]$ in the upper-half λ_l-plane. We shall evaluate the two contributions separately.

Since we assumed that the discrete eigenvalues k_n in $\mathbb{K}_r^+$ are simple poles of $1/c(k)$, and the reflection and transmission coefficients are continuous for $k \in \partial\mathbb{K}_r^+$, taking into account that $\psi(x, k_n) = \phi(x, k_n)/b_n$, we obtain

$$\tilde{\mathcal{I}}_1 = i\sum_{n=1}^{N} e^{-i\lambda_l(k_n)y}\tilde{C}_n\phi(x, k_n), \qquad \tilde{C}_n = \tilde{\tau}_n/b_n,$$

where $\tilde{\tau}_n$ is the residue of $1/c(k)$ at $\lambda_l = \lambda_l(k_n)$, and $\tilde{C}_n$ is the associated norming constant. Note that (2.28c) implies the residues $\tilde{\tau}_n$ and τ_n, and hence the norming constants $\tilde{C}_n$ and C_n, are related as follows:

$$\tilde{\tau}_n = \frac{\lambda_r(k_n) + k_n}{\lambda_l(k_n) + k_n}\tau_n, \qquad \tilde{C}_n\, C_n = \tau_n^2\frac{\lambda_r(k_n) + k_n}{\lambda_l(k_n) + k_n}.$$

Therefore, we have

$$\tilde{\mathcal{I}}_1 = G_1(x + y) + \int_{-\infty}^{x} ds\, \boldsymbol{J}(x, s)G_1(s + y), \tag{3.22}$$

where

$$G_1(z) = i \sum_{n=1}^{N} e^{-i\lambda_l(k_n)z} \tilde{C}_n W_{l,1}(k_n) .$$

Let us now look into the second contribution $\tilde{\mathcal{I}}_2$, which arises for $\lambda_l \in \left[0, i\sqrt{A_r^2 - A_l^2}\right]$, corresponding to $k \in [iA_l, iA_r]$ and $\lambda_r \in \mathbb{R}$. Introducing $\Delta = \sqrt{A_r^2 - A_l^2}$, we have

$$\begin{aligned}\tilde{\mathcal{I}}_2 &= \lim_{\epsilon\to 0} \frac{1}{2\pi} \left(\int_{i0-\epsilon}^{i\Delta-\epsilon} - \int_{i0+\epsilon}^{i\Delta+\epsilon} \right) d\lambda_l \left[\frac{\psi(x,k)}{c(k)}\, e^{-i\lambda_l x} - W_{l,2}(k) \right] e^{i\lambda_l(x-y)} \\ &= \frac{1}{2\pi} \int_0^{i\Delta} d\lambda_l \left[\frac{\psi^-(x,k)}{c^-(k)} - \frac{\psi^+(x,k)}{c^+(k)} \right] e^{-i\lambda_l y} . \end{aligned} \tag{3.23}$$

In the above integral along the cut $[0, i\Delta]$ on the positive imaginary axis, as usual, superscripts $\pm$ denote the limiting values from the right/left edge of the cut, respectively, and we have used that λ_l, and hence $W_{l,1}$, are continuous across it. Using (2.28c), we can write $\tilde{\mathcal{I}}_2$ as

$$\tilde{\mathcal{I}}_2 = \frac{1}{2\pi} \int_0^{i\Delta} d\lambda_l \left[\frac{|\lambda_r| - k}{|\lambda_r|} \frac{\psi^-(x,k)}{a^-(k)} - \frac{|\lambda_r| + k}{|\lambda_r|} \frac{\psi^+(x,k)}{a^+(k)} \right] \frac{\lambda_l}{\lambda_l + k} e^{-i\lambda_l y} ,$$

and the symmetry relations (2.34) and (2.36c) allow to express $(|\lambda_r| - k)\psi^-/a^- = -(|\lambda_r| + k)\bar{\psi}^+/b^+$, so that

$$\tilde{\mathcal{I}}_2 = -\frac{1}{2\pi} \int_0^{i\Delta} d\lambda_l \frac{\lambda_l(|\lambda_r| + k)}{|\lambda_r|(\lambda_l + k)} \left[\frac{\bar{\psi}^+(x,k)}{b^+(k)} + \frac{\psi^+(x,k)}{a^+(k)} \right] e^{-i\lambda_l y} . \tag{3.24}$$

Using first the scattering equation (3.8b), and then again the symmetry relation (2.36c), we finally have

$$\tilde{\mathcal{I}}_2 = -\frac{1}{2\pi} \int_0^{i\Delta} d\lambda_l \frac{\lambda_l(|\lambda_r| + k)}{|\lambda_r|(\lambda_l + k)} \frac{\phi^+(x,k)}{a^+(k)\, b^+(k)}\, e^{-i\lambda_l y} = \frac{iq_r}{2\pi} \int_0^{i\Delta} d\lambda_l \frac{\lambda_l}{|\lambda_r|(\lambda_l + k)} \frac{\phi^+(x,k)}{a^-(k)a^+(k)} e^{-i\lambda_l y} . \tag{3.25}$$

We can now insert into the last expression the triangular representation (3.2a), and obtain

$$\tilde{\mathcal{I}}_2 = G_2(x+y) + \int_{-\infty}^{x} ds\, \boldsymbol{J}(x,s) G_2(s+y) , \tag{3.26}$$

with

$$G_2(z) = \frac{iq_r}{2\pi} \int_0^{i\Delta} d\lambda_l \frac{\lambda_l}{|\lambda_r|(\lambda_l + k)} \frac{1}{a^+(k)a^-(k)} W_{l,1}(k)\, e^{-i\lambda_l z} .$$

If we now define

$$\begin{aligned}\Omega_l(z) =& G(z) - G_1(z) - G_2(z) \equiv \frac{1}{2\pi} \int_{-\infty}^{\infty} d\lambda_l\, r(k) W_{l,1}(k)\, e^{-i\lambda_l z} + \\ & - i \sum_{n=1}^{N} e^{-i\lambda_l(k_n)z} \tilde{C}_n W_{l,1}(k_n) - \frac{iq_r}{2\pi} \int_0^{i\Delta} d\lambda_l \frac{\lambda_l}{|\lambda_r|(\lambda_l + k)} \frac{1}{a^+(k)a^-(k)} W_{l,1}(k)\, e^{-i\lambda_l z} ,\end{aligned} \tag{3.27}$$

where in the first integral, corresponding to $\lambda_l \in \mathbb{R}$, one has $k = k(\lambda_l) \in \Gamma_l^+$, and use (3.22) and (3.26) to compute $\tilde{\mathcal{I}} = \tilde{\mathcal{I}}_1 + \tilde{\mathcal{I}}_2$ and introduce it into (3.21), we finally arrive at the left Marchenko integral equation

$$\boldsymbol{J}(x,y) \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \Omega_l(x+y) + \int_{-\infty}^{x} ds\, \boldsymbol{J}(x,s) \Omega_l(s+y) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{3.28}$$

In a similar way, starting from (3.19c), one can derive the "adjoint" left Marchenko equation

$$\boldsymbol{J}(x,y) \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \bar{\Omega}_l(x+y) + \int_{-\infty}^{x} ds\, \boldsymbol{J}(x,s) \bar{\Omega}_l(s+y) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{3.29}$$

where

$$\bar{\Omega}_l(z) = i\sigma_2 \Omega_l^*(z)\,.$$

The two Marchenko equations can be written in a compact matrix form as follows:

$$\boldsymbol{J}(x,y) + \boldsymbol{\Omega}_l(x+y) + \int_{-\infty}^{x} ds\; \boldsymbol{J}(x,s)\boldsymbol{\Omega}_l(s+y) = 0_{2\times 2}\,, \tag{3.30}$$

where

$$\boldsymbol{\Omega}_l(z) = \begin{pmatrix} \bar{\Omega}_l(z) & \Omega_l(z) \end{pmatrix}, \qquad \boldsymbol{\Omega}_l^*(z) = \sigma_2 \boldsymbol{\Omega}_l(z) \sigma_2\,. \tag{3.31}$$

It is worth making some remarks on the left and right Marchenko integral equations (3.16) and (3.30) derived above. First of all, notice that the asymmetry between left/right Marchenko equations is due to the asymmetry of the BCs, and explicitly to the choice $A_r \geq A_l$ (with $A_r \leq A_l$ the roles of the two integral equations is reversed). In the Marchenko integral equations from the left, $\boldsymbol{\Omega}_l(z)$ in (3.27) has three separate contributions: one from the discrete spectrum, one from the reflection coefficients from the left, $r(k)$ and $\bar{r}(k)$, integrated over values of k in the continuous spectrum, i.e., $k \in \mathbb{R} \cup \Sigma_l$, and a third contribution (sometimes referred to as the dispersive shock wave contribution, or DSW) which contains an integral over imaginary values of λ_l where the product of transmission coefficients $1/(a^+(k)a^-(k))$ appears. On the other hand, $\boldsymbol{\Omega}_r(z)$ in the integral equations from the right (cf. Eq. (3.12)) has only two contributions: one from the discrete spectrum, and one from the reflection coefficients from the right, $\rho(k)$ and $\bar{\rho}(k)$. However, it should be noted that in the latter the reflection coefficients are integrated over all $\lambda_r \in \mathbb{R}$, which means that the integral includes, in addition to the continuous spectrum $\mathbb{R} \cup \Sigma_l$, also a contribution from $\Sigma_r \setminus \Sigma_l$. Moreover, the integrand over $\Sigma_r \setminus \Sigma_l$ can never be identically zero, as, according to (2.37a), in the absence of spectral singularities $\rho^{\pm}(k) \neq 0$ for all $k \in [iA_l, iA_r]$, and $\bar{\rho}^{\pm}(k) \neq 0$ for all $k \in [-iA_r, -iA_l]$. In turn, this implies that when $\Sigma_r \setminus \Sigma_l \neq \emptyset$ (i.e., whenever one deals with asymmetric boundary conditions with $A_r \neq A_l$), no pure soliton solutions exist.

The Marchenko integral equations obtained here provide the necessary setup for the study of the long-time behavior of the solutions by means of matched asymptotics, as was recently done for KdV in Ref. 1.

C. Riemann-Hilbert problem

The purpose of this section is to provide an alternate formulation of the inverse problem, which is posed as a Riemann-Hilbert (RH) problem for the eigenfunctions, with jumps expressed in terms of the scattering data. Once the RH problem is solved, the large k expansion of the eigenfunctions then provides the reconstruction of the potential. We consider the following matrix of eigenfunctions:

$$M(x,k) = \begin{cases} \left[\frac{\phi(x,k)}{a(k)}\, e^{i\lambda_l x} \quad \psi(x,k)\, e^{-i\lambda_r x}\right], & k \in \mathbb{K}_r^+ \\[2mm] \left[\bar{\psi}(x,k)\, e^{i\lambda_r x} \quad \frac{\bar{\phi}(x,k)}{\bar{a}(k)}\, e^{-i\lambda_l x}\right], & k \in \mathbb{K}_r^- \end{cases}, \tag{3.32}$$

such that $M(x,k) \to I_2$ as $k \to \infty$, and formulate the inverse problem as a Riemann-Hilbert problem for the sectionally meromorphic matrix $M(x,k)$ across $\partial\mathbb{K}_r^+ \cup \partial\mathbb{K}_r^-$. Explicitly, we determine the five jump matrices illustrated in Fig. 5: V_0 is the jump matrix across the real axis of the complex k-plane; V_1 across $\Sigma_l^+ = [0, iA_l]$; V_2 across $\Sigma_l^- = [-iA_l, 0)$; V_3 across $\Sigma_r^+ \setminus \Sigma_l^+ = (iA_l, iA_r]$, and V_4 across $\Sigma_r^- \setminus \Sigma_l^- = [-iA_r, -iA_l)$. All jump matrices depend on k along the appropriate contour in the complex plane, as well as, parametrically, on $(x,t) \in \mathbb{R} \times \mathbb{R}^+$ [the x-dependence is explicit, while the time dependence is "hidden" in that of the corresponding reflection coefficients, see Sec. IV, and will be omitted for brevity].

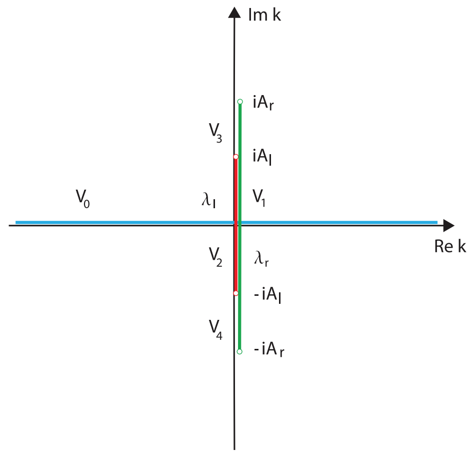

FIG. 5. The jump matrices V_j, $j = 0, 1 \cdots, 4$, of the RH problem across $\mathbb{R} \cup \Sigma_l^+ \cup \Sigma_l^- \cup (\Sigma_r^+ \setminus \Sigma_l^+) \cup (\Sigma_r^- \setminus \Sigma_l^-)$.

The RH problem across the real axis can be written in matrix form as: $M^+(x,k) = M^-(x,k)V_0(x,k)$, i.e.,

$$\left[\frac{\phi^+(x,k)}{a^+(k)}\, e^{i\lambda_l x} \quad \psi^+(x,k)\, e^{-i\lambda_r x}\right] = \left[\bar{\psi}^-(x,k)\, e^{i\lambda_r x} \quad \frac{\bar{\phi}^-(x,k)}{\bar{a}^-(k)}\, e^{-i\lambda_l x}\right] V_0(x,k)\,, \qquad k \in \mathbb{R}, \tag{3.33}$$

where in this case the superscripts $^\pm$ denote limiting values from the upper/lower complex plane, respectively. The jump matrix across the real axis can be easily computed from (2.24a), and it is given by

$$V_0(x,k) = \begin{pmatrix} [1 - \rho(k)\bar{\rho}(k)]\; e^{i(\lambda_l - \lambda_r)x} & -\bar{\rho}(k)\, e^{-2i\lambda_r x} \\ \rho(k)\, e^{2i\lambda_l x} & e^{i(\lambda_l - \lambda_r)x} \end{pmatrix}. \tag{3.34}$$

We then write the RH problem across Σ_l^+ as: $M^+(x,k) = M^-(x,k)V_1(x,k)$, $k \in \mathbb{C}^+$, where now $^\pm$ denote limiting values from the right/left edge of the cut across Σ_l^+ (Σ_l in the upper half plane). Taking into account that across Σ_l both λ_l and λ_r change sign, and using the notation

$$\lambda_l^+ = -\lambda_l^- = \lambda_l, \quad \lambda_r^+ = -\lambda_r^- = \lambda_r,$$

we have

$$\left[\frac{\phi^+(x,k)}{a^+(k)}\, e^{i\lambda_l^+ x} \quad \psi^+(x,k)\, e^{-i\lambda_r^+ x}\right] = \left[\frac{\phi^-(x,k)}{a^-(k)}\, e^{i\lambda_l^- x} \quad \psi^-(x,k)\, e^{-i\lambda_r^- x}\right] V_1(x,k), \tag{3.35}$$

and the jump matrix V_1 can easily be computed using (2.24a) and the symmetry relations (2.34),

$$V_1(x,k) = -\frac{iq_r}{\lambda_r + k} \begin{pmatrix} \rho^+(k)\, e^{2i\lambda_l x} & e^{i(\lambda_l - \lambda_r)x} \\ \left[\frac{q_r^*}{q_r} - \rho^+(k)\rho^-(k)\right] e^{i(\lambda_l - \lambda_r)x} & -\rho^-(k)\, e^{-2i\lambda_r x} \end{pmatrix}. \tag{3.36}$$

The RH problem across Σ_l^- will be written as $M^+(x,k) = M^-(x,k)V_2(x,k)$, $k \in \mathbb{C}^-$, with superscripts $^\pm$ denoting non-tangential limits from the right/left of the cut across Σ_l^-, i.e., Σ_l in the lower half plane. Explicitly, one has

$$\left[\bar{\psi}^+(x,k)\, e^{i\lambda_r^+ x} \quad \frac{\bar{\phi}^+(x,k)}{\bar{a}^+(k)}\, e^{-i\lambda_l^+ x}\right] = \left[\bar{\psi}^-(x,k)\, e^{i\lambda_r^- x} \quad \frac{\bar{\phi}^-(x,k)}{\bar{a}^-(k)}\, e^{-i\lambda_l^- x}\right] V_2(x,k). \tag{3.37}$$

As before, the jump matrix can be determined using (2.24a) and (2.34), and it is given by

$$V_2(x,k) = -\frac{iq_r^*}{\lambda_r + k} \begin{pmatrix} -\bar{\rho}^-(k)\, e^{2i\lambda_r x} & \left[\frac{q_r}{q_r^*} - \bar{\rho}^+(k)\bar{\rho}^-(k)\right] e^{i(\lambda_r - \lambda_l)x} \\ e^{i(\lambda_r - \lambda_l)x} & \bar{\rho}^+(k)\, e^{-2i\lambda_l x} \end{pmatrix}. \tag{3.38}$$

The RH problem across $\Sigma_r^+ \setminus \Sigma_l^+$ is written as $M^+(x,k) = M^-(x,k)V_3(x,k)$, $k \in \mathbb{C}^+$,

$$\left[\frac{\phi^+(x,k)}{a^+(k)}\,e^{i\lambda_l x} \quad \psi^+(x,k)\,e^{-i\lambda_r^+ x}\right] = \left[\frac{\phi^-(x,k)}{a^-(k)}\,e^{i\lambda_l x} \quad \psi^-\,e^{-i\lambda_r^- x}\right] V_3(x,k), \tag{3.39}$$

and taking into account that λ_r changes sign, while λ_l and ϕ are continuous, from (2.24a) and (2.34), one obtains

$$V_3(x,k) = -\frac{iq_r}{\lambda_r + k}\begin{pmatrix} \rho^+(k) & e^{-i(\lambda_l+\lambda_r)x} \\ \left[\frac{q_r^*}{q_r} - \rho^+(k)\rho^-(k)\right] e^{i(\lambda_l-\lambda_r)x} & -\rho^-(k)\,e^{-2i\lambda_r x}\end{pmatrix}.$$

The symmetry (2.37b) finally yields

$$V_3(x,k) = -\frac{iq_r}{\lambda_r + k}\begin{pmatrix} \rho^+(k) & e^{-i(\lambda_l+\lambda_r)x} \\ 0 & -\frac{q_r^*}{q_r}\frac{1}{\rho^+(k)}\,e^{-2i\lambda_r x}\end{pmatrix}. \tag{3.40}$$

Finally, we write the RH problem across $\Sigma_r^- \setminus \Sigma_l^-$, where λ_r changes sign but λ_l and $\bar{\phi}$ are continuous, as $M^+(x,k) = M^-(x,k)V_4(x,k)$, $k \in \mathbb{C}^-$,

$$\left[\bar{\psi}^+(x,k)\,e^{i\lambda_r^+ x} \quad \frac{\bar{\phi}^+(x,k)}{\bar{a}^+(k)}\,e^{-i\lambda_l x}\right] = \left[\bar{\psi}^-(x,k)\,e^{i\lambda_r^- x} \quad \frac{\bar{\phi}^-(x,k)}{\bar{a}^-(k)}\,e^{-i\lambda_l x}\right] V_4(x,k), \tag{3.41}$$

where

$$V_4(x,k) = -\frac{iq_r^*}{\lambda_r + k}\begin{pmatrix} -\bar{\rho}^-(k)\,e^{2i\lambda_r x} & 0 \\ e^{i(\lambda_r+\lambda_l)x} & \frac{q_r}{q_r^*}\frac{1}{\bar{\rho}^-(k)}\end{pmatrix}. \tag{3.42}$$

Note that the jump matrices satisfy the following upper/lower half plane symmetry:

$$V_2(x,k) = \sigma_2 V_1^*(x,k^*)\sigma_2\,, \qquad V_4(x,k) = \sigma_2 V_3^*(x,k^*)\sigma_2\,.$$

Solving the inverse problem as a RH problem (with poles, corresponding to the zeros of $a(k)$ and $\bar{a}(k)$ in the upper/lower half planes) then amounts to computing the sectionally meromorphic matrix $M(x,k)$ with the given jumps, and normalized to the identity as $k \to \infty$. Specifically, we can write the problem as $M^+ = M^- + (V - I_2)M^-$, where $V(x,k) = V_j(x,k)$ for $j = 0, \cdots, 4$ depending on which piece of the contour is being considered, and superscripts $\pm$ denote non-tangential limits from either side of the contour. Then, subtracting the behavior as $k \to \infty$, and the residues of $M^\pm$ at the poles in $\mathbb{K}_r^\pm$ from both sides we obtain

$$M^+ - I_2 - \sum_{n=1}^{N}\frac{1}{k-k_n}\mathrm{Res}_{k_n}M^+ - \sum_{n=1}^{N}\frac{1}{k-k_n^*}\mathrm{Res}_{k_n^*}M^- = \tag{3.43}$$

$$M^- - I_2 - \sum_{n=1}^{N}\frac{1}{k-k_n}\mathrm{Res}_{k_n}M^+ - \sum_{n=1}^{N}\frac{1}{k-k_n^*}\mathrm{Res}_{k_n^*}M^- + (V - I_2)M^-\,.$$

The left-hand side of the above equation is now analytic in $\mathbb{K}_r^+$, and it is $O(1/k)$ as $k \to \infty$ there, while the sum of all terms but the last one in the right-hand side is analytic in $\mathbb{K}_r^-$, and is $O(1/k)$ as $k \to \infty$ there. We then introduce projectors $P_\pm$ over $\Gamma_r^\pm \equiv \mathbb{R} \cup \Sigma_r^\pm$:

$$P_\pm[f](z) = \frac{1}{2\pi i}\int_{\Gamma_r^\pm}\frac{f(\xi)}{\xi - k}\,d\xi\,,$$

where $\int_{\Gamma_r^+}$ [resp. $\int_{\Gamma_r^-}$] denotes the integral along the oriented contours in Fig. 3, and when $k \in \Gamma_r^\pm \cap \mathbb{R}$ the limit is taken from the above/below. One can easily prove that if $f^\pm$ are analytic in $\mathbb{K}_r^\pm$ and are

$O(1/k)$ as $k \to \infty$, the following holds: $P_\pm f^\pm = \pm f^\pm$ and $P_+ f^- = P_- f^+ = 0$. Then, applying $P_\pm$ to both sides of (3.43), we find

$$M(k) = I_2 + \sum_{n=1}^{N} \frac{\mathrm{Res}_{k_n} M^+}{k - k_n} + \sum_{n=1}^{N} \frac{\mathrm{Res}_{k_n^*} M^-}{k - k_n^*} + \frac{1}{2\pi i} \int_{\Gamma_r^\pm} \frac{M^-(\xi)}{\xi - k} \left[V(\xi) - I_2\right] d\xi \,, \quad k \in \mathbb{C}^\pm \setminus \Sigma_r \,, \tag{3.44}$$

where the x-dependence in eigenfunctions and jump matrices has been omitted for brevity. Taking into account that the second column of $\mathrm{Res}_{k_n} M^+$ is zero for all n, while the first column is proportional to the second column of $M^+(x, k_n)$, and vice-versa the first column of $\mathrm{Res}_{k_n^*} M^-$ is zero for all n, while the second column is proportional to the second column of $M^-(x, k_n^*)$ according to (2.39a)–(2.39b), the above integral/algebraic system can be closed by evaluating it at each $k = k_n$ and $k = k_n^*$. The potential is then reconstructed by the large k expansion of the latter, since

$$M(x,k) = \left(I_2 + \frac{i}{2k} Q(x)\sigma_3\right)[1 + o(1)]\,.$$

Note that unlike what happens in the same-amplitude case, the above system cannot be reduced to a purely algebraic one: although the reflection coefficients can be chosen to be identically zero on the continuous spectrum, i.e., for $k \in \mathbb{R} \cup \Sigma_l$, the integrals appearing in the right-hand side of (3.44) always exhibit a non-zero contribution from the contours $\Sigma_r^\pm \setminus \Sigma_l^\pm$. In particular, this implies that no pure soliton solutions exist, and solitons are always accompanied by a radiative contribution of some sort. One could nonetheless solve the system iteratively, assuming the reflection coefficients are small for $k \in \Sigma_l^\pm$ (and/or for $k \in \Sigma_r^\pm \setminus \Sigma_l^\pm$), and thus obtaining NLS solutions comprising solitons superimposed to small radiation. Moreover, the RH problem formulated here provides the key setup for the investigation of the long-time asymptotic behavior by the Deift-Zhou steepest descent method.[13,14,17] The time dependence in the system is simply accounted for by the time dependence of the scattering coefficients, as described in Sec. IV. When one is interested only in capturing the leading order behavior of the solution for large t, the jumps across the contours illustrated in Fig. 5 and determined above can be simplified by suitable factorizations and contour deformations, and reduced to certain model problems for which "explicit" solutions (often expressed in terms of Riemann theta functions) can be sought for. This study obviously goes beyond the scope of the present paper, and will be the subject of future investigation.

The RH problem could also be formulated in terms of left scattering data, introducing the sectionally meromorphic matrix of eigenfunctions

$$\tilde{M}(x,k) = \begin{cases} \left[\phi(x,k)\, e^{i\lambda_l x} \quad \frac{\psi(x,k)}{c(k)}\, e^{-i\lambda_r x}\right], & k \in \mathbb{K}_r^+ \\ \left[\frac{\bar\psi(x,k)}{\bar c(k)}\, e^{i\lambda_r x} \quad \bar\phi(x,k)\, e^{-i\lambda_l x}\right], & k \in \mathbb{K}_r^- \,. \end{cases}$$

The RH problem across the real axis can be written in matrix form as: $\tilde{M}^+(x,k) = \tilde{M}^-(x,k)\tilde{V}_0(x,k)$, i.e., for $k \in \mathbb{R}$,

$$\left[\phi^+(x,k)\, e^{i\lambda_l x} \quad \frac{\psi^+(x,k)}{c^+(k)}\, e^{-i\lambda_r x}\right] = \left[\frac{\bar\psi^-(x,k)}{\bar c^-(k)}\, e^{i\lambda_r x} \quad \bar\phi^-(x,k)\, e^{-i\lambda_l x}\right] \tilde{V}_0(x,k)\,, \tag{3.45}$$

where in this case the superscripts $\pm$ denote limiting values from the upper/lower complex plane, respectively. The jump matrix across the real axis can be easily computed from (2.24b), and it is given by

$$\tilde{V}_0(x,k) = \begin{pmatrix} e^{i(\lambda_l - \lambda_r)x} & r(k)\, e^{-2i\lambda_r x} \\ -\bar r(k)\, e^{2i\lambda_l x} & [1 - r(k)\bar r(k)]\; e^{i(\lambda_l - \lambda_r)x} \end{pmatrix}. \tag{3.46a}$$

In a similar way, one can obtain the jump matrices $\tilde{V}_1$ across Σ_l^+ and $\tilde{V}_2$ across Σ_l^-, respectively,

$$\tilde{V}_1(x,k) = -\frac{i q_l^*}{\lambda_l + k} \begin{pmatrix} -r^-(k)\, e^{2i\lambda_l x} & \left[\frac{q_l}{q_l^*} - r^+(k) r^-(k)\right] e^{i(\lambda_l - \lambda_r)x} \\ e^{i(\lambda_l - \lambda_r)x} & r^+(k)\, e^{-2i\lambda_r x} \end{pmatrix} \quad \text{across} \quad \Sigma_l^+\,, \tag{3.46b}$$

$$\tilde{V}_2(x,k) = -\frac{iq_l}{\lambda_l + k} \begin{pmatrix} \bar{r}^+(k)\, e^{2i\lambda_r x} & e^{i(\lambda_r - \lambda_l)x} \\ \left[\frac{q_l^*}{q_l} - \bar{r}^+(k)\bar{r}^-(k)\right] e^{i(\lambda_r-\lambda_l)x} & -\bar{r}^-(k)\, e^{-2i\lambda_l x} \end{pmatrix} \quad \text{across} \quad \Sigma_l^- . \qquad (3.46c)$$

As for the RH problem in terms of scattering coefficients from the right, the jump matrices $\tilde{V}_1(x,k)$ and $\tilde{V}_2(x,k)$ satisfy the following upper/lower half plane symmetry:

$$\tilde{V}_2(x,k) = \sigma_2 \tilde{V}_1^*(x,k^*)\sigma_2 .$$

In the RH problem across $\Sigma_r^+ \setminus \Sigma_l^+$, one has

$$\tilde{M}^+(x,k) = \left[\phi^+(x,k)\, e^{i\lambda_l x} \quad \frac{\psi^+(x,k)}{c^+(k)}\, e^{-i\lambda_r^+ x}\right],$$

$$\tilde{M}^-(x,k) = \left[\phi^-(x,k)\, e^{i\lambda_l x} \quad \frac{\psi^-(x,k)}{c^-(k)}\, e^{-i\lambda_r^- x}\right].$$

Note, however, that unlike what happens in the RH problem from the right, here one cannot use (3.19a)–(3.19d) to determine the jump. The same holds for the RH problem on $\Sigma_r^- \setminus \Sigma_l^-$. In fact, in both Eqs. (3.19b) and (3.19d), the right-hand sides are only simultaneously defined for $k \in \mathbb{R} \cup \Sigma_l$, and cannot be extended on either $\Sigma_r^+ \setminus \Sigma_l^+$ or $\Sigma_r^- \setminus \Sigma_l^-$. This is also evident from (2.30c), where it is clear that, unlike $\rho^\pm(k)$ and $\bar{\rho}^\pm(k)$, which can be respectively continued on $\Sigma_r^+ \setminus \Sigma_l^+$ and $\Sigma_r^- \setminus \Sigma_l^-$, the reflection coefficients from the left, $r^\pm(k)$ and $\bar{r}^\pm(k)$, are only generically defined on the continuous spectrum, i.e., for $k \in \mathbb{R} \cup \Sigma_l$.

In order to formulate the RH problem from the left on $\Sigma_r \setminus \Sigma_l$, one has to consider both pieces of the cut $\Sigma_r^+ \setminus \Sigma_l^+$ and $\Sigma_r^- \setminus \Sigma_l^-$ simultaneously, and take into account that: (i) λ_r changes sign across $\Sigma_r^+ \setminus \Sigma_l^+$ and $\Sigma_r^- \setminus \Sigma_l^-$; (ii) $\phi^+(x,k) = \phi^-(x,k)$ for $k \in \Sigma_r^+ \setminus \Sigma_l^+$, and $\bar{\phi}^+(x,k) = \bar{\phi}^-(x,k)$ for $k \in \Sigma_r^- \setminus \Sigma_l^-$; (iii) $\psi^\pm(x,k)/c^\pm(k)$ and $\bar{\psi}^\pm(x,k)/\bar{c}^\pm(x,k)$ are related to each other via the symmetry relations (2.34), (2.38a) and (2.38b), i.e.,

$$\frac{\psi^\pm(x,k)}{c^\pm(x,k)} = \frac{\bar{\psi}^\mp(x,k)}{\bar{d}^\mp(x,k)} \qquad k \in \Sigma_r^+ \setminus \Sigma_l^+ , \qquad (3.47a)$$

$$\frac{\bar{\psi}^\pm(x,k)}{\bar{c}^\pm(x,k)} = \frac{\psi^\mp(x,k)}{d^\mp(x,k)} \qquad k \in \Sigma_r^- \setminus \Sigma_l^- . \qquad (3.47b)$$

Solving the RH problem from the left (with poles, corresponding to the zeros of $c(k)$ and $\bar{c}(k)$ in the upper/lower half planes, which, by (2.28d) are the same as the ones from the right) amounts to computing the sectionally meromorphic matrix $\tilde{M}(x,k)$ with the given jumps $\tilde{V}_j$, and normalized to the identity as $k \to \infty$. The potential is then reconstructed by the large k expansion of the latter, since

$$\tilde{M}(x,k) = \left(I_2 + \frac{i}{2k} Q(x)\sigma_3\right)[1 + o(1)] .$$

IV. TIME EVOLUTION OF THE SCATTERING DATA

According to the second of (2.1), the time-evolution of the eigenfunctions is given by

$$v_t = \begin{pmatrix} 2ik^2 - i|q|^2 & -2kq - iq_x \\ 2kq^* - iq_x^* & -2ik^2 + i|q|^2 \end{pmatrix} v , \qquad (4.1)$$

and asymptotically, taking into account $q(x,t) \to q_{l/r}(t) = A_{l/r} e^{i\theta_{l/r}(t)}$ as $x \to \mp\infty$,

$$v_t \simeq \begin{pmatrix} 2ik^2 - iA_{l/r}^2 & -2kq_{l/r} \\ 2kq_{l/r}^* & -2ik^2 + iA_{l/r}^2 \end{pmatrix} v \qquad \text{as } x \to \mp\infty\,. \tag{4.2}$$

The scattering problem in (2.1) as $x \to \mp\infty$ gives for the two components of any eigenfunction $v(x,t)$,

$$q_{l/r}^* v^{(1)} \simeq -v_x^{(2)} + ikv^{(2)}\,, \qquad q_{l/r} v^{(2)} \simeq v_x^{(1)} + ikv^{(1)}\,.$$

Introducing these equations into (4.2), we obtain as $x \to \mp\infty$,

$$v_t^{(1)} \simeq -iA_{l/r}^2 v^{(1)} - 2kv_x^{(1)}\,, \qquad v_t^{(2)} \simeq iA_{l/r}^2 v^{(2)} - 2kv_x^{(2)}\,. \tag{4.3}$$

The Jost solutions, whose boundary values as $x \to \mp\infty$ are given by (2.20a)–(2.20b), are not compatible with the above time evolution. Therefore, we introduce time-dependent eigenfunctions to be solutions of the evolution equation (4.1). For instance, let

$$\varphi(x,k,t) = e^{iA_\infty t}\phi(x,k,t), \tag{4.4}$$

so that

$$\varphi_t = iA_\infty \varphi + e^{iA_\infty t}\phi_t\,. \tag{4.5}$$

Taking into account that the components of φ satisfy asymptotically the system (4.3) as $x \to -\infty$, and that

$$\phi \simeq \begin{pmatrix} 1 \\ \frac{-iq_l^*(t)}{\lambda_l + k} \end{pmatrix} e^{-i\lambda_l x}\,, \quad \phi_t \simeq \begin{pmatrix} 0 \\ -\dot\theta_l(t)\frac{q_l^*(t)}{\lambda_l + k} \end{pmatrix} e^{-i\lambda_l x}, \quad x \to -\infty\,,$$

where dot denotes differentiation with respect to time, substituting into (4.5), the first component yields

$$A_\infty = 2k\lambda_l - A_l^2\,,$$

and from the second component we have: $\dot\theta_l(t) = -2A_l^2$, so that

$$\theta_l(t) = -2A_l^2 t + \theta_l(0)\,.$$

In a similar way, we can determine the evolution of the asymptotic phase as $x \to +\infty$: $\theta_r(t) = -2A_r^2 t + \theta_r(0)$, as well as the time evolution of the other Jost solutions, obtaining for $\Phi = (\phi \;\; \bar\phi)$ and $\Psi = (\bar\psi \;\; \psi)$,

$$\partial_t \Phi = \left[i(2k^2 - |q|^2 + Q_x)\sigma_3 - 2kQ\right]\Phi - i(2k\lambda_l - A_l^2)\,\Phi\,\sigma_3\,, \tag{4.6a}$$

$$\partial_t \Psi = \left[i(2k^2 - |q|^2 + Q_x)\sigma_3 - 2kQ\right]\Psi - i(2k\lambda_r - A_r^2)\Psi\,\sigma_3\,. \tag{4.6b}$$

Differentiating (2.24a) with respect to t and taking into account the time evolution of the Jost solutions (4.6a)–(4.6b), we obtain for the scattering matrix

$$\partial_t S = i(2k\lambda_r - A_r^2)\sigma_3\, S - i(2k\lambda_l - A_l^2) S\,\sigma_3\,. \tag{4.7}$$

Explicitly, this yields the following expressions for the time evolution of the scattering coefficients $a(k,t)$ and $b(k,t)$, and the reflection coefficient from the right $\rho(k,t) = b(k,t)/a(k,t)$,

$$a(k,t) = a(k,0)\, e^{i[2k(\lambda_r - \lambda_l) - A_r^2 + A_l^2]t}\,, \tag{4.8a}$$

$$b(k,t) = b(k,0)\, e^{i[-2k(\lambda_r + \lambda_l) + A_r^2 + A_l^2]t}\,, \tag{4.8b}$$

$$\rho(k,t) = \rho(k,0)\, e^{-2i(2k\lambda_r - A_r^2)t}\,. \tag{4.8c}$$

The first equation shows that the discrete eigenvalues k_n are time independent, and given by the zeros of $a(k, 0)$. Note that in the symmetric case $A_l = A_r$, and therefore $a(k, t) = a(k, 0)$, i.e., the transmission coefficient is time-independent. Moreover, for the large k behavior of $a(k, t)$, taking into account (A11), one still finds from (4.8a) that, consistently with (2.47a), $a(k, t) \sim 1$ as $|k| \to \infty$ for $k \in \mathbb{K}_r^+ \cup \mathbb{R}$ and for all $t \geq 0$; it ensures that the inverse problem is well posed. Similarly, we can find the evolution of the other scattering coefficients $c(k, t)$, $d(k, t)$ etc., and of the reflection coefficient from the left $r(k, t) = d(k, t)/c(k, t)$,

$$r(k, t) = r(k, 0)\, e^{2i(2k\lambda_l - A_l^2)t}\,. \tag{4.9}$$

Finally, we need to determine the time dependence of the norming constants. Differentiating $\phi(x, k_n) = b_n \psi(x, k_n)$ with respect to time and evaluating the first column of (4.6a) and the second column of (4.6b) at $k = k_n$, we get

$$b_n(t) = b_n(0) e^{-i[2k_n(\lambda_l(k_n)+\lambda_r(k_n)) - A_l^2 - A_r^2]t}, \quad n = 1, \ldots, N.$$

Then from the definition of the norming constants in (2.39a), we obtain

$$C_n(t) = C_n(0) e^{-2i[2k_n\lambda_r(k_n) - A_r^2]t}\,, \quad n = 1, \ldots, N. \tag{4.10}$$

V. CONCLUSIONS

We have developed the IST for the focusing NLS with fully asymmetric non-zero boundary conditions as $x \to \pm\infty$. This is a highly nontrivial generalization of the case where the amplitudes of the background field are taken to be the same at both space infinities (see Ref. 12), and it involves dealing with additional technical difficulties, the most important of which being the fact that when the amplitudes of the NLS solutions as $x \to \pm\infty$ are different, in the spectral domain one cannot introduce a uniformization variable that allows mapping the multiply sheeted Riemann surface for the scattering parameter to a single complex plane. Important differences with respect to the symmetric case obviously also arise in the inverse problem, where, in addition to solitons (corresponding to the discrete eigenvalues of the scattering problem), and to radiation (corresponding to the continuous spectrum of the scattering operator, and represented in the inverse problem by the reflection coefficients for $k \in \mathbb{R} \cup (-iA_l, iA_l)$), one also has a nontrivial contribution from the transmission coefficients for $k \in (-iA_r, -iA_l) \cup (iA_l, iA_r)$, as shown by the last term is (3.27), contributing to the left Marchenko equations. Correspondingly, (2.37a) and (2.37c) show that in the right Marchenko equations one always has a nontrivial contribution from the integral terms in (3.12) and (3.14), since $\rho(k)$ [resp. $\bar{\rho}(k)$] cannot vanish for $k \in (iA_l, iA_r)$ [resp. $k \in (-iA_r, -iA_l)$]. In particular, this implies that no pure soliton solutions exist, and solitons are always accompanied by a radiative contribution of some sort. As a consequence, unlike the symmetric case, here no explicit solution can be obtained by simply reducing the inverse problem to a set of algebraic equations.

The results presented in this paper will pave the way for the investigation of the long-time asymptotic behavior of fairly general NLS solutions with nontrivial boundary conditions via the nonlinear steepest descent method, in analogy to what was done, for instance, in Ref. 17 for the modified KdV equation, or in Refs. 13 and 14 for the focusing NLS with step-like initial conditions. Moreover, the Marchenko integral equations obtained here will provide an alternative setup for the study of the long-time behavior of the solutions by means of matched asymptotics, as was recently done for KdV in Ref. 1.

The study of the long-time asymptotics, as well as the derivation of solutions describing solitons superimposed to small radiation, will be the subject of future investigation.

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APPENDIX: NORM ESTIMATES OF THE GROUPS $e^{x\,\Lambda_{l/r}(k)}$

Taking into account that $\Lambda^2_{r/l}(k) = -\lambda^2_{r/l} I_2$, it is straightforward to obtain from its series representation the explicit expression of $e^{x\,\Lambda_{r/l}(k)}$ as

$$e^{x\Lambda_{r/l}(k)} = \cos(\lambda_{r/l}x)\,I_2 + \frac{\sin(\lambda_{r/l}x)}{\lambda_{r/l}}Q_{r/l} - ik\frac{\sin(\lambda_{r/l}x)}{\lambda_{r/l}}\sigma_3\,. \tag{A1}$$

We recall that the Hilbert-Schmidt norm of a matrix A is defined as: $\|A\|^2_{\rm HS} = \operatorname{trace}(A^\dagger A)$, while the spectral norm is the square root of the largest singular value of $A^\dagger A$. Clearly, $\det e^{x\Lambda_{r/l}(k)} = 1$ for any $k \in \mathbb{C}$, and for the Hilbert-Schmidt norm of the group for $\lambda_{r/l} = \sqrt{k^2 + A^2_{r/l}} \in \mathbb{R}$, i.e., for $k \in \mathbb{R} \cup [-iA_{r/l}, iA_{r/l}]$, one has

$$\|e^{x\Lambda_{r/l}(k)}\|^2_{\rm HS} =: 2F(x,k)\,, \qquad F(x,k) \equiv \cos^2(\lambda_{r/l}x) + \frac{|k|^2 + A^2_{r/l}}{\lambda^2_{r/l}}\sin^2(\lambda_{r/l}x)\,.$$

Note that $F(x,k) \equiv 1$ for $k \in \mathbb{R}$ [since in this case $|k|^2 + A^2_{r/l} = k^2 + A^2_{r/l} = \lambda^2_{r/l}$], and $F(x, \pm iA_{r/l}) \equiv 1 + 2A^2_{r/l}x^2$, which is obtained from the above in the limit $\lambda_{r/l} \to 0$. Also, note that for $k \in (-iA_{r/l}, iA_{r/l})$ it is $|k|^2 = -k^2$ and $\lambda^2_{r/l} \le A^2_{r/l}$. Hence for $k \in (-iA_{r/l}, iA_{r/l})$, one has

$$F(x,k) = \cos^2(\lambda_{r/l}x) + \frac{|k|^2 + A^2_{r/l}}{\lambda^2_{r/l}}\sin^2(\lambda_{r/l}x) \equiv 1 + \frac{2(A^2_{r/l} - \lambda^2_{r/l})}{\lambda^2_{r/l}}\sin^2(\lambda_{r/l}x) \ge 1\,, \tag{A2a}$$

and for $k \in [-iA_{r/l}, iA_{r/l}]$,

$$F(x,k) \le 1 + 2A^2_{r/l}x^2\,. \tag{A2b}$$

The bounds (A2) obviously also hold for $k \in \mathbb{R}$. Using the identity

$$\|A\|^2 = \tfrac{1}{2}\left[\|A\|^2_{\rm HS} + \sqrt{\|A\|^4_{\rm HS} - 4|\det A|^2}\right]$$

for the squared spectral norm of a 2×2 matrix A, we then get

$$\|e^{x\Lambda_{r/l}(k)}\|^2 = F(x,k) + \sqrt{F^2(x,k) - 1}, \qquad k \in \mathbb{R} \cup (-iA_{r/l}, iA_{r/l}),$$

yielding

$$C_{r/l}(k) = \sup_{x\in\mathbb{R}} \|e^{x\Lambda_{r/l}(k)}\| = \begin{cases} 1, & k \in \mathbb{R}, \\ \sqrt{F(k) + \sqrt{F^2(k) - 1}}, & k \in (-iA_{r/l}, iA_{r/l}), \end{cases} \tag{A3}$$

where

$$F(k) = 1 + \frac{2(A^2_{r/l} - \lambda^2_{r/l})}{\lambda^2_{r/l}} = 1 + \frac{2|k|^2}{\lambda^2_{r/l}} \qquad k \in (-iA_{r/l}, iA_{r/l})\,.$$

Finally, for $k \in [-iA_{r/l}, iA_{r/l}] \cup \mathbb{R}$ and for each $x \in \mathbb{R}$, from (A2b) we have

$$\|e^{x\Lambda_{r/l}(\pm iA_{r/l})}\|^2 = 1 + 2A^2_{r/l}x^2 + \sqrt{(1 + 2A^2_{r/l}x^2)^2 - 1} \le \tilde{C}^2_{r/l}\,(1 + |x|)^2\,, \tag{A4}$$

where $\tilde{C}_{r/l}$ is a positive constant, independent of $x \in \mathbb{R}$.

Gronwall's inequality. Consider the integral equation

$$U(x) = A(x) + \int_x^\infty dy\, B(x, y)U(y), \tag{A5}$$

where $A(x)$ and $B(x, y)$ are, respectively, vector-valued and matrix-valued continuous functions for all $x \in \mathbb{R}$, such that

$$\|A(x)\| \le \alpha(x), \qquad \|B(x, y)\| \le \beta(y)\,, \tag{A6}$$

for some $\alpha(x)$, $\beta(x)$ real-valued, continuous, non-negative functions; moreover, $\alpha(x)$ is assumed to be non-increasing for all $x \in \mathbb{R}$, and $\beta(x) \in L^1(a, \infty)$ for all $a \in \mathbb{R}$. Then the Neumann series

$$U(x) = \sum_{n=0}^\infty U_n(x)\,, \qquad U_0(x) = A(x)\,, \qquad U_{n+1}(x) = \int_x^\infty dy\, B(x, y)U_n(y)\,, \tag{A7}$$

is uniformly convergent and it provides the (unique) solution of the integral equation (A5). Since the Neumann series is a uniformly convergent series of continuous functions, $U(x)$ is itself a continuous function for all $x \in \mathbb{R}$, and it satisfies Gronwall's inequality

$$\|U(x)\| \le \alpha(x) \exp\left(\int_x^\infty dz\, \beta(z)\right). \tag{A8}$$

The above results can be generalized in the obvious way to the case when the integral equation (A5) and, consequently, the bounds (A6) contain a parametric dependence on k in $A(x, k)$, $B(x, y, k)$, and $U(x, k)$.

Proof of Gronwall's inequality. The proof of the above results relies on the fact that from the bounds (A6) it follows that for all non-negative integers n,

$$\|U_n(x)\| \le \alpha(x)\frac{1}{n!}\left(\int_x^\infty dz\, \beta(z)\right)^n \quad \Rightarrow \quad \sum_{n=0}^\infty \|U_n(x)\| \le \alpha(x)\sum_{n=0}^\infty \frac{1}{n!}\left(\int_x^\infty dz\, \beta(z)\right)^n.$$

Indeed, the above estimate is obviously true for $n = 0$. Assuming it to be true for a certain n, we obtain

$$\begin{aligned}
\|U_{n+1}(x)\| &\le \int_x^\infty dy\, \|B(x, y)\|\|U_n(y)\| \\
&\le \int_x^\infty dy\, \alpha(y)\beta(y)\frac{1}{n!}\left(\int_y^\infty dz\, \beta(z)\right)^n \\
&\le \alpha(x)\int_x^\infty dy\, \frac{1}{n!}\beta(y)\left(\int_y^\infty dz\, \beta(z)\right)^n \\
&= \alpha(x)\left[\frac{-1}{(n+1)!}\left(\int_y^\infty dz\, \beta(z)\right)^{n+1}\right]_{y=x}^\infty \\
&= \alpha(x)\,\frac{1}{(n+1)!}\left(\int_x^\infty dz\, \beta(z)\right)^{n+1},
\end{aligned}$$

as claimed. Note that we have used that $\alpha(x)$ is non-increasing.

Proof of Proposition 2.1. For $k \in \mathbb{R} \cup (-iA_r, iA_r)$, the estimate (A3) shows that $\|e^{x\Lambda_r(k)}\| \le C_r(k)$ for each $x \in \mathbb{R}$. Then the result for (2.8a) follows by applying Gronwall's inequality with $\alpha(x, k) = C_r(k)$, $\beta(y, k) = C_r(k)\|Q(y) - Q_r\|$, yielding

$$\|\tilde{\Psi}(x, k)\| \le C_r(k)e^{C_r(k)\int_x^\infty dy\,\|Q(y)-Q_r\|}.$$

For $k \in \mathbb{R} \cup [-iA_r, iA_r]$, from the estimate (A4) it follows that there exists a constant $\tilde{C}_r$ such that $\|e^{x\Lambda_r(k)}\| \leq \tilde{C}_r(1+|x|)$ for each $x \in \mathbb{R}$. By applying Gronwall's inequality we then arrive at

$$\|\tilde{\Psi}(x,k)\| \leq \tilde{C}_r(1+|x|)e^{\tilde{C}_r \int_x^\infty dy\,(1+|y-x|)\|Q(y)-Q_r\|} \leq \tilde{C}_r(x_0)(1+|x|)e^{\tilde{C}_r(x_0)\int_x^\infty dy(1+|y|)\|Q(y)-Q_r\|},$$

where $\tilde{C}_r(x_0) = \tilde{C}_r\,(1+\max(0,-x_0))$ and $x \geq x_0$.

Proof of Proposition 2.2. Multiplying (2.8a) and (2.8b) from the right by the appropriate columns of the matrices $W_{r/l}(k)$, respectively, and using the explicit expressions (A1) for $e^{(x-y)A_{r/l}(k)}$, we obtain the following Volterra integral equations for the Jost solutions:

$$e^{i\lambda_r x}\bar{\psi}(x,k) = W_{r,1}(k) - \int_x^\infty dy\; \Xi_r^-(y-x,k)[Q(y)-Q_r]e^{i\lambda_r y}\bar{\psi}(y,k), \tag{A9a}$$

$$e^{-i\lambda_r x}\psi(x,k) = W_{r,2}(k) - \int_x^\infty dy\; \Xi_r^+(y-x,k)[Q(y)-Q_r]e^{-i\lambda_r y}\psi(y,k), \tag{A9b}$$

$$e^{i\lambda_l x}\phi(x,k) = W_{l,1}(k) + \int_{-\infty}^x dy\; \Xi_l^+(x-y,k)[Q(y)-Q_l]e^{i\lambda_l y}\phi(y,k), \tag{A9c}$$

$$e^{-i\lambda_l x}\bar{\phi}(x,k) = W_{l,2}(k) + \int_{-\infty}^x dy\; \Xi_l^-(x-y,k)[Q(y)-Q_l]e^{-i\lambda_l y}\bar{\phi}(y,k), \tag{A9d}$$

where the subscripts $j = 1, 2$ in the matrices $W_{l/r}(k)$ denote their jth column, and

$$\Xi_r^-(x,k) = \begin{pmatrix} 1+\frac{\lambda_r-k}{2\lambda_r}\left[e^{-2i\lambda_r x}-1\right] & -\frac{iq_r}{2\lambda_r}\left[e^{-2i\lambda_r x}-1\right] \\ \frac{iq_r^*}{2\lambda_r}\left[e^{-2i\lambda_r x}-1\right] & e^{-2i\lambda_r x}-\frac{\lambda_r-k}{2\lambda_r}\left[e^{-2i\lambda_r x}-1\right] \end{pmatrix}, \tag{A10a}$$

$$\Xi_r^+(x,k) = \begin{pmatrix} e^{2i\lambda_r x}-\frac{\lambda_r-k}{2\lambda_r}\left[e^{2i\lambda_r x}-1\right] & \frac{iq_r}{2\lambda_r}\left[e^{2i\lambda_r x}-1\right] \\ -\frac{iq_r^*}{2\lambda_r}\left[e^{2i\lambda_r x}-1\right] & 1+\frac{\lambda_r-k}{2\lambda_r}\left[e^{2i\lambda_r x}-1\right] \end{pmatrix}, \tag{A10b}$$

$$\Xi_l^+(x,k) = \begin{pmatrix} 1+\frac{\lambda_l-k}{2\lambda_l}\left[e^{2i\lambda_l x}-1\right] & -\frac{iq_l}{2\lambda_l}\left[e^{2i\lambda_l x}-1\right] \\ \frac{iq_l^*}{2\lambda_l}\left[e^{2i\lambda_l x}-1\right] & e^{2i\lambda_l x}-\frac{\lambda_l-k}{2\lambda_l}\left[e^{2i\lambda_l x}-1\right] \end{pmatrix}, \tag{A10c}$$

$$\Xi_l^-(x,k) = \begin{pmatrix} e^{-2i\lambda_l x}-\frac{\lambda_l-k}{2\lambda_l}\left[e^{-2i\lambda_l x}-1\right] & \frac{iq_l}{2\lambda_l}\left[e^{-2i\lambda_l x}-1\right] \\ -\frac{iq_l^*}{2\lambda_l}\left[e^{-2i\lambda_l x}-1\right] & 1+\frac{\lambda_l-k}{2\lambda_l}\left[e^{-2i\lambda_l x}-1\right] \end{pmatrix}. \tag{A10d}$$

With the given choice of the branch cuts, we easily derive the following expressions for the behavior of $\lambda_{r/l}$ as $k \to \infty$:

$$\lambda_r - k = \frac{A_r^2}{2k}\left[1+O(k^{-2})\right], \qquad \lambda_l - k = \frac{A_l^2}{2k}\left[1+O(k^{-2})\right]. \tag{A11}$$

Moreover, for $x \geq 0$ we have

$$\left|\frac{e^{\pm 2i\lambda_r x}-1}{2\lambda_r}\right| \equiv \left|\int_0^x dz\, e^{\pm 2i\lambda_r z}\right| \leq \min(x, 1/|\lambda_r|), \qquad k \in \mathbb{K}_r^\pm \cup \partial\mathbb{K}_r^\pm, \tag{A12}$$

and similarly for the l-subscripted quantities. By the maximum modulus principle, we also get for $k \in \mathbb{K}_r^\pm \cup \partial\mathbb{K}_r^\pm$,

$$\left\|(1\;,\, -iq_r/(\lambda_r+k))^T\right\| = \left\|\left(1\;,\, -i(\lambda_r-k)/q_r^*\right)^T\right\| = \left[1+\left(\frac{1}{|q_r|}\max_{k\in\partial\mathbb{K}_r^\pm}|\lambda_r-k|\right)^2\right]^{1/2} = \sqrt{2}, \tag{A13}$$

where the spectral norm is used. Now applying (A11)–(A13) to estimate

$$\left\|\Xi_r^-(x,k)\right\| \leq 1 + 2|q_r|\min(x, 1/|\lambda_r|), \qquad k \in \mathbb{K}_r^- \cup \partial\mathbb{K}_r^-, \tag{A14}$$

we obtain with the help of Gronwall's inequality

$$\left\|e^{i\lambda_r x}\bar{\psi}(x,k)\right\| \leq \sqrt{2}\, e^{\int_x^\infty dy\,[1+2|q_r|(y-x)]\|Q(y)-Q_r\|}, \tag{A15}$$

where the estimate is uniform for $(x,k) \in [x_0,+\infty) \times [\mathbb{K}_r^- \cup \partial\mathbb{K}_r^- \cup \partial\mathbb{K}_r^+]$ under the hypothesis $(\boldsymbol{H}_1)$. Thus, under the hypothesis $(\boldsymbol{H}_1)$ and for each $x \in \mathbb{R}$, the Jost solution $\bar{\psi}(x,k)$ is continuous in $k \in \mathbb{K}_r^- \cup \partial\mathbb{K}_r^- \cup \mathbb{K}_r^+$ and analytic in $k \in \mathbb{K}_r^-$. Likewise we obtain estimates for the other three Jost solutions, which proves the continuity and analyticity properties mentioned above.

Proof of Proposition 2.3. Differentiating (A9a) with respect to k we obtain the integral equation

$$\begin{aligned}\frac{\partial}{\partial k}\left[e^{i\lambda_r x}\bar{\psi}(x,k) - W_{r,1}(k)\right] &= -\frac{\partial}{\partial k}\int_x^\infty dy\,\Xi_r^-(y-x,k)[Q(y)-Q_r]W_{r,1}(k)\\ &- \int_x^\infty dy\,\frac{\partial}{\partial k}\left[\Xi_r^-(y-x,k)\right][Q(y)-Q_r]\left\{e^{i\lambda_r y}\bar{\psi}(y,k) - W_{r,1}(k)\right\}\\ &- \int_x^\infty dy\,\Xi_r^-(y-x,k)[Q(y)-Q_r]\frac{\partial}{\partial k}\left[e^{i\lambda_r y}\bar{\psi}(y,k) - W_{r,1}(k)\right],\end{aligned}$$

where, from (A10a),

$$\begin{aligned}\frac{\partial}{\partial k}\Xi_r^-(y-x,k) = &-2i\frac{k}{\lambda_r}(y-x)e^{-2i\lambda_r(y-x)}\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\\ &- \frac{k}{2\lambda_r^2}\begin{pmatrix}\lambda_r - k & -iq_r\\ iq_r^* & -(\lambda_r-k)\end{pmatrix}\left[\frac{e^{-2i\lambda_r(y-x)}-1}{\lambda_r} + 2i(y-x)e^{-2i\lambda_r(y-x)}\right]\\ &- \frac{\lambda_r - k}{\lambda_r}\frac{(e^{-2i\lambda_r(y-x)}-1)}{2\lambda_r}\sigma_3.\end{aligned}$$

Using (A12), we obtain

$$\left\|\frac{\partial}{\partial k}\Xi_r^-(y-x,k)\right\| \leq 2\frac{|k|}{|\lambda_r|}(y-x)\left[1 + \frac{|\lambda_r - k| + |q_r|}{|\lambda_r|}\right] + \frac{|\lambda_r - k|}{|\lambda_r|}(y-x). \tag{A16}$$

A simple Gronwall argument then suffices to estimate its solution (for $k \neq \pm iA_r$). A similar result holds for $\psi(x,k)$, as well as for the other two Jost solutions, under the hypothesis $k \neq \pm iA_l$.

Proof of the inversion formulas (3.3). Consider the representation (3.1), which we can write as

$$\begin{aligned}\tilde{\Psi}(x,k) &= e^{x\Lambda_r(k)} + \int_x^\infty dy\,\boldsymbol{K}(x,y)e^{y\Lambda_r(k)}\\ &= \cos(\lambda_r x)I_2 + \frac{\sin(\lambda_r x)}{\lambda_r}Q_r + \int_x^\infty dy\,\boldsymbol{K}(x,y)\left[\cos(\lambda_r y) + \frac{\sin(\lambda_r y)}{\lambda_r}Q_r\right]\\ &- ik\frac{\sin(\lambda_r x)}{\lambda_r}\sigma_3 - ik\int_x^\infty dy\,\boldsymbol{K}(x,y)\frac{\sin(\lambda_r y)}{\lambda_r}\sigma_3\\ &= \cos(\lambda_r x)I_2 + \frac{\sin(\lambda_r x)}{\lambda_r}Q_r + \boldsymbol{\Psi}_c(x,\lambda_r) + \boldsymbol{\Psi}_s(x,\lambda_r)Q_r\\ &- ik\frac{\sin(\lambda_r x)}{\lambda_r}\sigma_3 - ik\boldsymbol{\Psi}_s(x,\lambda_r)\sigma_3,\end{aligned}$$

where

$$\boldsymbol{\Psi}_c(x,\lambda_r) = \int_x^\infty dy\,\boldsymbol{K}(x,y)\cos(\lambda_r y), \qquad \boldsymbol{\Psi}_s(x,\lambda_r) = \int_x^\infty dy\,\boldsymbol{K}(x,y)\frac{\sin(\lambda_r y)}{\lambda_r}.$$

Separating the parts that are even and odd (in λ_r), we obtain

$$\frac{\tilde{\Psi}(x,k)+\tilde{\Psi}(x,-k)}{2} = \cos(\lambda_r x) I_2 + \frac{\sin(\lambda_r x)}{\lambda_r} Q_r + \boldsymbol{\Psi}_c(x,\lambda_r) + \boldsymbol{\Psi}_c(x,\lambda_r) Q_r \,,$$
$$\frac{\tilde{\Psi}(x,k)-\tilde{\Psi}(x,-k)}{2} = -ik\frac{\sin(\lambda_r x)}{\lambda_r}\sigma_3 - ik\boldsymbol{\Psi}_s(x,\lambda_r)\sigma_3 \,,$$

and as a result,

$$\boldsymbol{\Psi}_s(x,\lambda_r) = \frac{\tilde{\Psi}(x,k)-\tilde{\Psi}(x,-k)}{-2ik}\sigma_3 - \frac{\sin(\lambda_r x)}{\lambda_r} I_2 \,,$$
$$\boldsymbol{\Psi}_c(x,\lambda_r) = \frac{\tilde{\Psi}(x,k)+\tilde{\Psi}(x,-k)}{2} - \cos(\lambda_r x) I_2 + \frac{\tilde{\Psi}(x,k)-\tilde{\Psi}(x,-k)}{2ik}\sigma_3 Q_r \,.$$

If we then depart from the identity

$$\int_x^\infty dy\, \boldsymbol{K}(x,y) e^{i\lambda_r y} = \boldsymbol{\Psi}_c(x,\lambda_r) + i\lambda_r \boldsymbol{\Psi}_s(x,\lambda_r) \,,$$

we can write

$$\int_x^\infty dy\, \boldsymbol{K}(x,y) e^{i\lambda_r y} = \frac{\tilde{\Psi}(x,k)+\tilde{\Psi}(x,-k)}{2} - \cos(\lambda_r x) I_2 + \frac{\tilde{\Psi}(x,k)-\tilde{\Psi}(x,-k)}{2ik}\sigma_3 Q_r$$
$$- \frac{\lambda_r}{2k}[\tilde{\Psi}(x,k)-\tilde{\Psi}(x,-k)]\sigma_3 - i\sin(\lambda_r x) I_2 \,,$$

i.e.,

$$\int_x^\infty dy\, \boldsymbol{K}(x,y) e^{i\lambda_r y} = \tfrac{1}{2}\tilde{\Psi}(x,k)\left[I_2 - \frac{i}{k}\sigma_3 Q_r - \frac{\lambda_r}{k}\sigma_3\right]$$
$$+\tfrac{1}{2}\tilde{\Psi}(x,-k)\left[I_2 + \frac{i}{k}\sigma_3 Q_r + \frac{\lambda_r}{k}\sigma_3\right] - e^{i\lambda_r x} I_2 \,. \tag{A17}$$

Now, the following identities can be easily verified:

$$e^{x\Lambda_r(k)}\left[I_2 - \frac{i}{k}\sigma_3 Q_r - \frac{\lambda_r}{k}\sigma_3\right] = e^{i\lambda_r x}\left[I_2 - \frac{i}{k}\sigma_3 Q_r - \frac{\lambda_r}{k}\sigma_3\right],$$
$$e^{x\Lambda_r(-k)}\left[I_2 + \frac{i}{k}\sigma_3 Q_r + \frac{\lambda_r}{k}\sigma_3\right] = e^{i\lambda_r x}\left[I_2 + \frac{i}{k}\sigma_3 Q_r + \frac{\lambda_r}{k}\sigma_3\right].$$

Multiplying either side of (A17) by $e^{-i\lambda_r x}$, and using the above identities, we find

$$\int_x^\infty dy\, \boldsymbol{K}(x,y) e^{i\lambda_r (y-x)} = \tfrac{1}{2}\tilde{\Psi}(x,k) e^{-x\Lambda_r(k)}\left[I_2 - \frac{i}{k}\sigma_3 Q_r - \frac{\lambda_r}{k}\sigma_3\right]$$
$$+ \tfrac{1}{2}\tilde{\Psi}(x,-k) e^{-x\Lambda_r(-k)}\left[I_2 + \frac{i}{k}\sigma_3 Q_r + \frac{\lambda_r}{k}\sigma_3\right] - I_2$$
$$= \tfrac{1}{2}\left[\tilde{\Psi}(x,k) e^{-x\Lambda_r(k)} - I_2\right]\left[I_2 - \frac{i}{k}\sigma_3 Q_r - \frac{\lambda_r}{k}\sigma_3\right]$$
$$+ \tfrac{1}{2}\left[\tilde{\Psi}(x,-k) e^{-x\Lambda_r(-k)} - I_2\right]\left[I_2 + \frac{i}{k}\sigma_3 Q_r + \frac{\lambda_r}{k}\sigma_3\right].$$

Consequently, since $\int_x^\infty dy\, \boldsymbol{K}(x,y)e^{i\lambda_r(y-x)} \equiv \int_{-\infty}^\infty dy\, \boldsymbol{K}(x,y)e^{i\lambda_r(y-x)}$ because $\boldsymbol{K}(x,y)\equiv 0$ for $y<x$, we arrive at

$$\boldsymbol{K}(x,y) = \frac{1}{4\pi}\int_{-\infty}^\infty d\lambda_r e^{-i\lambda_r(y-x)}\left\{\left[\tilde\Psi(x,k)e^{-x\Lambda_r(k)} - I_2\right]\left[I_2 - \frac{i}{k}\sigma_3 Q_r - \frac{\lambda_r}{k}\sigma_3\right]\right.$$
$$\left. + \left[\tilde\Psi(x,-k)e^{-x\Lambda_r(-k)} - I_2\right]\left[I_2 + \frac{i}{k}\sigma_3 Q_r + \frac{\lambda_r}{k}\sigma_3\right]\right\}.$$

Proof of Proposition 3.1. Substituting (3.2a) into (2.1) and multiplying the resulting equation from the right by $e^{i\lambda_r\sigma_3 x}$, we get

$$\partial_x\left[\Psi(x,k)e^{i\lambda_r\sigma_3 x}\right] - i\lambda_r\Psi(x,k)\sigma_3 e^{i\lambda_r\sigma_3 x}$$
$$= (-ik\sigma_3 + Q(x))\left\{W_r(k) + \int_x^\infty ds\, \boldsymbol{K}(x,s)W_r(k)e^{-i\lambda_r\sigma_3(s-x)}\right\}.$$

By using (2.18), (2.45a)–(2.45b), (2.46) (where in (2.46) we assume that $\partial_x q \in L^1(\mathbb{R})$), we obtain

$$\left\{\frac{i\partial_x Q(x)\sigma_3}{2k} - i\lambda_r\left(I_2 + \frac{iQ(x)\sigma_3}{2k}\right)\sigma_3\right\}[1+o(1)]$$
$$= (-ik\sigma_3 + Q(x))\left\{I_2 + \int_x^\infty ds\, \boldsymbol{K}(x,s)e^{-i\lambda_r\sigma_3(s-x)}\right\}$$
$$+ \frac{i}{\lambda_r+k}(-ik\sigma_3 + Q(x))\left\{Q_r + \int_x^\infty ds\, \boldsymbol{K}(x,s)Q_r e^{-i\lambda_r\sigma_3(s-x)}\right\}\sigma_3,$$

i.e.,

$$ik\sigma_3 + \tfrac{1}{2}Q(x) + o(1/k)$$
$$= -ik\sigma_3 + Q(x) + Q(x)\int_x^\infty ds\, \boldsymbol{K}(x,s)e^{-i\lambda_r\sigma_3(s-x)}$$
$$- i\lambda_r\sigma_3\int_x^\infty ds\, \boldsymbol{K}(x,s)e^{-i\lambda_r\sigma_3(s-x)} + \frac{iA_r^2}{\lambda_r+k}\sigma_3\int_x^\infty ds\, \boldsymbol{K}(x,s)e^{-i\lambda_r\sigma_3(s-x)}$$
$$+ \tfrac{1}{2}\left[1 - \frac{A_r^2}{(\lambda_r+k)^2}\right]\sigma_3 Q_r\sigma_3 + \frac{i}{\lambda_r+k}Q(x)Q_r\sigma_3$$
$$+ \frac{i}{\lambda_r+k}(-ik\sigma_3 + Q(x))\int_x^\infty ds\, \boldsymbol{K}(x,s)Q_r e^{-i\lambda_r\sigma_3(s-x)}\sigma_3.$$

Now on the right-hand side, the third, fifth, and last terms involve Fourier integrals of matrix functions with entries in $L^2(\mathbb{R}; d\lambda_r)$, multiplied by factors which are at least bounded at large k; therefore all such terms vanish as $k, \lambda_r \to \infty$. Next, we write for the fourth term

$$- i\lambda_r\sigma_3\int_x^\infty ds\, \boldsymbol{K}(x,s)e^{-i\lambda_r\sigma_3(s-x)} =$$
$$- i\lambda_r\sigma_3\int_x^\infty ds\, \boldsymbol{K}(x,s)W_r(k)e^{-i\lambda_r\sigma_3(s-x)} - \frac{\lambda_r}{\lambda_r+k}\sigma_3\int_x^\infty ds\, \boldsymbol{K}(x,s)Q_r\sigma_3 e^{-i\lambda_r\sigma_3(s-x)},$$

where the second term on the right has its entries in $L^2(\mathbb{R}; d\lambda_r)$. The first term on the right is written as

$$\begin{aligned}
& -i\lambda_r\sigma_3 \int_x^\infty ds\, \boldsymbol{K}(x,s)W_r(k)e^{-i\lambda_r\sigma_3(s-x)} \\
&= \sigma_3 \left\{ -i\lambda_r \int_x^\infty ds\, \boldsymbol{K}(x,s)W_r(k)e^{-i\lambda_r\sigma_3(s-x)}\sigma_3 \right\} \sigma_3 \\
&= \sigma_3 \left\{ -\boldsymbol{K}(x,x)W_r(k) + \int_x^\infty ds\, [\partial_x \boldsymbol{K}(x,s)]\, W_r(k)e^{-i\lambda_r\sigma_3(s-x)} \right. \\
&\left. - \partial_x \int_x^\infty ds\, \boldsymbol{K}(x,s)W_r(k)e^{-i\lambda_r\sigma_3(s-x)} \right\} \sigma_3 \\
&= \sigma_3 \left\{ -\boldsymbol{K}(x,x)W_r(k) + \int_x^\infty ds\, [\partial_x \boldsymbol{K}(x,s)]\, e^{-i\lambda_r\sigma_3(s-x)} \right. \\
&\left. + \frac{i}{\lambda_r + k} \int_x^\infty ds\, [\partial_x \boldsymbol{K}(x,s)]\, Q_r\sigma_3 e^{-i\lambda_r\sigma_3(s-x)} - \partial_x \left[\Psi(x,k)e^{i\lambda_r\sigma_3 x} - W_r(k) \right] \right\} \sigma_3.
\end{aligned}$$

Now let us examine the various terms within brackets on right-hand side of this last identity. The first term is simply $-\boldsymbol{K}(x,x) + o(1)$. According to (2.46), the last term vanishes as $k \to \pm\infty$, the second term has its entries in $L^2(\mathbb{R}; d\lambda_r)$ (being the Fourier transform of an L^2 matrix function), and the third term of the last member is L^2 multiplied by a bounded factor. Thus, dropping all contributions that vanish as $k \to \pm\infty$, and using that $\sigma_3 Q_r \sigma_3 = -Q_r$, we obtain

$$-ik\sigma_3 + \tfrac{1}{2}Q(x) = -ik\sigma_3 + Q(x) - \sigma_3 \boldsymbol{K}(x,x)\sigma_3 - \tfrac{1}{2}Q_r,$$

i.e., the first of (3.4). The second equality in (3.4) can be proved in a similar way.

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