1

SET and You

1.1 A GAME OF SET

Three students, **S**tefan, **E**mily, and **T**anya, are playing SET, a game played with a special deck of cards. Each card in the game of SET has symbols characterized by four different attributes:

- Number: 1, 2, or 3 symbols
- Color: red, green, or purple symbols
- Shading: empty, striped, or solid symbols
- Shape: ovals, squiggles, or diamonds

The game is new to Tanya. Stefan is the dealer, but before he can deal the first cards, Tanya starts asking questions.

TANYA: *How many cards are in the deck?*
STEFAN: *That sounds like a math question.*
EMILY: starts counting the cards. *Give me a minute and I'll know!*
STEFAN: to Emily. *Cheater! Don't count. We can figure it out, using... ∗math!∗*
TANYA: *How did you do that?!*
STEFAN: *Do what?*
TANYA: *How did you speak asterisks like that?*
STEFAN: *I do not understand the question.*
EMILY: *He did it because we are in a book, obviously! We can speak all the symbols we want!* ☺

Tanya has asked the first math question that most people ask about the game, and Stefan's advice is directed to Tanya and Emily, but also

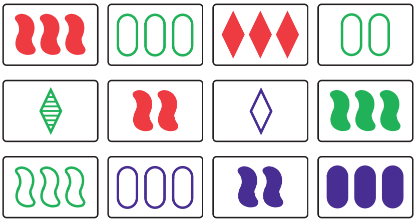


Figure 1.1. First layout of cards.

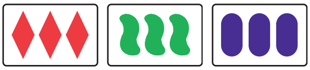


Figure 1.2. Emily's *SET*.

Figure 1.3. The *SET* not taken.

to you. Of course, if you have a deck in your hands, it's easy to answer this question the way Emily started to.

In the spirit of self-discovery, though, we will postpone answering this (and other) questions until later in the book. We encourage you to try to figure out the answers on your own. But the questions Stefan, Emily, and Tanya ask here will motivate much of what you will see in the coming chapters.

Stefan deals 12 cards—see figure 1.1.

TANYA: *How do you play the game?*

STEFAN: *You find three cards that are either all the same or all different in each of the four attributes. That's called a "SET."*

EMILY: grabbing three cards. *Like this!* (See figure 1.2.)

TANYA: *I see—all the cards have three symbols, there are three different colors, all are solid, and the three different shapes appear.*

STEFAN: *That's right! In fact, there was another SET containing one of Emily's cards, 3 Green Solid Squiggles.* (See figure 1.3.)

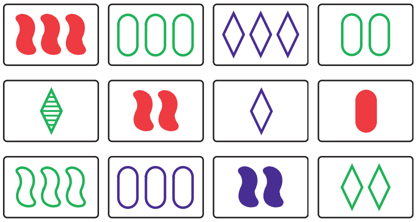


Figure 1.4. Second layout of cards.

Figure 1.5. Tanya's non-*SET*.

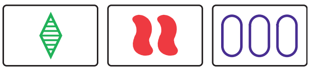


Figure 1.6. Tanya's actual *SET*.

STEFAN: *These are all green, but for each of the other attributes, they are all different.*

TANYA: *There were two SETs in the first layout. Is that weird?*

STEFAN: *No. There's a nice probability calculation that tells you the average number of SETs in the first layout.* (See chapter 3.) *Now, since we want to have 12 cards, I need to replace the three cards Emily took.* (Stefan deals out three more cards. See figure 1.4.)

TANYA: pointing to the three solid red cards. *Hey—is this a SET?* (See figure 1.5.)

STEFAN: *Almost! But you see there's a problem with shape: two are squiggles and one is an oval. Any time you can say "two are x and one is y," you're out of luck.*

TANYA: *Oh, now I see. How about this?* (See figure 1.6.)

EMILY: *Great! Some people find these SETs the hardest to see—all four attributes are different.*

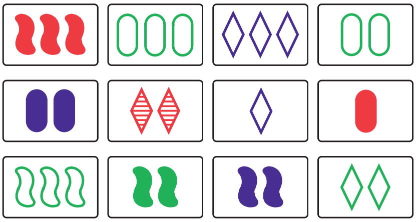


Figure 1.7. Third layout of cards.

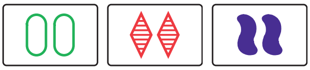


Figure 1.8. Stefan's *SET*.

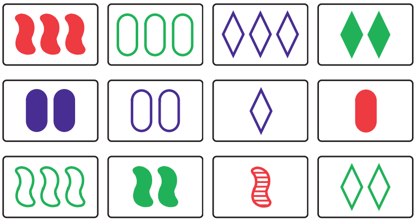


Figure 1.9. Fourth layout of cards. There are no *SET*s!

STEFAN: *Tanya's SET was actually in the original layout, but no one saw it till now.*

Tanya takes her cards and Stefan deals three more. See figure 1.7.

STEFAN: grabbing three cards. *The dealer finds a SET! That's allowed, you know.* (See figure 1.8.)

Stefan deals another three cards, and the players stare at the layout in figure 1.9 for a while.

EMILY: *I don't think there's a SET in these 12 cards. I can't find one.*
STEFAN: *Ow—my head hurts!*
TANYA: *Emily! You're hitting Stefan in the head!*
EMILY: *Sorry! Sometimes I swing my arms wildly when I'm thinking. It's pretty dangerous.*
TANYA: *Are there any SETs in here? How can we be sure?*

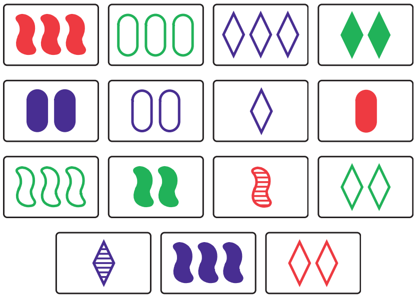


Figure 1.10. Fourth layout of cards with three cards added.

EMILY: *There aren't. There are a few ways we could check this, but for the most part, if everyone's been staring for a while and no one has found anything, we put out three more cards.*

TANYA: *How often does it happen that there are no SETs among the 12 cards?*

STEFAN: *I think that's hard to calculate. But people have estimated how often this happens by using computer simulations.*

The interlude (following chapter 5) includes methods to verify that there are no *SET*s in a given layout, and chapter 10 deals[1] with simulations. When everyone agrees there are no *SET*s, three cards are added to the layout. (See figure 1.10.)

EMILY: *Now there's a SET!*[2]

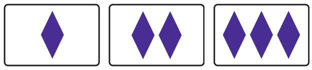
TANYA: *Is it possible for 15 cards to have no SETs?*

STEFAN: *Yes. In fact, you can have as many as 20 cards without a SET, and that's the most you can have. This turns out to be a question related to finite geometry.* (See chapter 5.)

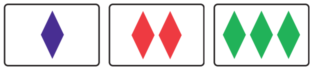
TANYA: *Cool. So, I now know it's possible for 12 cards to have no SETs. What's the maximum number of SETs 12 cards can have?*

STEFAN: *Well, it's kind of amazing, but this is also a geometry question.* (There's a project at the end of chapter 5 devoted to constructing collections of 12 cards that have a prescribed number of *SET*s.)

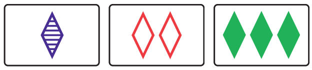
[1] This is a pun, but it was unintentional.
[2] See if you can find one. Then, see if you can find a second one.



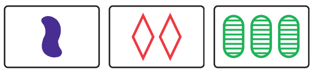
(a) Three attributes the same and one different.



(b) Two attributes the same and two different.



(c) One attribute the same and three different.



(d) All four attributes different.

Figure 1.11. The four different kinds of *SET*s.

TANYA: *Is it known how many SETs there are in the entire deck?*
EMILY: *Yes. That's a fun calculation.* (The answer appears in chapter 2, where lots of things get counted.)
STEFAN: *And every card is in the same number of SETs!*
TANYA: *Thanks for answering a question I didn't ask. And I can't help but notice that you aren't actually answering any of my questions. Something else I noticed: the first few SETs we found didn't always have the same number of attributes that were the same. How many different kinds of SETs are there?*
EMILY: looking through the deck. *Four. Here are examples of every possibility for how many attributes are the same.* (See figure 1.11.)

At this point, the game proceeds as before, with the players taking *SET*s and Stefan dealing more cards. After taking as many *SET*s as they can find, there are six cards left. (See figure 1.12.)

TANYA: *Are there any SETs remaining in the final layout?*
EMILY: *Nope. Game over!*
TANYA: *Is it typical for there to be six cards at the end?*

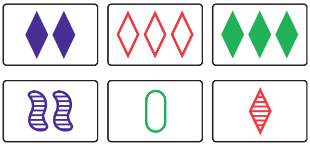


Figure 1.12. Six cards remain at the end of the game.

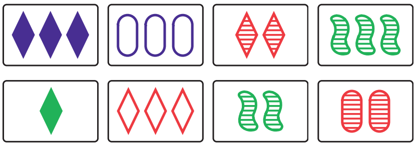


Figure 1.13. One card at the end of the game is missing—can you find it?

STEFAN: *This is a hard probability to calculate exactly, but based on computer simulations, it's usually true that either six or nine cards remain. Sometimes we can clear the deck, but that's fairly uncommon.*

EMILY: *It's also possible for 12, 15, or 18 cards to be left, but we've never actually played a game with either 15 or 18 cards left at the end.* (There are several simulations in chapter 10 that explore this topic.)

TANYA: *So always multiples of three—that makes sense. How often are there exactly three cards left?*

STEFAN *and* EMILY, together: *Never!*

At this point, the game ends, the players count their *SET*s, and Emily wins. But Tanya wonders why there can't be three cards left at the end of the game. A complete explanation uses modular arithmetic, which you will find in chapter 4.

TANYA: *That was fun! Can we play another game?*

The group plays a second game, and this time, there are eight cards left at the end of the game. (See figure 1.13.)

TANYA: *Wait! How can there be eight cards left? I thought it had to be a multiple of three! Is there a card missing or something?*

EMILY: *Yes, precisely! We hid one card at the beginning of the game, then played the usual way.*

TANYA: *Why would you do that? What is the missing card?*
STEFAN: *The amazing thing is that you can figure it out!*
TANYA: *But I didn't memorize every card that's been played!*
EMILY: *You don't need to—you can determine the missing card from the cards left on the table!*
TANYA: *How!?*

Rather than answer Tanya's impassioned plea, Emily explains what she and Stefan are doing. They (and we) call this the *End Game*.

The End Game

1. At the beginning of the game, remove one card from the deck (without looking at it!) and put it aside.
2. Now deal 12 cards face up, and play the game as usual, removing *SET*s and replacing the cards you took.
3. At the end of the game, you can determine the hidden card using just the cards left on the table.
4. Finally, now that you've determined the hidden card, you might be lucky enough to find a *SET* using the hidden card and two of the cards that are left on the table.

We'll explain how this procedure works in detail in chapter 4; we'll also discuss how often the missing card makes a *SET* in chapter 10. For now, see if you can find the missing card in the configuration in figure 1.13. [Hint: Concentrate on each attribute separately: first, determine the color of the missing card, then the number, and so on. The answer is at the end of the chapter.]

STEFAN: *OK Tanya, here's how to find the missing card.*
(∗Inaudible whispers∗*)*
TANYA: *Now I get it! This is so cool! The missing card is . . .* (Tanya shouts the missing card so loudly that we couldn't hear it.)
EMILY: *Perfect! It gets better. Does the missing card form a SET with two of the eight cards left on the table?*
TANYA: *Yes! In fact, it's in two different SETs.*

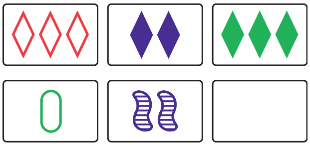


Figure 1.14. Find the missing card at the end of the game.

STEFAN: *That's right. It's really impressive when you yell "SET!," then take two of the cards on the table and finally turn over the hidden card.*

EMILY: *Yeah, it really looks like a magic trick!*

For practice (without any instruction), see if you can determine the missing card in figure 1.14. The identity of the missing card also appears at the end of this chapter.

TANYA: *Does this trick always work? Can you find the missing card if there is a different number of cards left?*

EMILY: *Yes, it always works, but you won't necessarily be able to form a SET with two of the cards on the table.*

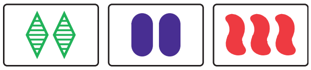
STEFAN: *By the way, when we play the End Game, if there are five cards left, then the missing card will never form a SET with two of the cards on the table.*

TANYA: *I think I understand why. Is it related to the fact that it's impossible to have just three cards left at the end of the game?*

STEFAN: *Yes—this uses modular arithmetic.*

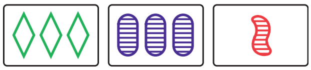
TANYA: *This is all so cool, and oddly foreshadowing! Let's play another game.*

Stefan, Emily, and Tanya play another game. Even though she's new to the game, Tanya does well, partly because Stefan and Emily give themselves a handicap by not immediately taking *SET*s they find. (See the interlude for some possible ways that experienced players can play with new players so that the game is fun for everyone.) When the game is over, the players have another discussion.



(a) Three cards that aren't a *SET*; call them *A*, *B*, and *C*.

(b) These three cards make *SET*s with the three pairs of cards in (a): The first card makes a *SET* with *A* and *B*, the second makes a *SET* with *A* and *C*, and the third makes a *SET* with *B* and *C*.



(c) These three cards complete *SET*s with the cards in (b).

Figure 1.15. Completing as many *SET*s as possible.

Tanya: *So, you two know so much about this game, there's gotta be another trick you can teach me.*

Stefan: *Indeed, there is. I'm going to hand you three random cards that aren't a SET. For each pair of those cards, find the third card that makes a SET with that pair.* (Stefan puts out the three cards in figure 1.15(a).)

Tanya, hunts through the deck and finds the three cards in figure 1.15(b): *OK, these are the three cards I found.*

Emily: *Good. Now do the same thing with the three cards you just found.*

Tanya: *Done.* (She lays down the three cards in figure 1.15(c).)

Stefan: *Now do the same thing with the three cards you just found.*

Tanya: *You've got to be kidding. This could go on forever!*

Emily: *It could, but it doesn't. Keep going.*

Tanya: *What is happening?!? It's the same cards we started with!*

Stefan: *Now, look at these nine cards. Can you organize them nicely so that you can see all the SETs?*

After some reorganizing, Tanya lays out the cards in figure 1.16.

Tanya: *You're right, that was a great trick. And look at how pretty this is!*

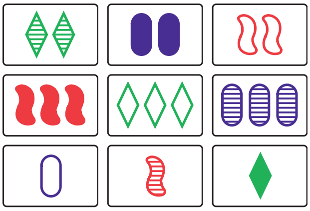


Figure 1.16. The nine cards in figure 1.15 nicely organized.

EMILY: *This is what the Set Enterprises website calls a magic square. Notice that no matter which two cards you pick, the third card that makes a SET is in there!*

We'll return to these special layouts in the next section. In the meantime, try doing this trick yourself.

1.2 MORE QUESTIONS AND A PREVIEW

In this section, we give an overview of several of the ways SET and math are related. We hope this whets your appetite for much of what follows. We will pose many questions, but just like Stefan and Emily, we will answer very few of them, at least for now. We encourage you to read actively, thinking about the questions and trying to find your own solutions. But first, a word from our sponsor.

History

SET was invented in 1974 by Marsha Falco, a population geneticist studying epilepsy in German shepherd dogs. She had a card for each dog, and she placed symbols on the card to represent that dog's expressions of various genes. As she looked at the cards, she realized she could make a game out of them. At first, she played with her family, and then in 1990, she founded Set Enterprises, Inc., to develop and market the game.

SET has repeatedly been recognized as an outstanding game, winning the TD*monthly* (ToyDirect) Top-10 Most Wanted Card Games every year from 2006 to 2015, the Mensa Select Award (1991), and the

Figure 1.17. The winner (left), Marsha Falco (right).

Parents' Choice Best 25 games of the past 25 years (2004). There has been one National SET Competition (in August 2006) advertised on the Set Enterprises' website www.setgame.com, which was won by one of the authors of this book.[3] See figure 1.17.

Counting Questions

People (including mathematicians, who are also people) started asking questions about SET as soon as it appeared in toy stores. But, as is typical in mathematics, just knowing the answer to a question is not enough. Answers often lead to more questions, and, in the case of SET, these new questions expose deeper connections between the game and math.

We begin with the first question Tanya asked.

- How many cards are needed to make the deck?

The lazy solution is to get a deck and count all the cards. You should get 81. A mathy explanation for this is the following: since there are four attributes, and each attribute has three possibilities, there are $3 \times 3 \times 3 \times 3 = 3^4 = 81$ possible cards. Why do we multiply (instead of adding, for instance)? Because we need to choose a number AND

[3] Just ask Hannah which one.

Figure 1.18. The fundamental theorem of SET tells you that there is a unique card that makes a *SET* with these two cards. What is it?

a color AND a shading AND a shape. Replacing AND with × is sometimes called the *multiplication principle* in textbooks on discrete math. We explain this fundamental idea more carefully in chapter 2.

- How many *SET*s are there?

A short answer: Enough to make the game interesting. We answer this question in chapter 2.

- What percentage of the *SET*s differ in all four attributes? Three attributes? Two attributes? Only one attribute?

The very clean answer to this question is explained rather carefully in chapter 2. The calculation uses some basic counting techniques. For now, you might enjoy trying to guess which kind of *SET* is most common, and which is least common.

- How many different *SET*s contain a given card?

Stefan mentioned that each card is in the same number of *SET*s. But this counting question introduces an important idea, so we'll answer it now. We'll return to it in chapter 2.

Finding the number of *SET*s that contain a given card uses a principle so important that we call it the fundamental theorem of SET.

FUNDAMENTAL THEOREM OF SET

Given any pair of cards, there is a unique card that completes a *SET* with the pair.

Two cards are shown in figure 1.18. It should be clear that there is a unique card that completes a *SET* with those two cards.

Here's how we can apply this theorem. First, choose a card C. Then the other 80 cards can be split up into 40 pairs, each of which makes a *SET* with C. This tells us that there are 40 *SET*s that contain any given card.

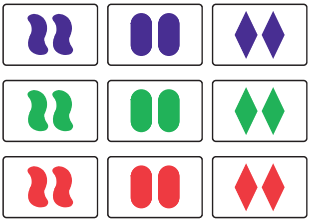


Figure 1.19. Lots of *SET*s.

By the way, we can use this fact to calculate the total number of *SET*s. Your argument might begin, "Since there are 81 cards and 40 *SET*s that contain that card, we get $81 \times 40 = 3240$. But this overcounts the number of *SET*s because we've counted each *SET* three times. So. . . ."

Geometry Questions

The connection between the game and geometry is surprising. The game is played with a finite deck of cards, and standard Euclidean geometry isn't finite (there are an infinite number of points on a line, lines in a plane, and so on). But the connection to *finite* geometry is fundamental. We explore this in chapters 5 and 9. For a warm-up, try this:

- How many *SET*s can you find in the collection of nine cards in figure 1.19?

The answer is below.[4] As Emily mentioned, the Set Enterprises website calls a configuration like this a *magic square*. Unfortunately, the term magic square means something else to mathematicians.[5] It has the largest possible number of *SET*s that can be found in nine cards. We (and most mathematicians) call this configuration a *plane*, for reasons that will become clear in chapter 5.

How is this related to geometry? Think of the nine cards as "points" and the *SET*s as "lines." Then we can redraw this picture as in

[4] There are 12 *SET*s, although this may not be the answer to the question "How many can you find?"

[5] And to Ben Franklin, who made a study of magic squares. A *magic square* is a square array of distinct integers where each column, each row, and both main diagonals add up to the same sum. There is no relation between these magic squares and SET.

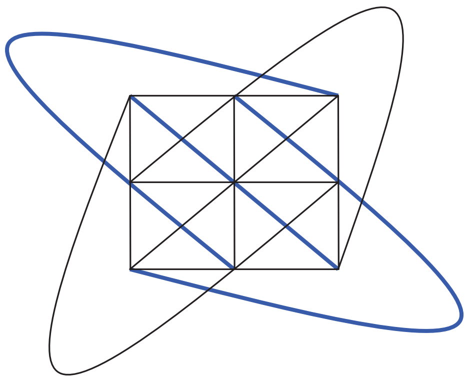


Figure 1.20. Schematic diagram of *SET*s in figure 1.19. This is the affine plane with three points per line.

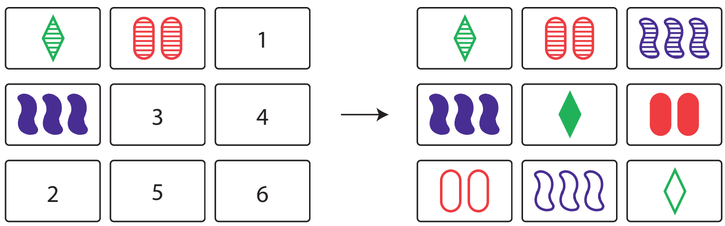


Figure 1.21. Complete the plane.

figure 1.20. (Those swoopy curves are "lines" passing through three points, as you can see by looking at the corresponding cards in figure 1.19. In this geometry, lines don't have to be straight!)

If you look back at the nine cards that Tanya organized in figure 1.16, you'll see that the *SET*s in that figure are in exactly the same relative positions as the *SET*s in figure 1.20. So her nine cards also form a plane.

In fact, you can make a plane like this from any three cards that do not form a *SET*, like Tanya did (see exercise 1.3). She used a two-step procedure, where first she found the cards and then she organized them. You can also do this in one step, by following these simple instructions: take three cards that aren't a *SET*, and put them in the corner of a square, as in the left side of figure 1.21. The numbered spaces give you one possible order to add cards that complete *SET*s. The finished result is shown in the same figure on the right.

There are two things that seem like "magic" about planes. First, if you take any pair of cards from the plane, the third card that completes the *SET* is also in the plane. Second, if you take any three cards that aren't a *SET* from the plane, and follow the procedure in figure 1.21, you'll get the same nine cards.

We'll revisit this topic in chapter 2 when we count the number of planes in the deck, and also in chapter 5 when we investigate the game using the axioms of geometry. These planes are also good starting places for creating 12-card layouts that contain the maximum possible number of *SET*s. You're asked to do that in exercise 1.4.

There are a few observations that have a geometric flavor; these are so important, we can't wait to tell you about them:

- Given any two cards, there is a unique third card that completes a *SET* with them. (The fundamental theorem of SET.)
- Given any three cards that don't form a *SET*, there is a unique plane (up to reordering the cards) containing them.

Compare these statements with the fundamental facts you (may have) learned in high-school geometry:

- Given any two points, there is a unique line containing them.
- Given any three non-collinear points, there is a unique plane containing them.

Our geometry is different from Euclidean geometry (in particular, "lines" have only three points, and don't need to be straight). But a large and somewhat surprising amount of Euclidean geometry will apply to SET. In chapter 5, we'll learn that the SET cards form the *affine geometry* AG(4, 3), and this will have some important consequences.

There is one interesting consequence of the geometric approach:

- All *SET*s are the same.

But this is crazy. We know that there are four *different* kinds of *SET*s (see figure 1.11). However, from the geometric point of view, *SET*s are simply lines in a finite geometry, and all lines are the "same." This can be made more precise using linear algebra, which we do when we revisit this question in chapter 8.

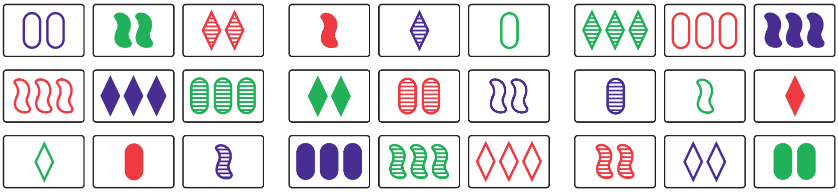


Figure 1.22. A hyperplane.

One final comment on geometry, for now. Thinking of our *SET*s as one-dimensional lines and the planes as two-dimensional objects, we are naturally led to create three-dimensional hyperplanes, as in figure 1.22. These consist of 27 cards that can be split up into three parallel two-dimensional planes. For comparison, note that the plane of figure 1.19 can be split up into three parallel lines. (Parallel *SET*s are discussed in chapters 5 and 8.)

What's special about *SET*s, planes, and hyperplanes? These collections of cards are *closed*. This means that given any pair of cards in the collection, the card that completes the *SET* with that pair is also in the collection. In chapter 8, we'll see that there are only five types of closed collections of cards: single cards, *SET*s, planes, hyperplanes, or the entire deck.

For entertainment, try to find the locations of some *SET*s in the hyperplane shown in figure 1.22. You should notice fairly quickly that if you pick any two cards, the *SET* that contains them lies within the hyperplane. As you're looking, see if you can make some sense out of the positions that the cards in a *SET* occupy in the array. If you like looking for patterns, this should offer you lots of practice.[6] We'll count the number of *SET*s in a hyperplane in chapter 6 when we consider generalizations of the game.

Finally, the entire deck of SET cards is a four-dimensional geometry, consisting of 81 points and lots of lines, planes, and hyperplanes. This will be displayed in a rather striking way in chapter 5.

[6] You, as a human, are wired for pattern recognition. This game, and much of mathematics, is really an elaborate pattern recognition game.

Probability and Simulations

Most games have an element of chance, because luck helps to spice things up. SET is a game of skill, but the order the cards are dealt obviously introduces uncertainty into the game. How the game unfolds also depends on which *SET*s are taken along the way, which introduces a second level of chance.

Here are a few more questions that might occur to you.

- Suppose three cards are chosen at random. What are the chances they form a *SET*?

Many probability problems are just counting problems in disguise. We'll describe two different ways to do this in chapter 3.

- Why does the game begin with 12 cards?

Well, those are the rules. But it's worth figuring out why 12 is the "right" number for playing the game. We will do so in chapter 3, when we calculate the expected number of *SET*s among 12 randomly chosen cards.

The next few probability questions are frequently asked by people who have played the game a lot. Unfortunately, they seem to be quite difficult to answer precisely. But it's possible to estimate the answers by playing the game millions of times.[7] We give the results of some simulations in chapter 10.

- What is the probability that there are no *SET*s in the initial layout of 12 cards?

Simulations indicate that this happens approximately 3.2% of the time.

- What is the probability that there are no cards left at the end?

Simulations suggest that this happens even more infrequently, approximately 1.2% of the time.

- Suppose you have a shuffled deck in your hands. Is it always possible that you could take *SET*s in such a way that there are no cards left at the end?

[7] Better yet, ask a computer to do this. Ask politely.

TABLE 1.1.
Assignment of coordinates to cards.

Attribute	*Value*		*Coordinate*
Number	3, 1, 2	↔	0, 1, 2
Color	green, purple, red	↔	0, 1, 2
Shading	empty, striped, solid	↔	0, 1, 2
Shape	diamonds, ovals, squiggles	↔	0, 1, 2

In playing the game many times, we can backtrack (changing the game by taking a different *SET* earlier) to get a different number of cards at the end. In fact, people who have played online games where multiple people play the same deck may have noticed that different games (with the same deck) end with different numbers of cards left on the table. Does every deck have a way to clear it? We give an answer in chapter 10.[8]

Coordinates and Modular Arithmetic

The game of SET is intimately tied to the number 3: there are 3 cards in a *SET*, $3^2 = 9$ cards in a plane, $3^3 = 27$ cards in a hyperplane, and $3^4 = 81$ cards in the deck. This connection is best understood using coordinates, where each card will have its number, color, shading, and shape specified. Converting those attributes to numbers will allow us to perform arithmetic on the deck.

We will need to encode each card as an ordered list of four numbers. We make an (arbitrary) choice in table 1.1.

Using this setup,[9] the card consisting of 3 Purple Empty Squiggles will be represented by the coordinates $(0, 1, 0, 2)$. Which card is represented by (0, 0, 0, 0)? It's 3 Green Empty Diamonds, which is not special in any way. This illustrates the arbitrary nature of this process, but we will stick to these assignments throughout this book.

[8] Skip ahead, and you'll finish the book rather quickly.

[9] If you are curious, we ordered color and shape alphabetically. That's the kind of people we are.

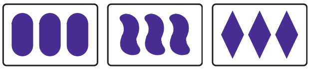


Figure 1.23. A *SET*.

Modular arithmetic is sometimes called *clock* arithmetic. Here's a standard problem that you may have seen before:

- It's currently 10:30 a.m. What time will it be in 100 hours?

Here's a solution. (Avert your eyes to do this yourself!) First, let's pretend that it's 10:00 a.m. (we'll add in the half hour at the end of the problem). We know 10:00 is 10 hours after midnight. In 100 hours, it will be 110 hours after (that same) midnight. Then divide by 24 and compute the remainder: since $110 = 4 \times 24 + 14$, we know it's now 14 hours after midnight (4 days later). We will write $110 = 14 \pmod{24}$.[10] So the time will be 2:30 p.m. (Alternatively, adding $100 = 4 \times 24 + 4$ hours adds 4 days and 4 hours to the current time. This also uses remainders after division by 24.)

In the clock problem, we are working mod 24 since there are 24 hours in a day. The key step to force our final answer to be a time between 0 and 23 is to first divide by 24, then find the remainder. Modular arithmetic concentrates solely on remainders.

Here's how modular arithmetic, specifically mod 3, is useful to the game of SET. Choose your favorite *SET*, which might be the one shown in figure 1.23.

What are the coordinates for the cards in this *SET*? Using our assignments from table 1.1, we get (0, 1, 2, 1), (0, 1, 2, 2), and (0, 1, 2, 0), from left to right. What happens when we add these coordinates one at a time?

1. Adding the first coordinates (which correspond to the number of attributes on the card) gives us $0 + 0 + 0 = 0 \pmod{3}$.
2. Adding the second coordinates (which correspond to the color of the card) gives us $1 + 1 + 1 = 3 = 0 \pmod{3}$, since 3 has remainder 0 when you divide by 3.

[10] Math books usually use an "equals" sign with three bars here: $100 \equiv 4 \pmod{24}$. For now, we won't.

Figure 1.24. Not a *SET*.

3. Adding the third coordinates (the shading attribute) gives $2 + 2 + 2 = 6 = 0 \pmod 3$, since the remainder when you divide 6 by 3 is also 0.
4. Adding the fourth coordinates (shape attribute) gives you $1 + 2 + 0 = 3 = 0 \pmod 3$.

So each sum is $0 \pmod 3$, and we get that the sum of the three cards is just $(0, 0, 0, 0) \pmod 3$.

What happens if we do this for three cards that are not a *SET*? Try this yourself for the three cards shown in figure 1.24.

What's the takeaway message from these two examples? It's the following striking result:

***Takeaway Message*:**

- Suppose A, B, and C are the vectors for three cards that form a *SET*. Then $A + B + C = (0, 0, 0, 0) \pmod 3$.
- Conversely, suppose A, B, and C are the vectors for three cards that do not form a *SET*. Then $A + B + C \neq (0, 0, 0, 0) \pmod 3$.

This is true *regardless* of how we assign our coordinates, as long as we are consistent (and use the numbers 0, 1, and 2, and work mod 3). Modular arithmetic will be very useful for us throughout the book.

We have one final comment about the power of modular arithmetic. Why are the cards in figure 1.24 not a *SET*? The problem is shading: two cards are empty, but one is solid. The sum of those coordinates is $(0, 0, 2, 0) \pmod 3$, and the nonzero coordinate occurs in the third spot, which corresponds to shading. This will connect SET with error-correcting codes in chapter 8.

Advanced Topics

The second half of the book is devoted to more advanced topics. Here is a (very) brief overview. *Affine geometry* is an important area of mathematics, and many of its classical theorems have interpretations through the game. Some of this will involve extending the game by adding more attributes:

- SET has four attributes: number, color, shading, and shape. What if we add more attributes to the game?

One way to play a five-attribute version of the game is to buy three decks, and mark each card in one of the decks with polka dots and each card in another deck with stripes, for example.[11] But adding attributes is an easy thing to do abstractly.

- Suppose there are $n > 4$ attributes. How many *SET*s are there? How many planes? Higher-dimensional hyperplanes?

We answer these questions in chapter 6. Considering more than four attributes will lead to general formulas, and those formulas have connections to classical counting problems.

When there are n attributes, there are n different kinds of *SET*s: all attributes different, all but one attribute different, and so on.

- How many *SET*s of each kind are there?

This is not too difficult to calculate exactly, and we can figure out which kind of *SET* is most common, and which is least common. We answer these questions in chapters 6 and 7.

Finally, there are famous unsolved problems that we can interpret in terms of the n-attribute game:

- In n-attribute SET, what is the maximum number of cards you can have with no *SET*s?

This number is known when $n \leq 6$ (at present), but not for any larger values. This question is the focus of some very high powered research, and the problem has attracted the interest of some of the top mathematicians in the world. We explore this question in chapter 9.

We love this game and its mathematics, and we hope this book motivates you to think about SET (and other games) from a mathematical perspective. Like everything in math (and the rest of life), you will understand things best by working out the details yourself.

[11] Warning: Actually playing this five-attribute game will give you a headache. In the head.

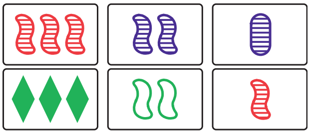


Figure 1.25. Exercise 1.2.

The exercises at the end of each chapter give you a chance to explore, on your own, some of the ideas we've introduced. Some are designed for practice and some lead to deeper topics that we'll return to later in the book. Enjoy!

EXERCISES

Exercise 1.1. The people who manufacture SET use a shorthand notation when they encode specific cards of the game as pdf files. For instance, the card with 3 Red Empty Squiggles is abbreviated 3ROS. (For some reason, they use "O" for "open" instead of "E" for "empty.") With this shorthand scheme, what card has a code that forms the basis of many Western religions?

Exercise 1.2. Suppose the six cards in figure 1.25 are left at the end of the game.

Here's a Stupid SET Trick:[12]

- Arbitrarily break up the six cards into three pairs; for example, you could make the pairs AB, CD, and EF.
- Figure out the three cards X, Y, and Z that complete these three pairs to make three *SET*s (so ABX and CDY and EFZ are all *SET*s).
- Then XYZ is a *SET*!

Try this for different ways of breaking up the six cards into three pairs (there are 15 different ways to pair them up, but you don't need to try this for all of them). [We'll see why this works in chapter 4.]

Exercise 1.3. There are three cards in figure 1.26. Add six cards to complete a plane.

[12] It's actually not stupid, it's great. Back in the day, David Letterman hosted a late night talk show that occasionally had Stupid Pet Tricks, so we borrowed the title.

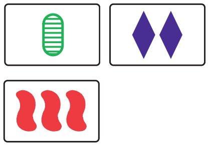


Figure 1.26. Exercise 1.3.

Figure 1.27. Build a SET ladder from Emily's *SET* to Tanya's, in order from left to right.

Exercise 1.4. Find a collection of 12 cards that contain 14 different *SET*s. [Hint: Start with 9 cards that form a plane, as in figure 1.19. This will reappear as part of project 5.1.]

Exercise 1.5. The 27 red cards in the deck form a hyperplane. How many *SET*s are there? [We'll return to this question and some generalizations in chapter 6.]

Exercise 1.6. *Word ladders* are games where you transform one word to another by changing one letter at a time. A standard example changes the word COLD to WARM in four steps:

$$\text{COLD} \to \text{CORD} \to \text{CARD} \to \text{WARD} \to \text{WARM}.$$

A *SET ladder* connects one *SET* to another, changing one attribute at a time. For SET, however, we'll insist that exactly one card stays the same at each step. See figure 1.11 for an example, where the *SET* at the top of the figure is changed to the *SET* at the bottom by first changing color, then shading, then shape.

a. Find a SET ladder joining the two *SET*s in figure 1.27. (The first is Emily's first *SET* from section 1.1, and the second is Tanya's.)
b. Suppose you want your favorite *SET* to be the first *SET* in a ladder. How many different *SET*s can be the next *SET* in the ladder?
c. What is the largest number of steps a SET ladder can need? Give an example of two *SET*s that achieve this maximum.
d. Change the rules! You can change the rules in any way you like, and then ask the same questions as above. For example, you could allow a color change without requiring that one card stay the same, so an all-red *SET*

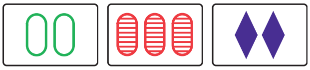


Figure 1.28. Exercise 1.7.

could become an all-green *SET*, or a *SET* with all colors different could have the colors cycle through the cards. Another choice might be to allow a rearrangement of the cards. Armed with your new rules, find a *SET* ladder between the two *SET*s in figure 1.27. How many *SET*s could be the next *SET* for a particular *SET*? What's the longest distance between two *SET*s?

Exercise 1.7. There is a non-*SET* in figure 1.28.

a. Which attribute (or attributes) are wrong for this non-*SET*? (An attribute is "wrong" if you can say "two are one thing, while one isn't.")
b. Find the coordinates for the three cards.
c. Add the coordinates for those cards, mod 3, and call the result X. Find the coordinate positions of X that are not 0. What is the connection between those positions and the attribute (or attributes) that are wrong?
d. Do you think that it's likely that these three cards could mistakenly be taken as a *SET* during play of the game? Explain.

Exercise 1.8. The three cards of figure 1.28 are not a *SET*. Call these three cards A, B, and C (in left-to-right order).

a. Replace the first card A in this non-*SET* (2 Green Empty Ovals) with a card D so that BCD forms a *SET*. Find the number of attributes the cards A and D differ in.
b. Now repeat part (a) for the second card B (finding a card E so that ACE is a *SET*) and the last card C (finding a card F with ABF a *SET*). How many attributes do B and E differ in? How about C and F?
c. True/False:

   i. The 3 pairs AD, BE, and CF all differ in the same number of attributes.
   ii. The cards D, E, and F form a *SET*.

Answers to End Game questions

- Figure 1.13: The missing card is 2 Purple Striped Diamonds, and there are two *SET*s that can be formed using this card and the remaining cards on the table.
- Figure 1.14: The missing card is 1 Red Striped Diamond, and there are no *SET*s that can be formed using this card and those on the table.