The Laplace Transform of step functions (Sect. 6.3).

- Overview and notation.
- The definition of a step function.
- Piecewise discontinuous functions.
- The Laplace Transform of discontinuous functions.
- Properties of the Laplace Transform.

Overview and notation.

Overview: The Laplace Transform method can be used to solve constant coefficients differential equations with *discontinuous source functions*.

Overview and notation.

Overview: The Laplace Transform method can be used to solve constant coefficients differential equations with *discontinuous source functions*.

Notation:
If $\mathcal{L}[f(t)] = F(s)$, then we denote $\mathcal{L}^{-1}[F(s)] = f(t)$.

Overview and notation.

Overview: The Laplace Transform method can be used to solve constant coefficients differential equations with *discontinuous source functions*.

Notation:
If $\mathcal{L}[f(t)] = F(s)$, then we denote $\mathcal{L}^{-1}[F(s)] = f(t)$.

Remark: One can show that for a particular type of functions f, that includes all functions we work with in this Section, the notation above is well-defined.

Overview and notation.

Overview: The Laplace Transform method can be used to solve constant coefficients differential equations with *discontinuous source functions*.

Notation:
If $\mathcal{L}[f(t)] = F(s)$, then we denote $\mathcal{L}^{-1}[F(s)] = f(t)$.

Remark: One can show that for a particular type of functions f, that includes all functions we work with in this Section, the notation above is well-defined.

Example

From the Laplace Transform table we know that $\mathcal{L}\left[e^{at}\right] = \dfrac{1}{s-a}$.

Overview and notation.

Overview: The Laplace Transform method can be used to solve constant coefficients differential equations with *discontinuous source functions*.

Notation:
If $\mathcal{L}[f(t)] = F(s)$, then we denote $\mathcal{L}^{-1}[F(s)] = f(t)$.

Remark: One can show that for a particular type of functions f, that includes all functions we work with in this Section, the notation above is well-defined.

Example
From the Laplace Transform table we know that $\mathcal{L}\big[e^{at}\big] = \dfrac{1}{s-a}$.

Then also holds that $\mathcal{L}^{-1}\Big[\dfrac{1}{s-a}\Big] = e^{at}$. $\triangleleft$

The Laplace Transform of step functions (Sect. 6.3).

- Overview and notation.
- **The definition of a step function.**
- Piecewise discontinuous functions.
- The Laplace Transform of discontinuous functions.
- Properties of the Laplace Transform.

The definition of a step function.

Definition

A function u is called a *step function* at $t = 0$ iff holds

$$u(t) = \begin{cases} 0 & \text{for } t < 0, \\ 1 & \text{for } t \geqslant 0. \end{cases}$$

The definition of a step function.

Definition

A function u is called a *step function* at $t = 0$ iff holds

$$u(t) = \begin{cases} 0 & \text{for } t < 0, \\ 1 & \text{for } t \geqslant 0. \end{cases}$$

Example

Graph the step function values $u(t)$ above, and the translations $u(t - c)$ and $u(t + c)$ with $c > 0$.

The definition of a step function.

Definition

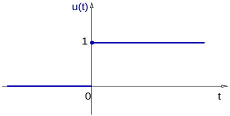
A function u is called a *step function* at $t = 0$ iff holds

$$u(t) = \begin{cases} 0 & \text{for } t < 0, \\ 1 & \text{for } t \geqslant 0. \end{cases}$$

Example

Graph the step function values $u(t)$ above, and the translations $u(t - c)$ and $u(t + c)$ with $c > 0$.

Solution:



The definition of a step function.

Definition

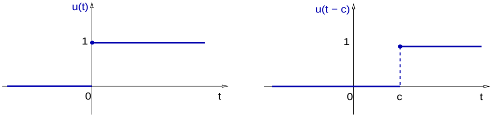
A function u is called a *step function* at $t = 0$ iff holds

$$u(t) = \begin{cases} 0 & \text{for } t < 0, \\ 1 & \text{for } t \geqslant 0. \end{cases}$$

Example

Graph the step function values $u(t)$ above, and the translations $u(t-c)$ and $u(t+c)$ with $c > 0$.

Solution:



The definition of a step function.

Definition

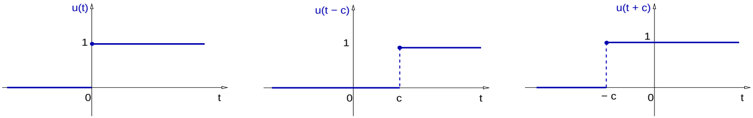
A function u is called a *step function* at $t = 0$ iff holds

$$u(t) = \begin{cases} 0 & \text{for } t < 0, \\ 1 & \text{for } t \geqslant 0. \end{cases}$$

Example

Graph the step function values $u(t)$ above, and the translations $u(t-c)$ and $u(t+c)$ with $c > 0$.

Solution:



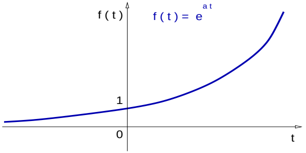
The definition of a step function.

Remark: Given any function values $f(t)$ and $c > 0$, then $f(t - c)$ is a right translation of f and $f(t + c)$ is a left translation of f.

The definition of a step function.

Remark: Given any function values $f(t)$ and $c > 0$, then $f(t - c)$ is a right translation of f and $f(t + c)$ is a left translation of f.

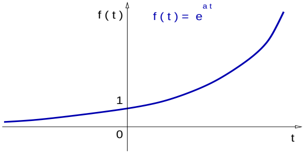
Example

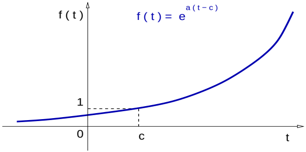


The definition of a step function.

Remark: Given any function values $f(t)$ and $c > 0$, then $f(t-c)$ is a right translation of f and $f(t+c)$ is a left translation of f.

Example

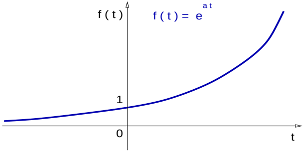


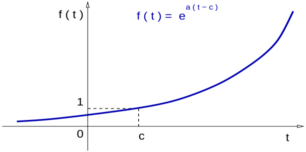


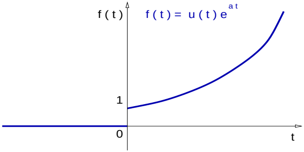
The definition of a step function.

Remark: Given any function values $f(t)$ and $c > 0$, then $f(t - c)$ is a right translation of f and $f(t + c)$ is a left translation of f.

Example



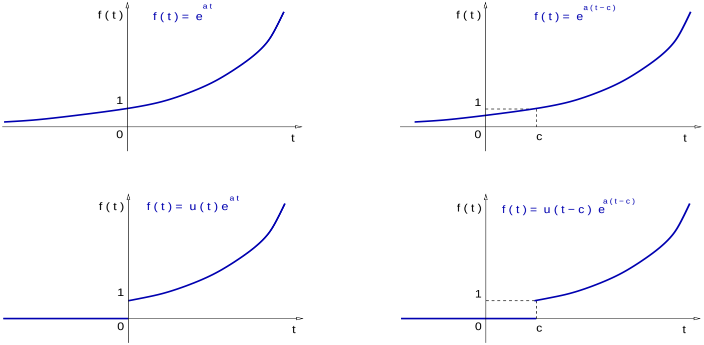




The definition of a step function.

Remark: Given any function values $f(t)$ and $c > 0$, then $f(t-c)$ is a right translation of f and $f(t+c)$ is a left translation of f.

Example



The Laplace Transform of step functions (Sect. 6.3).

- Overview and notation.
- The definition of a step function.
- **Piecewise discontinuous functions.**
- The Laplace Transform of discontinuous functions.
- Properties of the Laplace Transform.

Piecewise discontinuous functions.

Example

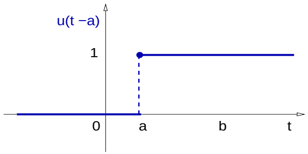
Graph of the function $b(t) = u(t - a) - u(t - b)$, with $0 < a < b$.

Piecewise discontinuous functions.

Example

Graph of the function $b(t) = u(t-a) - u(t-b)$, with $0 < a < b$.

Solution: The bump function b can be graphed as follows:

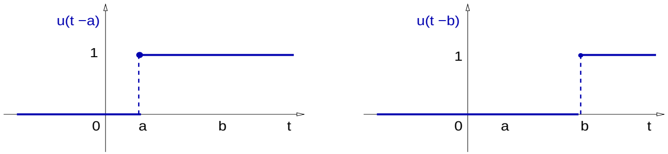


Piecewise discontinuous functions.

Example

Graph of the function $b(t) = u(t-a) - u(t-b)$, with $0 < a < b$.

Solution: The bump function b can be graphed as follows:

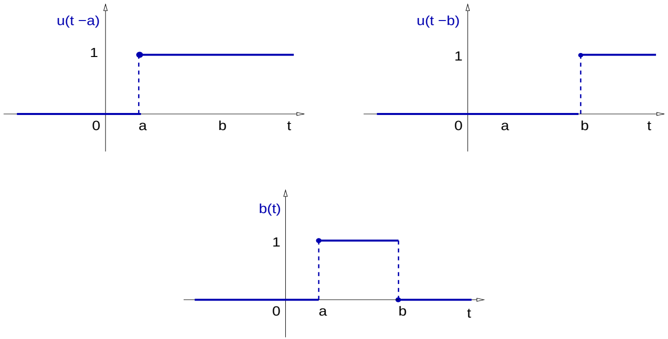


Piecewise discontinuous functions.

Example

Graph of the function $b(t) = u(t-a) - u(t-b)$, with $0 < a < b$.

Solution: The bump function b can be graphed as follows:



◁

Piecewise discontinuous functions.

Example

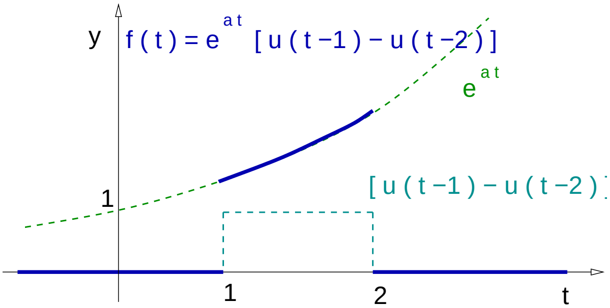
Graph of the function $f(t) = e^{at}\left[u(t-1) - u(t-2)\right]$.

Piecewise discontinuous functions.

Example

Graph of the function $f(t) = e^{at}\left[u(t-1) - u(t-2)\right]$.

Solution:

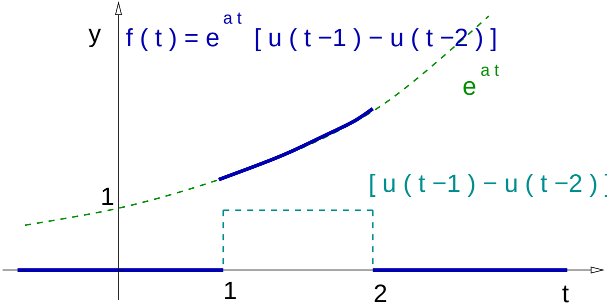


Piecewise discontinuous functions.

Example

Graph of the function $f(t) = e^{at}\left[u(t-1) - u(t-2)\right]$.

Solution:



Notation: The function values $u(t-c)$ are denoted in the textbook as $u_c(t)$.

The Laplace Transform of step functions (Sect. 6.3).

- Overview and notation.
- The definition of a step function.
- Piecewise discontinuous functions.
- **The Laplace Transform of discontinuous functions.**
- Properties of the Laplace Transform.

The Laplace Transform of discontinuous functions.

Theorem

Given any real number $c \geqslant 0$, the following equation holds,

$$\mathcal{L}[u(t-c)] = \frac{e^{-cs}}{s}, \qquad s > 0.$$

The Laplace Transform of discontinuous functions.

Theorem

Given any real number $c \geqslant 0$, the following equation holds,

$$\mathcal{L}[u(t-c)] = \frac{e^{-cs}}{s}, \qquad s > 0.$$

Proof:

$$\mathcal{L}[u(t-c)] = \int_0^\infty e^{-st} u(t-c)\, dt$$

The Laplace Transform of discontinuous functions.

Theorem

Given any real number $c \geqslant 0$, the following equation holds,

$$\mathcal{L}[u(t-c)] = \frac{e^{-cs}}{s}, \qquad s > 0.$$

Proof:

$$\mathcal{L}[u(t-c)] = \int_0^\infty e^{-st} u(t-c)\,dt = \int_c^\infty e^{-st}\,dt,$$

The Laplace Transform of discontinuous functions.

Theorem

Given any real number $c \geqslant 0$, the following equation holds,

$$\mathcal{L}[u(t-c)] = \frac{e^{-cs}}{s}, \qquad s > 0.$$

Proof:

$$\mathcal{L}[u(t-c)] = \int_0^\infty e^{-st} u(t-c)\, dt = \int_c^\infty e^{-st}\, dt,$$

$$\mathcal{L}[u(t-c)] = \lim_{N\to\infty} -\frac{1}{s}\left(e^{-Ns} - e^{-cs}\right)$$

The Laplace Transform of discontinuous functions.

Theorem

Given any real number $c \geqslant 0$, the following equation holds,

$$\mathcal{L}[u(t-c)] = \frac{e^{-cs}}{s}, \qquad s > 0.$$

Proof:

$$\mathcal{L}[u(t-c)] = \int_0^\infty e^{-st} u(t-c)\, dt = \int_c^\infty e^{-st}\, dt,$$

$$\mathcal{L}[u(t-c)] = \lim_{N\to\infty} -\frac{1}{s}\left(e^{-Ns} - e^{-cs}\right) = \frac{e^{-cs}}{s}, \quad s > 0.$$

We conclude that $\mathcal{L}[u(t-c)] = \dfrac{e^{-cs}}{s}$. □

The Laplace Transform of discontinuous functions.

Example

Compute $\mathcal{L}[3u(t-2)]$.

The Laplace Transform of discontinuous functions.

Example

Compute $\mathcal{L}[3u(t-2)]$.

Solution: $\mathcal{L}[3u(t-2)] = 3\,\mathcal{L}[u(t-2)]$

The Laplace Transform of discontinuous functions.

Example

Compute $\mathcal{L}[3u(t-2)]$.

Solution: $\mathcal{L}[3u(t-2)] = 3\,\mathcal{L}[u(t-2)] = 3\,\frac{e^{-2s}}{s}.$

The Laplace Transform of discontinuous functions.

Example

Compute $\mathcal{L}[3u(t-2)]$.

Solution: $$\mathcal{L}[3u(t-2)] = 3\,\mathcal{L}[u(t-2)] = 3\,\frac{e^{-2s}}{s}.$$

We conclude: $\mathcal{L}[3u(t-2)] = \dfrac{3e^{-2s}}{s}$. $\lhd$

The Laplace Transform of discontinuous functions.

Example

Compute $\mathcal{L}[3u(t-2)]$.

Solution: $\mathcal{L}[3u(t-2)] = 3\,\mathcal{L}[u(t-2)] = 3\,\dfrac{e^{-2s}}{s}$.

We conclude: $\mathcal{L}[3u(t-2)] = \dfrac{3e^{-2s}}{s}$. $\lhd$

Example

Compute $\mathcal{L}^{-1}\Big[\dfrac{e^{-3s}}{s}\Big]$.

The Laplace Transform of discontinuous functions.

Example

Compute $\mathcal{L}[3u(t-2)]$.

Solution: $\mathcal{L}[3u(t-2)] = 3\,\mathcal{L}[u(t-2)] = 3\,\dfrac{e^{-2s}}{s}$.

We conclude: $\mathcal{L}[3u(t-2)] = \dfrac{3e^{-2s}}{s}$. ◁

Example

Compute $\mathcal{L}^{-1}\left[\dfrac{e^{-3s}}{s}\right]$.

Solution: $\mathcal{L}^{-1}\left[\dfrac{e^{-3s}}{s}\right]$

The Laplace Transform of discontinuous functions.

Example

Compute $\mathcal{L}[3u(t-2)]$.

Solution: $\mathcal{L}[3u(t-2)] = 3\,\mathcal{L}[u(t-2)] = 3\,\dfrac{e^{-2s}}{s}$.

We conclude: $\mathcal{L}[3u(t-2)] = \dfrac{3e^{-2s}}{s}$. ◁

Example

Compute $\mathcal{L}^{-1}\Big[\dfrac{e^{-3s}}{s}\Big]$.

Solution: $\mathcal{L}^{-1}\Big[\dfrac{e^{-3s}}{s}\Big] = u(t-3)$.

The Laplace Transform of discontinuous functions.

Example

Compute $\mathcal{L}[3u(t-2)]$.

Solution: $\mathcal{L}[3u(t-2)] = 3\,\mathcal{L}[u(t-2)] = 3\,\dfrac{e^{-2s}}{s}$.

We conclude: $\mathcal{L}[3u(t-2)] = \dfrac{3e^{-2s}}{s}$. ◁

Example

Compute $\mathcal{L}^{-1}\left[\dfrac{e^{-3s}}{s}\right]$.

Solution: $\mathcal{L}^{-1}\left[\dfrac{e^{-3s}}{s}\right] = u(t-3)$.

We conclude: $\mathcal{L}^{-1}\left[\dfrac{e^{-3s}}{s}\right] = u(t-3)$. ◁

The Laplace Transform of step functions (Sect. 6.3).

- Overview and notation.
- The definition of a step function.
- Piecewise discontinuous functions.
- The Laplace Transform of discontinuous functions.
- **Properties of the Laplace Transform.**

Properties of the Laplace Transform.

Theorem (Translations)

If $F(s) = \mathcal{L}[f(t)]$ *exists for* $s > a \geqslant 0$ *and* $c \geqslant 0$*, then holds*

$$\mathcal{L}[u(t-c)f(t-c)] = e^{-cs} F(s), \qquad s > a.$$

Furthermore,

$$\mathcal{L}[e^{ct} f(t)] = F(s-c), \qquad s > a + c.$$

Properties of the Laplace Transform.

Theorem (Translations)

If $F(s) = \mathcal{L}[f(t)]$ *exists for* $s > a \geqslant 0$ *and* $c \geqslant 0$, *then holds*

$$\mathcal{L}[u(t-c)f(t-c)] = e^{-cs}F(s), \qquad s > a.$$

Furthermore,

$$\mathcal{L}[e^{ct}f(t)] = F(s-c), \qquad s > a + c.$$

Remark:

- $\mathcal{L}\big[\text{translation } (uf)\big] = (\exp)\,\big(\mathcal{L}[f]\big).$

Properties of the Laplace Transform.

Theorem (Translations)

If $F(s) = \mathcal{L}[f(t)]$ *exists for* $s > a \geqslant 0$ *and* $c \geqslant 0$, *then holds*

$$\mathcal{L}[u(t-c)f(t-c)] = e^{-cs}\,F(s), \qquad s > a.$$

Furthermore,

$$\mathcal{L}[e^{ct}f(t)] = F(s-c), \qquad s > a + c.$$

Remark:

- $\mathcal{L}\big[\text{translation }(uf)\big] = (\exp)\,\big(\mathcal{L}[f]\big)$.
- $\mathcal{L}\big[(\exp)\,(f)\big] = \text{translation}\big(\mathcal{L}[f]\big)$.

Properties of the Laplace Transform.

Theorem (Translations)

If $F(s) = \mathcal{L}[f(t)]$ *exists for* $s > a \geqslant 0$ *and* $c \geqslant 0$*, then holds*

$$\mathcal{L}[u(t-c)f(t-c)] = e^{-cs}\,F(s), \qquad s > a.$$

Furthermore,

$$\mathcal{L}[e^{ct}f(t)] = F(s-c), \qquad s > a + c.$$

Remark:

- $\mathcal{L}\big[\text{translation}\ (uf)\big] = (\exp)\,\big(\mathcal{L}[f]\big)$.
- $\mathcal{L}\big[(\exp)\,(f)\big] = \text{translation}\big(\mathcal{L}[f]\big)$.

Equivalent notation:

- $\mathcal{L}[u(t-c)f(t-c)] = e^{-cs}\,\mathcal{L}[f(t)]$,

Properties of the Laplace Transform.

Theorem (Translations)

If $F(s) = \mathcal{L}[f(t)]$ *exists for* $s > a \geqslant 0$ *and* $c \geqslant 0$, *then holds*

$$\mathcal{L}[u(t-c)f(t-c)] = e^{-cs}\,F(s), \qquad s > a.$$

Furthermore,

$$\mathcal{L}[e^{ct}f(t)] = F(s-c), \qquad s > a + c.$$

Remark:

- $\mathcal{L}\big[\text{translation }(uf)\big] = (\exp)\,\big(\mathcal{L}[f]\big)$.
- $\mathcal{L}\big[(\exp)\,(f)\big] = \text{translation}\big(\mathcal{L}[f]\big)$.

Equivalent notation:

- $\mathcal{L}[u(t-c)f(t-c)] = e^{-cs}\,\mathcal{L}[f(t)]$,
- $\mathcal{L}[e^{ct}f(t)] = \mathcal{L}[f](s-c)$.

Properties of the Laplace Transform.

Example

Compute $\mathcal{L}\big[u(t-2)\,\sin(a(t-2))\big]$.

Properties of the Laplace Transform.

Example

Compute $\mathcal{L}\big[u(t-2)\,\sin(a(t-2))\big]$.

Solution: $\mathcal{L}[\sin(at)] = \dfrac{a}{s^2+a^2}$,

Properties of the Laplace Transform.

Example

Compute $\mathcal{L}\big[u(t-2)\,\sin(a(t-2))\big]$.

Solution: $\mathcal{L}[\sin(at)] = \dfrac{a}{s^2+a^2}$, $\mathcal{L}[u(t-c)f(t-c)] = e^{-cs}\,\mathcal{L}[f(t)]$.

Properties of the Laplace Transform.

Example

Compute $\mathcal{L}\big[u(t-2)\,\sin(a(t-2))\big]$.

Solution: $\mathcal{L}[\sin(at)] = \dfrac{a}{s^2+a^2}$, $\mathcal{L}[u(t-c)f(t-c)] = e^{-cs}\,\mathcal{L}[f(t)]$.

$$\mathcal{L}\big[u(t-2)\,\sin(a(t-2))\big] = e^{-2s}\,\mathcal{L}[\sin(at)]$$

Properties of the Laplace Transform.

Example

Compute $\mathcal{L}\big[u(t-2)\,\sin(a(t-2))\big]$.

Solution: $\mathcal{L}[\sin(at)] = \dfrac{a}{s^2+a^2}$, $\mathcal{L}[u(t-c)f(t-c)] = e^{-cs}\,\mathcal{L}[f(t)]$.

$$\mathcal{L}\big[u(t-2)\,\sin(a(t-2))\big] = e^{-2s}\,\mathcal{L}[\sin(at)] = e^{-2s}\,\frac{a}{s^2+a^2}.$$

Properties of the Laplace Transform.

Example

Compute $\mathcal{L}\big[u(t-2)\,\sin(a(t-2))\big]$.

Solution: $\mathcal{L}[\sin(at)] = \dfrac{a}{s^2+a^2}$, $\mathcal{L}[u(t-c)f(t-c)] = e^{-cs}\,\mathcal{L}[f(t)]$.

$$\mathcal{L}\big[u(t-2)\,\sin(a(t-2))\big] = e^{-2s}\,\mathcal{L}[\sin(at)] = e^{-2s}\,\frac{a}{s^2+a^2}.$$

We conclude: $\mathcal{L}\big[u(t-2)\,\sin(a(t-2))\big] = e^{-2s}\,\dfrac{a}{s^2+a^2}$. $\lhd$

Properties of the Laplace Transform.

Example

Compute $\mathcal{L}\big[u(t-2)\,\sin(a(t-2))\big]$.

Solution: $\mathcal{L}[\sin(at)] = \dfrac{a}{s^2+a^2}$, $\mathcal{L}[u(t-c)f(t-c)] = e^{-cs}\,\mathcal{L}[f(t)]$.

$$\mathcal{L}\big[u(t-2)\,\sin(a(t-2))\big] = e^{-2s}\,\mathcal{L}[\sin(at)] = e^{-2s}\,\frac{a}{s^2+a^2}.$$

We conclude: $\mathcal{L}\big[u(t-2)\,\sin(a(t-2))\big] = e^{-2s}\,\dfrac{a}{s^2+a^2}$. $\triangleleft$

Example

Compute $\mathcal{L}\big[e^{3t}\,\sin(at)\big]$.

Properties of the Laplace Transform.

Example

Compute $\mathcal{L}\big[u(t-2)\,\sin(a(t-2))\big]$.

Solution: $\mathcal{L}[\sin(at)] = \dfrac{a}{s^2+a^2}$, $\mathcal{L}[u(t-c)f(t-c)] = e^{-cs}\,\mathcal{L}[f(t)]$.

$$\mathcal{L}\big[u(t-2)\,\sin(a(t-2))\big] = e^{-2s}\,\mathcal{L}[\sin(at)] = e^{-2s}\,\frac{a}{s^2+a^2}.$$

We conclude: $\mathcal{L}\big[u(t-2)\,\sin(a(t-2))\big] = e^{-2s}\,\dfrac{a}{s^2+a^2}$. $\triangleleft$

Example

Compute $\mathcal{L}\big[e^{3t}\,\sin(at)\big]$.

Solution: Recall: $\mathcal{L}[e^{ct}f(t)] = \mathcal{L}[f](s-c)$.

Properties of the Laplace Transform.

Example

Compute $\mathcal{L}\big[u(t-2)\,\sin(a(t-2))\big]$.

Solution: $\mathcal{L}[\sin(at)] = \dfrac{a}{s^2+a^2}$, $\mathcal{L}[u(t-c)f(t-c)] = e^{-cs}\,\mathcal{L}[f(t)]$.

$$\mathcal{L}\big[u(t-2)\,\sin(a(t-2))\big] = e^{-2s}\,\mathcal{L}[\sin(at)] = e^{-2s}\,\frac{a}{s^2+a^2}.$$

We conclude: $\mathcal{L}\big[u(t-2)\,\sin(a(t-2))\big] = e^{-2s}\,\dfrac{a}{s^2+a^2}$. $\lhd$

Example

Compute $\mathcal{L}\big[e^{3t}\,\sin(at)\big]$.

Solution: Recall: $\mathcal{L}[e^{ct}f(t)] = \mathcal{L}[f](s-c)$.

We conclude: $\mathcal{L}\big[e^{3t}\,\sin(at)\big] = \dfrac{a}{(s-3)^2+a^2}$, with $s > 3$. $\lhd$

Properties of the Laplace Transform.

Example

Find the Laplace transform of $f(t) = \begin{cases} 0, & t < 1, \\ (t^2 - 2t + 2), & t \geqslant 1. \end{cases}$

Properties of the Laplace Transform.

Example

Find the Laplace transform of $f(t) = \begin{cases} 0, & t < 1, \\ (t^2 - 2t + 2), & t \geqslant 1. \end{cases}$

Solution: Using step function notation,

Properties of the Laplace Transform.

Example

Find the Laplace transform of $f(t) = \begin{cases} 0, & t < 1, \\ (t^2 - 2t + 2), & t \geqslant 1. \end{cases}$

Solution: Using step function notation,

$$f(t) = u(t-1)(t^2 - 2t + 2).$$

Properties of the Laplace Transform.

Example

Find the Laplace transform of $f(t) = \begin{cases} 0, & t < 1, \\ (t^2 - 2t + 2), & t \geqslant 1. \end{cases}$

Solution: Using step function notation,

$$f(t) = u(t-1)(t^2 - 2t + 2).$$

Completing the square we obtain,

$$t^2 - 2t + 2 = (t^2 - 2t + 1) - 1 + 2$$

Properties of the Laplace Transform.

Example

Find the Laplace transform of $f(t) = \begin{cases} 0, & t < 1, \\ (t^2 - 2t + 2), & t \geqslant 1. \end{cases}$

Solution: Using step function notation,

$$f(t) = u(t-1)(t^2 - 2t + 2).$$

Completing the square we obtain,

$$t^2 - 2t + 2 = (t^2 - 2t + 1) - 1 + 2 = (t-1)^2 + 1.$$

Properties of the Laplace Transform.

Example

Find the Laplace transform of $f(t) = \begin{cases} 0, & t < 1, \\ (t^2 - 2t + 2), & t \geqslant 1. \end{cases}$

Solution: Using step function notation,

$$f(t) = u(t-1)(t^2 - 2t + 2).$$

Completing the square we obtain,

$$t^2 - 2t + 2 = (t^2 - 2t + 1) - 1 + 2 = (t-1)^2 + 1.$$

This is a parabola t^2 translated to the right by 1 and up by one. This is a discontinuous function.

Properties of the Laplace Transform.

Example

Find the Laplace transform of $f(t) = \begin{cases} 0, & t < 1, \\ (t^2 - 2t + 2), & t \geqslant 1. \end{cases}$

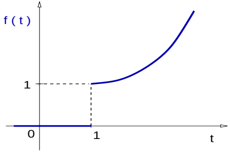
Solution: Using step function notation,

$$f(t) = u(t-1)(t^2 - 2t + 2).$$

Completing the square we obtain,

$$t^2 - 2t + 2 = (t^2 - 2t + 1) - 1 + 2 = (t-1)^2 + 1.$$

This is a parabola t^2 translated to the right by 1 and up by one. This is a discontinuous function.



Properties of the Laplace Transform.

Example

Find the Laplace transform of $f(t) = \begin{cases} 0, & t < 1, \\ (t^2 - 2t + 2), & t \geqslant 1. \end{cases}$

Solution: Recall: $f(t) = u(t-1)\left[(t-1)^2 + 1\right].$

Properties of the Laplace Transform.

Example

Find the Laplace transform of $f(t) = \begin{cases} 0, & t < 1, \\ (t^2 - 2t + 2), & t \geqslant 1. \end{cases}$

Solution: Recall: $f(t) = u(t-1)\left[(t-1)^2 + 1\right]$.

This is equivalent to

$$f(t) = u(t-1)\,(t-1)^2 + u(t-1).$$

Properties of the Laplace Transform.

Example

Find the Laplace transform of $f(t) = \begin{cases} 0, & t < 1, \\ (t^2 - 2t + 2), & t \geqslant 1. \end{cases}$

Solution: Recall: $f(t) = u(t-1)\left[(t-1)^2 + 1\right]$.

This is equivalent to

$$f(t) = u(t-1)\,(t-1)^2 + u(t-1).$$

Since $\mathcal{L}[t^2] = 2/s^3$,

Properties of the Laplace Transform.

Example

Find the Laplace transform of $f(t) = \begin{cases} 0, & t < 1, \\ (t^2 - 2t + 2), & t \geqslant 1. \end{cases}$

Solution: Recall: $f(t) = u(t-1)\left[(t-1)^2 + 1\right]$.

This is equivalent to

$$f(t) = u(t-1)\,(t-1)^2 + u(t-1).$$

Since $\mathcal{L}[t^2] = 2/s^3$, and $\mathcal{L}[u(t-c)g(t-c)] = e^{-cs}\,\mathcal{L}[g(t)]$,

Properties of the Laplace Transform.

Example

Find the Laplace transform of $f(t) = \begin{cases} 0, & t < 1, \\ (t^2 - 2t + 2), & t \geqslant 1. \end{cases}$

Solution: Recall: $f(t) = u(t-1)\left[(t-1)^2 + 1\right]$.

This is equivalent to

$$f(t) = u(t-1)\,(t-1)^2 + u(t-1).$$

Since $\mathcal{L}[t^2] = 2/s^3$, and $\mathcal{L}[u(t-c)g(t-c)] = e^{-cs}\,\mathcal{L}[g(t)]$, then

$$\mathcal{L}[f(t)] = \mathcal{L}[u(t-1)\,(t-1)^2] + \mathcal{L}[u(t-1)]$$

Properties of the Laplace Transform.

Example

Find the Laplace transform of $f(t) = \begin{cases} 0, & t < 1, \\ (t^2 - 2t + 2), & t \geqslant 1. \end{cases}$

Solution: Recall: $f(t) = u(t-1)\left[(t-1)^2 + 1\right]$.

This is equivalent to

$$f(t) = u(t-1)\,(t-1)^2 + u(t-1).$$

Since $\mathcal{L}[t^2] = 2/s^3$, and $\mathcal{L}[u(t-c)g(t-c)] = e^{-cs}\,\mathcal{L}[g(t)]$, then

$$\mathcal{L}[f(t)] = \mathcal{L}[u(t-1)\,(t-1)^2] + \mathcal{L}[u(t-1)] = e^{-s}\frac{2}{s^3} + e^{-s}\frac{1}{s}.$$

Properties of the Laplace Transform.

Example

Find the Laplace transform of $f(t) = \begin{cases} 0, & t < 1, \\ (t^2 - 2t + 2), & t \geqslant 1. \end{cases}$

Solution: Recall: $f(t) = u(t-1)\left[(t-1)^2 + 1\right]$.

This is equivalent to

$$f(t) = u(t-1)\,(t-1)^2 + u(t-1).$$

Since $\mathcal{L}[t^2] = 2/s^3$, and $\mathcal{L}[u(t-c)g(t-c)] = e^{-cs}\,\mathcal{L}[g(t)]$, then

$$\mathcal{L}[f(t)] = \mathcal{L}[u(t-1)\,(t-1)^2] + \mathcal{L}[u(t-1)] = e^{-s}\frac{2}{s^3} + e^{-s}\frac{1}{s}.$$

We conclude: $\mathcal{L}[f(t)] = \frac{e^{-s}}{s^3}\left(2 + s^2\right)$. $\triangleleft$

Properties of the Laplace Transform.

Remark: The inverse of the formulas in the Theorem above are:

$$\mathcal{L}^{-1}\big[e^{-cs}\,F(s)\big] = u(t-c)\,f(t-c),$$

Properties of the Laplace Transform.

Remark: The inverse of the formulas in the Theorem above are:

$$\mathcal{L}^{-1}\big[e^{-cs}\,F(s)\big] = u(t-c)\,f(t-c),$$

$$\mathcal{L}^{-1}\big[F(s-c)\big] = e^{ct}\,f(t).$$

Properties of the Laplace Transform.

Remark: The inverse of the formulas in the Theorem above are:

$$\mathcal{L}^{-1}\big[e^{-cs}\,F(s)\big] = u(t-c)\,f(t-c),$$

$$\mathcal{L}^{-1}\big[F(s-c)\big] = e^{ct}\,f(t).$$

Example

Find $\mathcal{L}^{-1}\Big[\dfrac{e^{-4s}}{s^2+9}\Big]$.

Properties of the Laplace Transform.

Remark: The inverse of the formulas in the Theorem above are:

$$\mathcal{L}^{-1}\big[e^{-cs}\,F(s)\big] = u(t-c)\,f(t-c),$$

$$\mathcal{L}^{-1}\big[F(s-c)\big] = e^{ct}\,f(t).$$

Example

Find $\mathcal{L}^{-1}\Big[\dfrac{e^{-4s}}{s^2+9}\Big]$.

Solution: $\mathcal{L}^{-1}\Big[\dfrac{e^{-4s}}{s^2+9}\Big] = \dfrac{1}{3}\,\mathcal{L}^{-1}\Big[e^{-4s}\,\dfrac{3}{s^2+9}\Big]$.

Properties of the Laplace Transform.

Remark: The inverse of the formulas in the Theorem above are:

$$\mathcal{L}^{-1}\big[e^{-cs}\,F(s)\big] = u(t-c)\,f(t-c),$$

$$\mathcal{L}^{-1}\big[F(s-c)\big] = e^{ct}\,f(t).$$

Example

Find $\mathcal{L}^{-1}\Big[\dfrac{e^{-4s}}{s^2+9}\Big]$.

Solution: $\mathcal{L}^{-1}\Big[\dfrac{e^{-4s}}{s^2+9}\Big] = \dfrac{1}{3}\,\mathcal{L}^{-1}\Big[e^{-4s}\,\dfrac{3}{s^2+9}\Big]$.

Recall: $\mathcal{L}^{-1}\Big[\dfrac{a}{s^2+a^2}\Big] = \sin(at)$.

Properties of the Laplace Transform.

Remark: The inverse of the formulas in the Theorem above are:

$$\mathcal{L}^{-1}\big[e^{-cs}\,F(s)\big] = u(t-c)\,f(t-c),$$

$$\mathcal{L}^{-1}\big[F(s-c)\big] = e^{ct}\,f(t).$$

Example

Find $\mathcal{L}^{-1}\Big[\dfrac{e^{-4s}}{s^2+9}\Big]$.

Solution: $\mathcal{L}^{-1}\Big[\dfrac{e^{-4s}}{s^2+9}\Big] = \dfrac{1}{3}\,\mathcal{L}^{-1}\Big[e^{-4s}\,\dfrac{3}{s^2+9}\Big]$.

Recall: $\mathcal{L}^{-1}\Big[\dfrac{a}{s^2+a^2}\Big] = \sin(at)$. Then, we conclude that

$$\mathcal{L}^{-1}\Big[\frac{e^{-4s}}{s^2+9}\Big] = \frac{1}{3}\,u(t-4)\,\sin\big(3(t-4)\big). \qquad \triangleleft$$

Properties of the Laplace Transform.

Example

Find $\mathcal{L}^{-1}\left[\dfrac{(s-2)}{(s-2)^2+9}\right]$.

Properties of the Laplace Transform.

Example

Find $\mathcal{L}^{-1}\left[\dfrac{(s-2)}{(s-2)^2+9}\right]$.

Solution: $\mathcal{L}^{-1}\left[\dfrac{s}{s^2+a^2}\right] = \cos(at)$,

Properties of the Laplace Transform.

Example

Find $\mathcal{L}^{-1}\left[\frac{(s-2)}{(s-2)^2+9}\right]$.

Solution: $\mathcal{L}^{-1}\left[\frac{s}{s^2+a^2}\right] = \cos(at)$, $\mathcal{L}^{-1}\left[F(s-c)\right] = e^{ct} f(t)$.

Properties of the Laplace Transform.

Example

Find $\mathcal{L}^{-1}\Big[\dfrac{(s-2)}{(s-2)^2+9}\Big]$.

Solution: $\mathcal{L}^{-1}\Big[\dfrac{s}{s^2+a^2}\Big] = \cos(at)$, $\mathcal{L}^{-1}\big[F(s-c)\big] = e^{ct}\,f(t)$.

We conclude: $\mathcal{L}^{-1}\Big[\dfrac{(s-2)}{(s-2)^2+9}\Big] = e^{2t}\cos(3t)$. $\triangleleft$

Properties of the Laplace Transform.

Example

Find $\mathcal{L}^{-1}\Big[\dfrac{(s-2)}{(s-2)^2+9}\Big]$.

Solution: $\mathcal{L}^{-1}\Big[\dfrac{s}{s^2+a^2}\Big] = \cos(at)$, $\mathcal{L}^{-1}\big[F(s-c)\big] = e^{ct} f(t)$.

We conclude: $\mathcal{L}^{-1}\Big[\dfrac{(s-2)}{(s-2)^2+9}\Big] = e^{2t}\cos(3t)$. $\triangleleft$

Example

Find $\mathcal{L}^{-1}\Big[\dfrac{2e^{-3s}}{s^2-4}\Big]$.

Properties of the Laplace Transform.

Example

Find $\mathcal{L}^{-1}\Big[\dfrac{(s-2)}{(s-2)^2+9}\Big]$.

Solution: $\mathcal{L}^{-1}\Big[\dfrac{s}{s^2+a^2}\Big] = \cos(at)$, $\mathcal{L}^{-1}\big[F(s-c)\big] = e^{ct}\, f(t)$.

We conclude: $\mathcal{L}^{-1}\Big[\dfrac{(s-2)}{(s-2)^2+9}\Big] = e^{2t}\,\cos(3t)$. $\triangleleft$

Example

Find $\mathcal{L}^{-1}\Big[\dfrac{2e^{-3s}}{s^2-4}\Big]$.

Solution: Recall: $\mathcal{L}^{-1}\Big[\dfrac{a}{s^2-a^2}\Big] = \sinh(at)$

Properties of the Laplace Transform.

Example

Find $\mathcal{L}^{-1}\left[\dfrac{(s-2)}{(s-2)^2+9}\right]$.

Solution: $\mathcal{L}^{-1}\left[\dfrac{s}{s^2+a^2}\right] = \cos(at)$, $\mathcal{L}^{-1}\big[F(s-c)\big] = e^{ct}\, f(t)$.

We conclude: $\mathcal{L}^{-1}\left[\dfrac{(s-2)}{(s-2)^2+9}\right] = e^{2t}\cos(3t)$. $\triangleleft$

Example

Find $\mathcal{L}^{-1}\left[\dfrac{2e^{-3s}}{s^2-4}\right]$.

Solution: Recall: $\mathcal{L}^{-1}\left[\dfrac{a}{s^2-a^2}\right] = \sinh(at)$

and $\mathcal{L}^{-1}\big[e^{-cs}\,F(s)\big] = u(t-c)\, f(t-c)$.

Properties of the Laplace Transform.

Example

Find $\mathcal{L}^{-1}\left[\dfrac{2e^{-3s}}{s^2-4}\right]$.

Solution: Recall:

$$\mathcal{L}^{-1}\left[\frac{a}{s^2-a^2}\right] = \sinh(at), \quad \mathcal{L}^{-1}\left[e^{-cs}\,F(s)\right] = u(t-c)\,f(t-c).$$

Properties of the Laplace Transform.

Example

Find $\mathcal{L}^{-1}\Big[\dfrac{2e^{-3s}}{s^2-4}\Big]$.

Solution: Recall:

$$\mathcal{L}^{-1}\Big[\frac{a}{s^2-a^2}\Big] = \sinh(at), \quad \mathcal{L}^{-1}\big[e^{-cs}\,F(s)\big] = u(t-c)\,f(t-c).$$

$$\mathcal{L}^{-1}\Big[\frac{2e^{-3s}}{s^2-4}\Big] = \mathcal{L}^{-1}\Big[e^{-3s}\,\frac{2}{s^2-4}\Big].$$

Properties of the Laplace Transform.

Example

Find $\mathcal{L}^{-1}\left[\dfrac{2e^{-3s}}{s^2-4}\right]$.

Solution: Recall:

$$\mathcal{L}^{-1}\left[\frac{a}{s^2-a^2}\right] = \sinh(at), \quad \mathcal{L}^{-1}\left[e^{-cs}\,F(s)\right] = u(t-c)\,f(t-c).$$

$$\mathcal{L}^{-1}\left[\frac{2e^{-3s}}{s^2-4}\right] = \mathcal{L}^{-1}\left[e^{-3s}\,\frac{2}{s^2-4}\right].$$

We conclude: $\mathcal{L}^{-1}\left[\dfrac{2e^{-3s}}{s^2-4}\right] = u(t-3)\,\sinh\big(2(t-3)\big)$. $\triangleleft$

Properties of the Laplace Transform.

Example

Find $\mathcal{L}^{-1}\left[\dfrac{e^{-2s}}{s^2 + s - 2}\right]$.

Properties of the Laplace Transform.

Example

Find $\mathcal{L}^{-1}\left[\dfrac{e^{-2s}}{s^2+s-2}\right]$.

Solution: Find the roots of the denominator:

$$s_{\pm} = \frac{1}{2}\left[-1 \pm \sqrt{1+8}\right]$$

Properties of the Laplace Transform.

Example

Find $\mathcal{L}^{-1}\Big[\dfrac{e^{-2s}}{s^2+s-2}\Big]$.

Solution: Find the roots of the denominator:

$$s_{\pm} = \frac{1}{2}\big[-1 \pm \sqrt{1+8}\big] \quad \Rightarrow \quad \begin{cases} s_+ = 1, \\ s_- = -2. \end{cases}$$

Properties of the Laplace Transform.

Example

Find $\mathcal{L}^{-1}\left[\dfrac{e^{-2s}}{s^2+s-2}\right]$.

Solution: Find the roots of the denominator:

$$s_\pm = \frac{1}{2}\left[-1 \pm \sqrt{1+8}\right] \quad \Rightarrow \quad \begin{cases} s_+ = 1, \\ s_- = -2. \end{cases}$$

Therefore, $s^2 + s - 2 = (s-1)\,(s+2)$.

Properties of the Laplace Transform.

Example

Find $\mathcal{L}^{-1}\Big[\dfrac{e^{-2s}}{s^2+s-2}\Big]$.

Solution: Find the roots of the denominator:

$$s_{\pm} = \frac{1}{2}\big[-1 \pm \sqrt{1+8}\big] \quad \Rightarrow \quad \begin{cases} s_+ = 1, \\ s_- = -2. \end{cases}$$

Therefore, $s^2 + s - 2 = (s-1)\,(s+2)$.

Use partial fractions to simplify the rational function:

$$\frac{1}{s^2+s-2} = \frac{1}{(s-1)\,(s+2)}$$

Properties of the Laplace Transform.

Example

Find $\mathcal{L}^{-1}\Big[\dfrac{e^{-2s}}{s^2+s-2}\Big]$.

Solution: Find the roots of the denominator:

$$s_{\pm} = \frac{1}{2}\big[-1 \pm \sqrt{1+8}\big] \quad \Rightarrow \quad \begin{cases} s_+ = 1, \\ s_- = -2. \end{cases}$$

Therefore, $s^2 + s - 2 = (s-1)\,(s+2)$.

Use partial fractions to simplify the rational function:

$$\frac{1}{s^2+s-2} = \frac{1}{(s-1)\,(s+2)} = \frac{a}{(s-1)} + \frac{b}{(s+2)},$$

Properties of the Laplace Transform.

Example

Find $\mathcal{L}^{-1}\Big[\dfrac{e^{-2s}}{s^2+s-2}\Big]$.

Solution: Find the roots of the denominator:

$$s_\pm = \frac{1}{2}\big[-1 \pm \sqrt{1+8}\big] \quad \Rightarrow \quad \begin{cases} s_+ = 1, \\ s_- = -2. \end{cases}$$

Therefore, $s^2+s-2=(s-1)\,(s+2)$.

Use partial fractions to simplify the rational function:

$$\frac{1}{s^2+s-2} = \frac{1}{(s-1)\,(s+2)} = \frac{a}{(s-1)} + \frac{b}{(s+2)},$$

$$\frac{1}{s^2+s-2} = a(s+2) + b(s-1)$$

Properties of the Laplace Transform.

Example

Find $\mathcal{L}^{-1}\Big[\frac{e^{-2s}}{s^2+s-2}\Big]$.

Solution: Find the roots of the denominator:

$$s_{\pm} = \frac{1}{2}\big[-1 \pm \sqrt{1+8}\big] \quad \Rightarrow \quad \begin{cases} s_+ = 1, \\ s_- = -2. \end{cases}$$

Therefore, $s^2+s-2 = (s-1)\,(s+2)$.

Use partial fractions to simplify the rational function:

$$\frac{1}{s^2+s-2} = \frac{1}{(s-1)\,(s+2)} = \frac{a}{(s-1)} + \frac{b}{(s+2)},$$

$$\frac{1}{s^2+s-2} = a(s+2) + b(s-1) = \frac{(a+b)\,s + (2a-b)}{(s-1)\,(s+2)}.$$

Properties of the Laplace Transform.

Example

Find $\mathcal{L}^{-1}\Big[\dfrac{e^{-2s}}{s^2+s-2}\Big]$.

Solution: Recall: $\dfrac{1}{s^2+s-2} = \dfrac{(a+b)\,s+(2a-b)}{(s-1)\,(s+2)}$

Properties of the Laplace Transform.

Example

Find $\mathcal{L}^{-1}\left[\dfrac{e^{-2s}}{s^2+s-2}\right]$.

Solution: Recall: $\dfrac{1}{s^2+s-2} = \dfrac{(a+b)\,s + (2a-b)}{(s-1)\,(s+2)}$

$$a + b = 0,$$

Properties of the Laplace Transform.

Example

Find $\mathcal{L}^{-1}\left[\dfrac{e^{-2s}}{s^2+s-2}\right]$.

Solution: Recall: $\dfrac{1}{s^2+s-2} = \dfrac{(a+b)\,s + (2a-b)}{(s-1)\,(s+2)}$

$$a + b = 0, \quad 2a - b = 1,$$

Properties of the Laplace Transform.

Example

Find $\mathcal{L}^{-1}\left[\dfrac{e^{-2s}}{s^2+s-2}\right]$.

Solution: Recall: $\dfrac{1}{s^2+s-2} = \dfrac{(a+b)\,s + (2a-b)}{(s-1)\,(s+2)}$

$$a+b=0, \quad 2a-b=1, \quad \Rightarrow \quad a=\frac{1}{3}, \quad b=-\frac{1}{3}.$$

Properties of the Laplace Transform.

Example

Find $\mathcal{L}^{-1}\Big[\dfrac{e^{-2s}}{s^2+s-2}\Big]$.

Solution: Recall: $\dfrac{1}{s^2+s-2} = \dfrac{(a+b)\,s + (2a-b)}{(s-1)\,(s+2)}$

$$a+b=0, \quad 2a-b=1, \quad \Rightarrow \quad a=\frac{1}{3}, \quad b=-\frac{1}{3}.$$

$$\mathcal{L}^{-1}\Big[\frac{e^{-2s}}{s^2+s-2}\Big] = \frac{1}{3}\,\mathcal{L}^{-1}\Big[e^{-2s}\,\frac{1}{s-1}\Big] - \frac{1}{3}\,\mathcal{L}^{-1}\Big[e^{-2s}\,\frac{1}{s+2}\Big].$$

Properties of the Laplace Transform.

Example

Find $\mathcal{L}^{-1}\Big[\dfrac{e^{-2s}}{s^2+s-2}\Big]$.

Solution: Recall: $\dfrac{1}{s^2+s-2} = \dfrac{(a+b)\,s + (2a-b)}{(s-1)\,(s+2)}$

$$a+b=0, \quad 2a-b=1, \quad \Rightarrow \quad a=\frac{1}{3}, \quad b=-\frac{1}{3}.$$

$$\mathcal{L}^{-1}\Big[\frac{e^{-2s}}{s^2+s-2}\Big] = \frac{1}{3}\,\mathcal{L}^{-1}\Big[e^{-2s}\,\frac{1}{s-1}\Big] - \frac{1}{3}\,\mathcal{L}^{-1}\Big[e^{-2s}\,\frac{1}{s+2}\Big].$$

Recall: $\mathcal{L}^{-1}\Big[\dfrac{1}{s-a}\Big] = e^{at}$,

Properties of the Laplace Transform.

Example

Find $\mathcal{L}^{-1}\Big[\dfrac{e^{-2s}}{s^2+s-2}\Big]$.

Solution: Recall: $\dfrac{1}{s^2+s-2} = \dfrac{(a+b)\,s + (2a-b)}{(s-1)\,(s+2)}$

$$a+b=0, \quad 2a-b=1, \quad \Rightarrow \quad a=\frac{1}{3}, \quad b=-\frac{1}{3}.$$

$$\mathcal{L}^{-1}\Big[\frac{e^{-2s}}{s^2+s-2}\Big] = \frac{1}{3}\,\mathcal{L}^{-1}\Big[e^{-2s}\,\frac{1}{s-1}\Big] - \frac{1}{3}\,\mathcal{L}^{-1}\Big[e^{-2s}\,\frac{1}{s+2}\Big].$$

Recall: $\mathcal{L}^{-1}\Big[\dfrac{1}{s-a}\Big] = e^{at}$, $\mathcal{L}^{-1}\big[e^{-cs}\,F(s)\big] = u(t-c)\,f(t-c)$,

Properties of the Laplace Transform.

Example

Find $\mathcal{L}^{-1}\Big[\dfrac{e^{-2s}}{s^2+s-2}\Big]$.

Solution: Recall: $\dfrac{1}{s^2+s-2} = \dfrac{(a+b)\,s+(2a-b)}{(s-1)\,(s+2)}$

$$a+b=0, \quad 2a-b=1, \quad \Rightarrow \quad a=\frac{1}{3}, \quad b=-\frac{1}{3}.$$

$$\mathcal{L}^{-1}\Big[\frac{e^{-2s}}{s^2+s-2}\Big] = \frac{1}{3}\,\mathcal{L}^{-1}\Big[e^{-2s}\,\frac{1}{s-1}\Big] - \frac{1}{3}\,\mathcal{L}^{-1}\Big[e^{-2s}\,\frac{1}{s+2}\Big].$$

Recall: $\mathcal{L}^{-1}\Big[\dfrac{1}{s-a}\Big] = e^{at}$, $\mathcal{L}^{-1}\big[e^{-cs}\,F(s)\big] = u(t-c)\,f(t-c)$,

$$\mathcal{L}^{-1}\Big[\frac{e^{-2s}}{s^2+s-2}\Big] = \frac{1}{3}\,u(t-2)\,e^{(t-2)} - \frac{1}{3}\,u(t-2)\,e^{-2(t-2)}.$$

Properties of the Laplace Transform.

Example

Find $\mathcal{L}^{-1}\Big[\dfrac{e^{-2s}}{s^2+s-2}\Big]$.

Solution: Recall: $\dfrac{1}{s^2+s-2} = \dfrac{(a+b)\,s + (2a-b)}{(s-1)\,(s+2)}$

$$a+b=0, \quad 2a-b=1, \quad \Rightarrow \quad a=\frac{1}{3}, \quad b=-\frac{1}{3}.$$

$$\mathcal{L}^{-1}\Big[\frac{e^{-2s}}{s^2+s-2}\Big] = \frac{1}{3}\,\mathcal{L}^{-1}\Big[e^{-2s}\,\frac{1}{s-1}\Big] - \frac{1}{3}\,\mathcal{L}^{-1}\Big[e^{-2s}\,\frac{1}{s+2}\Big].$$

Recall: $\mathcal{L}^{-1}\Big[\dfrac{1}{s-a}\Big] = e^{at}$, $\mathcal{L}^{-1}\big[e^{-cs}\,F(s)\big] = u(t-c)\,f(t-c)$,

$$\mathcal{L}^{-1}\Big[\frac{e^{-2s}}{s^2+s-2}\Big] = \frac{1}{3}\,u(t-2)\,e^{(t-2)} - \frac{1}{3}\,u(t-2)\,e^{-2(t-2)}.$$

Hence: $\mathcal{L}^{-1}\Big[\dfrac{e^{-2s}}{s^2+s-2}\Big] = \dfrac{1}{3}\,u(t-2)\Big[e^{(t-2)} - e^{-2(t-2)}\Big]$. $\triangleleft$

Equations with discontinuous sources (Sect. 6.4).

- Differential equations with discontinuous sources.
- We solve the IVPs:
  (a) Example 1:

$$y' + 2y = u(t-4), \qquad y(0) = 3.$$

  (b) Example 2:

$$y'' + y' + \frac{5}{4}\,y = b(t), \qquad \begin{matrix} y(0) = 0, \\ y'(0) = 0, \end{matrix} \qquad b(t) = \begin{cases} 1, & t \in [0, \pi) \\ 0, & t \in [\pi, \infty). \end{cases}$$

  (c) Example 3:

$$y'' + y' + \frac{5}{4}\,y = g(t), \qquad \begin{matrix} y(0) = 0, \\ y'(0) = 0, \end{matrix} \qquad g(t) = \begin{cases} \sin(t), & t \in [0, \pi) \\ 0, & t \in [\pi, \infty). \end{cases}$$

Equations with discontinuous sources (Sect. 6.4).

- Differential equations with discontinuous sources.
- We solve the IVPs:

(a) **Example 1:**

$$y' + 2y = u(t-4), \qquad y(0) = 3.$$

(b) Example 2:

$$y'' + y' + \frac{5}{4}\,y = b(t), \qquad \begin{array}{c} y(0) = 0, \\ y'(0) = 0, \end{array} \qquad b(t) = \begin{cases} 1, & t \in [0, \pi) \\ 0, & t \in [\pi, \infty). \end{cases}$$

(c) Example 3:

$$y'' + y' + \frac{5}{4}\,y = g(t), \quad \begin{array}{c} y(0) = 0, \\ y'(0) = 0, \end{array} \quad g(t) = \begin{cases} \sin(t), & t \in [0, \pi) \\ 0, & t \in [\pi, \infty). \end{cases}$$

Differential equations with discontinuous sources.

Example

Use the Laplace transform to find the solution of the IVP

$$y' + 2y = u(t-4), \qquad y(0) = 3.$$

Differential equations with discontinuous sources.

Example

Use the Laplace transform to find the solution of the IVP

$$y' + 2y = u(t-4), \qquad y(0) = 3.$$

Solution: Compute the Laplace transform of the whole equation,

$$\mathcal{L}[y'] + 2\,\mathcal{L}[y] = \mathcal{L}[u(t-4)]$$

Differential equations with discontinuous sources.

Example

Use the Laplace transform to find the solution of the IVP

$$y' + 2y = u(t-4), \qquad y(0) = 3.$$

Solution: Compute the Laplace transform of the whole equation,

$$\mathcal{L}[y'] + 2\,\mathcal{L}[y] = \mathcal{L}[u(t-4)] = \frac{e^{-4s}}{s}.$$

Differential equations with discontinuous sources.

Example

Use the Laplace transform to find the solution of the IVP

$$y' + 2y = u(t-4), \qquad y(0) = 3.$$

Solution: Compute the Laplace transform of the whole equation,

$$\mathcal{L}[y'] + 2\,\mathcal{L}[y] = \mathcal{L}[u(t-4)] = \frac{e^{-4s}}{s}.$$

From the previous Section we know that

$$\big[s\,\mathcal{L}[y] - y(0)\big] + 2\,\mathcal{L}[y] = \frac{e^{-4s}}{s}$$

Differential equations with discontinuous sources.

Example

Use the Laplace transform to find the solution of the IVP

$$y' + 2y = u(t-4), \qquad y(0) = 3.$$

Solution: Compute the Laplace transform of the whole equation,

$$\mathcal{L}[y'] + 2\,\mathcal{L}[y] = \mathcal{L}[u(t-4)] = \frac{e^{-4s}}{s}.$$

From the previous Section we know that

$$\big[s\,\mathcal{L}[y] - y(0)\big] + 2\,\mathcal{L}[y] = \frac{e^{-4s}}{s} \quad \Rightarrow \quad (s+2)\,\mathcal{L}[y] = y(0) + \frac{e^{-4s}}{s}.$$

Differential equations with discontinuous sources.

Example

Use the Laplace transform to find the solution of the IVP

$$y' + 2y = u(t-4), \qquad y(0) = 3.$$

Solution: Compute the Laplace transform of the whole equation,

$$\mathcal{L}[y'] + 2\,\mathcal{L}[y] = \mathcal{L}[u(t-4)] = \frac{e^{-4s}}{s}.$$

From the previous Section we know that

$$\big[s\,\mathcal{L}[y] - y(0)\big] + 2\,\mathcal{L}[y] = \frac{e^{-4s}}{s} \quad \Rightarrow \quad (s+2)\,\mathcal{L}[y] = y(0) + \frac{e^{-4s}}{s}.$$

Introduce the initial condition,

Differential equations with discontinuous sources.

Example

Use the Laplace transform to find the solution of the IVP

$$y' + 2y = u(t-4), \qquad y(0) = 3.$$

Solution: Compute the Laplace transform of the whole equation,

$$\mathcal{L}[y'] + 2\,\mathcal{L}[y] = \mathcal{L}[u(t-4)] = \frac{e^{-4s}}{s}.$$

From the previous Section we know that

$$\big[s\,\mathcal{L}[y] - y(0)\big] + 2\,\mathcal{L}[y] = \frac{e^{-4s}}{s} \quad \Rightarrow \quad (s+2)\,\mathcal{L}[y] = y(0) + \frac{e^{-4s}}{s}.$$

Introduce the initial condition, $\mathcal{L}[y] = \dfrac{3}{(s+2)} + e^{-4s}\,\dfrac{1}{s(s+2)},$

Differential equations with discontinuous sources.

Example

Use the Laplace transform to find the solution of the IVP

$$y' + 2y = u(t-4), \qquad y(0) = 3.$$

Solution: Compute the Laplace transform of the whole equation,

$$\mathcal{L}[y'] + 2\,\mathcal{L}[y] = \mathcal{L}[u(t-4)] = \frac{e^{-4s}}{s}.$$

From the previous Section we know that

$$\big[s\,\mathcal{L}[y] - y(0)\big] + 2\,\mathcal{L}[y] = \frac{e^{-4s}}{s} \quad \Rightarrow \quad (s+2)\,\mathcal{L}[y] = y(0) + \frac{e^{-4s}}{s}.$$

Introduce the initial condition, $\mathcal{L}[y] = \dfrac{3}{(s+2)} + e^{-4s}\,\dfrac{1}{s(s+2)}$,

Use the table: $\mathcal{L}[y] = 3\,\mathcal{L}\big[e^{-2t}\big] + e^{-4s}\,\dfrac{1}{s(s+2)}$.

Differential equations with discontinuous sources.

Example

Use the Laplace transform to find the solution of the IVP

$$y' + 2y = u(t-4), \qquad y(0) = 3.$$

Solution: Recall: $\mathcal{L}[y] = 3\,\mathcal{L}\big[e^{-2t}\big] + e^{-4s}\,\dfrac{1}{s(s+2)}.$

Differential equations with discontinuous sources.

Example

Use the Laplace transform to find the solution of the IVP

$$y' + 2y = u(t-4), \qquad y(0) = 3.$$

Solution: Recall: $\mathcal{L}[y] = 3\,\mathcal{L}\big[e^{-2t}\big] + e^{-4s}\,\dfrac{1}{s(s+2)}.$

We need to invert the Laplace transform on the last term.

Differential equations with discontinuous sources.

Example

Use the Laplace transform to find the solution of the IVP

$$y' + 2y = u(t-4), \qquad y(0) = 3.$$

Solution: Recall: $\mathcal{L}[y] = 3\,\mathcal{L}\left[e^{-2t}\right] + e^{-4s}\,\dfrac{1}{s(s+2)}.$

We need to invert the Laplace transform on the last term.
Partial fractions:

Differential equations with discontinuous sources.

Example

Use the Laplace transform to find the solution of the IVP

$$y' + 2y = u(t-4), \qquad y(0) = 3.$$

Solution: Recall: $\mathcal{L}[y] = 3\,\mathcal{L}\big[e^{-2t}\big] + e^{-4s}\,\dfrac{1}{s(s+2)}.$

We need to invert the Laplace transform on the last term.
Partial fractions:

$$\frac{1}{s(s+2)} = \frac{a}{s} + \frac{b}{(s+2)}$$

Differential equations with discontinuous sources.

Example

Use the Laplace transform to find the solution of the IVP

$$y' + 2y = u(t-4), \qquad y(0) = 3.$$

Solution: Recall: $\mathcal{L}[y] = 3\,\mathcal{L}\big[e^{-2t}\big] + e^{-4s}\,\dfrac{1}{s(s+2)}$.

We need to invert the Laplace transform on the last term.
Partial fractions:

$$\frac{1}{s(s+2)} = \frac{a}{s} + \frac{b}{(s+2)} = \frac{a(s+2)+bs}{s(s+2)}$$

Differential equations with discontinuous sources.

Example

Use the Laplace transform to find the solution of the IVP

$$y' + 2y = u(t-4), \qquad y(0) = 3.$$

Solution: Recall: $\mathcal{L}[y] = 3\,\mathcal{L}\big[e^{-2t}\big] + e^{-4s}\,\dfrac{1}{s(s+2)}$.

We need to invert the Laplace transform on the last term.
Partial fractions:

$$\frac{1}{s(s+2)} = \frac{a}{s} + \frac{b}{(s+2)} = \frac{a(s+2)+bs}{s(s+2)} = \frac{(a+b)\,s + (2a)}{s(s+2)}$$

Differential equations with discontinuous sources.

Example

Use the Laplace transform to find the solution of the IVP

$$y' + 2y = u(t-4), \qquad y(0) = 3.$$

Solution: Recall: $\mathcal{L}[y] = 3\,\mathcal{L}\left[e^{-2t}\right] + e^{-4s}\,\dfrac{1}{s(s+2)}.$

We need to invert the Laplace transform on the last term.
Partial fractions:

$$\frac{1}{s(s+2)} = \frac{a}{s} + \frac{b}{(s+2)} = \frac{a(s+2)+bs}{s(s+2)} = \frac{(a+b)\,s + (2a)}{s(s+2)}$$

We get, $a + b = 0$, $2a = 1$.

Differential equations with discontinuous sources.

Example

Use the Laplace transform to find the solution of the IVP

$$y' + 2y = u(t-4), \qquad y(0) = 3.$$

Solution: Recall: $\mathcal{L}[y] = 3\,\mathcal{L}\big[e^{-2t}\big] + e^{-4s}\,\dfrac{1}{s(s+2)}.$

We need to invert the Laplace transform on the last term.
Partial fractions:

$$\frac{1}{s(s+2)} = \frac{a}{s} + \frac{b}{(s+2)} = \frac{a(s+2)+bs}{s(s+2)} = \frac{(a+b)\,s + (2a)}{s(s+2)}$$

We get, $a + b = 0$, $2a = 1$. We obtain: $a = \dfrac{1}{2}$, $b = -\dfrac{1}{2}$.

Differential equations with discontinuous sources.

Example

Use the Laplace transform to find the solution of the IVP

$$y' + 2y = u(t-4), \qquad y(0) = 3.$$

Solution: Recall: $\mathcal{L}[y] = 3\,\mathcal{L}\big[e^{-2t}\big] + e^{-4s}\,\dfrac{1}{s(s+2)}$.

We need to invert the Laplace transform on the last term.
Partial fractions:

$$\frac{1}{s(s+2)} = \frac{a}{s} + \frac{b}{(s+2)} = \frac{a(s+2)+bs}{s(s+2)} = \frac{(a+b)\,s + (2a)}{s(s+2)}$$

We get, $a + b = 0$, $2a = 1$. We obtain: $a = \dfrac{1}{2}$, $b = -\dfrac{1}{2}$. Hence,

$$\frac{1}{s(s+2)} = \frac{1}{2}\Big[\frac{1}{s} - \frac{1}{(s+2)}\Big].$$

Differential equations with discontinuous sources.

Example

Use the Laplace transform to find the solution of the IVP

$$y' + 2y = u(t-4), \qquad y(0) = 3.$$

Solution: Recall: $\dfrac{1}{s(s+2)} = \dfrac{1}{2}\left[\dfrac{1}{s} - \dfrac{1}{(s+2)}\right].$

Differential equations with discontinuous sources.

Example

Use the Laplace transform to find the solution of the IVP

$$y' + 2y = u(t-4), \qquad y(0) = 3.$$

Solution: Recall: $\dfrac{1}{s(s+2)} = \dfrac{1}{2}\Big[\dfrac{1}{s} - \dfrac{1}{(s+2)}\Big].$

The algebraic equation for $\mathcal{L}[y]$ has the form,

$$\mathcal{L}[y] = 3\,\mathcal{L}\big[e^{-2t}\big] + \frac{1}{2}\Big[e^{-4s}\,\frac{1}{s} - e^{-4s}\,\frac{1}{(s+2)}\Big].$$

Differential equations with discontinuous sources.

Example

Use the Laplace transform to find the solution of the IVP

$$y' + 2y = u(t-4), \qquad y(0) = 3.$$

Solution: Recall: $\dfrac{1}{s(s+2)} = \dfrac{1}{2}\Big[\dfrac{1}{s} - \dfrac{1}{(s+2)}\Big]$.

The algebraic equation for $\mathcal{L}[y]$ has the form,

$$\mathcal{L}[y] = 3\,\mathcal{L}\big[e^{-2t}\big] + \frac{1}{2}\Big[e^{-4s}\,\frac{1}{s} - e^{-4s}\,\frac{1}{(s+2)}\Big].$$

$$\mathcal{L}[y] = 3\,\mathcal{L}\big[e^{-2t}\big]$$

Differential equations with discontinuous sources.

Example

Use the Laplace transform to find the solution of the IVP

$$y' + 2y = u(t-4), \qquad y(0) = 3.$$

Solution: Recall: $\dfrac{1}{s(s+2)} = \dfrac{1}{2}\left[\dfrac{1}{s} - \dfrac{1}{(s+2)}\right].$

The algebraic equation for $\mathcal{L}[y]$ has the form,

$$\mathcal{L}[y] = 3\,\mathcal{L}\left[e^{-2t}\right] + \frac{1}{2}\left[e^{-4s}\,\frac{1}{s} - e^{-4s}\,\frac{1}{(s+2)}\right].$$

$$\mathcal{L}[y] = 3\,\mathcal{L}\left[e^{-2t}\right] + \frac{1}{2}\Big(\mathcal{L}[u(t-4)]$$

Differential equations with discontinuous sources.

Example

Use the Laplace transform to find the solution of the IVP

$$y' + 2y = u(t-4), \qquad y(0) = 3.$$

Solution: Recall: $\dfrac{1}{s(s+2)} = \dfrac{1}{2}\left[\dfrac{1}{s} - \dfrac{1}{(s+2)}\right].$

The algebraic equation for $\mathcal{L}[y]$ has the form,

$$\mathcal{L}[y] = 3\,\mathcal{L}\left[e^{-2t}\right] + \frac{1}{2}\left[e^{-4s}\,\frac{1}{s} - e^{-4s}\,\frac{1}{(s+2)}\right].$$

$$\mathcal{L}[y] = 3\,\mathcal{L}\left[e^{-2t}\right] + \frac{1}{2}\Big(\mathcal{L}[u(t-4)] - \mathcal{L}\left[u(t-4)\,e^{-2(t-4)}\right]\Big).$$

Differential equations with discontinuous sources.

Example

Use the Laplace transform to find the solution of the IVP

$$y' + 2y = u(t-4), \qquad y(0) = 3.$$

Solution: Recall: $\dfrac{1}{s(s+2)} = \dfrac{1}{2}\Big[\dfrac{1}{s} - \dfrac{1}{(s+2)}\Big].$

The algebraic equation for $\mathcal{L}[y]$ has the form,

$$\mathcal{L}[y] = 3\,\mathcal{L}\big[e^{-2t}\big] + \frac{1}{2}\Big[e^{-4s}\,\frac{1}{s} - e^{-4s}\,\frac{1}{(s+2)}\Big].$$

$$\mathcal{L}[y] = 3\,\mathcal{L}\big[e^{-2t}\big] + \frac{1}{2}\Big(\mathcal{L}[u(t-4)] - \mathcal{L}\big[u(t-4)\,e^{-2(t-4)}\big]\Big).$$

We conclude that

$$y(t) = 3e^{-2t} + \frac{1}{2}\,u(t-4)\Big[1 - e^{-2(t-4)}\Big]. \qquad \triangleleft$$

Equations with discontinuous sources (Sect. 6.4).

- Differential equations with discontinuous sources.
- We solve the IVPs:
  (a) Example 1:

$$y' + 2y = u(t-4), \qquad y(0) = 3.$$

  (b) **Example 2:**

$$y'' + y' + \frac{5}{4}\,y = b(t), \quad \begin{array}{r} y(0) = 0, \\ y'(0) = 0, \end{array} \quad b(t) = \begin{cases} 1, & t \in [0, \pi) \\ 0, & t \in [\pi, \infty). \end{cases}$$

  (c) Example 3:

$$y'' + y' + \frac{5}{4}\,y = g(t), \quad \begin{array}{r} y(0) = 0, \\ y'(0) = 0, \end{array} \quad g(t) = \begin{cases} \sin(t), & t \in [0, \pi) \\ 0, & t \in [\pi, \infty). \end{cases}$$

Differential equations with discontinuous sources.

Example

Use the Laplace transform to find the solution of the IVP

$$y'' + y' + \frac{5}{4}\,y = b(t), \qquad \begin{aligned} y(0) &= 0, \\ y'(0) &= 0, \end{aligned} \qquad b(t) = \begin{cases} 1, & t \in [0, \pi) \\ 0, & t \in [\pi, \infty). \end{cases}$$

Differential equations with discontinuous sources.

Example

Use the Laplace transform to find the solution of the IVP

$$y'' + y' + \frac{5}{4}\,y = b(t), \qquad \begin{array}{r} y(0) = 0, \\ y'(0) = 0, \end{array} \qquad b(t) = \begin{cases} 1, & t \in [0, \pi) \\ 0, & t \in [\pi, \infty). \end{cases}$$

Solution:

Rewrite the source function using step functions.

Differential equations with discontinuous sources.

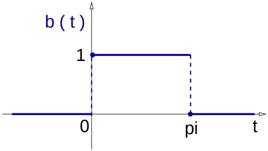
Example

Use the Laplace transform to find the solution of the IVP

$$y'' + y' + \frac{5}{4}\,y = b(t), \qquad \begin{matrix} y(0) = 0, \\ y'(0) = 0, \end{matrix} \qquad b(t) = \begin{cases} 1, & t \in [0, \pi) \\ 0, & t \in [\pi, \infty). \end{cases}$$

Solution:

Rewrite the source function using step functions.



Differential equations with discontinuous sources.

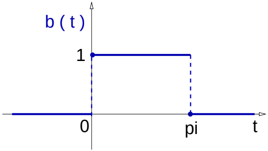
Example

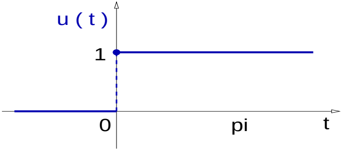
Use the Laplace transform to find the solution of the IVP

$$y'' + y' + \frac{5}{4}\,y = b(t), \qquad \begin{aligned} y(0) &= 0, \\ y'(0) &= 0, \end{aligned} \qquad b(t) = \begin{cases} 1, & t \in [0, \pi) \\ 0, & t \in [\pi, \infty). \end{cases}$$

Solution:

Rewrite the source function using step functions.





Differential equations with discontinuous sources.

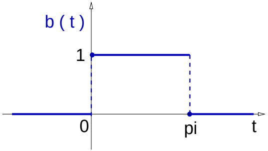
Example

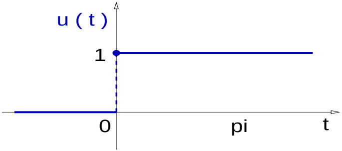
Use the Laplace transform to find the solution of the IVP

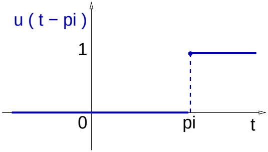
$$y'' + y' + \frac{5}{4}\,y = b(t), \qquad \begin{array}{l} y(0) = 0, \\ y'(0) = 0, \end{array} \qquad b(t) = \begin{cases} 1, & t \in [0, \pi) \\ 0, & t \in [\pi, \infty). \end{cases}$$

Solution:

Rewrite the source function using step functions.







Differential equations with discontinuous sources.

Example

Use the Laplace transform to find the solution of the IVP

$$y'' + y' + \frac{5}{4}\,y = b(t), \qquad \begin{matrix} y(0) = 0, \\ y'(0) = 0, \end{matrix} \qquad b(t) = \begin{cases} 1, & t \in [0, \pi) \\ 0, & t \in [\pi, \infty). \end{cases}$$

Solution: The graphs imply: $b(t) = u(t) - u(t - \pi)$

Differential equations with discontinuous sources.

Example

Use the Laplace transform to find the solution of the IVP

$$y'' + y' + \frac{5}{4}\,y = b(t), \qquad \begin{array}{r} y(0) = 0, \\ y'(0) = 0, \end{array} \qquad b(t) = \begin{cases} 1, & t \in [0, \pi) \\ 0, & t \in [\pi, \infty). \end{cases}$$

Solution: The graphs imply: $b(t) = u(t) - u(t - \pi)$

Now is simple to find $\mathcal{L}[b]$,

Differential equations with discontinuous sources.

Example

Use the Laplace transform to find the solution of the IVP

$$y'' + y' + \frac{5}{4}\,y = b(t), \qquad \begin{aligned} y(0) &= 0, \\ y'(0) &= 0, \end{aligned} \qquad b(t) = \begin{cases} 1, & t \in [0, \pi) \\ 0, & t \in [\pi, \infty). \end{cases}$$

Solution: The graphs imply: $b(t) = u(t) - u(t - \pi)$

Now is simple to find $\mathcal{L}[b]$, since

$$\mathcal{L}[b(t)] = \mathcal{L}[u(t)] - \mathcal{L}[u(t - \pi)]$$

Differential equations with discontinuous sources.

Example

Use the Laplace transform to find the solution of the IVP

$$y'' + y' + \frac{5}{4}\,y = b(t), \qquad \begin{array}{r} y(0) = 0, \\ y'(0) = 0, \end{array} \qquad b(t) = \begin{cases} 1, & t \in [0, \pi) \\ 0, & t \in [\pi, \infty). \end{cases}$$

Solution: The graphs imply: $b(t) = u(t) - u(t - \pi)$

Now is simple to find $\mathcal{L}[b]$, since

$$\mathcal{L}[b(t)] = \mathcal{L}[u(t)] - \mathcal{L}[u(t-\pi)] = \frac{1}{s} - \frac{e^{-\pi s}}{s}.$$

Differential equations with discontinuous sources.

Example

Use the Laplace transform to find the solution of the IVP

$$y'' + y' + \frac{5}{4}\,y = b(t), \qquad \begin{array}{c} y(0) = 0, \\ y'(0) = 0, \end{array} \qquad b(t) = \begin{cases} 1, & t \in [0, \pi) \\ 0, & t \in [\pi, \infty). \end{cases}$$

Solution: The graphs imply: $b(t) = u(t) - u(t - \pi)$

Now is simple to find $\mathcal{L}[b]$, since

$$\mathcal{L}[b(t)] = \mathcal{L}[u(t)] - \mathcal{L}[u(t - \pi)] = \frac{1}{s} - \frac{e^{-\pi s}}{s}.$$

So, the source is $\mathcal{L}[b(t)] = \left(1 - e^{-\pi s}\right) \dfrac{1}{s}$,

Differential equations with discontinuous sources.

Example

Use the Laplace transform to find the solution of the IVP

$$y'' + y' + \frac{5}{4}\,y = b(t), \qquad \begin{array}{r} y(0) = 0, \\ y'(0) = 0, \end{array} \qquad b(t) = \begin{cases} 1, & t \in [0, \pi) \\ 0, & t \in [\pi, \infty). \end{cases}$$

Solution: The graphs imply: $b(t) = u(t) - u(t - \pi)$

Now is simple to find $\mathcal{L}[b]$, since

$$\mathcal{L}[b(t)] = \mathcal{L}[u(t)] - \mathcal{L}[u(t - \pi)] = \frac{1}{s} - \frac{e^{-\pi s}}{s}.$$

So, the source is $\mathcal{L}[b(t)] = \left(1 - e^{-\pi s}\right)\frac{1}{s}$, and the equation is

$$\mathcal{L}[y''] + \mathcal{L}[y'] + \frac{5}{4}\,\mathcal{L}[y] = \left(1 - e^{-\pi s}\right)\frac{1}{s}.$$

Differential equations with discontinuous sources.

Example

Use the Laplace transform to find the solution of the IVP

$$y'' + y' + \frac{5}{4}\,y = b(t), \qquad \begin{array}{r} y(0) = 0, \\ y'(0) = 0, \end{array} \qquad b(t) = \begin{cases} 1, & t \in [0, \pi) \\ 0, & t \in [\pi, \infty). \end{cases}$$

Solution: So: $\mathcal{L}[y''] + \mathcal{L}[y'] + \frac{5}{4}\,\mathcal{L}[y] = \big(1 - e^{-\pi s}\big)\,\frac{1}{s}.$

Differential equations with discontinuous sources.

Example

Use the Laplace transform to find the solution of the IVP

$$y'' + y' + \frac{5}{4}\,y = b(t), \qquad \begin{array}{r} y(0) = 0, \\ y'(0) = 0, \end{array} \qquad b(t) = \begin{cases} 1, & t \in [0, \pi) \\ 0, & t \in [\pi, \infty). \end{cases}$$

Solution: So: $\mathcal{L}[y''] + \mathcal{L}[y'] + \frac{5}{4}\,\mathcal{L}[y] = \left(1 - e^{-\pi s}\right)\frac{1}{s}.$

The initial conditions imply:

Differential equations with discontinuous sources.

Example

Use the Laplace transform to find the solution of the IVP

$$y'' + y' + \frac{5}{4}\,y = b(t), \qquad \begin{array}{r} y(0) = 0, \\ y'(0) = 0, \end{array} \qquad b(t) = \begin{cases} 1, & t \in [0, \pi) \\ 0, & t \in [\pi, \infty). \end{cases}$$

Solution: So: $\mathcal{L}[y''] + \mathcal{L}[y'] + \frac{5}{4}\,\mathcal{L}[y] = \left(1 - e^{-\pi s}\right)\frac{1}{s}.$

The initial conditions imply: $\mathcal{L}[y''] = s^2\,\mathcal{L}[y]$

Differential equations with discontinuous sources.

Example

Use the Laplace transform to find the solution of the IVP

$$y'' + y' + \frac{5}{4}\,y = b(t), \qquad \begin{array}{r} y(0) = 0, \\ y'(0) = 0, \end{array} \qquad b(t) = \begin{cases} 1, & t \in [0, \pi) \\ 0, & t \in [\pi, \infty). \end{cases}$$

Solution: So: $\mathcal{L}[y''] + \mathcal{L}[y'] + \frac{5}{4}\,\mathcal{L}[y] = \left(1 - e^{-\pi s}\right)\frac{1}{s}$.

The initial conditions imply: $\mathcal{L}[y''] = s^2\,\mathcal{L}[y]$ and $\mathcal{L}[y'] = s\,\mathcal{L}[y]$.

Differential equations with discontinuous sources.

Example

Use the Laplace transform to find the solution of the IVP

$$y'' + y' + \frac{5}{4}\,y = b(t), \qquad \begin{array}{r} y(0) = 0, \\ y'(0) = 0, \end{array} \qquad b(t) = \begin{cases} 1, & t \in [0, \pi) \\ 0, & t \in [\pi, \infty). \end{cases}$$

Solution: So: $\mathcal{L}[y''] + \mathcal{L}[y'] + \frac{5}{4}\,\mathcal{L}[y] = \left(1 - e^{-\pi s}\right)\frac{1}{s}.$

The initial conditions imply: $\mathcal{L}[y''] = s^2\,\mathcal{L}[y]$ and $\mathcal{L}[y'] = s\,\mathcal{L}[y]$.

Therefore, $\left(s^2 + s + \frac{5}{4}\right)\mathcal{L}[y] = \left(1 - e^{-\pi s}\right)\frac{1}{s}.$

Differential equations with discontinuous sources.

Example

Use the Laplace transform to find the solution of the IVP

$$y'' + y' + \frac{5}{4}\,y = b(t), \qquad \begin{array}{r} y(0) = 0, \\ y'(0) = 0, \end{array} \qquad b(t) = \begin{cases} 1, & t \in [0, \pi) \\ 0, & t \in [\pi, \infty). \end{cases}$$

Solution: So: $\mathcal{L}[y''] + \mathcal{L}[y'] + \frac{5}{4}\,\mathcal{L}[y] = \left(1 - e^{-\pi s}\right)\frac{1}{s}$.

The initial conditions imply: $\mathcal{L}[y''] = s^2\,\mathcal{L}[y]$ and $\mathcal{L}[y'] = s\,\mathcal{L}[y]$.

Therefore, $\left(s^2 + s + \frac{5}{4}\right)\mathcal{L}[y] = \left(1 - e^{-\pi s}\right)\frac{1}{s}$.

We arrive at the expression: $\mathcal{L}[y] = \left(1 - e^{-\pi s}\right)\dfrac{1}{s\left(s^2 + s + \frac{5}{4}\right)}$.

Differential equations with discontinuous sources.

Example

Use the Laplace transform to find the solution of the IVP

$$y'' + y' + \frac{5}{4}\,y = b(t), \qquad \begin{array}{c} y(0) = 0, \\ y'(0) = 0, \end{array} \qquad b(t) = \begin{cases} 1, & t \in [0, \pi) \\ 0, & t \in [\pi, \infty). \end{cases}$$

Solution: Recall: $\mathcal{L}[y] = \left(1 - e^{-\pi s}\right) \dfrac{1}{s\left(s^2 + s + \frac{5}{4}\right)}.$

Differential equations with discontinuous sources.

Example

Use the Laplace transform to find the solution of the IVP

$$y'' + y' + \frac{5}{4}\,y = b(t), \qquad \begin{array}{r} y(0) = 0, \\ y'(0) = 0, \end{array} \qquad b(t) = \begin{cases} 1, & t \in [0, \pi) \\ 0, & t \in [\pi, \infty). \end{cases}$$

Solution: Recall: $\mathcal{L}[y] = \left(1 - e^{-\pi s}\right) \dfrac{1}{s\left(s^2 + s + \frac{5}{4}\right)}$.

Denoting: $H(s) = \dfrac{1}{s\left(s^2 + s + \frac{5}{4}\right)}$,

Differential equations with discontinuous sources.

Example

Use the Laplace transform to find the solution of the IVP

$$y'' + y' + \frac{5}{4}\,y = b(t), \qquad \begin{aligned} y(0) &= 0, \\ y'(0) &= 0, \end{aligned} \qquad b(t) = \begin{cases} 1, & t \in [0, \pi) \\ 0, & t \in [\pi, \infty). \end{cases}$$

Solution: Recall: $\mathcal{L}[y] = \left(1 - e^{-\pi s}\right) \dfrac{1}{s\left(s^2 + s + \frac{5}{4}\right)}$.

Denoting: $H(s) = \dfrac{1}{s\left(s^2 + s + \frac{5}{4}\right)}$,

we obtain, $\mathcal{L}[y] = \left(1 - e^{-\pi s}\right) H(s)$.

Differential equations with discontinuous sources.

Example

Use the Laplace transform to find the solution of the IVP

$$y'' + y' + \frac{5}{4}\,y = b(t), \qquad \begin{aligned} y(0) &= 0, \\ y'(0) &= 0, \end{aligned} \qquad b(t) = \begin{cases} 1, & t \in [0, \pi) \\ 0, & t \in [\pi, \infty). \end{cases}$$

Solution: Recall: $\mathcal{L}[y] = \left(1 - e^{-\pi s}\right) \dfrac{1}{s\left(s^2 + s + \frac{5}{4}\right)}$.

Denoting: $H(s) = \dfrac{1}{s\left(s^2 + s + \frac{5}{4}\right)}$,

we obtain, $\mathcal{L}[y] = \left(1 - e^{-\pi s}\right) H(s)$.

In other words: $y(t) = \mathcal{L}^{-1}\big[H(s)\big] - \mathcal{L}^{-1}\big[e^{-\pi s}\, H(s)\big]$.

Differential equations with discontinuous sources.

Example

Use the Laplace transform to find the solution of the IVP

$$y'' + y' + \frac{5}{4}\,y = b(t), \qquad \begin{aligned} y(0) &= 0, \\ y'(0) &= 0, \end{aligned} \qquad b(t) = \begin{cases} 1, & t \in [0, \pi) \\ 0, & t \in [\pi, \infty). \end{cases}$$

Solution: Recall: $y(t) = \mathcal{L}^{-1}\big[H(s)\big] - \mathcal{L}^{-1}\big[e^{-\pi s}\,H(s)\big]$.

Differential equations with discontinuous sources.

Example

Use the Laplace transform to find the solution of the IVP

$$y'' + y' + \frac{5}{4}\,y = b(t), \qquad \begin{array}{r} y(0) = 0, \\ y'(0) = 0, \end{array} \qquad b(t) = \begin{cases} 1, & t \in [0, \pi) \\ 0, & t \in [\pi, \infty). \end{cases}$$

Solution: Recall: $y(t) = \mathcal{L}^{-1}\big[H(s)\big] - \mathcal{L}^{-1}\big[e^{-\pi s}\,H(s)\big]$.

Denoting: $h(t) = \mathcal{L}^{-1}\big[H(s)\big]$,

Differential equations with discontinuous sources.

Example

Use the Laplace transform to find the solution of the IVP

$$y'' + y' + \frac{5}{4}\,y = b(t), \qquad \begin{array}{c} y(0) = 0, \\ y'(0) = 0, \end{array} \qquad b(t) = \begin{cases} 1, & t \in [0, \pi) \\ 0, & t \in [\pi, \infty). \end{cases}$$

Solution: Recall: $y(t) = \mathcal{L}^{-1}\big[H(s)\big] - \mathcal{L}^{-1}\big[e^{-\pi s}\,H(s)\big]$.

Denoting: $h(t) = \mathcal{L}^{-1}\big[H(s)\big]$, the $\mathcal{L}[\ ]$ properties imply

$$\mathcal{L}^{-1}\big[e^{-\pi s} H(s)\big] = u(t - \pi)\,h(t - \pi).$$

Differential equations with discontinuous sources.

Example

Use the Laplace transform to find the solution of the IVP

$$y'' + y' + \frac{5}{4}\,y = b(t), \qquad \begin{aligned} y(0) &= 0, \\ y'(0) &= 0, \end{aligned} \qquad b(t) = \begin{cases} 1, & t \in [0, \pi) \\ 0, & t \in [\pi, \infty). \end{cases}$$

Solution: Recall: $y(t) = \mathcal{L}^{-1}\big[H(s)\big] - \mathcal{L}^{-1}\big[e^{-\pi s}\,H(s)\big]$.

Denoting: $h(t) = \mathcal{L}^{-1}\big[H(s)\big]$, the $\mathcal{L}[\ ]$ properties imply

$$\mathcal{L}^{-1}\big[e^{-\pi s} H(s)\big] = u(t-\pi)\, h(t-\pi).$$

Therefore, the solution has the form

$$y(t) = h(t) - u(t-\pi)\, h(t-\pi).$$

Differential equations with discontinuous sources.

Example

Use the Laplace transform to find the solution of the IVP

$$y'' + y' + \frac{5}{4}\,y = b(t), \qquad \begin{array}{r} y(0) = 0, \\ y'(0) = 0, \end{array} \qquad b(t) = \begin{cases} 1, & t \in [0, \pi) \\ 0, & t \in [\pi, \infty). \end{cases}$$

Solution: Recall: $y(t) = \mathcal{L}^{-1}\big[H(s)\big] - \mathcal{L}^{-1}\big[e^{-\pi s}\,H(s)\big]$.

Denoting: $h(t) = \mathcal{L}^{-1}\big[H(s)\big]$, the $\mathcal{L}[\;]$ properties imply

$$\mathcal{L}^{-1}\big[e^{-\pi s} H(s)\big] = u(t - \pi)\,h(t - \pi).$$

Therefore, the solution has the form

$$y(t) = h(t) - u(t - \pi)\,h(t - \pi).$$

We only need to find $h(t) = \mathcal{L}^{-1}\Big[\dfrac{1}{s\left(s^2 + s + \frac{5}{4}\right)}\Big]$.

Differential equations with discontinuous sources.

Example

Use the Laplace transform to find the solution of the IVP

$$y'' + y' + \frac{5}{4}\,y = b(t), \qquad \begin{array}{r} y(0) = 0, \\ y'(0) = 0, \end{array} \qquad b(t) = \begin{cases} 1, & t \in [0, \pi) \\ 0, & t \in [\pi, \infty). \end{cases}$$

Solution: Recall: $h(t) = \mathcal{L}^{-1}\Big[\dfrac{1}{s\left(s^2 + s + \frac{5}{4}\right)}\Big].$

Differential equations with discontinuous sources.

Example

Use the Laplace transform to find the solution of the IVP

$$y'' + y' + \frac{5}{4}\,y = b(t), \qquad \begin{matrix} y(0) = 0, \\ y'(0) = 0, \end{matrix} \qquad b(t) = \begin{cases} 1, & t \in [0, \pi) \\ 0, & t \in [\pi, \infty). \end{cases}$$

Solution: Recall: $h(t) = \mathcal{L}^{-1}\Big[\dfrac{1}{s\left(s^2 + s + \frac{5}{4}\right)}\Big].$

Partial fractions:

Differential equations with discontinuous sources.

Example

Use the Laplace transform to find the solution of the IVP

$$y'' + y' + \frac{5}{4}\,y = b(t), \qquad \begin{aligned} y(0) &= 0, \\ y'(0) &= 0, \end{aligned} \qquad b(t) = \begin{cases} 1, & t \in [0, \pi) \\ 0, & t \in [\pi, \infty). \end{cases}$$

Solution: Recall: $h(t) = \mathcal{L}^{-1}\Big[\frac{1}{s\left(s^2 + s + \frac{5}{4}\right)}\Big].$

Partial fractions: Find the zeros of the denominator,

Differential equations with discontinuous sources.

Example

Use the Laplace transform to find the solution of the IVP

$$y'' + y' + \frac{5}{4}\,y = b(t), \qquad \begin{aligned} y(0) &= 0, \\ y'(0) &= 0, \end{aligned} \qquad b(t) = \begin{cases} 1, & t \in [0, \pi) \\ 0, & t \in [\pi, \infty). \end{cases}$$

Solution: Recall: $h(t) = \mathcal{L}^{-1}\Big[\dfrac{1}{s\left(s^2 + s + \frac{5}{4}\right)}\Big].$

Partial fractions: Find the zeros of the denominator,

$$s_\pm = \frac{1}{2}\big[-1 \pm \sqrt{1-5}\big]$$

Differential equations with discontinuous sources.

Example

Use the Laplace transform to find the solution of the IVP

$$y'' + y' + \frac{5}{4}\,y = b(t), \qquad \begin{array}{r} y(0) = 0, \\ y'(0) = 0, \end{array} \qquad b(t) = \begin{cases} 1, & t \in [0, \pi) \\ 0, & t \in [\pi, \infty). \end{cases}$$

Solution: Recall: $h(t) = \mathcal{L}^{-1}\Big[\dfrac{1}{s\left(s^2 + s + \frac{5}{4}\right)}\Big]$.

Partial fractions: Find the zeros of the denominator,

$$s_\pm = \frac{1}{2}\big[-1 \pm \sqrt{1-5}\big] \quad \Rightarrow \quad \text{Complex roots.}$$

Differential equations with discontinuous sources.

Example

Use the Laplace transform to find the solution of the IVP

$$y'' + y' + \frac{5}{4}\,y = b(t), \qquad \begin{array}{r} y(0) = 0, \\ y'(0) = 0, \end{array} \qquad b(t) = \begin{cases} 1, & t \in [0, \pi) \\ 0, & t \in [\pi, \infty). \end{cases}$$

Solution: Recall: $h(t) = \mathcal{L}^{-1}\Big[\frac{1}{s\left(s^2 + s + \frac{5}{4}\right)}\Big].$

Partial fractions: Find the zeros of the denominator,

$$s_{\pm} = \frac{1}{2}\big[-1 \pm \sqrt{1-5}\big] \quad \Rightarrow \quad \text{Complex roots.}$$

The partial fraction decomposition is:

Differential equations with discontinuous sources.

Example

Use the Laplace transform to find the solution of the IVP

$$y'' + y' + \frac{5}{4}\,y = b(t), \qquad \begin{aligned} y(0) &= 0, \\ y'(0) &= 0, \end{aligned} \qquad b(t) = \begin{cases} 1, & t \in [0, \pi) \\ 0, & t \in [\pi, \infty). \end{cases}$$

Solution: Recall: $h(t) = \mathcal{L}^{-1}\Big[\frac{1}{s\left(s^2 + s + \frac{5}{4}\right)}\Big].$

Partial fractions: Find the zeros of the denominator,

$$s_{\pm} = \frac{1}{2}\big[-1 \pm \sqrt{1-5}\big] \quad \Rightarrow \quad \text{Complex roots.}$$

The partial fraction decomposition is:

$$H(s) = \frac{1}{\left(s^2 + s + \frac{5}{4}\right)s}$$

Differential equations with discontinuous sources.

Example

Use the Laplace transform to find the solution of the IVP

$$y'' + y' + \frac{5}{4}\,y = b(t), \qquad \begin{aligned} y(0) &= 0, \\ y'(0) &= 0, \end{aligned} \qquad b(t) = \begin{cases} 1, & t \in [0, \pi) \\ 0, & t \in [\pi, \infty). \end{cases}$$

Solution: Recall: $h(t) = \mathcal{L}^{-1}\Big[\dfrac{1}{s\left(s^2 + s + \frac{5}{4}\right)}\Big]$.

Partial fractions: Find the zeros of the denominator,

$$s_\pm = \frac{1}{2}\big[-1 \pm \sqrt{1-5}\big] \quad \Rightarrow \quad \text{Complex roots.}$$

The partial fraction decomposition is:

$$H(s) = \frac{1}{\left(s^2 + s + \frac{5}{4}\right) s} = \frac{a}{s} + \frac{(bs + c)}{\left(s^2 + s + \frac{5}{4}\right)}$$

Differential equations with discontinuous sources.

Example

Use the Laplace transform to find the solution of the IVP

$$y'' + y' + \frac{5}{4}\,y = b(t), \qquad \begin{array}{r} y(0) = 0, \\ y'(0) = 0, \end{array} \qquad b(t) = \begin{cases} 1, & t \in [0, \pi) \\ 0, & t \in [\pi, \infty). \end{cases}$$

Solution: Recall: $H(s) = \dfrac{1}{\left(s^2 + s + \frac{5}{4}\right) s} = \dfrac{a}{s} + \dfrac{(bs + c)}{\left(s^2 + s + \frac{5}{4}\right)}.$

Differential equations with discontinuous sources.

Example

Use the Laplace transform to find the solution of the IVP

$$y'' + y' + \frac{5}{4}\,y = b(t), \qquad \begin{aligned} y(0) &= 0, \\ y'(0) &= 0, \end{aligned} \qquad b(t) = \begin{cases} 1, & t \in [0, \pi) \\ 0, & t \in [\pi, \infty). \end{cases}$$

Solution: Recall: $H(s) = \dfrac{1}{\left(s^2 + s + \frac{5}{4}\right) s} = \dfrac{a}{s} + \dfrac{(bs + c)}{\left(s^2 + s + \frac{5}{4}\right)}.$

The partial fraction decomposition is:

Differential equations with discontinuous sources.

Example

Use the Laplace transform to find the solution of the IVP

$$y'' + y' + \frac{5}{4}\, y = b(t), \qquad \begin{aligned} y(0) &= 0, \\ y'(0) &= 0, \end{aligned} \qquad b(t) = \begin{cases} 1, & t \in [0, \pi) \\ 0, & t \in [\pi, \infty). \end{cases}$$

Solution: Recall: $H(s) = \dfrac{1}{\left(s^2 + s + \frac{5}{4}\right) s} = \dfrac{a}{s} + \dfrac{(bs + c)}{\left(s^2 + s + \frac{5}{4}\right)}.$

The partial fraction decomposition is:

$$1 = a\left(s^2 + s + \frac{5}{4}\right) + s\,(bs + c)$$

Differential equations with discontinuous sources.

Example

Use the Laplace transform to find the solution of the IVP

$$y'' + y' + \frac{5}{4}\,y = b(t), \qquad \begin{array}{r} y(0) = 0, \\ y'(0) = 0, \end{array} \qquad b(t) = \begin{cases} 1, & t \in [0, \pi) \\ 0, & t \in [\pi, \infty). \end{cases}$$

Solution: Recall: $H(s) = \dfrac{1}{\left(s^2 + s + \frac{5}{4}\right) s} = \dfrac{a}{s} + \dfrac{(bs + c)}{\left(s^2 + s + \frac{5}{4}\right)}$.

The partial fraction decomposition is:

$$1 = a\left(s^2 + s + \frac{5}{4}\right) + s\,(bs + c) = (a + b)\,s^2 + (a + c)\,s + \frac{5}{4}a.$$

Differential equations with discontinuous sources.

Example

Use the Laplace transform to find the solution of the IVP

$$y'' + y' + \frac{5}{4}\,y = b(t), \qquad \begin{matrix} y(0) = 0, \\ y'(0) = 0, \end{matrix} \qquad b(t) = \begin{cases} 1, & t \in [0, \pi) \\ 0, & t \in [\pi, \infty). \end{cases}$$

Solution: Recall: $H(s) = \dfrac{1}{\left(s^2 + s + \frac{5}{4}\right) s} = \dfrac{a}{s} + \dfrac{(bs + c)}{\left(s^2 + s + \frac{5}{4}\right)}.$

The partial fraction decomposition is:

$$1 = a\left(s^2 + s + \frac{5}{4}\right) + s\,(bs + c) = (a + b)\,s^2 + (a + c)\,s + \frac{5}{4}a.$$

This equation implies that a, b, and c, are solutions of

$$a + b = 0, \quad a + c = 0, \quad \frac{5}{4}\,a = 1.$$

Differential equations with discontinuous sources.

Example

Use the Laplace transform to find the solution of the IVP

$$y'' + y' + \frac{5}{4}\,y = b(t), \qquad \begin{array}{c} y(0) = 0, \\ y'(0) = 0, \end{array} \quad b(t) = \begin{cases} 1, & t \in [0, \pi) \\ 0, & t \in [\pi, \infty). \end{cases}$$

Solution: So: $a = \frac{4}{5}, \;\; b = -\frac{4}{5}, \;\; c = -\frac{4}{5}.$

Differential equations with discontinuous sources.

Example

Use the Laplace transform to find the solution of the IVP

$$y'' + y' + \frac{5}{4}\,y = b(t), \qquad \begin{array}{r} y(0) = 0, \\ y'(0) = 0, \end{array} \qquad b(t) = \begin{cases} 1, & t \in [0, \pi) \\ 0, & t \in [\pi, \infty). \end{cases}$$

Solution: So: $a = \frac{4}{5}$, $b = -\frac{4}{5}$, $c = -\frac{4}{5}$.

Hence, we have found that,

$$H(s) = \frac{1}{\left(s^2 + s + \frac{5}{4}\right) s}$$

Differential equations with discontinuous sources.

Example

Use the Laplace transform to find the solution of the IVP

$$y'' + y' + \frac{5}{4}\,y = b(t), \qquad \begin{array}{r} y(0) = 0, \\ y'(0) = 0, \end{array} \qquad b(t) = \begin{cases} 1, & t \in [0,\pi) \\ 0, & t \in [\pi,\infty). \end{cases}$$

Solution: So: $a = \frac{4}{5}$, $b = -\frac{4}{5}$, $c = -\frac{4}{5}$.

Hence, we have found that,

$$H(s) = \frac{1}{\left(s^2 + s + \frac{5}{4}\right) s} = \frac{4}{5}\left[\frac{1}{s} - \frac{(s+1)}{\left(s^2 + s + \frac{5}{4}\right)}\right]$$

Differential equations with discontinuous sources.

Example

Use the Laplace transform to find the solution of the IVP

$$y'' + y' + \frac{5}{4}\,y = b(t), \qquad \begin{array}{c} y(0) = 0, \\ y'(0) = 0, \end{array} \qquad b(t) = \begin{cases} 1, & t \in [0, \pi) \\ 0, & t \in [\pi, \infty). \end{cases}$$

Solution: So: $a = \frac{4}{5}, \;\; b = -\frac{4}{5}, \;\; c = -\frac{4}{5}.$

Hence, we have found that,

$$H(s) = \frac{1}{\left(s^2 + s + \frac{5}{4}\right) s} = \frac{4}{5}\left[\frac{1}{s} - \frac{(s+1)}{\left(s^2 + s + \frac{5}{4}\right)}\right]$$

We have to compute the inverse Laplace Transform

Differential equations with discontinuous sources.

Example

Use the Laplace transform to find the solution of the IVP

$$y'' + y' + \frac{5}{4}\,y = b(t), \qquad \begin{array}{r} y(0) = 0, \\ y'(0) = 0, \end{array} \qquad b(t) = \begin{cases} 1, & t \in [0, \pi) \\ 0, & t \in [\pi, \infty). \end{cases}$$

Solution: So: $a = \frac{4}{5}$, $b = -\frac{4}{5}$, $c = -\frac{4}{5}$.

Hence, we have found that,

$$H(s) = \frac{1}{\left(s^2 + s + \frac{5}{4}\right)s} = \frac{4}{5}\left[\frac{1}{s} - \frac{(s+1)}{\left(s^2 + s + \frac{5}{4}\right)}\right]$$

We have to compute the inverse Laplace Transform

$$h(t) = \frac{4}{5}\,\mathcal{L}^{-1}\left[\frac{1}{s} - \frac{(s+1)}{\left(s^2 + s + \frac{5}{4}\right)}\right]$$

Differential equations with discontinuous sources.

Example

Use the Laplace transform to find the solution of the IVP

$$y'' + y' + \frac{5}{4}\,y = b(t), \qquad \begin{array}{r} y(0) = 0, \\ y'(0) = 0, \end{array} \qquad b(t) = \begin{cases} 1, & t \in [0, \pi) \\ 0, & t \in [\pi, \infty). \end{cases}$$

Solution: Recall: $h(t) = \dfrac{4}{5}\,\mathcal{L}^{-1}\Big[\dfrac{1}{s} - \dfrac{(s+1)}{\big(s^2 + s + \frac{5}{4}\big)}\Big].$

Differential equations with discontinuous sources.

Example

Use the Laplace transform to find the solution of the IVP

$$y'' + y' + \frac{5}{4}\,y = b(t), \qquad \begin{aligned} y(0) &= 0, \\ y'(0) &= 0, \end{aligned} \qquad b(t) = \begin{cases} 1, & t \in [0, \pi) \\ 0, & t \in [\pi, \infty). \end{cases}$$

Solution: Recall: $h(t) = \dfrac{4}{5}\,\mathcal{L}^{-1}\Big[\dfrac{1}{s} - \dfrac{(s+1)}{\big(s^2 + s + \frac{5}{4}\big)}\Big].$

In this case we complete the square in the denominator,

Differential equations with discontinuous sources.

Example

Use the Laplace transform to find the solution of the IVP

$$y'' + y' + \frac{5}{4}\,y = b(t), \qquad \begin{array}{c} y(0) = 0, \\ y'(0) = 0, \end{array} \qquad b(t) = \begin{cases} 1, & t \in [0, \pi) \\ 0, & t \in [\pi, \infty). \end{cases}$$

Solution: Recall: $h(t) = \dfrac{4}{5}\,\mathcal{L}^{-1}\Big[\dfrac{1}{s} - \dfrac{(s+1)}{\big(s^2 + s + \frac{5}{4}\big)}\Big].$

In this case we complete the square in the denominator,

$$s^2 + s + \frac{5}{4} = \Big[s^2 + 2\Big(\frac{1}{2}\Big)\,s + \frac{1}{4}\Big] - \frac{1}{4} + \frac{5}{4}$$

Differential equations with discontinuous sources.

Example

Use the Laplace transform to find the solution of the IVP

$$y'' + y' + \frac{5}{4}\,y = b(t), \qquad \begin{array}{r} y(0) = 0, \\ y'(0) = 0, \end{array} \qquad b(t) = \begin{cases} 1, & t \in [0, \pi) \\ 0, & t \in [\pi, \infty). \end{cases}$$

Solution: Recall: $h(t) = \frac{4}{5}\,\mathcal{L}^{-1}\Big[\frac{1}{s} - \frac{(s+1)}{\big(s^2 + s + \frac{5}{4}\big)}\Big].$

In this case we complete the square in the denominator,

$$s^2 + s + \frac{5}{4} = \Big[s^2 + 2\Big(\frac{1}{2}\Big)\,s + \frac{1}{4}\Big] - \frac{1}{4} + \frac{5}{4} = \Big(s + \frac{1}{2}\Big)^2 + 1.$$

Differential equations with discontinuous sources.

Example

Use the Laplace transform to find the solution of the IVP

$$y'' + y' + \frac{5}{4}\,y = b(t), \qquad \begin{aligned} y(0) &= 0, \\ y'(0) &= 0, \end{aligned} \qquad b(t) = \begin{cases} 1, & t \in [0, \pi) \\ 0, & t \in [\pi, \infty). \end{cases}$$

Solution: Recall: $h(t) = \frac{4}{5}\,\mathcal{L}^{-1}\Big[\frac{1}{s} - \frac{(s+1)}{\left(s^2 + s + \frac{5}{4}\right)}\Big].$

In this case we complete the square in the denominator,

$$s^2 + s + \frac{5}{4} = \Big[s^2 + 2\Big(\frac{1}{2}\Big)\,s + \frac{1}{4}\Big] - \frac{1}{4} + \frac{5}{4} = \Big(s + \frac{1}{2}\Big)^2 + 1.$$

So: $h(t) = \frac{4}{5}\,\mathcal{L}^{-1}\Big[\frac{1}{s} - \frac{(s+1)}{\left[\left(s + \frac{1}{2}\right)^2 + 1\right]}\Big].$

Differential equations with discontinuous sources.

Example

Use the Laplace transform to find the solution of the IVP

$$y'' + y' + \frac{5}{4}\,y = b(t), \qquad \begin{array}{r} y(0) = 0, \\ y'(0) = 0, \end{array} \qquad b(t) = \begin{cases} 1, & t \in [0, \pi) \\ 0, & t \in [\pi, \infty). \end{cases}$$

Solution: Recall: $h(t) = \frac{4}{5}\,\mathcal{L}^{-1}\Big[\frac{1}{s} - \frac{(s+1)}{\left(s^2 + s + \frac{5}{4}\right)}\Big].$

In this case we complete the square in the denominator,

$$s^2 + s + \frac{5}{4} = \Big[s^2 + 2\Big(\frac{1}{2}\Big)\,s + \frac{1}{4}\Big] - \frac{1}{4} + \frac{5}{4} = \Big(s + \frac{1}{2}\Big)^2 + 1.$$

So: $h(t) = \frac{4}{5}\,\mathcal{L}^{-1}\Big[\frac{1}{s} - \frac{(s+1)}{\left[\left(s + \frac{1}{2}\right)^2 + 1\right]}\Big].$

That is, $h(t) = \frac{4}{5}\,\mathcal{L}^{-1}\Big[\frac{1}{s}\Big] - \frac{4}{5}\,\mathcal{L}^{-1}\Big[\frac{\left(s + \frac{1}{2}\right) + \frac{1}{2}}{\left[\left(s + \frac{1}{2}\right)^2 + 1\right]}\Big].$

Differential equations with discontinuous sources.

Example

Use the Laplace transform to find the solution of the IVP

$$y'' + y' + \frac{5}{4}\,y = b(t), \qquad \begin{array}{c} y(0) = 0, \\ y'(0) = 0, \end{array} \qquad b(t) = \begin{cases} 1, & t \in [0, \pi) \\ 0, & t \in [\pi, \infty). \end{cases}$$

Solution: Recall: $h(t) = \frac{4}{5}\,\mathcal{L}^{-1}\Big[\frac{1}{s}\Big] - \frac{4}{5}\,\mathcal{L}^{-1}\Big[\frac{\big(s+\frac{1}{2}\big)+\frac{1}{2}}{\big[\big(s+\frac{1}{2}\big)^2+1\big]}\Big].$

Differential equations with discontinuous sources.

Example

Use the Laplace transform to find the solution of the IVP

$$y'' + y' + \frac{5}{4}\,y = b(t), \qquad \begin{array}{r} y(0) = 0, \\ y'(0) = 0, \end{array} \qquad b(t) = \begin{cases} 1, & t \in [0, \pi) \\ 0, & t \in [\pi, \infty). \end{cases}$$

Solution: Recall: $h(t) = \frac{4}{5}\,\mathcal{L}^{-1}\Big[\frac{1}{s}\Big] - \frac{4}{5}\,\mathcal{L}^{-1}\Big[\frac{\big(s + \frac{1}{2}\big) + \frac{1}{2}}{\big[\big(s + \frac{1}{2}\big)^2 + 1\big]}\Big].$

$$h(t) = \frac{4}{5}\,\mathcal{L}^{-1}\Big[\frac{1}{s}\Big] - \frac{4}{5}\,\mathcal{L}^{-1}\Big[\frac{\big(s + \frac{1}{2}\big)}{\big[\big(s + \frac{1}{2}\big)^2 + 1\big]}\Big] - \frac{2}{5}\,\mathcal{L}^{-1}\Big[\frac{1}{\big[\big(s + \frac{1}{2}\big)^2 + 1\big]}\Big].$$

Differential equations with discontinuous sources.

Example

Use the Laplace transform to find the solution of the IVP

$$y'' + y' + \frac{5}{4}\,y = b(t), \qquad \begin{array}{r} y(0) = 0, \\ y'(0) = 0, \end{array} \qquad b(t) = \begin{cases} 1, & t \in [0, \pi) \\ 0, & t \in [\pi, \infty). \end{cases}$$

Solution: Recall: $h(t) = \frac{4}{5}\,\mathcal{L}^{-1}\Big[\frac{1}{s}\Big] - \frac{4}{5}\,\mathcal{L}^{-1}\Bigg[\frac{\big(s + \frac{1}{2}\big) + \frac{1}{2}}{\big[\big(s + \frac{1}{2}\big)^2 + 1\big]}\Bigg].$

$$h(t) = \frac{4}{5}\,\mathcal{L}^{-1}\Big[\frac{1}{s}\Big] - \frac{4}{5}\,\mathcal{L}^{-1}\Bigg[\frac{\big(s + \frac{1}{2}\big)}{\big[\big(s + \frac{1}{2}\big)^2 + 1\big]}\Bigg] - \frac{2}{5}\,\mathcal{L}^{-1}\Bigg[\frac{1}{\big[\big(s + \frac{1}{2}\big)^2 + 1\big]}\Bigg].$$

Recall: $\mathcal{L}^{-1}\big[F(s - c)\big] = e^{ct}\,f(t).$

Differential equations with discontinuous sources.

Example

Use the Laplace transform to find the solution of the IVP

$$y'' + y' + \frac{5}{4}\,y = b(t), \qquad \begin{array}{r} y(0) = 0, \\ y'(0) = 0, \end{array} \qquad b(t) = \begin{cases} 1, & t \in [0, \pi) \\ 0, & t \in [\pi, \infty). \end{cases}$$

Solution: Recall: $h(t) = \frac{4}{5}\,\mathcal{L}^{-1}\Big[\frac{1}{s}\Big] - \frac{4}{5}\,\mathcal{L}^{-1}\Big[\frac{\big(s + \frac{1}{2}\big) + \frac{1}{2}}{\big[\big(s + \frac{1}{2}\big)^2 + 1\big]}\Big].$

$$h(t) = \frac{4}{5}\,\mathcal{L}^{-1}\Big[\frac{1}{s}\Big] - \frac{4}{5}\,\mathcal{L}^{-1}\Big[\frac{\big(s + \frac{1}{2}\big)}{\big[\big(s + \frac{1}{2}\big)^2 + 1\big]}\Big] - \frac{2}{5}\,\mathcal{L}^{-1}\Big[\frac{1}{\big[\big(s + \frac{1}{2}\big)^2 + 1\big]}\Big].$$

Recall: $\mathcal{L}^{-1}\big[F(s - c)\big] = e^{ct}\,f(t)$. Hence,

$$h(t) = \frac{4}{5}\,\Big[1 - e^{-t/2}\,\cos(t) - \frac{1}{2}\,e^{-t/2}\,\sin(t)\Big].$$

Differential equations with discontinuous sources.

Example

Use the Laplace transform to find the solution of the IVP

$$y'' + y' + \frac{5}{4}\,y = b(t), \qquad \begin{array}{r} y(0) = 0, \\ y'(0) = 0, \end{array} \qquad b(t) = \begin{cases} 1, & t \in [0, \pi) \\ 0, & t \in [\pi, \infty). \end{cases}$$

Solution: Recall: $h(t) = \frac{4}{5}\,\mathcal{L}^{-1}\Big[\frac{1}{s}\Big] - \frac{4}{5}\,\mathcal{L}^{-1}\Big[\frac{\big(s+\frac{1}{2}\big)+\frac{1}{2}}{\big[\big(s+\frac{1}{2}\big)^2+1\big]}\Big].$

$$h(t) = \frac{4}{5}\,\mathcal{L}^{-1}\Big[\frac{1}{s}\Big] - \frac{4}{5}\,\mathcal{L}^{-1}\Big[\frac{\big(s+\frac{1}{2}\big)}{\big[\big(s+\frac{1}{2}\big)^2+1\big]}\Big] - \frac{2}{5}\,\mathcal{L}^{-1}\Big[\frac{1}{\big[\big(s+\frac{1}{2}\big)^2+1\big]}\Big].$$

Recall: $\mathcal{L}^{-1}\big[F(s-c)\big] = e^{ct}\,f(t)$. Hence,

$$h(t) = \frac{4}{5}\,\Big[1 - e^{-t/2}\,\cos(t) - \frac{1}{2}\,e^{-t/2}\,\sin(t)\Big].$$

We conclude: $y(t) = h(t) + u(t-\pi)h(t-\pi)$. $\lhd$

Equations with discontinuous sources (Sect. 6.4).

- Differential equations with discontinuous sources.
- We solve the IVPs:
  (a) Example 1:

$$y' + 2y = u(t-4), \qquad y(0) = 3.$$

  (b) Example 2:

$$y'' + y' + \frac{5}{4}\,y = b(t), \quad \begin{array}{c} y(0) = 0, \\ y'(0) = 0, \end{array} \quad b(t) = \begin{cases} 1, & t \in [0,\pi) \\ 0, & t \in [\pi,\infty). \end{cases}$$

  (c) **Example 3:**

$$y'' + y' + \frac{5}{4}\,y = g(t), \quad \begin{array}{c} y(0) = 0, \\ y'(0) = 0, \end{array} \quad g(t) = \begin{cases} \sin(t), & t \in [0,\pi) \\ 0, & t \in [\pi,\infty). \end{cases}$$

Differential equations with discontinuous sources.

Example

Use the Laplace transform to find the solution of the IVP

$$y'' + y' + \frac{5}{4}\, y = g(t), \qquad \begin{array}{r} y(0) = 0, \\ y'(0) = 0, \end{array} \qquad g(t) = \begin{cases} \sin(t) & t \in [0, \pi) \\ 0 & t \in [\pi, \infty). \end{cases}$$

Differential equations with discontinuous sources.

Example

Use the Laplace transform to find the solution of the IVP

$$y'' + y' + \frac{5}{4}\,y = g(t), \qquad \begin{aligned} y(0) &= 0, \\ y'(0) &= 0, \end{aligned} \qquad g(t) = \begin{cases} \sin(t) & t \in [0, \pi) \\ 0 & t \in [\pi, \infty). \end{cases}$$

Solution:

Rewrite the source function using step functions.

Differential equations with discontinuous sources.

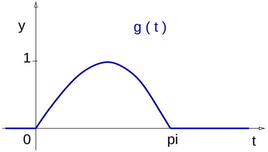
Example

Use the Laplace transform to find the solution of the IVP

$$y'' + y' + \frac{5}{4}\,y = g(t), \qquad \begin{array}{c} y(0) = 0, \\ y'(0) = 0, \end{array} \qquad g(t) = \begin{cases} \sin(t) & t \in [0, \pi) \\ 0 & t \in [\pi, \infty). \end{cases}$$

Solution:

Rewrite the source function using step functions.



Differential equations with discontinuous sources.

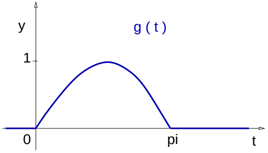
Example

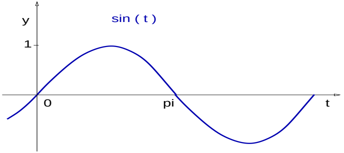
Use the Laplace transform to find the solution of the IVP

$$y'' + y' + \frac{5}{4}\,y = g(t), \qquad \begin{array}{r} y(0) = 0, \\ y'(0) = 0, \end{array} \qquad g(t) = \begin{cases} \sin(t) & t \in [0, \pi) \\ 0 & t \in [\pi, \infty). \end{cases}$$

Solution:

Rewrite the source function using step functions.





Differential equations with discontinuous sources.

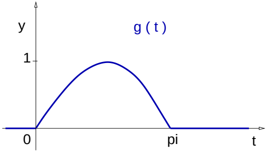
Example

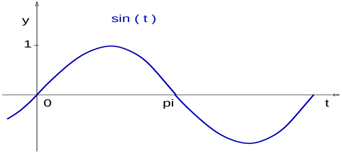
Use the Laplace transform to find the solution of the IVP

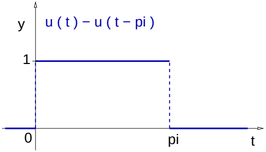
$$y'' + y' + \frac{5}{4}\,y = g(t), \qquad \begin{array}{r} y(0) = 0, \\ y'(0) = 0, \end{array} \qquad g(t) = \begin{cases} \sin(t) & t \in [0, \pi) \\ 0 & t \in [\pi, \infty). \end{cases}$$

Solution:

Rewrite the source function using step functions.







Differential equations with discontinuous sources.

Example

Use the Laplace transform to find the solution of the IVP

$$y'' + y' + \frac{5}{4}\,y = g(t), \qquad \begin{array}{c} y(0) = 0, \\ y'(0) = 0, \end{array} \quad g(t) = \begin{cases} \sin(t) & t \in [0, \pi) \\ 0 & t \in [\pi, \infty). \end{cases}$$

Solution: The graphs imply: $g(t) = \big[u(t) - u(t - \pi)\big] \sin(t)$.

Differential equations with discontinuous sources.

Example

Use the Laplace transform to find the solution of the IVP

$$y'' + y' + \frac{5}{4}\,y = g(t), \qquad \begin{array}{c} y(0) = 0, \\ y'(0) = 0, \end{array} \quad g(t) = \begin{cases} \sin(t) & t \in [0, \pi) \\ 0 & t \in [\pi, \infty). \end{cases}$$

Solution: The graphs imply: $g(t) = \big[u(t) - u(t - \pi)\big] \sin(t)$.

Recall the identity: $\sin(t) = -\sin(t - \pi)$.

Differential equations with discontinuous sources.

Example

Use the Laplace transform to find the solution of the IVP

$$y'' + y' + \frac{5}{4}\,y = g(t), \qquad \begin{array}{c} y(0) = 0, \\ y'(0) = 0, \end{array} \quad g(t) = \begin{cases} \sin(t) & t \in [0, \pi) \\ 0 & t \in [\pi, \infty). \end{cases}$$

Solution: The graphs imply: $g(t) = \big[u(t) - u(t-\pi)\big]\,\sin(t)$.

Recall the identity: $\sin(t) = -\sin(t-\pi)$. Then,

$$g(t) = u(t)\,\sin(t) - u(t-\pi)\,\sin(t),$$

Differential equations with discontinuous sources.

Example

Use the Laplace transform to find the solution of the IVP

$$y'' + y' + \frac{5}{4}\,y = g(t), \qquad \begin{array}{c} y(0) = 0, \\ y'(0) = 0, \end{array} \quad g(t) = \begin{cases} \sin(t) & t \in [0, \pi) \\ 0 & t \in [\pi, \infty). \end{cases}$$

Solution: The graphs imply: $g(t) = \big[u(t) - u(t-\pi)\big]\,\sin(t)$.

Recall the identity: $\sin(t) = -\sin(t-\pi)$. Then,

$$g(t) = u(t)\,\sin(t) - u(t-\pi)\,\sin(t),$$

$$g(t) = u(t)\,\sin(t) + u(t-\pi)\,\sin(t-\pi).$$

Differential equations with discontinuous sources.

Example

Use the Laplace transform to find the solution of the IVP

$$y'' + y' + \frac{5}{4}\,y = g(t), \qquad \begin{aligned} y(0) &= 0, \\ y'(0) &= 0, \end{aligned} \qquad g(t) = \begin{cases} \sin(t) & t \in [0, \pi) \\ 0 & t \in [\pi, \infty). \end{cases}$$

Solution: The graphs imply: $g(t) = \big[u(t) - u(t-\pi)\big]\,\sin(t)$.

Recall the identity: $\sin(t) = -\sin(t-\pi)$. Then,

$$g(t) = u(t)\,\sin(t) - u(t-\pi)\,\sin(t),$$

$$g(t) = u(t)\,\sin(t) + u(t-\pi)\,\sin(t-\pi).$$

Now is simple to find $\mathcal{L}[g]$,

Differential equations with discontinuous sources.

Example

Use the Laplace transform to find the solution of the IVP

$$y'' + y' + \frac{5}{4}\,y = g(t), \qquad \begin{aligned} y(0) &= 0, \\ y'(0) &= 0, \end{aligned} \qquad g(t) = \begin{cases} \sin(t) & t \in [0, \pi) \\ 0 & t \in [\pi, \infty). \end{cases}$$

Solution: The graphs imply: $g(t) = \big[u(t) - u(t - \pi)\big]\,\sin(t)$.

Recall the identity: $\sin(t) = -\sin(t - \pi)$. Then,

$$g(t) = u(t)\,\sin(t) - u(t - \pi)\,\sin(t),$$

$$g(t) = u(t)\,\sin(t) + u(t - \pi)\,\sin(t - \pi).$$

Now is simple to find $\mathcal{L}[g]$, since

$$\mathcal{L}[g(t)] = \mathcal{L}[u(t)\,\sin(t)] + \mathcal{L}[u(t - \pi)\,\sin(t - \pi)].$$

Differential equations with discontinuous sources.

Example

Use the Laplace transform to find the solution of the IVP

$$y'' + y' + \frac{5}{4}\,y = g(t), \qquad \begin{array}{c} y(0) = 0, \\ y'(0) = 0, \end{array} \quad g(t) = \begin{cases} \sin(t) & t \in [0, \pi) \\ 0 & t \in [\pi, \infty). \end{cases}$$

Solution: So: $\mathcal{L}[g(t)] = \mathcal{L}[u(t)\,\sin(t)] + \mathcal{L}[u(t-\pi)\,\sin(t-\pi)]$.

Differential equations with discontinuous sources.

Example

Use the Laplace transform to find the solution of the IVP

$$y'' + y' + \frac{5}{4}\,y = g(t), \qquad \begin{array}{r} y(0) = 0, \\ y'(0) = 0, \end{array} \qquad g(t) = \begin{cases} \sin(t) & t \in [0, \pi) \\ 0 & t \in [\pi, \infty). \end{cases}$$

Solution: So: $\mathcal{L}[g(t)] = \mathcal{L}[u(t)\,\sin(t)] + \mathcal{L}[u(t-\pi)\,\sin(t-\pi)]$.

$$\mathcal{L}[g(t)] = \frac{1}{(s^2+1)} + e^{-\pi s}\,\frac{1}{(s^2+1)}.$$

Differential equations with discontinuous sources.

Example

Use the Laplace transform to find the solution of the IVP

$$y'' + y' + \frac{5}{4}\,y = g(t), \qquad \begin{array}{r} y(0) = 0, \\ y'(0) = 0, \end{array} \qquad g(t) = \begin{cases} \sin(t) & t \in [0, \pi) \\ 0 & t \in [\pi, \infty). \end{cases}$$

Solution: So: $\mathcal{L}[g(t)] = \mathcal{L}[u(t)\,\sin(t)] + \mathcal{L}[u(t-\pi)\,\sin(t-\pi)]$.

$$\mathcal{L}[g(t)] = \frac{1}{(s^2+1)} + e^{-\pi s}\,\frac{1}{(s^2+1)}.$$

Recall the Laplace transform of the differential equation

$$\mathcal{L}[y''] + \mathcal{L}[y'] + \frac{5}{4}\,\mathcal{L}[y] = \mathcal{L}[g].$$

Differential equations with discontinuous sources.

Example

Use the Laplace transform to find the solution of the IVP

$$y'' + y' + \frac{5}{4}\,y = g(t), \qquad \begin{array}{r} y(0) = 0, \\ y'(0) = 0, \end{array} \qquad g(t) = \begin{cases} \sin(t) & t \in [0, \pi) \\ 0 & t \in [\pi, \infty). \end{cases}$$

Solution: So: $\mathcal{L}[g(t)] = \mathcal{L}[u(t)\,\sin(t)] + \mathcal{L}[u(t-\pi)\,\sin(t-\pi)]$.

$$\mathcal{L}[g(t)] = \frac{1}{(s^2+1)} + e^{-\pi s}\,\frac{1}{(s^2+1)}.$$

Recall the Laplace transform of the differential equation

$$\mathcal{L}[y''] + \mathcal{L}[y'] + \frac{5}{4}\,\mathcal{L}[y] = \mathcal{L}[g].$$

The initial conditions imply:

Differential equations with discontinuous sources.

Example

Use the Laplace transform to find the solution of the IVP

$$y'' + y' + \frac{5}{4}\,y = g(t), \qquad \begin{array}{r} y(0) = 0, \\ y'(0) = 0, \end{array} \quad g(t) = \begin{cases} \sin(t) & t \in [0, \pi) \\ 0 & t \in [\pi, \infty). \end{cases}$$

Solution: So: $\mathcal{L}[g(t)] = \mathcal{L}[u(t)\,\sin(t)] + \mathcal{L}[u(t-\pi)\,\sin(t-\pi)]$.

$$\mathcal{L}[g(t)] = \frac{1}{(s^2+1)} + e^{-\pi s}\,\frac{1}{(s^2+1)}.$$

Recall the Laplace transform of the differential equation

$$\mathcal{L}[y''] + \mathcal{L}[y'] + \frac{5}{4}\,\mathcal{L}[y] = \mathcal{L}[g].$$

The initial conditions imply: $\mathcal{L}[y''] = s^2\,\mathcal{L}[y]$

Differential equations with discontinuous sources.

Example

Use the Laplace transform to find the solution of the IVP

$$y'' + y' + \frac{5}{4}\,y = g(t), \qquad \begin{array}{r} y(0) = 0, \\ y'(0) = 0, \end{array} \qquad g(t) = \begin{cases} \sin(t) & t \in [0, \pi) \\ 0 & t \in [\pi, \infty). \end{cases}$$

Solution: So: $\mathcal{L}[g(t)] = \mathcal{L}[u(t)\,\sin(t)] + \mathcal{L}[u(t-\pi)\,\sin(t-\pi)]$.

$$\mathcal{L}[g(t)] = \frac{1}{(s^2+1)} + e^{-\pi s}\,\frac{1}{(s^2+1)}.$$

Recall the Laplace transform of the differential equation

$$\mathcal{L}[y''] + \mathcal{L}[y'] + \frac{5}{4}\,\mathcal{L}[y] = \mathcal{L}[g].$$

The initial conditions imply: $\mathcal{L}[y''] = s^2\,\mathcal{L}[y]$ and $\mathcal{L}[y'] = s\,\mathcal{L}[y]$.

Differential equations with discontinuous sources.

Example

Use the Laplace transform to find the solution of the IVP

$$y'' + y' + \frac{5}{4}\,y = g(t), \qquad \begin{aligned} y(0) &= 0, \\ y'(0) &= 0, \end{aligned} \quad g(t) = \begin{cases} \sin(t) & t \in [0,\pi) \\ 0 & t \in [\pi,\infty). \end{cases}$$

Solution: So: $\mathcal{L}[g(t)] = \mathcal{L}[u(t)\,\sin(t)] + \mathcal{L}[u(t-\pi)\,\sin(t-\pi)]$.

$$\mathcal{L}[g(t)] = \frac{1}{(s^2+1)} + e^{-\pi s}\,\frac{1}{(s^2+1)}.$$

Recall the Laplace transform of the differential equation

$$\mathcal{L}[y''] + \mathcal{L}[y'] + \frac{5}{4}\,\mathcal{L}[y] = \mathcal{L}[g].$$

The initial conditions imply: $\mathcal{L}[y''] = s^2\,\mathcal{L}[y]$ and $\mathcal{L}[y'] = s\,\mathcal{L}[y]$.

Therefore, $\left(s^2 + s + \frac{5}{4}\right)\mathcal{L}[y] = \left(1 + e^{-\pi s}\right)\frac{1}{(s^2+1)}$.

Differential equations with discontinuous sources.

Example

Use the Laplace transform to find the solution of the IVP

$$y'' + y' + \frac{5}{4}\, y = g(t), \qquad \begin{aligned} y(0) &= 0, \\ y'(0) &= 0, \end{aligned} \quad g(t) = \begin{cases} \sin(t) & t \in [0, \pi) \\ 0 & t \in [\pi, \infty). \end{cases}$$

Solution: Recall: $\left(s^2 + s + \frac{5}{4}\right) \mathcal{L}[y] = \left(1 + e^{-\pi s}\right) \frac{1}{(s^2+1)}.$

Differential equations with discontinuous sources.

Example

Use the Laplace transform to find the solution of the IVP

$$y'' + y' + \frac{5}{4}\,y = g(t), \qquad \begin{array}{r} y(0) = 0, \\ y'(0) = 0, \end{array} \qquad g(t) = \begin{cases} \sin(t) & t \in [0, \pi) \\ 0 & t \in [\pi, \infty). \end{cases}$$

Solution: Recall: $\left(s^2 + s + \dfrac{5}{4}\right) \mathcal{L}[y] = \left(1 + e^{-\pi s}\right) \dfrac{1}{(s^2+1)}$.

$$\mathcal{L}[y] = \left(1 + e^{-\pi s}\right) \frac{1}{\left(s^2 + s + \frac{5}{4}\right)(s^2+1)}.$$

Differential equations with discontinuous sources.

Example

Use the Laplace transform to find the solution of the IVP

$$y'' + y' + \frac{5}{4}\,y = g(t), \qquad \begin{aligned} y(0) &= 0, \\ y'(0) &= 0, \end{aligned} \qquad g(t) = \begin{cases} \sin(t) & t \in [0, \pi) \\ 0 & t \in [\pi, \infty). \end{cases}$$

Solution: Recall: $\left(s^2 + s + \dfrac{5}{4}\right) \mathcal{L}[y] = \left(1 + e^{-\pi s}\right) \dfrac{1}{(s^2+1)}.$

$$\mathcal{L}[y] = \left(1 + e^{-\pi s}\right) \frac{1}{\left(s^2 + s + \frac{5}{4}\right)(s^2+1)}.$$

Introduce the function $H(s) = \dfrac{1}{\left(s^2 + s + \frac{5}{4}\right)(s^2+1)}.$

Differential equations with discontinuous sources.

Example

Use the Laplace transform to find the solution of the IVP

$$y'' + y' + \frac{5}{4}\,y = g(t), \qquad \begin{aligned} y(0) &= 0, \\ y'(0) &= 0, \end{aligned} \qquad g(t) = \begin{cases} \sin(t) & t \in [0, \pi) \\ 0 & t \in [\pi, \infty). \end{cases}$$

Solution: Recall: $\left(s^2 + s + \frac{5}{4}\right) \mathcal{L}[y] = \left(1 + e^{-\pi s}\right) \dfrac{1}{(s^2 + 1)}$.

$$\mathcal{L}[y] = \left(1 + e^{-\pi s}\right) \frac{1}{\left(s^2 + s + \frac{5}{4}\right)(s^2 + 1)}.$$

Introduce the function $H(s) = \dfrac{1}{\left(s^2 + s + \frac{5}{4}\right)(s^2 + 1)}$.

Then, $y(t) = \mathcal{L}^{-1}[H(s)] + \mathcal{L}^{-1}[e^{-\pi s}\, H(s)]$.

Differential equations with discontinuous sources.

Example

Use the Laplace transform to find the solution of the IVP

$$y'' + y' + \frac{5}{4}\,y = g(t), \qquad \begin{array}{r} y(0) = 0, \\ y'(0) = 0, \end{array} \qquad g(t) = \begin{cases} \sin(t) & t \in [0, \pi) \\ 0 & t \in [\pi, \infty). \end{cases}$$

Solution: Recall: $y(t) = \mathcal{L}^{-1}[H(s)] + \mathcal{L}^{-1}[e^{-\pi s}\,H(s)]$, and

$$H(s) = \frac{1}{\left(s^2 + s + \frac{5}{4}\right)(s^2 + 1)}.$$

Differential equations with discontinuous sources.

Example

Use the Laplace transform to find the solution of the IVP

$$y'' + y' + \frac{5}{4}\,y = g(t), \qquad \begin{array}{r} y(0) = 0, \\ y'(0) = 0, \end{array} \quad g(t) = \begin{cases} \sin(t) & t \in [0,\pi) \\ 0 & t \in [\pi,\infty). \end{cases}$$

Solution: Recall: $y(t) = \mathcal{L}^{-1}[H(s)] + \mathcal{L}^{-1}[e^{-\pi s}\,H(s)]$, and

$$H(s) = \frac{1}{\left(s^2 + s + \frac{5}{4}\right)(s^2+1)}.$$

Partial fractions:

Differential equations with discontinuous sources.

Example

Use the Laplace transform to find the solution of the IVP

$$y'' + y' + \frac{5}{4}\,y = g(t), \qquad \begin{array}{r} y(0) = 0, \\ y'(0) = 0, \end{array} \quad g(t) = \begin{cases} \sin(t) & t \in [0, \pi) \\ 0 & t \in [\pi, \infty). \end{cases}$$

Solution: Recall: $y(t) = \mathcal{L}^{-1}[H(s)] + \mathcal{L}^{-1}[e^{-\pi s}\,H(s)]$, and

$$H(s) = \frac{1}{\left(s^2 + s + \frac{5}{4}\right)(s^2 + 1)}.$$

Partial fractions: Find the zeros of the denominator,

Differential equations with discontinuous sources.

Example

Use the Laplace transform to find the solution of the IVP

$$y'' + y' + \frac{5}{4}\,y = g(t), \qquad \begin{array}{r} y(0) = 0, \\ y'(0) = 0, \end{array} \quad g(t) = \begin{cases} \sin(t) & t \in [0, \pi) \\ 0 & t \in [\pi, \infty). \end{cases}$$

Solution: Recall: $y(t) = \mathcal{L}^{-1}[H(s)] + \mathcal{L}^{-1}[e^{-\pi s}\,H(s)]$, and

$$H(s) = \frac{1}{\left(s^2 + s + \frac{5}{4}\right)(s^2 + 1)}.$$

Partial fractions: Find the zeros of the denominator,

$$s_{\pm} = \frac{1}{2}\left[-1 \pm \sqrt{1 - 5}\right]$$

Differential equations with discontinuous sources.

Example

Use the Laplace transform to find the solution of the IVP

$$y'' + y' + \frac{5}{4}\,y = g(t), \qquad \begin{array}{c} y(0) = 0, \\ y'(0) = 0, \end{array} \qquad g(t) = \begin{cases} \sin(t) & t \in [0,\pi) \\ 0 & t \in [\pi,\infty). \end{cases}$$

Solution: Recall: $y(t) = \mathcal{L}^{-1}[H(s)] + \mathcal{L}^{-1}[e^{-\pi s}\,H(s)]$, and

$$H(s) = \frac{1}{\left(s^2 + s + \frac{5}{4}\right)(s^2 + 1)}.$$

Partial fractions: Find the zeros of the denominator,

$$s_{\pm} = \frac{1}{2}\left[-1 \pm \sqrt{1-5}\right] \quad \Rightarrow \quad \text{Complex roots.}$$

Differential equations with discontinuous sources.

Example

Use the Laplace transform to find the solution of the IVP

$$y'' + y' + \frac{5}{4}\,y = g(t), \qquad \begin{aligned} y(0) &= 0, \\ y'(0) &= 0, \end{aligned} \qquad g(t) = \begin{cases} \sin(t) & t \in [0, \pi) \\ 0 & t \in [\pi, \infty). \end{cases}$$

Solution: Recall: $y(t) = \mathcal{L}^{-1}[H(s)] + \mathcal{L}^{-1}[e^{-\pi s}\,H(s)]$, and

$$H(s) = \frac{1}{\left(s^2 + s + \frac{5}{4}\right)(s^2 + 1)}.$$

Partial fractions: Find the zeros of the denominator,

$$s_{\pm} = \frac{1}{2}\left[-1 \pm \sqrt{1 - 5}\right] \quad \Rightarrow \quad \text{Complex roots.}$$

The partial fraction decomposition is:

Differential equations with discontinuous sources.

Example

Use the Laplace transform to find the solution of the IVP

$$y'' + y' + \frac{5}{4}\,y = g(t), \qquad \begin{aligned} y(0) &= 0, \\ y'(0) &= 0, \end{aligned} \qquad g(t) = \begin{cases} \sin(t) & t \in [0, \pi) \\ 0 & t \in [\pi, \infty). \end{cases}$$

Solution: Recall: $y(t) = \mathcal{L}^{-1}[H(s)] + \mathcal{L}^{-1}[e^{-\pi s}\,H(s)]$, and

$$H(s) = \frac{1}{\left(s^2 + s + \frac{5}{4}\right)(s^2 + 1)}.$$

Partial fractions: Find the zeros of the denominator,

$$s_{\pm} = \frac{1}{2}\left[-1 \pm \sqrt{1 - 5}\right] \quad \Rightarrow \quad \text{Complex roots.}$$

The partial fraction decomposition is:

$$\frac{1}{\left(s^2 + s + \frac{5}{4}\right)(s^2 + 1)} = \frac{(as + b)}{\left(s^2 + s + \frac{5}{4}\right)} + \frac{(cs + d)}{(s^2 + 1)}.$$

Differential equations with discontinuous sources.

Example

Use the Laplace transform to find the solution of the IVP

$$y'' + y' + \frac{5}{4}\,y = g(t), \qquad \begin{aligned} y(0) &= 0, \\ y'(0) &= 0, \end{aligned} \qquad g(t) = \begin{cases} \sin(t) & t \in [0, \pi) \\ 0 & t \in [\pi, \infty). \end{cases}$$

Solution: So: $$\frac{1}{\left(s^2 + s + \frac{5}{4}\right)(s^2 + 1)} = \frac{(as + b)}{\left(s^2 + s + \frac{5}{4}\right)} + \frac{(cs + d)}{(s^2 + 1)}.$$

Differential equations with discontinuous sources.

Example

Use the Laplace transform to find the solution of the IVP

$$y'' + y' + \frac{5}{4}\,y = g(t), \qquad \begin{aligned} y(0) &= 0, \\ y'(0) &= 0, \end{aligned} \qquad g(t) = \begin{cases} \sin(t) & t \in [0, \pi) \\ 0 & t \in [\pi, \infty). \end{cases}$$

Solution: So: $\dfrac{1}{\left(s^2 + s + \frac{5}{4}\right)\left(s^2 + 1\right)} = \dfrac{(as + b)}{\left(s^2 + s + \frac{5}{4}\right)} + \dfrac{(cs + d)}{(s^2 + 1)}.$

Therefore, we get

$$1 = (as + b)(s^2 + 1) + (cs + d)\Big(s^2 + s + \frac{5}{4}\Big),$$

Differential equations with discontinuous sources.

Example

Use the Laplace transform to find the solution of the IVP

$$y'' + y' + \frac{5}{4}\,y = g(t), \qquad \begin{aligned} y(0) &= 0, \\ y'(0) &= 0, \end{aligned} \quad g(t) = \begin{cases} \sin(t) & t \in [0, \pi) \\ 0 & t \in [\pi, \infty). \end{cases}$$

Solution: So: $$\frac{1}{\left(s^2 + s + \frac{5}{4}\right)\left(s^2 + 1\right)} = \frac{(as + b)}{\left(s^2 + s + \frac{5}{4}\right)} + \frac{(cs + d)}{(s^2 + 1)}.$$

Therefore, we get

$$1 = (as + b)(s^2 + 1) + (cs + d)\Big(s^2 + s + \frac{5}{4}\Big),$$

$$1 = (a + c)\,s^3 + (b + c + d)\,s^2 + \Big(a + \frac{5}{4}\,c + d\Big)\,s + \Big(b + \frac{5}{4}\,d\Big).$$

Differential equations with discontinuous sources.

Example

Use the Laplace transform to find the solution of the IVP

$$y'' + y' + \frac{5}{4}\,y = g(t), \qquad \begin{array}{c} y(0) = 0, \\ y'(0) = 0, \end{array} \quad g(t) = \begin{cases} \sin(t) & t \in [0, \pi) \\ 0 & t \in [\pi, \infty). \end{cases}$$

Solution: So: $\dfrac{1}{\left(s^2 + s + \frac{5}{4}\right)\left(s^2 + 1\right)} = \dfrac{(as + b)}{\left(s^2 + s + \frac{5}{4}\right)} + \dfrac{(cs + d)}{(s^2 + 1)}.$

Therefore, we get

$$1 = (as + b)(s^2 + 1) + (cs + d)\Big(s^2 + s + \frac{5}{4}\Big),$$

$$1 = (a + c)\,s^3 + (b + c + d)\,s^2 + \Big(a + \frac{5}{4}\,c + d\Big)\,s + \Big(b + \frac{5}{4}\,d\Big).$$

This equation implies that a, b, c, and d, are solutions of

$$a + c = 0, \quad b + c + d = 0, \quad a + \frac{5}{4}\,c + d = 0, \quad b + \frac{5}{4}\,d = 1.$$

Differential equations with discontinuous sources.

Example

Use the Laplace transform to find the solution of the IVP

$$y'' + y' + \frac{5}{4}\,y = g(t), \qquad \begin{aligned} y(0) &= 0, \\ y'(0) &= 0, \end{aligned} \qquad g(t) = \begin{cases} \sin(t) & t \in [0, \pi) \\ 0 & t \in [\pi, \infty). \end{cases}$$

Solution: So: $a = \frac{16}{17}$, $b = \frac{12}{17}$, $c = -\frac{16}{17}$, $d = \frac{4}{17}$.

Differential equations with discontinuous sources.

Example

Use the Laplace transform to find the solution of the IVP

$$y'' + y' + \frac{5}{4}\,y = g(t), \qquad \begin{matrix} y(0) = 0, \\ y'(0) = 0, \end{matrix} \qquad g(t) = \begin{cases} \sin(t) & t \in [0, \pi) \\ 0 & t \in [\pi, \infty). \end{cases}$$

Solution: So: $a = \dfrac{16}{17}$, $b = \dfrac{12}{17}$, $c = -\dfrac{16}{17}$, $d = \dfrac{4}{17}$.

We have found: $H(s) = \dfrac{4}{17}\left[\dfrac{(4s+3)}{\left(s^2 + s + \frac{5}{4}\right)} + \dfrac{(-4s+1)}{(s^2+1)}\right].$

Differential equations with discontinuous sources.

Example

Use the Laplace transform to find the solution of the IVP

$$y'' + y' + \frac{5}{4}\,y = g(t), \qquad \begin{aligned} y(0) &= 0, \\ y'(0) &= 0, \end{aligned} \qquad g(t) = \begin{cases} \sin(t) & t \in [0, \pi) \\ 0 & t \in [\pi, \infty). \end{cases}$$

Solution: So: $a = \frac{16}{17}$, $b = \frac{12}{17}$, $c = -\frac{16}{17}$, $d = \frac{4}{17}$.

We have found: $H(s) = \frac{4}{17}\Big[\frac{(4s+3)}{\big(s^2+s+\frac{5}{4}\big)} + \frac{(-4s+1)}{(s^2+1)}\Big]$.

Complete the square in the denominator,

Differential equations with discontinuous sources.

Example

Use the Laplace transform to find the solution of the IVP

$$y'' + y' + \frac{5}{4}\,y = g(t), \qquad \begin{aligned} y(0) &= 0, \\ y'(0) &= 0, \end{aligned} \qquad g(t) = \begin{cases} \sin(t) & t \in [0, \pi) \\ 0 & t \in [\pi, \infty). \end{cases}$$

Solution: So: $a = \frac{16}{17}$, $b = \frac{12}{17}$, $c = -\frac{16}{17}$, $d = \frac{4}{17}$.

We have found: $H(s) = \frac{4}{17}\left[\frac{(4s+3)}{(s^2 + s + \frac{5}{4})} + \frac{(-4s+1)}{(s^2+1)}\right]$.

Complete the square in the denominator,

$$s^2 + s + \frac{5}{4} = \left[s^2 + 2\left(\frac{1}{2}\right)s + \frac{1}{4}\right] - \frac{1}{4} + \frac{5}{4}$$

Differential equations with discontinuous sources.

Example

Use the Laplace transform to find the solution of the IVP

$$y'' + y' + \frac{5}{4}\,y = g(t), \qquad \begin{matrix} y(0) = 0, \\ y'(0) = 0, \end{matrix} \qquad g(t) = \begin{cases} \sin(t) & t \in [0, \pi) \\ 0 & t \in [\pi, \infty). \end{cases}$$

Solution: So: $a = \frac{16}{17}$, $b = \frac{12}{17}$, $c = -\frac{16}{17}$, $d = \frac{4}{17}$.

We have found: $H(s) = \frac{4}{17}\Big[\frac{(4s+3)}{(s^2+s+\frac{5}{4})} + \frac{(-4s+1)}{(s^2+1)}\Big]$.

Complete the square in the denominator,

$$s^2 + s + \frac{5}{4} = \Big[s^2 + 2\Big(\frac{1}{2}\Big)\,s + \frac{1}{4}\Big] - \frac{1}{4} + \frac{5}{4} = \Big(s + \frac{1}{2}\Big)^2 + 1.$$

Differential equations with discontinuous sources.

Example

Use the Laplace transform to find the solution of the IVP

$$y'' + y' + \frac{5}{4}\,y = g(t), \qquad \begin{array}{c} y(0) = 0, \\ y'(0) = 0, \end{array} \quad g(t) = \begin{cases} \sin(t) & t \in [0, \pi) \\ 0 & t \in [\pi, \infty). \end{cases}$$

Solution: So: $a = \dfrac{16}{17}$, $b = \dfrac{12}{17}$, $c = -\dfrac{16}{17}$, $d = \dfrac{4}{17}$.

We have found: $H(s) = \dfrac{4}{17}\left[\dfrac{(4s+3)}{\left(s^2+s+\frac{5}{4}\right)} + \dfrac{(-4s+1)}{(s^2+1)}\right].$

Complete the square in the denominator,

$$s^2 + s + \frac{5}{4} = \left[s^2 + 2\left(\frac{1}{2}\right)s + \frac{1}{4}\right] - \frac{1}{4} + \frac{5}{4} = \left(s + \frac{1}{2}\right)^2 + 1.$$

$$H(s) = \frac{4}{17}\left[\frac{(4s+3)}{\left[\left(s+\frac{1}{2}\right)^2 + 1\right]} + \frac{(-4s+1)}{(s^2+1)}\right].$$

Differential equations with discontinuous sources.

Example

Use the Laplace transform to find the solution of the IVP

$$y'' + y' + \frac{5}{4}\,y = g(t), \qquad \begin{matrix} y(0) = 0, \\ y'(0) = 0, \end{matrix} \quad g(t) = \begin{cases} \sin(t) & t \in [0, \pi) \\ 0 & t \in [\pi, \infty). \end{cases}$$

Solution: So: $H(s) = \dfrac{4}{17} \Big[\dfrac{(4s+3)}{\big[\big(s+\frac{1}{2}\big)^2 + 1\big]} + \dfrac{(-4s+1)}{(s^2+1)} \Big].$

Differential equations with discontinuous sources.

Example

Use the Laplace transform to find the solution of the IVP

$$y'' + y' + \frac{5}{4}\,y = g(t), \qquad \begin{array}{r} y(0) = 0, \\ y'(0) = 0, \end{array} \qquad g(t) = \begin{cases} \sin(t) & t \in [0, \pi) \\ 0 & t \in [\pi, \infty). \end{cases}$$

Solution: So: $H(s) = \frac{4}{17}\Big[\frac{(4s+3)}{\big[\big(s+\frac{1}{2}\big)^2 + 1\big]} + \frac{(-4s+1)}{(s^2+1)}\Big].$

Rewrite the polynomial in the numerator,

Differential equations with discontinuous sources.

Example

Use the Laplace transform to find the solution of the IVP

$$y'' + y' + \frac{5}{4}\,y = g(t), \qquad \begin{aligned} y(0) &= 0, \\ y'(0) &= 0, \end{aligned} \qquad g(t) = \begin{cases} \sin(t) & t \in [0, \pi) \\ 0 & t \in [\pi, \infty). \end{cases}$$

Solution: So: $H(s) = \frac{4}{17}\left[\frac{(4s+3)}{\left[\left(s+\frac{1}{2}\right)^2+1\right]} + \frac{(-4s+1)}{(s^2+1)}\right].$

Rewrite the polynomial in the numerator,

$$(4s+3) = 4\Big(s + \frac{1}{2} - \frac{1}{2}\Big) + 3$$

Differential equations with discontinuous sources.

Example

Use the Laplace transform to find the solution of the IVP

$$y'' + y' + \frac{5}{4}\,y = g(t), \qquad \begin{array}{r} y(0) = 0, \\ y'(0) = 0, \end{array} \quad g(t) = \begin{cases} \sin(t) & t \in [0, \pi) \\ 0 & t \in [\pi, \infty). \end{cases}$$

Solution: So: $H(s) = \dfrac{4}{17}\left[\dfrac{(4s+3)}{\left[\left(s+\frac{1}{2}\right)^2 + 1\right]} + \dfrac{(-4s+1)}{(s^2+1)}\right].$

Rewrite the polynomial in the numerator,

$$(4s+3) = 4\Big(s + \frac{1}{2} - \frac{1}{2}\Big) + 3 = 4\Big(s + \frac{1}{2}\Big) + 1,$$

Differential equations with discontinuous sources.

Example

Use the Laplace transform to find the solution of the IVP

$$y'' + y' + \frac{5}{4}\,y = g(t), \qquad \begin{array}{r} y(0) = 0, \\ y'(0) = 0, \end{array} \qquad g(t) = \begin{cases} \sin(t) & t \in [0, \pi) \\ 0 & t \in [\pi, \infty). \end{cases}$$

Solution: So: $H(s) = \dfrac{4}{17}\Big[\dfrac{(4s+3)}{\big[\big(s+\frac{1}{2}\big)^2+1\big]} + \dfrac{(-4s+1)}{(s^2+1)}\Big].$

Rewrite the polynomial in the numerator,

$$(4s+3) = 4\Big(s + \frac{1}{2} - \frac{1}{2}\Big) + 3 = 4\Big(s + \frac{1}{2}\Big) + 1,$$

$$H(s) = \frac{4}{17}\Big[4\,\frac{\big(s+\frac{1}{2}\big)}{\big[\big(s+\frac{1}{2}\big)^2+1\big]} + \frac{1}{\big[\big(s+\frac{1}{2}\big)^2+1\big]} - 4\,\frac{s}{(s^2+1)} + \frac{1}{(s^2+1)}\Big],$$

Differential equations with discontinuous sources.

Example

Use the Laplace transform to find the solution of the IVP

$$y'' + y' + \frac{5}{4}\,y = g(t), \qquad \begin{array}{r} y(0) = 0, \\ y'(0) = 0, \end{array} \qquad g(t) = \begin{cases} \sin(t) & t \in [0, \pi) \\ 0 & t \in [\pi, \infty). \end{cases}$$

Solution:

$$H(s) = \frac{4}{17}\left[4\,\frac{\left(s+\frac{1}{2}\right)}{\left[\left(s+\frac{1}{2}\right)^2+1\right]} + \frac{1}{\left[\left(s+\frac{1}{2}\right)^2+1\right]} - 4\,\frac{s}{(s^2+1)} + \frac{1}{(s^2+1)}\right],$$

Differential equations with discontinuous sources.

Example

Use the Laplace transform to find the solution of the IVP

$$y'' + y' + \frac{5}{4}\,y = g(t), \qquad \begin{array}{c} y(0) = 0, \\ y'(0) = 0, \end{array} \qquad g(t) = \begin{cases} \sin(t) & t \in [0, \pi) \\ 0 & t \in [\pi, \infty). \end{cases}$$

Solution:

$$H(s) = \frac{4}{17}\left[4\,\frac{\left(s+\frac{1}{2}\right)}{\left[\left(s+\frac{1}{2}\right)^2+1\right]} + \frac{1}{\left[\left(s+\frac{1}{2}\right)^2+1\right]} - 4\,\frac{s}{(s^2+1)} + \frac{1}{(s^2+1)}\right],$$

Use the Laplace Transform table to get $H(s)$ equal to

$$H(s) = \frac{4}{17}\Big[4\,\mathcal{L}\big[e^{-t/2}\cos(t)\big] + \mathcal{L}\big[e^{-t/2}\sin(t)\big] - 4\,\mathcal{L}[\cos(t)] + \mathcal{L}[\sin(t)]\Big].$$

Differential equations with discontinuous sources.

Example

Use the Laplace transform to find the solution of the IVP

$$y'' + y' + \frac{5}{4}\,y = g(t), \qquad \begin{aligned} y(0) &= 0, \\ y'(0) &= 0, \end{aligned} \qquad g(t) = \begin{cases} \sin(t) & t \in [0, \pi) \\ 0 & t \in [\pi, \infty). \end{cases}$$

Solution:

$$H(s) = \frac{4}{17}\left[4\,\frac{\left(s+\frac{1}{2}\right)}{\left[\left(s+\frac{1}{2}\right)^2+1\right]} + \frac{1}{\left[\left(s+\frac{1}{2}\right)^2+1\right]} - 4\,\frac{s}{(s^2+1)} + \frac{1}{(s^2+1)}\right],$$

Use the Laplace Transform table to get $H(s)$ equal to

$$H(s) = \frac{4}{17}\left[4\,\mathcal{L}\left[e^{-t/2}\cos(t)\right] + \mathcal{L}\left[e^{-t/2}\sin(t)\right] - 4\,\mathcal{L}[\cos(t)] + \mathcal{L}[\sin(t)]\right].$$

$$H(s) = \mathcal{L}\left[\frac{4}{17}\Big(4e^{-t/2}\cos(t) + e^{-t/2}\sin(t) - 4\cos(t) + \sin(t)\Big)\right].$$

Differential equations with discontinuous sources.

Example

Use the Laplace transform to find the solution of the IVP

$$y'' + y' + \frac{5}{4}\,y = g(t), \qquad \begin{aligned} y(0) &= 0, \\ y'(0) &= 0, \end{aligned} \qquad g(t) = \begin{cases} \sin(t) & t \in [0, \pi) \\ 0 & t \in [\pi, \infty). \end{cases}$$

Solution: Recall:

$$H(s) = \mathcal{L}\Big[\frac{4}{17}\Big(4e^{-t/2}\cos(t) + e^{-t/2}\sin(t) - 4\cos(t) + \sin(t)\Big)\Big].$$

Differential equations with discontinuous sources.

Example

Use the Laplace transform to find the solution of the IVP

$$y'' + y' + \frac{5}{4}\,y = g(t), \qquad \begin{array}{r} y(0) = 0, \\ y'(0) = 0, \end{array} \qquad g(t) = \begin{cases} \sin(t) & t \in [0, \pi) \\ 0 & t \in [\pi, \infty). \end{cases}$$

Solution: Recall:

$$H(s) = \mathcal{L}\Big[\frac{4}{17}\Big(4e^{-t/2}\cos(t) + e^{-t/2}\sin(t) - 4\cos(t) + \sin(t)\Big)\Big].$$

Denote:

$$h(t) = \frac{4}{17}\Big[4e^{-t/2}\cos(t) + e^{-t/2}\sin(t) - 4\cos(t) + \sin(t)\Big].$$

Differential equations with discontinuous sources.

Example

Use the Laplace transform to find the solution of the IVP

$$y'' + y' + \frac{5}{4}\,y = g(t), \qquad \begin{array}{r} y(0) = 0, \\ y'(0) = 0, \end{array} \qquad g(t) = \begin{cases} \sin(t) & t \in [0, \pi) \\ 0 & t \in [\pi, \infty). \end{cases}$$

Solution: Recall:

$$H(s) = \mathcal{L}\Big[\frac{4}{17}\Big(4e^{-t/2}\cos(t) + e^{-t/2}\sin(t) - 4\cos(t) + \sin(t)\Big)\Big].$$

Denote:

$$h(t) = \frac{4}{17}\Big[4e^{-t/2}\cos(t) + e^{-t/2}\sin(t) - 4\cos(t) + \sin(t)\Big].$$

Then, $H(s) = \mathcal{L}[h(t)]$.

Differential equations with discontinuous sources.

Example

Use the Laplace transform to find the solution of the IVP

$$y'' + y' + \frac{5}{4}\,y = g(t), \qquad \begin{aligned} y(0) &= 0, \\ y'(0) &= 0, \end{aligned} \qquad g(t) = \begin{cases} \sin(t) & t \in [0, \pi) \\ 0 & t \in [\pi, \infty). \end{cases}$$

Solution: Recall:

$$H(s) = \mathcal{L}\Big[\frac{4}{17}\Big(4e^{-t/2}\cos(t) + e^{-t/2}\sin(t) - 4\cos(t) + \sin(t)\Big)\Big].$$

Denote:

$$h(t) = \frac{4}{17}\Big[4e^{-t/2}\cos(t) + e^{-t/2}\sin(t) - 4\cos(t) + \sin(t)\Big].$$

Then, $H(s) = \mathcal{L}[h(t)]$. Recalling: $\mathcal{L}[y(t)] = H(s) + e^{-\pi s}\,H(s)$,

Differential equations with discontinuous sources.

Example

Use the Laplace transform to find the solution of the IVP

$$y'' + y' + \frac{5}{4}\,y = g(t), \qquad \begin{array}{r} y(0) = 0, \\ y'(0) = 0, \end{array} \quad g(t) = \begin{cases} \sin(t) & t \in [0, \pi) \\ 0 & t \in [\pi, \infty). \end{cases}$$

Solution: Recall:

$$H(s) = \mathcal{L}\Big[\frac{4}{17}\Big(4e^{-t/2}\cos(t) + e^{-t/2}\sin(t) - 4\cos(t) + \sin(t)\Big)\Big].$$

Denote:

$$h(t) = \frac{4}{17}\Big[4e^{-t/2}\cos(t) + e^{-t/2}\sin(t) - 4\cos(t) + \sin(t)\Big].$$

Then, $H(s) = \mathcal{L}[h(t)]$. Recalling: $\mathcal{L}[y(t)] = H(s) + e^{-\pi s}\,H(s)$,

$$\mathcal{L}[y(t)] = \mathcal{L}[h(t)] + e^{-\pi s}\,\mathcal{L}[h(t)].$$

Differential equations with discontinuous sources.

Example

Use the Laplace transform to find the solution of the IVP

$$y'' + y' + \frac{5}{4}\,y = g(t), \qquad \begin{matrix} y(0) = 0, \\ y'(0) = 0, \end{matrix} \qquad g(t) = \begin{cases} \sin(t) & t \in [0, \pi) \\ 0 & t \in [\pi, \infty). \end{cases}$$

Solution: Recall:

$$H(s) = \mathcal{L}\Big[\frac{4}{17}\Big(4e^{-t/2}\cos(t) + e^{-t/2}\sin(t) - 4\cos(t) + \sin(t)\Big)\Big].$$

Denote:

$$h(t) = \frac{4}{17}\Big[4e^{-t/2}\cos(t) + e^{-t/2}\sin(t) - 4\cos(t) + \sin(t)\Big].$$

Then, $H(s) = \mathcal{L}[h(t)]$. Recalling: $\mathcal{L}[y(t)] = H(s) + e^{-\pi s}\,H(s)$,

$$\mathcal{L}[y(t)] = \mathcal{L}[h(t)] + e^{-\pi s}\,\mathcal{L}[h(t)].$$

We conclude: $y(t) = h(t) + u(t-\pi)h(t-\pi)$. $\triangleleft$

Generalized sources (Sect. 6.5).

- The Dirac delta generalized function.
- Properties of Dirac's delta.
- Relation between deltas and steps.
- Dirac's delta in Physics.
- The Laplace Transform of Dirac's delta.
- Differential equations with Dirac's delta sources.

Generalized sources (Sect. 6.5).

- **The Dirac delta generalized function.**
- Properties of Dirac's delta.
- Relation between deltas and steps.
- Dirac's delta in Physics.
- The Laplace Transform of Dirac's delta.
- Differential equations with Dirac's delta sources.

The Dirac delta generalized function.

Definition

Consider the sequence of functions for $n \geqslant 1$,

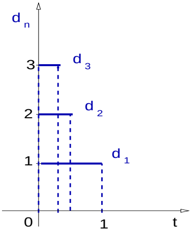
$$\delta_n(t) = \begin{cases} 0, & t < 0 \\ n, & 0 \leqslant t \leqslant \dfrac{1}{n} \\ 0, & t > \dfrac{1}{n}. \end{cases}$$

The Dirac delta generalized function.

Definition

Consider the sequence of functions for $n \geqslant 1$,

$$\delta_n(t) = \begin{cases} 0, & t < 0 \\ n, & 0 \leqslant t \leqslant \dfrac{1}{n} \\ 0, & t > \dfrac{1}{n}. \end{cases}$$

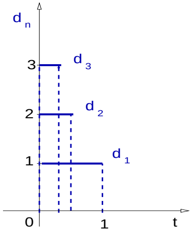


The Dirac delta generalized function.

Definition

Consider the sequence of functions for $n \geqslant 1$,

$$\delta_n(t) = \begin{cases} 0, & t < 0 \\ n, & 0 \leqslant t \leqslant \dfrac{1}{n} \\ 0, & t > \dfrac{1}{n}. \end{cases}$$



The Dirac delta generalized function is given by

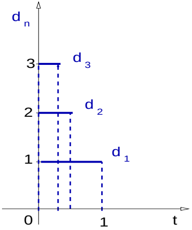
$$\lim_{n \to \infty} \delta_n(t) = \delta(t), \qquad t \in \mathbb{R}.$$

The Dirac delta generalized function.

Definition

Consider the sequence of functions for $n \geqslant 1$,

$$\delta_n(t) = \begin{cases} 0, & t < 0 \\ n, & 0 \leqslant t \leqslant \dfrac{1}{n} \\ 0, & t > \dfrac{1}{n}. \end{cases}$$



The Dirac delta generalized function is given by

$$\lim_{n\to\infty} \delta_n(t) = \delta(t), \qquad t \in \mathbb{R}.$$

Remarks:

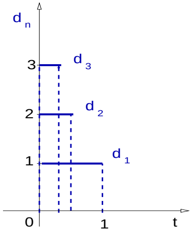
(a) There exist infinitely many sequences δ_n that define the same generalized function δ.

The Dirac delta generalized function.

Definition

Consider the sequence of functions for $n \geqslant 1$,

$$\delta_n(t) = \begin{cases} 0, & t < 0 \\ n, & 0 \leqslant t \leqslant \frac{1}{n} \\ 0, & t > \frac{1}{n}. \end{cases}$$



The Dirac delta generalized function is given by

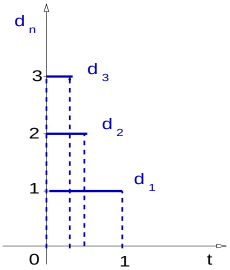
$$\lim_{n \to \infty} \delta_n(t) = \delta(t), \qquad t \in \mathbb{R}.$$

Remarks:

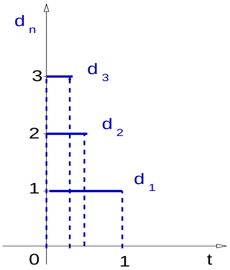
(a) There exist infinitely many sequences δ_n that define the same generalized function δ.

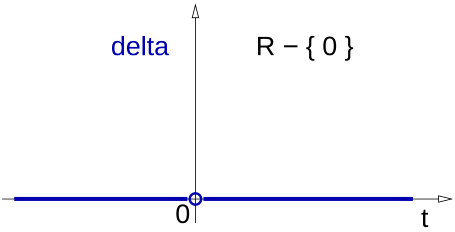
(b) For example, compare with the sequence δ_n in the textbook.

The Dirac delta generalized function.

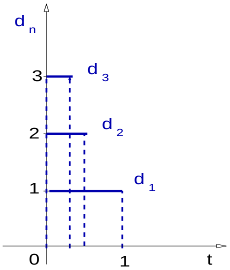


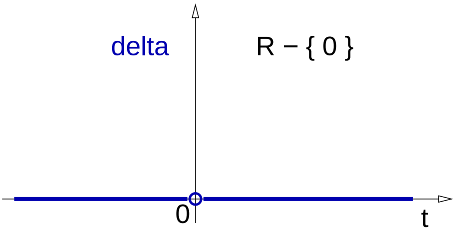
The Dirac delta generalized function.





The Dirac delta generalized function.

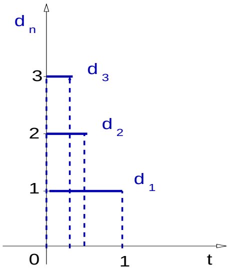


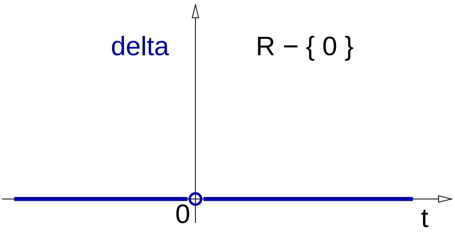


Remarks:

(a) The Dirac δ is a function on the domain $\mathbb{R} - \{0\}$, and $\delta(t) = 0$ for $t \in \mathbb{R} - \{0\}$.

The Dirac delta generalized function.



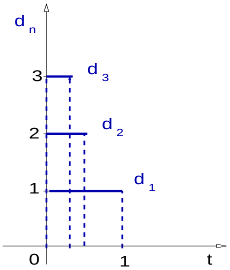


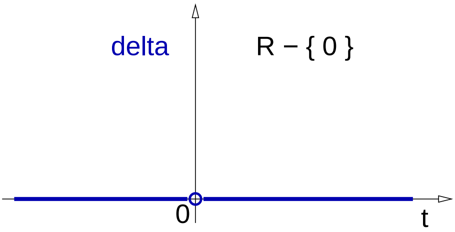
Remarks:

(a) The Dirac δ is a function on the domain $\mathbb{R} - \{0\}$, and $\delta(t) = 0$ for $t \in \mathbb{R} - \{0\}$.

(b) δ at $t = 0$ is not defined, since $\delta(0) = \lim_{n\to\infty} n = +\infty$.

The Dirac delta generalized function.





Remarks:

(a) The Dirac δ is a function on the domain $\mathbb{R} - \{0\}$, and $\delta(t) = 0$ for $t \in \mathbb{R} - \{0\}$.

(b) δ at $t = 0$ is not defined, since $\delta(0) = \lim_{n\to\infty} n = +\infty$.

(c) δ is not a function on $\mathbb{R}$.

Generalized sources (Sect. 6.5).

- The Dirac delta generalized function.
- **Properties of Dirac's delta.**
- Relation between deltas and steps.
- Dirac's delta in Physics.
- The Laplace Transform of Dirac's delta.
- Differential equations with Dirac's delta sources.

Properties of Dirac's delta.

Remark: The Dirac δ is not a function.

Properties of Dirac's delta.

Remark: The Dirac δ is not a function.

We define operations on Dirac's δ as limits $n \to \infty$ of the operation on the sequence elements δ_n.

Properties of Dirac's delta.

Remark: The Dirac δ is not a function.

We define operations on Dirac's δ as limits $n \to \infty$ of the operation on the sequence elements δ_n.

Definition

$$\delta(t-c) = \lim_{n\to\infty} \delta_n(t-c),$$

Properties of Dirac's delta.

Remark: The Dirac δ is not a function.

We define operations on Dirac's δ as limits $n \to \infty$ of the operation on the sequence elements δ_n.

Definition

$$\delta(t - c) = \lim_{n\to\infty} \delta_n(t - c),$$

$$a\,\delta(t) + b\,\delta(t) = \lim_{n\to\infty} \big[a\,\delta_n(t) + b\,\delta_n(t)\big],$$

Properties of Dirac's delta.

Remark: The Dirac δ is not a function.

We define operations on Dirac's δ as limits $n \to \infty$ of the operation on the sequence elements δ_n.

Definition

$$\delta(t - c) = \lim_{n \to \infty} \delta_n(t - c),$$

$$a\,\delta(t) + b\,\delta(t) = \lim_{n \to \infty} \big[a\,\delta_n(t) + b\,\delta_n(t) \big],$$

$$f(t)\,\delta(t) = \lim_{n \to \infty} \big[f(t)\,\delta_n(t) \big],$$

Properties of Dirac's delta.

Remark: The Dirac δ is not a function.

We define operations on Dirac's δ as limits $n \to \infty$ of the operation on the sequence elements δ_n.

Definition

$$\delta(t-c) = \lim_{n\to\infty} \delta_n(t-c),$$

$$a\,\delta(t) + b\,\delta(t) = \lim_{n\to\infty} \big[a\,\delta_n(t) + b\,\delta_n(t)\big],$$

$$f(t)\,\delta(t) = \lim_{n\to\infty} \big[f(t)\,\delta_n(t)\big],$$

$$\int_a^b \delta(t)\,dt = \lim_{n\to\infty} \int_a^b \delta_n(t)\,dt,$$

Properties of Dirac's delta.

Remark: The Dirac δ is not a function.

We define operations on Dirac's δ as limits $n \to \infty$ of the operation on the sequence elements δ_n.

Definition

$$\delta(t-c) = \lim_{n\to\infty} \delta_n(t-c),$$

$$a\,\delta(t) + b\,\delta(t) = \lim_{n\to\infty} \big[a\,\delta_n(t) + b\,\delta_n(t)\big],$$

$$f(t)\,\delta(t) = \lim_{n\to\infty} \big[f(t)\,\delta_n(t)\big],$$

$$\int_a^b \delta(t)\,dt = \lim_{n\to\infty} \int_a^b \delta_n(t)\,dt,$$

$$\mathcal{L}[\delta] = \lim_{n\to\infty} \mathcal{L}[\delta_n].$$

Properties of Dirac's delta.

Theorem

$$\int_{-a}^{a} \delta(t)\,dt = 1, \qquad a > 0.$$

Properties of Dirac's delta.

Theorem

$$\int_{-a}^{a} \delta(t)\, dt = 1, \qquad a > 0.$$

Proof:

$$\int_{-a}^{a} \delta(t)\, dt$$

Properties of Dirac's delta.

Theorem

$$\int_{-a}^{a} \delta(t)\,dt = 1, \qquad a > 0.$$

Proof:

$$\int_{-a}^{a} \delta(t)\,dt = \lim_{n\to\infty} \int_{-a}^{a} \delta_n(t)\,dt$$

Properties of Dirac's delta.

Theorem

$$\int_{-a}^{a} \delta(t)\, dt = 1, \qquad a > 0.$$

Proof:

$$\int_{-a}^{a} \delta(t)\, dt = \lim_{n\to\infty} \int_{-a}^{a} \delta_n(t)\, dt = \lim_{n\to\infty} \int_{0}^{1/n} n\, dt$$

Properties of Dirac's delta.

Theorem

$$\int_{-a}^{a} \delta(t)\,dt = 1, \qquad a > 0.$$

Proof:

$$\int_{-a}^{a} \delta(t)\,dt = \lim_{n\to\infty} \int_{-a}^{a} \delta_n(t)\,dt = \lim_{n\to\infty} \int_{0}^{1/n} n\,dt$$

$$\int_{-a}^{a} \delta(t)\,dt = \lim_{n\to\infty} \Big[n \Big(t \Big|_{0}^{1/n} \Big) \Big]$$

Properties of Dirac's delta.

Theorem

$$\int_{-a}^{a} \delta(t)\,dt = 1, \qquad a > 0.$$

Proof:

$$\int_{-a}^{a} \delta(t)\,dt = \lim_{n\to\infty} \int_{-a}^{a} \delta_n(t)\,dt = \lim_{n\to\infty} \int_{0}^{1/n} n\,dt$$

$$\int_{-a}^{a} \delta(t)\,dt = \lim_{n\to\infty} \Big[n \Big(t \Big|_0^{1/n} \Big) \Big] = \lim_{n\to\infty} \Big[n\,\frac{1}{n} \Big].$$

Properties of Dirac's delta.

Theorem

$$\int_{-a}^{a} \delta(t)\, dt = 1, \qquad a > 0.$$

Proof:

$$\int_{-a}^{a} \delta(t)\, dt = \lim_{n\to\infty} \int_{-a}^{a} \delta_n(t)\, dt = \lim_{n\to\infty} \int_{0}^{1/n} n\, dt$$

$$\int_{-a}^{a} \delta(t)\, dt = \lim_{n\to\infty} \Big[n \Big(t \Big|_0^{1/n} \Big) \Big] = \lim_{n\to\infty} \Big[n\, \frac{1}{n} \Big].$$

We conclude: $\displaystyle\int_{-a}^{a} \delta(t)\, dt = 1.$ $\square$

Properties of Dirac's delta.

Theorem

If $f : \mathbb{R} \to \mathbb{R}$ *is continuous,* $t_0 \in \mathbb{R}$ *and* $a > 0$*, then*

$$\int_{t_0-a}^{t_0+a} \delta(t - t_0)\, f(t)\, dt = f(t_0).$$

Properties of Dirac's delta.

Theorem

If $f : \mathbb{R} \to \mathbb{R}$ is continuous, $t_0 \in \mathbb{R}$ and $a > 0$, then

$$\int_{t_0-a}^{t_0+a} \delta(t - t_0)\, f(t)\, dt = f(t_0).$$

Proof: Introduce the change of variable $\tau = t - t_0$,

$$I = \int_{t_0-a}^{t_0+a} \delta(t - t_0)\, f(t)\, dt$$

Properties of Dirac's delta.

Theorem

If $f : \mathbb{R} \to \mathbb{R}$ is continuous, $t_0 \in \mathbb{R}$ and $a > 0$, then

$$\int_{t_0-a}^{t_0+a} \delta(t - t_0)\, f(t)\, dt = f(t_0).$$

Proof: Introduce the change of variable $\tau = t - t_0$,

$$I = \int_{t_0-a}^{t_0+a} \delta(t - t_0)\, f(t)\, dt = \int_{-a}^{a} \delta(\tau)\, f(\tau + t_0)\, d\tau,$$

Properties of Dirac's delta.

Theorem

If $f : \mathbb{R} \to \mathbb{R}$ is continuous, $t_0 \in \mathbb{R}$ and $a > 0$, then

$$\int_{t_0-a}^{t_0+a} \delta(t - t_0)\, f(t)\, dt = f(t_0).$$

Proof: Introduce the change of variable $\tau = t - t_0$,

$$I = \int_{t_0-a}^{t_0+a} \delta(t - t_0)\, f(t)\, dt = \int_{-a}^{a} \delta(\tau)\, f(\tau + t_0)\, d\tau,$$

$$I = \lim_{n\to\infty} \int_{-a}^{a} \delta_n(\tau)\, f(\tau + t_0)\, d\tau$$

Properties of Dirac's delta.

Theorem

If $f : \mathbb{R} \to \mathbb{R}$ is continuous, $t_0 \in \mathbb{R}$ and $a > 0$, then

$$\int_{t_0-a}^{t_0+a} \delta(t - t_0)\, f(t)\, dt = f(t_0).$$

Proof: Introduce the change of variable $\tau = t - t_0$,

$$I = \int_{t_0-a}^{t_0+a} \delta(t - t_0)\, f(t)\, dt = \int_{-a}^{a} \delta(\tau)\, f(\tau + t_0)\, d\tau,$$

$$I = \lim_{n\to\infty} \int_{-a}^{a} \delta_n(\tau)\, f(\tau + t_0)\, d\tau = \lim_{n\to\infty} \int_{0}^{1/n} n\, f(\tau + t_0)\, d\tau$$

Properties of Dirac's delta.

Theorem

If $f : \mathbb{R} \to \mathbb{R}$ is continuous, $t_0 \in \mathbb{R}$ and $a > 0$, then

$$\int_{t_0-a}^{t_0+a} \delta(t - t_0)\, f(t)\, dt = f(t_0).$$

Proof: Introduce the change of variable $\tau = t - t_0$,

$$I = \int_{t_0-a}^{t_0+a} \delta(t - t_0)\, f(t)\, dt = \int_{-a}^{a} \delta(\tau)\, f(\tau + t_0)\, d\tau,$$

$$I = \lim_{n\to\infty} \int_{-a}^{a} \delta_n(\tau)\, f(\tau + t_0)\, d\tau = \lim_{n\to\infty} \int_{0}^{1/n} n\, f(\tau + t_0)\, d\tau$$

Therefore, $I = \lim_{n\to\infty} n \int_{0}^{1/n} F'(\tau + t_0)\, d\tau,$

Properties of Dirac's delta.

Theorem

If $f : \mathbb{R} \to \mathbb{R}$ is continuous, $t_0 \in \mathbb{R}$ and $a > 0$, then

$$\int_{t_0-a}^{t_0+a} \delta(t - t_0)\, f(t)\, dt = f(t_0).$$

Proof: Introduce the change of variable $\tau = t - t_0$,

$$I = \int_{t_0-a}^{t_0+a} \delta(t - t_0)\, f(t)\, dt = \int_{-a}^{a} \delta(\tau)\, f(\tau + t_0)\, d\tau,$$

$$I = \lim_{n\to\infty} \int_{-a}^{a} \delta_n(\tau)\, f(\tau + t_0)\, d\tau = \lim_{n\to\infty} \int_{0}^{1/n} n\, f(\tau + t_0)\, d\tau$$

Therefore, $I = \lim_{n\to\infty} n \int_0^{1/n} F'(\tau + t_0)\, d\tau$, where we introduced the primitive $F(t) = \int f(t)\, dt$, that is, $f(t) = F'(t)$.

Properties of Dirac's delta.

Theorem

If $f : \mathbb{R} \to \mathbb{R}$ is continuous, $t_0 \in \mathbb{R}$ and $a > 0$, then

$$\int_{t_0-a}^{t_0+a} \delta(t - t_0)\, f(t)\, dt = f(t_0).$$

Proof: So, $I = \lim_{n\to\infty} n \int_0^{1/n} F'(\tau + t_0)\, d\tau$, with $f(t) = F'(t)$.

Properties of Dirac's delta.

Theorem

If $f : \mathbb{R} \to \mathbb{R}$ *is continuous,* $t_0 \in \mathbb{R}$ *and* $a > 0$*, then*

$$\int_{t_0 - a}^{t_0 + a} \delta(t - t_0)\, f(t)\, dt = f(t_0).$$

Proof: So, $I = \lim_{n \to \infty} n \int_0^{1/n} F'(\tau + t_0)\, d\tau$, with $f(t) = F'(t)$.

$$I = \lim_{n \to \infty} n \left[F(\tau + t_0)\Big|_0^{1/n} \right]$$

Properties of Dirac's delta.

Theorem

If $f : \mathbb{R} \to \mathbb{R}$ is continuous, $t_0 \in \mathbb{R}$ and $a > 0$, then

$$\int_{t_0-a}^{t_0+a} \delta(t - t_0)\, f(t)\, dt = f(t_0).$$

Proof: So, $I = \lim_{n\to\infty} n \int_0^{1/n} F'(\tau + t_0)\, d\tau$, with $f(t) = F'(t)$.

$$I = \lim_{n\to\infty} n \left[F(\tau + t_0)\Big|_0^{1/n} \right] = \lim_{n\to\infty} n \left[F\Big(t_0 + \frac{1}{n}\Big) - F(t_0) \right].$$

Properties of Dirac's delta.

Theorem

If $f : \mathbb{R} \to \mathbb{R}$ is continuous, $t_0 \in \mathbb{R}$ and $a > 0$, then

$$\int_{t_0-a}^{t_0+a} \delta(t - t_0)\, f(t)\, dt = f(t_0).$$

Proof: So, $I = \lim_{n\to\infty} n \int_0^{1/n} F'(\tau + t_0)\, d\tau$, with $f(t) = F'(t)$.

$$I = \lim_{n\to\infty} n \left[F(\tau + t_0)\Big|_0^{1/n} \right] = \lim_{n\to\infty} n \left[F\left(t_0 + \frac{1}{n}\right) - F(t_0) \right].$$

$$I = \lim_{n\to\infty} \frac{\left[F\left(t_0 + \frac{1}{n}\right) - F(t_0) \right]}{\frac{1}{n}}$$

Properties of Dirac's delta.

Theorem

If $f : \mathbb{R} \to \mathbb{R}$ *is continuous,* $t_0 \in \mathbb{R}$ *and* $a > 0$*, then*

$$\int_{t_0 - a}^{t_0 + a} \delta(t - t_0)\, f(t)\, dt = f(t_0).$$

Proof: So, $I = \lim_{n\to\infty} n \int_0^{1/n} F'(\tau + t_0)\, d\tau$, with $f(t) = F'(t)$.

$$I = \lim_{n\to\infty} n \left[F(\tau + t_0)\Big|_0^{1/n} \right] = \lim_{n\to\infty} n \left[F\left(t_0 + \frac{1}{n}\right) - F(t_0) \right].$$

$$I = \lim_{n\to\infty} \frac{\left[F\left(t_0 + \frac{1}{n}\right) - F(t_0) \right]}{\frac{1}{n}} = F'(t_0)$$

Properties of Dirac's delta.

Theorem

If $f : \mathbb{R} \to \mathbb{R}$ *is continuous,* $t_0 \in \mathbb{R}$ *and* $a > 0$*, then*

$$\int_{t_0-a}^{t_0+a} \delta(t - t_0)\, f(t)\, dt = f(t_0).$$

Proof: So, $I = \lim_{n\to\infty} n \int_0^{1/n} F'(\tau + t_0)\, d\tau$, with $f(t) = F'(t)$.

$$I = \lim_{n\to\infty} n \Big[F(\tau + t_0)\Big|_0^{1/n} \Big] = \lim_{n\to\infty} n \Big[F\Big(t_0 + \frac{1}{n}\Big) - F(t_0) \Big].$$

$$I = \lim_{n\to\infty} \frac{\Big[F\Big(t_0 + \frac{1}{n}\Big) - F(t_0) \Big]}{\frac{1}{n}} = F'(t_0) = f(t_0).$$

Properties of Dirac's delta.

Theorem

If $f : \mathbb{R} \to \mathbb{R}$ is continuous, $t_0 \in \mathbb{R}$ and $a > 0$, then

$$\int_{t_0-a}^{t_0+a} \delta(t - t_0)\, f(t)\, dt = f(t_0).$$

Proof: So, $I = \lim_{n\to\infty} n \int_0^{1/n} F'(\tau + t_0)\, d\tau$, with $f(t) = F'(t)$.

$$I = \lim_{n\to\infty} n \left[F(\tau + t_0)\Big|_0^{1/n} \right] = \lim_{n\to\infty} n \left[F\Big(t_0 + \frac{1}{n}\Big) - F(t_0) \right].$$

$$I = \lim_{n\to\infty} \frac{\left[F\big(t_0 + \frac{1}{n}\big) - F(t_0) \right]}{\frac{1}{n}} = F'(t_0) = f(t_0).$$

We conclude: $\displaystyle\int_{t_0-a}^{t_0+a} \delta(t - t_0)\, f(t)\, dt = f(t_0).$ □

Generalized sources (Sect. 6.5).

- The Dirac delta generalized function.
- Properties of Dirac's delta.
- **Relation between deltas and steps.**
- Dirac's delta in Physics.
- The Laplace Transform of Dirac's delta.
- Differential equations with Dirac's delta sources.

Relation between deltas and steps.

Theorem

The sequence of functions for $n \geqslant 1$*,*

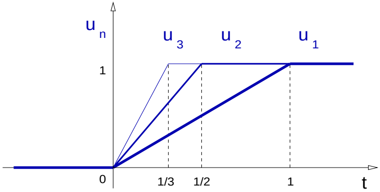
$$u_n(t) = \begin{cases} 0, & t < 0 \\ nt, & 0 \leqslant t \leqslant \dfrac{1}{n} \\ 1, & t > \dfrac{1}{n}. \end{cases}$$

Relation between deltas and steps.

Theorem

The sequence of functions for $n \geqslant 1$,

$$u_n(t) = \begin{cases} 0, & t < 0 \\ nt, & 0 \leqslant t \leqslant \dfrac{1}{n} \\ 1, & t > \dfrac{1}{n}. \end{cases}$$

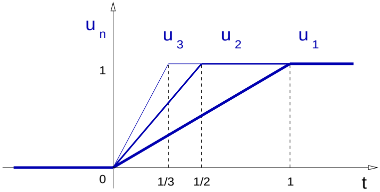


Relation between deltas and steps.

Theorem

The sequence of functions for $n \geqslant 1$,

$$u_n(t) = \begin{cases} 0, & t < 0 \\ nt, & 0 \leqslant t \leqslant \dfrac{1}{n} \\ 1, & t > \dfrac{1}{n}. \end{cases}$$



satisfies, for $t \in (-\infty, 0) \cup (0, 1/n) \cup (1/n, \infty)$, both equations,

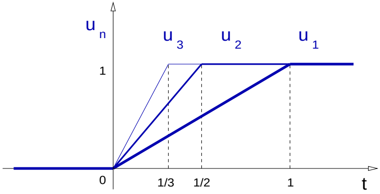
$$u_n'(t) = \delta_n(t), \qquad \lim_{n\to\infty} u_n(t) = u(t), \qquad t \in \mathbb{R}.$$

Relation between deltas and steps.

Theorem

The sequence of functions for $n \geqslant 1$,

$$u_n(t) = \begin{cases} 0, & t < 0 \\ nt, & 0 \leqslant t \leqslant \dfrac{1}{n} \\ 1, & t > \dfrac{1}{n}. \end{cases}$$



satisfies, for $t \in (-\infty, 0) \cup (0, 1/n) \cup (1/n, \infty)$, both equations,

$$u_n'(t) = \delta_n(t), \qquad \lim_{n\to\infty} u_n(t) = u(t), \qquad t \in \mathbb{R}.$$

Remark:

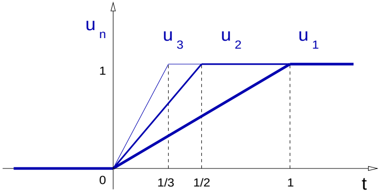
- If we generalize the notion of derivative as $u'(t) = \lim_{n\to\infty} \delta_n(t)$, then holds $u'(t) = \delta(t)$.

Relation between deltas and steps.

Theorem

The sequence of functions for $n \geqslant 1$,

$$u_n(t) = \begin{cases} 0, & t < 0 \\ nt, & 0 \leqslant t \leqslant \dfrac{1}{n} \\ 1, & t > \dfrac{1}{n}. \end{cases}$$



satisfies, for $t \in (-\infty, 0) \cup (0, 1/n) \cup (1/n, \infty)$, both equations,

$$u_n'(t) = \delta_n(t), \qquad \lim_{n\to\infty} u_n(t) = u(t), \qquad t \in \mathbb{R}.$$

Remark:

- If we generalize the notion of derivative as $u'(t) = \lim_{n\to\infty} \delta_n(t)$, then holds $u'(t) = \delta(t)$.
- Dirac's delta is a generalized derivative of the step function.

Generalized sources (Sect. 6.5).

- The Dirac delta generalized function.
- Properties of Dirac's delta.
- Relation between deltas and steps.
- **Dirac's delta in Physics.**
- The Laplace Transform of Dirac's delta.
- Differential equations with Dirac's delta sources.

Dirac's delta in Physics.

Remarks:

(a) Dirac's delta generalized function is useful to describe *impulsive forces* in mechanical systems.

Dirac's delta in Physics.

Remarks:

(a) Dirac's delta generalized function is useful to describe *impulsive forces* in mechanical systems.

(b) An impulsive force transmits a finite momentum in an infinitely short time.

Dirac's delta in Physics.

Remarks:

(a) Dirac's delta generalized function is useful to describe *impulsive forces* in mechanical systems.

(b) An impulsive force transmits a finite momentum in an infinitely short time.

(c) For example: The momentum transmitted to a pendulum when hit by a hammer.

Dirac's delta in Physics.

Remarks:

(a) Dirac's delta generalized function is useful to describe *impulsive forces* in mechanical systems.

(b) An impulsive force transmits a finite momentum in an infinitely short time.

(c) For example: The momentum transmitted to a pendulum when hit by a hammer. Newton's law of motion says,

$$m\, v'(t) = F(t),$$

Dirac's delta in Physics.

Remarks:

(a) Dirac's delta generalized function is useful to describe *impulsive forces* in mechanical systems.

(b) An impulsive force transmits a finite momentum in an infinitely short time.

(c) For example: The momentum transmitted to a pendulum when hit by a hammer. Newton's law of motion says,

$$m\,v'(t) = F(t), \quad \text{with} \quad F(t) = F_0\,\delta(t - t_0).$$

Dirac's delta in Physics.

Remarks:

(a) Dirac's delta generalized function is useful to describe *impulsive forces* in mechanical systems.

(b) An impulsive force transmits a finite momentum in an infinitely short time.

(c) For example: The momentum transmitted to a pendulum when hit by a hammer. Newton's law of motion says,

$$m\,v'(t) = F(t), \quad \text{with} \quad F(t) = F_0\,\delta(t - t_0).$$

The momentum transfer is:

$$\Delta I = \lim_{\Delta t \to 0} mv(t)\Big|_{t_0 - \Delta t}^{t_0 + \Delta t}$$

Dirac's delta in Physics.

Remarks:

(a) Dirac's delta generalized function is useful to describe *impulsive forces* in mechanical systems.

(b) An impulsive force transmits a finite momentum in an infinitely short time.

(c) For example: The momentum transmitted to a pendulum when hit by a hammer. Newton's law of motion says,

$$m\,v'(t) = F(t), \quad \text{with} \quad F(t) = F_0\,\delta(t - t_0).$$

The momentum transfer is:

$$\Delta I = \lim_{\Delta t \to 0} mv(t)\Big|_{t_0-\Delta t}^{t_0+\Delta t} = \lim_{\Delta t \to 0} \int_{t_0-\Delta t}^{t_0+\Delta t} F(t)\,dt$$

Dirac's delta in Physics.

Remarks:

(a) Dirac's delta generalized function is useful to describe *impulsive forces* in mechanical systems.

(b) An impulsive force transmits a finite momentum in an infinitely short time.

(c) For example: The momentum transmitted to a pendulum when hit by a hammer. Newton's law of motion says,

$$m\,v'(t) = F(t), \quad \text{with} \quad F(t) = F_0\,\delta(t - t_0).$$

The momentum transfer is:

$$\Delta I = \lim_{\Delta t \to 0} mv(t)\Big|_{t_0-\Delta t}^{t_0+\Delta t} = \lim_{\Delta t \to 0} \int_{t_0-\Delta t}^{t_0+\Delta t} F(t)\,dt = F_0.$$

Dirac's delta in Physics.

Remarks:

(a) Dirac's delta generalized function is useful to describe *impulsive forces* in mechanical systems.

(b) An impulsive force transmits a finite momentum in an infinitely short time.

(c) For example: The momentum transmitted to a pendulum when hit by a hammer. Newton's law of motion says,

$$m\, v'(t) = F(t), \quad \text{with} \quad F(t) = F_0\, \delta(t - t_0).$$

The momentum transfer is:

$$\Delta I = \lim_{\Delta t \to 0} mv(t)\Big|_{t_0 - \Delta t}^{t_0 + \Delta t} = \lim_{\Delta t \to 0} \int_{t_0 - \Delta t}^{t_0 + \Delta t} F(t)\, dt = F_0.$$

That is, $\Delta I = F_0$.

Generalized sources (Sect. 6.5).

- The Dirac delta generalized function.
- Properties of Dirac's delta.
- Relation between deltas and steps.
- Dirac's delta in Physics.
- **The Laplace Transform of Dirac's delta.**
- Differential equations with Dirac's delta sources.

The Laplace Transform of Dirac's delta.

Recall: The Laplace Transform can be generalized from functions to δ,

The Laplace Transform of Dirac's delta.

Recall: The Laplace Transform can be generalized from functions to δ, as follows, $\mathcal{L}[\delta(t-c)] = \lim_{n\to\infty} \mathcal{L}[\delta_n(t-c)]$.

The Laplace Transform of Dirac's delta.

Recall: The Laplace Transform can be generalized from functions to δ, as follows, $\mathcal{L}[\delta(t-c)] = \lim_{n\to\infty} \mathcal{L}[\delta_n(t-c)]$.

Theorem

$$\mathcal{L}[\delta(t-c)] = e^{-cs}.$$

The Laplace Transform of Dirac's delta.

Recall: The Laplace Transform can be generalized from functions to δ, as follows, $\mathcal{L}[\delta(t-c)] = \lim_{n\to\infty} \mathcal{L}[\delta_n(t-c)]$.

Theorem

$$\mathcal{L}[\delta(t-c)] = e^{-cs}.$$

Proof:

$$\mathcal{L}[\delta(t-c)] = \lim_{n\to\infty} \mathcal{L}[\delta_n(t-c)],$$

The Laplace Transform of Dirac's delta.

Recall: The Laplace Transform can be generalized from functions to δ, as follows, $\mathcal{L}[\delta(t-c)] = \lim_{n\to\infty} \mathcal{L}[\delta_n(t-c)]$.

Theorem

$$\mathcal{L}[\delta(t-c)] = e^{-cs}.$$

Proof:

$$\mathcal{L}[\delta(t-c)] = \lim_{n\to\infty} \mathcal{L}[\delta_n(t-c)], \qquad \delta_n(t) = n\left[u(t) - u\big(t - \frac{1}{n}\big)\right].$$

The Laplace Transform of Dirac's delta.

Recall: The Laplace Transform can be generalized from functions to δ, as follows, $\mathcal{L}[\delta(t-c)] = \lim_{n\to\infty} \mathcal{L}[\delta_n(t-c)]$.

Theorem

$$\mathcal{L}[\delta(t-c)] = e^{-cs}.$$

Proof:

$$\mathcal{L}[\delta(t-c)] = \lim_{n\to\infty} \mathcal{L}[\delta_n(t-c)], \qquad \delta_n(t) = n\left[u(t) - u\big(t - \frac{1}{n}\big)\right].$$

$$\mathcal{L}[\delta(t-c)] = \lim_{n\to\infty} n\Big(\mathcal{L}[u(t-c)] - \mathcal{L}\Big[u\big(t-c-\frac{1}{n}\big)\Big]\Big)$$

The Laplace Transform of Dirac's delta.

Recall: The Laplace Transform can be generalized from functions to δ, as follows, $\mathcal{L}[\delta(t-c)] = \lim_{n\to\infty} \mathcal{L}[\delta_n(t-c)]$.

Theorem

$$\mathcal{L}[\delta(t-c)] = e^{-cs}.$$

Proof:

$$\mathcal{L}[\delta(t-c)] = \lim_{n\to\infty} \mathcal{L}[\delta_n(t-c)], \qquad \delta_n(t) = n\left[u(t) - u\left(t - \frac{1}{n}\right)\right].$$

$$\mathcal{L}[\delta(t-c)] = \lim_{n\to\infty} n\Big(\mathcal{L}[u(t-c)] - \mathcal{L}\Big[u\Big(t-c-\frac{1}{n}\Big)\Big]\Big)$$

$$\mathcal{L}[\delta(t-c)] = \lim_{n\to\infty} n\Big(\frac{e^{-cs}}{s} - \frac{e^{-(c+\frac{1}{n})s}}{s}\Big)$$

The Laplace Transform of Dirac's delta.

Recall: The Laplace Transform can be generalized from functions to δ, as follows, $\mathcal{L}[\delta(t-c)] = \lim_{n\to\infty} \mathcal{L}[\delta_n(t-c)]$.

Theorem

$$\mathcal{L}[\delta(t-c)] = e^{-cs}.$$

Proof:

$$\mathcal{L}[\delta(t-c)] = \lim_{n\to\infty} \mathcal{L}[\delta_n(t-c)], \qquad \delta_n(t) = n\Big[u(t) - u\Big(t - \frac{1}{n}\Big)\Big].$$

$$\mathcal{L}[\delta(t-c)] = \lim_{n\to\infty} n\Big(\mathcal{L}[u(t-c)] - \mathcal{L}\Big[u\Big(t-c-\frac{1}{n}\Big)\Big]\Big)$$

$$\mathcal{L}[\delta(t-c)] = \lim_{n\to\infty} n\Big(\frac{e^{-cs}}{s} - \frac{e^{-(c+\frac{1}{n})s}}{s}\Big) = e^{-cs} \lim_{n\to\infty} \frac{(1 - e^{-\frac{s}{n}})}{\left(\frac{s}{n}\right)}.$$

The Laplace Transform of Dirac's delta.

Recall: The Laplace Transform can be generalized from functions to δ, as follows, $\mathcal{L}[\delta(t-c)] = \lim_{n\to\infty} \mathcal{L}[\delta_n(t-c)]$.

Theorem

$$\mathcal{L}[\delta(t-c)] = e^{-cs}.$$

Proof:

$$\mathcal{L}[\delta(t-c)] = \lim_{n\to\infty} \mathcal{L}[\delta_n(t-c)], \qquad \delta_n(t) = n\left[u(t) - u\Big(t - \frac{1}{n}\Big)\right].$$

$$\mathcal{L}[\delta(t-c)] = \lim_{n\to\infty} n\Big(\mathcal{L}[u(t-c)] - \mathcal{L}\Big[u\Big(t - c - \frac{1}{n}\Big)\Big]\Big)$$

$$\mathcal{L}[\delta(t-c)] = \lim_{n\to\infty} n\Big(\frac{e^{-cs}}{s} - \frac{e^{-(c+\frac{1}{n})s}}{s}\Big) = e^{-cs} \lim_{n\to\infty} \frac{(1 - e^{-\frac{s}{n}})}{\big(\frac{s}{n}\big)}.$$

This is a singular limit, $\frac{0}{0}$.

The Laplace Transform of Dirac's delta.

Recall: The Laplace Transform can be generalized from functions to δ, as follows, $\mathcal{L}[\delta(t-c)] = \lim_{n\to\infty} \mathcal{L}[\delta_n(t-c)]$.

Theorem

$$\mathcal{L}[\delta(t-c)] = e^{-cs}.$$

Proof:

$$\mathcal{L}[\delta(t-c)] = \lim_{n\to\infty} \mathcal{L}[\delta_n(t-c)], \qquad \delta_n(t) = n\left[u(t) - u\left(t - \frac{1}{n}\right)\right].$$

$$\mathcal{L}[\delta(t-c)] = \lim_{n\to\infty} n\Big(\mathcal{L}[u(t-c)] - \mathcal{L}\Big[u\Big(t-c-\frac{1}{n}\Big)\Big]\Big)$$

$$\mathcal{L}[\delta(t-c)] = \lim_{n\to\infty} n\left(\frac{e^{-cs}}{s} - \frac{e^{-(c+\frac{1}{n})s}}{s}\right) = e^{-cs} \lim_{n\to\infty} \frac{(1-e^{-\frac{s}{n}})}{\left(\frac{s}{n}\right)}.$$

This is a singular limit, $\frac{0}{0}$. Use l'Hôpital rule.

The Laplace Transform of Dirac's delta.

Proof: Recall: $\mathcal{L}[\delta(t-c)] = e^{-cs} \lim_{n\to\infty} \dfrac{(1 - e^{-\frac{s}{n}})}{\left(\frac{s}{n}\right)}$.

The Laplace Transform of Dirac's delta.

Proof: Recall: $\mathcal{L}[\delta(t-c)] = e^{-cs} \lim_{n\to\infty} \dfrac{(1 - e^{-\frac{s}{n}})}{\left(\frac{s}{n}\right)}$.

$$\lim_{n\to\infty} \frac{(1 - e^{-\frac{s}{n}})}{\left(\frac{s}{n}\right)}$$

The Laplace Transform of Dirac's delta.

Proof: Recall: $\mathcal{L}[\delta(t-c)] = e^{-cs} \lim_{n\to\infty} \frac{\left(1 - e^{-\frac{s}{n}}\right)}{\left(\frac{s}{n}\right)}$.

$$\lim_{n\to\infty} \frac{\left(1 - e^{-\frac{s}{n}}\right)}{\left(\frac{s}{n}\right)} = \lim_{n\to\infty} \frac{\left(-\frac{s}{n^2}\, e^{-\frac{s}{n}}\right)}{\left(-\frac{s}{n^2}\right)}$$

The Laplace Transform of Dirac's delta.

Proof: Recall: $\mathcal{L}[\delta(t-c)] = e^{-cs} \lim_{n\to\infty} \frac{\left(1 - e^{-\frac{s}{n}}\right)}{\left(\frac{s}{n}\right)}$.

$$\lim_{n\to\infty} \frac{\left(1 - e^{-\frac{s}{n}}\right)}{\left(\frac{s}{n}\right)} = \lim_{n\to\infty} \frac{\left(-\frac{s}{n^2}\, e^{-\frac{s}{n}}\right)}{\left(-\frac{s}{n^2}\right)} = \lim_{n\to\infty} e^{-\frac{s}{n}}$$

The Laplace Transform of Dirac's delta.

Proof: Recall: $\mathcal{L}[\delta(t-c)] = e^{-cs} \lim_{n\to\infty} \dfrac{\left(1 - e^{-\frac{s}{n}}\right)}{\left(\frac{s}{n}\right)}$.

$$\lim_{n\to\infty} \frac{\left(1 - e^{-\frac{s}{n}}\right)}{\left(\frac{s}{n}\right)} = \lim_{n\to\infty} \frac{\left(-\frac{s}{n^2}\, e^{-\frac{s}{n}}\right)}{\left(-\frac{s}{n^2}\right)} = \lim_{n\to\infty} e^{-\frac{s}{n}} = 1.$$

The Laplace Transform of Dirac's delta.

Proof: Recall: $\mathcal{L}[\delta(t-c)] = e^{-cs} \lim_{n\to\infty} \dfrac{\left(1 - e^{-\frac{s}{n}}\right)}{\left(\frac{s}{n}\right)}$.

$$\lim_{n\to\infty} \frac{\left(1 - e^{-\frac{s}{n}}\right)}{\left(\frac{s}{n}\right)} = \lim_{n\to\infty} \frac{\left(-\frac{s}{n^2}\, e^{-\frac{s}{n}}\right)}{\left(-\frac{s}{n^2}\right)} = \lim_{n\to\infty} e^{-\frac{s}{n}} = 1.$$

We therefore conclude that $\mathcal{L}[\delta(t-c)] = e^{-cs}$. □

The Laplace Transform of Dirac's delta.

Proof: Recall: $\mathcal{L}[\delta(t-c)] = e^{-cs} \lim_{n\to\infty} \dfrac{\left(1 - e^{-\frac{s}{n}}\right)}{\left(\frac{s}{n}\right)}$.

$$\lim_{n\to\infty} \frac{\left(1 - e^{-\frac{s}{n}}\right)}{\left(\frac{s}{n}\right)} = \lim_{n\to\infty} \frac{\left(-\frac{s}{n^2}\, e^{-\frac{s}{n}}\right)}{\left(-\frac{s}{n^2}\right)} = \lim_{n\to\infty} e^{-\frac{s}{n}} = 1.$$

We therefore conclude that $\mathcal{L}[\delta(t-c)] = e^{-cs}$. □

Remarks:

(a) This result is consistent with a previous result:

The Laplace Transform of Dirac's delta.

Proof: Recall: $\mathcal{L}[\delta(t-c)] = e^{-cs} \lim_{n\to\infty} \dfrac{\left(1 - e^{-\frac{s}{n}}\right)}{\left(\frac{s}{n}\right)}$.

$$\lim_{n\to\infty} \frac{\left(1 - e^{-\frac{s}{n}}\right)}{\left(\frac{s}{n}\right)} = \lim_{n\to\infty} \frac{\left(-\frac{s}{n^2}\, e^{-\frac{s}{n}}\right)}{\left(-\frac{s}{n^2}\right)} = \lim_{n\to\infty} e^{-\frac{s}{n}} = 1.$$

We therefore conclude that $\mathcal{L}[\delta(t-c)] = e^{-cs}$. □

Remarks:

(a) This result is consistent with a previous result:

$$\int_{t_0-a}^{t_0+a} \delta(t-t_0)\, f(t)\, dt = f(t_0).$$

The Laplace Transform of Dirac's delta.

Proof: Recall: $\mathcal{L}[\delta(t-c)] = e^{-cs} \lim_{n\to\infty} \dfrac{\left(1 - e^{-\frac{s}{n}}\right)}{\left(\frac{s}{n}\right)}$.

$$\lim_{n\to\infty} \frac{\left(1 - e^{-\frac{s}{n}}\right)}{\left(\frac{s}{n}\right)} = \lim_{n\to\infty} \frac{\left(-\frac{s}{n^2}\, e^{-\frac{s}{n}}\right)}{\left(-\frac{s}{n^2}\right)} = \lim_{n\to\infty} e^{-\frac{s}{n}} = 1.$$

We therefore conclude that $\mathcal{L}[\delta(t-c)] = e^{-cs}$. □

Remarks:

(a) This result is consistent with a previous result:
$$\int_{t_0-a}^{t_0+a} \delta(t-t_0)\, f(t)\, dt = f(t_0).$$

(b) $\mathcal{L}[\delta(t-c)] = \displaystyle\int_0^{\infty} \delta(t-c)\, e^{-st}\, dt = e^{-cs}$.

The Laplace Transform of Dirac's delta.

Proof: Recall: $\mathcal{L}[\delta(t-c)] = e^{-cs} \lim_{n\to\infty} \dfrac{\left(1 - e^{-\frac{s}{n}}\right)}{\left(\frac{s}{n}\right)}$.

$$\lim_{n\to\infty} \frac{\left(1 - e^{-\frac{s}{n}}\right)}{\left(\frac{s}{n}\right)} = \lim_{n\to\infty} \frac{\left(-\frac{s}{n^2}\, e^{-\frac{s}{n}}\right)}{\left(-\frac{s}{n^2}\right)} = \lim_{n\to\infty} e^{-\frac{s}{n}} = 1.$$

We therefore conclude that $\mathcal{L}[\delta(t-c)] = e^{-cs}$. □

Remarks:

(a) This result is consistent with a previous result:
$$\int_{t_0-a}^{t_0+a} \delta(t-t_0)\, f(t)\, dt = f(t_0).$$

(b) $\displaystyle \mathcal{L}[\delta(t-c)] = \int_0^\infty \delta(t-c)\, e^{-st}\, dt = e^{-cs}$.

(c) $\displaystyle \mathcal{L}[\delta(t-c)\, f(t)] = \int_0^\infty \delta(t-c)\, e^{-st}\, f(t)\, dt = e^{-cs}\, f(c)$.

Generalized sources (Sect. 6.5).

- The Dirac delta generalized function.
- Properties of Dirac's delta.
- Relation between deltas and steps.
- Dirac's delta in Physics.
- The Laplace Transform of Dirac's delta.
- **Differential equations with Dirac's delta sources.**

Differential equations with Dirac's delta sources.

Example

Find the solution y to the initial value problem

$$y'' - y = -20\,\delta(t-3), \qquad y(0) = 1, \qquad y'(0) = 0.$$

Differential equations with Dirac's delta sources.

Example

Find the solution y to the initial value problem

$$y'' - y = -20\,\delta(t-3), \qquad y(0) = 1, \qquad y'(0) = 0.$$

Solution: Compute: $\mathcal{L}[y''] - \mathcal{L}[y] = -20\,\mathcal{L}[\delta(t-3)]$.

Differential equations with Dirac's delta sources.

Example

Find the solution y to the initial value problem

$$y'' - y = -20\,\delta(t-3), \qquad y(0) = 1, \qquad y'(0) = 0.$$

Solution: Compute: $\mathcal{L}[y''] - \mathcal{L}[y] = -20\,\mathcal{L}[\delta(t-3)].$

$\mathcal{L}[y''] = s^2\,\mathcal{L}[y] - s\,y(0) - y'(0)$

Differential equations with Dirac's delta sources.

Example

Find the solution y to the initial value problem

$$y'' - y = -20\,\delta(t-3), \qquad y(0) = 1, \qquad y'(0) = 0.$$

Solution: Compute: $\mathcal{L}[y''] - \mathcal{L}[y] = -20\,\mathcal{L}[\delta(t-3)]$.

$$\mathcal{L}[y''] = s^2\,\mathcal{L}[y] - s\,y(0) - y'(0) \quad \Rightarrow \quad (s^2-1)\,\mathcal{L}[y] - s = -20\,e^{-3s},$$

Differential equations with Dirac's delta sources.

Example

Find the solution y to the initial value problem

$$y'' - y = -20\,\delta(t-3), \qquad y(0) = 1, \qquad y'(0) = 0.$$

Solution: Compute: $\mathcal{L}[y''] - \mathcal{L}[y] = -20\,\mathcal{L}[\delta(t-3)]$.

$$\mathcal{L}[y''] = s^2\,\mathcal{L}[y] - s\,y(0) - y'(0) \quad \Rightarrow \quad (s^2-1)\,\mathcal{L}[y] - s = -20\,e^{-3s},$$

We arrive to the equation $\mathcal{L}[y] = \dfrac{s}{(s^2-1)} - 20\,e^{-3s}\,\dfrac{1}{(s^2-1)},$

Differential equations with Dirac's delta sources.

Example

Find the solution y to the initial value problem

$$y'' - y = -20\,\delta(t-3), \qquad y(0) = 1, \qquad y'(0) = 0.$$

Solution: Compute: $\mathcal{L}[y''] - \mathcal{L}[y] = -20\,\mathcal{L}[\delta(t-3)]$.

$$\mathcal{L}[y''] = s^2\,\mathcal{L}[y] - s\,y(0) - y'(0) \quad \Rightarrow \quad (s^2 - 1)\,\mathcal{L}[y] - s = -20\,e^{-3s},$$

We arrive to the equation $\mathcal{L}[y] = \dfrac{s}{(s^2-1)} - 20\,e^{-3s}\,\dfrac{1}{(s^2-1)}$,

$$\mathcal{L}[y] = \mathcal{L}[\cosh(t)] - 20\,\mathcal{L}[u(t-3)\,\sinh(t-3)],$$

Differential equations with Dirac's delta sources.

Example

Find the solution y to the initial value problem

$$y'' - y = -20\,\delta(t-3), \qquad y(0) = 1, \qquad y'(0) = 0.$$

Solution: Compute: $\mathcal{L}[y''] - \mathcal{L}[y] = -20\,\mathcal{L}[\delta(t-3)]$.

$$\mathcal{L}[y''] = s^2\,\mathcal{L}[y] - s\,y(0) - y'(0) \quad \Rightarrow \quad (s^2-1)\,\mathcal{L}[y] - s = -20\,e^{-3s},$$

We arrive to the equation $\mathcal{L}[y] = \dfrac{s}{(s^2-1)} - 20\,e^{-3s}\,\dfrac{1}{(s^2-1)}$,

$$\mathcal{L}[y] = \mathcal{L}[\cosh(t)] - 20\,\mathcal{L}[u(t-3)\,\sinh(t-3)],$$

We conclude: $y(t) = \cosh(t) - 20\,u(t-3)\,\sinh(t-3)$. $\triangleleft$

Differential equations with Dirac's delta sources.

Example

Find the solution to the initial value problem

$$y'' + 4y = \delta(t - \pi) - \delta(t - 2\pi), \qquad y(0) = 0, \qquad y'(0) = 0.$$

Differential equations with Dirac's delta sources.

Example

Find the solution to the initial value problem

$$y'' + 4y = \delta(t - \pi) - \delta(t - 2\pi), \qquad y(0) = 0, \qquad y'(0) = 0.$$

Solution: Compute: $\mathcal{L}[y''] + 4\,\mathcal{L}[y] = \mathcal{L}[\delta(t - \pi)] - \mathcal{L}[\delta(t - 2\pi)]$,

Differential equations with Dirac's delta sources.

Example

Find the solution to the initial value problem

$$y'' + 4y = \delta(t-\pi) - \delta(t-2\pi), \qquad y(0) = 0, \qquad y'(0) = 0.$$

Solution: Compute: $\mathcal{L}[y''] + 4\,\mathcal{L}[y] = \mathcal{L}[\delta(t-\pi)] - \mathcal{L}[\delta(t-2\pi)]$,

$$(s^2+4)\,\mathcal{L}[y] = e^{-\pi s} - e^{-2\pi s}$$

Differential equations with Dirac's delta sources.

Example

Find the solution to the initial value problem

$$y'' + 4y = \delta(t-\pi) - \delta(t-2\pi), \qquad y(0) = 0, \qquad y'(0) = 0.$$

Solution: Compute: $\mathcal{L}[y''] + 4\,\mathcal{L}[y] = \mathcal{L}[\delta(t-\pi)] - \mathcal{L}[\delta(t-2\pi)]$,

$$(s^2+4)\,\mathcal{L}[y] = e^{-\pi s} - e^{-2\pi s} \quad \Rightarrow \quad \mathcal{L}[y] = \frac{e^{-\pi s}}{(s^2+4)} - \frac{e^{-2\pi s}}{(s^2+4)},$$

Differential equations with Dirac's delta sources.

Example

Find the solution to the initial value problem

$$y'' + 4y = \delta(t-\pi) - \delta(t-2\pi), \qquad y(0) = 0, \qquad y'(0) = 0.$$

Solution: Compute: $\mathcal{L}[y''] + 4\,\mathcal{L}[y] = \mathcal{L}[\delta(t-\pi)] - \mathcal{L}[\delta(t-2\pi)]$,

$$(s^2+4)\,\mathcal{L}[y] = e^{-\pi s} - e^{-2\pi s} \quad \Rightarrow \quad \mathcal{L}[y] = \frac{e^{-\pi s}}{(s^2+4)} - \frac{e^{-2\pi s}}{(s^2+4)},$$

that is, $\mathcal{L}[y] = \dfrac{e^{-\pi s}}{2}\,\dfrac{2}{(s^2+4)} - \dfrac{e^{-2\pi s}}{2}\,\dfrac{2}{(s^2+4)}$.

Differential equations with Dirac's delta sources.

Example

Find the solution to the initial value problem

$$y'' + 4y = \delta(t - \pi) - \delta(t - 2\pi), \qquad y(0) = 0, \qquad y'(0) = 0.$$

Solution: Compute: $\mathcal{L}[y''] + 4\,\mathcal{L}[y] = \mathcal{L}[\delta(t - \pi)] - \mathcal{L}[\delta(t - 2\pi)]$,

$$(s^2 + 4)\,\mathcal{L}[y] = e^{-\pi s} - e^{-2\pi s} \quad \Rightarrow \quad \mathcal{L}[y] = \frac{e^{-\pi s}}{(s^2 + 4)} - \frac{e^{-2\pi s}}{(s^2 + 4)},$$

that is, $\mathcal{L}[y] = \dfrac{e^{-\pi s}}{2}\,\dfrac{2}{(s^2 + 4)} - \dfrac{e^{-2\pi s}}{2}\,\dfrac{2}{(s^2 + 4)}$.

Recall: $e^{-cs}\,\mathcal{L}[f(t)] = \mathcal{L}[u(t - c)\,f(t - c)]$.

Differential equations with Dirac's delta sources.

Example

Find the solution to the initial value problem

$$y'' + 4y = \delta(t-\pi) - \delta(t-2\pi), \qquad y(0) = 0, \qquad y'(0) = 0.$$

Solution: Compute: $\mathcal{L}[y''] + 4\,\mathcal{L}[y] = \mathcal{L}[\delta(t-\pi)] - \mathcal{L}[\delta(t-2\pi)]$,

$$(s^2+4)\,\mathcal{L}[y] = e^{-\pi s} - e^{-2\pi s} \quad \Rightarrow \quad \mathcal{L}[y] = \frac{e^{-\pi s}}{(s^2+4)} - \frac{e^{-2\pi s}}{(s^2+4)},$$

that is, $\mathcal{L}[y] = \dfrac{e^{-\pi s}}{2}\,\dfrac{2}{(s^2+4)} - \dfrac{e^{-2\pi s}}{2}\,\dfrac{2}{(s^2+4)}$.

Recall: $e^{-cs}\,\mathcal{L}[f(t)] = \mathcal{L}[u(t-c)\,f(t-c)]$. Therefore,

$$\mathcal{L}[y] = \frac{1}{2}\,\mathcal{L}\Big[u(t-\pi)\,\sin\big[2(t-\pi)\big]\Big] - \frac{1}{2}\,\mathcal{L}\Big[u(t-2\pi)\,\sin\big[2(t-2\pi)\big]\Big].$$

Differential equations with Dirac's delta sources.

Example

Find the solution to the initial value problem

$$y'' + 4y = \delta(t-\pi) - \delta(t-2\pi), \qquad y(0) = 0, \qquad y'(0) = 0.$$

Solution: Recall:

$$\mathcal{L}[y] = \frac{1}{2}\,\mathcal{L}\Big[u(t-\pi)\,\sin\big[2(t-\pi)\big]\Big] - \frac{1}{2}\,\mathcal{L}\Big[u(t-2\pi)\,\sin\big[2(t-2\pi)\big]\Big].$$

Differential equations with Dirac's delta sources.

Example

Find the solution to the initial value problem

$$y'' + 4y = \delta(t-\pi) - \delta(t-2\pi), \qquad y(0) = 0, \qquad y'(0) = 0.$$

Solution: Recall:

$$\mathcal{L}[y] = \frac{1}{2}\,\mathcal{L}\Big[u(t-\pi)\,\sin\big[2(t-\pi)\big]\Big] - \frac{1}{2}\,\mathcal{L}\Big[u(t-2\pi)\,\sin\big[2(t-2\pi)\big]\Big].$$

This implies that,

$$y(t) = \frac{1}{2}\,u(t-\pi)\,\sin\big[2(t-\pi)\big] - \frac{1}{2}\,u(t-2\pi)\,\sin\big[2(t-2\pi)\big],$$

Differential equations with Dirac's delta sources.

Example

Find the solution to the initial value problem

$$y'' + 4y = \delta(t-\pi) - \delta(t-2\pi), \qquad y(0) = 0, \qquad y'(0) = 0.$$

Solution: Recall:

$$\mathcal{L}[y] = \frac{1}{2}\,\mathcal{L}\Big[u(t-\pi)\,\sin\big[2(t-\pi)\big]\Big] - \frac{1}{2}\,\mathcal{L}\Big[u(t-2\pi)\,\sin\big[2(t-2\pi)\big]\Big].$$

This implies that,

$$y(t) = \frac{1}{2}\,u(t-\pi)\,\sin\big[2(t-\pi)\big] - \frac{1}{2}\,u(t-2\pi)\,\sin\big[2(t-2\pi)\big],$$

We conclude: $y(t) = \dfrac{1}{2}\,\big[u(t-\pi) - u(t-2\pi)\big]\,\sin(2t).$ $\lhd$