The Laplacian in Polar Coordinates

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Partial Differential Equations
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Physical motivation

Consider a thin elastic membrane stretched tightly over a circular frame. We take the radius of the frame to be a and assume that the edges of the membrane are fixed to the frame.

- Our goal is to mathematically model the vibrations in this "drum head."
- We center the membrane at the origin in the xy-plane and let

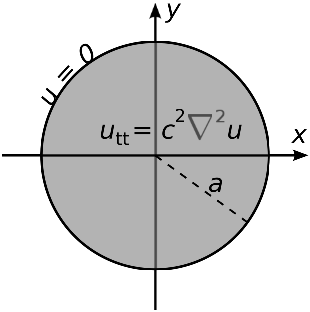
  $u(x, y, t) =$ deflection of membrane from equilibrium at position (x, y) and time t

  as before.

- Neglecting any initial conditions for the time being, we find that we are faced with the boundary value problem

$$u_{tt} = c^2\nabla^2 u, \qquad x^2+y^2<a^2, \qquad (1)$$
$$u(x,y,t) = 0, \qquad x^2+y^2=a^2. \qquad (2)$$

- **Picture:**



Separation of Variables?

Our eventual goal is to solve this problem (with appropriate initial conditions) using separation of variables and superposition.

As it stands, this is quite difficult. If we set

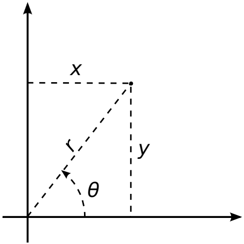
$$u(x, y, t) = X(x)Y(y)T(t)$$

then,

- the heat equation quickly reduces to the familiar separated equations for X, Y and T;
- however, because the boundary is given by $x^2 + y^2 = a^2$ (as opposed to simply $x = 0$, $x = a$, etc. in the rectangular case), it is not clear how to decouple the boundary conditions.

Polar coordinates

To alleviate this problem, we will switch from rectangular (x, y) to polar (r, θ) spatial coordinates:



$$x = r\cos\theta,$$

$$y = r\sin\theta,$$

$$x^2 + y^2 = r^2.$$

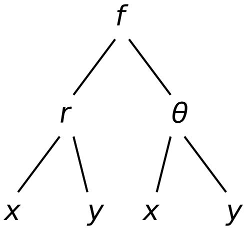
This requires us to express the rectangular Laplacian

$$\nabla^2 u = u_{xx} + u_{yy}$$

in terms of derivatives with respect to r and θ.

The chain rule

For any function $f(r,\theta)$, we have the familiar tree diagram and chain rule formulae:



$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial f}{\partial \theta}\frac{\partial \theta}{\partial x}$$

$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial f}{\partial \theta}\frac{\partial \theta}{\partial y}$$

or

$$f_x = f_r r_x + f_\theta \theta_x$$

$$f_y = f_r r_y + f_\theta \theta_y$$

First take $f = u$ to obtain

$$u_x = u_r r_x + u_\theta \theta_x.$$

Now differentiate with respect to x again and use the product rule to get

$$u_{xx} = u_r r_{xx} + (u_r)_x r_x + u_\theta \theta_{xx} + (u_\theta)_x \theta_x.$$

Applying the chain rule with $f = u_r$ and then with $f = u_\theta$ yields

$$\begin{aligned} u_{xx} &= u_r r_{xx} + \left(u_{rr} r_x + u_{r\theta}\theta_x\right) r_x + u_\theta \theta_{xx} + \left(u_{\theta r} r_x + u_{\theta\theta}\theta_x\right)\theta_x \\ &= u_r r_{xx} + u_{rr} r_x^2 + 2u_{r\theta} r_x \theta_x + u_\theta \theta_{xx} + u_{\theta\theta}\theta_x^2. \end{aligned}$$

An entirely similar computation using y instead of x also gives

$$u_{yy} = u_r r_{yy} + u_{rr} r_y^2 + 2u_{r\theta} r_y \theta_y + u_\theta \theta_{yy} + u_{\theta\theta}\theta_y^2.$$

If we add these expressions and collect like terms we get

$$\begin{aligned}\nabla^2 u =& u_{xx} + u_{yy} \\ =& u_r \left(r_{xx} + r_{yy}\right) + u_{rr}\left(r_x^2 + r_y^2\right) + 2u_{r\theta}\left(r_x\theta_x + r_y\theta_y\right) \\ &+ u_\theta\left(\theta_{xx} + \theta_{yy}\right) + u_{\theta\theta}\left(\theta_x^2 + \theta_y^2\right).\end{aligned}$$

If we differentiate the relationship $x^2 + y^2 = r^2$ with respect to x and then y we get

$$2x = 2rr_x \quad \longrightarrow \quad r_x = \frac{x}{r} \quad \longrightarrow \quad r_{xx} = \frac{r - xr_x}{r^2} = \frac{r^2 - x^2}{r^3} = \frac{y^2}{r^3}$$

and

$$r_y = \frac{y}{r}, \quad r_{yy} = \frac{x^2}{r^3},$$

so that

$$r_{xx} + r_{yy} = \frac{y^2 + x^2}{r^3} = \frac{1}{r} \quad \text{and} \quad r_x^2 + r_y^2 = \frac{x^2 + y^2}{r^2} = 1.$$

We now differentiate $\tan\theta = \dfrac{y}{x}$ with respect to x and then y to get

$$\sec^2\theta\,\theta_x = -\frac{y}{x^2} \;\longrightarrow\; \theta_x = -\frac{y\cos^2\theta}{x^2} = -\frac{y}{r^2} \;\longrightarrow\; \theta_{xx} = \frac{2y}{r^3}r_x = \frac{2xy}{r^4},$$

$$\sec^2\theta\,\theta_y = \frac{1}{x} \;\longrightarrow\; \theta_y = \frac{\cos^2\theta}{x} = \frac{x}{r^2} \;\longrightarrow\; \theta_{yy} = \frac{-2x}{r^3}r_y = -\frac{2xy}{r^4},$$

so that

$$\theta_{xx} + \theta_{yy} = \frac{2xy}{r^4} + \frac{-2xy}{r^4} = 0, \qquad \theta_x^2 + \theta_y^2 = \frac{y^2 + x^2}{r^4} = \frac{1}{r^2},$$

and

$$r_x\theta_x + r_y\theta_y = \frac{-xy}{r^3} + \frac{yx}{r^3} = 0.$$

Conclusion

We finally obtain

$$\begin{aligned} u_{xx}+u_{yy} =& u_r\left(r_{xx}+r_{yy}\right)+u_{rr}\left(r_x^2+r_y^2\right)+2u_{r\theta}\left(r_x\theta_x+r_y\theta_y\right) \\ &+u_\theta\left(\theta_{xx}+\theta_{yy}\right)+u_{\theta\theta}\left(\theta_x^2+\theta_y^2\right) \\ =& \frac{1}{r}u_r+u_{rr}+0+0+\frac{1}{r^2}u_{\theta\theta}, \end{aligned}$$

which we summarize as follows.

Theorem

The Laplacian in polar coordinates is given by

$$\nabla^2 u = u_{rr}+\frac{1}{r}u_r+\frac{1}{r^2}u_{\theta\theta}.$$

Example

Use polar coordinates to show that the function $u(x, y) = \dfrac{y}{x^2 + y^2}$ *is harmonic.*

We need to show that $\nabla^2 u = 0$. This would be tedious to verify using rectangular coordinates. However, in polar coordinates we have

$$u(r, \theta) = \frac{r \sin\theta}{r^2} = \frac{\sin\theta}{r}$$

so that

$$u_r = -\frac{\sin\theta}{r^2}, \qquad u_{rr} = \frac{2\sin\theta}{r^3}, \qquad u_{\theta\theta} = \frac{-\sin\theta}{r},$$

and thus

$$\nabla^2 u = u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = \frac{2\sin\theta}{r^3} - \frac{\sin\theta}{r^3} - \frac{\sin\theta}{r^3} = 0.$$