The Ledoux Criterion for Convection in a Star

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1 Mass Distribution and Gravitational Fields

The Eulerian Description

For gaseous, non-rotating, single stars, without strong magnetic fields, the only forces acting on a mass element are from pressure and gravity, resulting on a spherically symmetric configuration. If we use t and r as the independent variables, we have the *Eulerian description.* For instance, density would be $\rho = \rho(r, t)$.

We want to represent the *mass distribution* inside the stars and its effect on the gravitational field. We define the function $m(r, t)$:

$$dm(r, t) = 4\pi r^2 \rho dr - 4\pi r^2 \rho v dt.$$

Partially differentiating this equation, keeping either t or r, gives relations such as

$$\frac{\partial r}{\partial m} = (4\pi r^2 \rho)^{-1}. \tag{1.1}$$

The Lagrangian Description

If we take a *Lagrangian coordinate* instead of r, *i.e.,* one that is connected to the mass elements, we have the new independent variables as m and t. The partial derivatives with respect to the new derivatives are

$$\frac{\partial}{\partial m} = \frac{\partial}{\partial r} \cdot \frac{\partial r}{\partial m},$$

and

$$\left(\frac{\partial}{\partial t}\right)_m = \frac{\partial}{\partial r} \cdot \left(\frac{\partial r}{\partial t}\right)_m + \left(\frac{\partial}{\partial t}\right)_r.$$

The new recipe for the transformation between the two operators is then

$$\frac{\partial}{\partial m} = \frac{1}{4\pi r^2 \rho} \frac{\partial}{\partial r}.$$

Inside a spherically symmetric body, the absolute value g of the gravitational acceleration at a given distance r from the center does not depend on the mass elements outside of r, but it does depend on r and the mass inside:

$$g = \frac{Gm}{r^2}. \tag{1.2}$$

2 Conservation of Momentum

The mechanical equilibrium in a star is called *hydrostatic equilibrium* when without rotation or magnetic fields: they are in such long-lasting phases of their evolutions that no changes can be observed at all.

Considering a thin spherical mass shell with an infinitesimal thickness dr at radius r in the star. The mass per unit of area is ρdr, and the weight (gravitational force towards the center) of the shell is $-g\rho dr$. This is counterbalanced by the pressure of the same absolute value, outwards. Moreover, the shell must feel a larger pressure P_i at its interior than in the outer boundary P_o. The total net force per unit area acting on the shell due to this pressure is:

$$P_i - P_e = -\frac{\partial P}{\partial r} dr.$$

The sum of forces arising from pressure and gravity has to be zero,

$$\frac{\partial P}{\partial r} + g\rho = 0,$$

giving the condition of hydrostatic equilibrium,

$$\frac{\partial P}{\partial r} = -g\rho. \tag{2.1}$$

Together with Eq. 1.2, this equation becomes

$$\frac{\partial P}{\partial r} = -\frac{Gm}{r^2}\rho, \tag{2.2}$$

which is the second basic equation describing the stellar structure in the Eulerian form. If we take m as the independent variable, as in Eq. 1.1,

$$\frac{\partial P}{\partial m} = -\frac{Gm}{4\pi r^4}, \tag{2.3}$$

the second of the basic equations in the Lagrangian form.

3 Conservation of Energy

The first law of thermodynamics relating heat per unit mass is

$$dq = du + Pdv. \tag{3.1}$$

Assuming a general equation of state, $\rho = \rho(P, T)$ and $u = u(\rho, T)$ we can define the derivatives as

$$\alpha = \left(\frac{\partial \ln \rho}{\partial \ln P}\right)_{T,\mu} = -\frac{P}{v}\left(\frac{\partial v}{\partial P}\right)_{T,\mu}, \tag{3.2}$$

$$\delta = -\left(\frac{\partial \ln \rho}{\partial \ln T}\right)_{P,\mu} = \frac{T}{v}\left(\frac{\partial v}{\partial T}\right)_{P,\mu}, \tag{3.3}$$

and

$$\varphi = \left(\frac{\partial \ln \rho}{\partial \ln \mu}\right)_{P,T}. \tag{3.4}$$

For an ideal gas with $\rho = P\mu/T$, one has $\alpha = \delta = \varphi = 1$. The internal energy in the Eq. 3.1 is rather

$$du = \left(\frac{\partial u}{\partial v}\right)_T dv + \left(\frac{\partial u}{\partial T}\right)_v dT.$$

Plugging it back in that equation, we have

$$dq = \left(\frac{\partial u}{\partial T}\right)_v dT + \left[\left(\frac{\partial u}{\partial v}\right)_T + P\right] dv = \left(\frac{\partial u}{\partial T}\right)_v dT + T\left(\frac{\partial P}{\partial T}\right)_v dv.$$

4 Transport of Energy

The energy the star radiates away from its surface is generally replenished from reservoirs situated in the hot central region. This requires an effective transfer of energy through the stellar material, this is possible due the non-vanishing temperature gradient in the star. This transfer can occur due *radiation, conduction*, and *convection.*

Radioactive Transport of Energy

We first estimate the free path l_γ of a photon at some point in the star:

$$l_\gamma = \frac{1}{\kappa\rho},$$

where κ is a mean absorption coefficient (radioactive cross-section over frequency). For the sun, $l_\gamma \sim 2$ cm, *i.e.,* the matter is very opaque.

The typical temperature gradient in the star can be roughly estimated by averaging between center and surface,

$$\frac{\Delta T}{\Delta r} \sim \frac{T_c - T_s}{R_\odot}.$$

The radiation field at a given point is emitted from a small isothermal surrounding, where the difference of temperature is of the order of $\Delta T = l_\gamma(dT/dr)$. The energy of radiation is $u \sim T^4$, the relative anisotropy of the radiation at some point is $4\Delta T/T$. Stellar interiors are very close to thermal equilibrium, and the radiation very close to of a blackboard. However, the small anisotropy can be the carrier of the star's luminosity. Radioactive transport of energy occurs via the non-vanishing net flux.

Since the radioactive transport in stars is very small compared to the characteristic length,*i.e.,* the stellar radius, the transport can be treated as a diffusion process. The diffusive flux j of particles of different particle density n is

$$j = -D\nabla n,$$

where $D = \frac{1}{3}vl_p$ is the coefficient of diffusion, determined by the average values of mean velocity v and mean free path of the particles. We can replace n by the energy density of radiation $U = aT^4$, where a is the radiation-density constant.

Since we have spherical symmetry, F has only the radial component and we have

$$\frac{\partial U}{\partial r} = 4aT^3\frac{\partial T}{\partial r},$$

so that

$$F = -\frac{4ac}{3}\frac{T^3}{\kappa\rho}\frac{\partial T}{\partial r},$$

which is the equation for heat conduction

$$F = -\kappa_{rad}\nabla T.$$

If we replace F by the local luminosity $l = 4\pi r^2$,

$$\frac{\partial T}{\partial r} = -\frac{3}{16\pi ac}\frac{\kappa\rho l}{r^2T^3}.$$

If we use m as an independent variable instead of r, as in Eq. 1.1, the equation for radioactive transport of energy is

$$\frac{\partial T}{\partial m} = -\frac{3}{64\pi^2 ac}\frac{\kappa l}{r^4T^3}. \tag{4.1}$$

Dividing the last equation by Eq. 2.3, we have

$$\frac{\partial T/\partial m}{\partial P/\partial m} = \frac{3}{16\pi acG}\frac{\kappa l}{mT^3}. \tag{4.2}$$

The ration of the derivatives on the left, $(dT/dP)_{rad}$, is a gradient describing the temperature variation with depth,

$$\nabla_{rad} = \left(\frac{d\ln T}{d\ln P}\right)_{rad} = \frac{3}{16\pi acG}\frac{\kappa lP}{mT^4}. \tag{4.3}$$

5 Stability and the Ledoux Criterion

In stars, sometimes small perturbations may grow and give rise to macroscopic local (non-spherical) motions that are also statistically distributed over the sphere. These motions can have a strong influence in the stellar structure, such as mixing stellar material and transporting energy. Stability criteria will define whether small perturbation will grow or keep small.

Dynamical instability happens when the moving mass elements have no time to exchange large amounts of heat with the surroundings and move adiabatically. In the surface of a concentric sphere, physical quantities such as temperature, density, etc, may not be exactly constant, but show certain fluctuations [3].

For any physical quantity A, the difference between the element and its surroundings is defined as $DA = A_e - A_s$. In the case of pressure, we can assume that the element always remains in pressure balance with its surroundings, $DP = 0$.

If we assume $DT > 0$, for a ideal gas with $\rho \sim P/T \sim nR/V$, we have $D\rho < 0$. This means that the element is lighter than the surroundings and buoyancy forces will lift it upwards.

To test the instability of a layer we can take a radial shift $\Delta r > 0$ of the element. Considering this element to be lifted from r to $r + \Delta r$, its density will differ from the surroundings by

$$D\rho = \left[\left(\frac{d\rho}{dr} \right)_e - \left(\frac{d\rho}{dr} \right)_s \right] \Delta r.$$

A finite $D\rho$ gives the radial component of a buoyancy force (per volume),

$$K = -gD\rho,$$

where g is the absolute value of the acceleration of the gravity. Both cases can happen:

- $D\rho < 0$, the element is lighter and K is upwards, generating an unstable situation, the original perturbation being increased.
- $D\rho > 0$, the original element is heavier and K is directed downwards, the perturbation is removed, the layer is stable.

The condition for stability can be written as:

$$\left(\frac{d\rho}{dr} \right)_e - \left(\frac{d\rho}{dr} \right)_s > 0. \tag{5.1}$$

However these criteria is impractical and we need to rewrite it in terms of gradients of temperature. We can rewrite the equation of state $\rho = \rho(P, T, \mu)$ in the differential form, where α, δ are giving by Eqs. 3.2, 3.3, and 3.4.

$$\frac{d\rho}{\rho} = \alpha \frac{dP}{P} - \delta \frac{dT}{T} + \varphi \frac{d\mu}{\mu}. \tag{5.2}$$

Rewriting Eq. 5.1 with 5.2, we have

$$\left(\frac{\alpha}{P} \frac{dP}{dr} \right)_e - \left(\frac{\delta}{T} \frac{dT}{dr} \right)_e - \left(\frac{\alpha}{P} \frac{dP}{dr} \right)_s + \left(\frac{\delta}{T} \frac{dT}{dr} \right)_s - \left(\frac{\varphi}{\mu} \frac{d\mu}{dr} \right)_s > 0. \tag{5.3}$$

The terms containing the pressure gradient cancel each other since $DP = 0$. We then define the *scale height of pressure*,

$$H_p = -\frac{dr}{d \ln P} = -P \frac{dr}{dP},$$

where H_P has the dimension of length, being the length characteristic of the radial variation of P. With Eq. 2.1,

$$H_P = \frac{P}{\rho g} > 0,$$

since P decreases with increasing r.

Multiplying H_g back to Eq. 5.3 yields as a condition for stability:

$$\left(\frac{d\ln T}{d\ln P}\right)_s < \left(\frac{d\ln T}{d\ln P}\right)_e + \frac{\varphi}{\delta}\left(\frac{d\ln \mu}{d\ln P}\right)_s. \tag{5.4}$$

We can define three new derivatives,

$$\nabla = \left(\frac{d\ln T}{d\ln P}\right)_s, \nabla_e = \left(\frac{d\ln T}{d\ln P}\right)_e, \nabla_\mu = \left(\frac{d\ln \mu}{d\ln P}\right)_s,$$

where P is taken as a measure of depth. We can rewrite Eq. 5.3 as:

$$\nabla < \nabla_e + \frac{\varphi}{\delta}\nabla_\mu. \tag{5.5}$$

Recovering the definition of ∇_{rad} from Eq. 4.3, which describes the temperature gradient for the case that the energy is transported by radiation (or conduction) only. In a layer that transports all energy by radiation, $\nabla = \nabla_{rad}$. Recovering also $\nabla_e = \nabla_{ad}$, the radiation layer is stable if

$$\nabla_{rad} < \nabla_{ad} + \frac{\varphi}{\delta}\nabla_\mu,$$

the *Ledoux criterion*. In a region with homogeneous chemical composition, $\nabla_\mu = 0$, which is the *Schwarzschild criterion*,

$$\nabla_{rad} < \nabla_{ad}.$$

In the interior of evolving stars, the heavier elements are produced below the lighter ones, and the molecular weight μ increases inwards so $\nabla_\mu > 0$, so this element has an stabilizing effect. If these criteria admits stability, no convection motions will occur and the whole flux will be carried by radiation

6 How a Supernova Explodes

A supernova is an outcome in the sequence of nuclear fusion reactions which is the life story of a star. Heat given by the fusion generates pressure to counteracts the gravitational attraction that would make the star collapse [1].

Core collapse supernovae are explosions that mark the death of a massive star, releasing energy of order 10^{53} erg at rates of 10^{45-46} Watts. These events are the factory of most of the heavy nuclei found elsewhere the Universe. The *neutron-rich wind* that emanates from the *protoneutron* star after the explosion allows *trans-iron* elements to be synthesized by rapid neutron capture processes. In addition, nucleosynthesis may occur by neutrinos can cause protons and neutrons from heavier nuclei to produce rare isotopes. After the explosion, supernovae can cool down becoming neutron stars and black holes.

Stars' Chain of Fusion

The first series of fusion reactions have the net effect of four atoms of hydrogen into a single atom of helium:

$$4H \longrightarrow 1He^4 + \text{ Energy.}$$

When the core of the star runs out of H, it contracts due the gravitation. The core and the surrounding material are then heated, causing Hydrogen fusion to begin in the surrounding layers and other fusion reactions in the core:

$$He \rightarrow Ca \rightarrow Ne \rightarrow O \rightarrow Si \rightarrow^{56} Fe.$$

The iron nucleus is the most strongly bound of all nuclei and further fusion would absorb energy instead than releasing it. At this point the star has an onionlike structure, where the outer envelope is mostly hydrogen. How far in these elements depends on the star sizes, *e.g.,* the Sun would not burn further than He [1].

Chandrasekhar Mass

When the fusion ends, a small star will shrinks to a white dwarf. The quantity that defines whether the star will stop to burn further is called *Chandrasekhar* mass, m_C. This mass represents a limit to how much pressure can be resisted by the electrons' mutual repulsion: when the star contracts, the gravitational energy increases together with the energy of the electrons, and whether these two forces are in balance or not depends on the mass of the star, if it is larger than m_C, the star collapses [1]

The value of m_C depends in the relative number of electrons and nucleons, where the higher the proportion of electrons, the larger the electron pressure and the m_C. In small star with the chain of fusion reactions stopping at carbon, the radio is around 1/2 and the Chandrasekhar mass is 1.44 solar masses.

Type I Supernovae

White dwarfs in a binary star system are the origin of the Type I supernovae. Matter from the binary companion is attracted by the gravitational field of the dwarf star and gradually falls onto its surface, increasing the mass of carbon and oxygen core [1].

Type II Supernovae

The Type II supernovae arrives from very massive stars, where the lower limit is around eight solar masses. When the final fusion reaction begins, the core made up of iron and other few elements begins to form in the center of the star, within a shell of silicon [1]. The core now is inert under great pressure and it can resist contraction only by electron pressure, subject to the Chandrasekhar limit.

The Implosion of the Core

The compression raises the temperature of the core, however the pressure raised by it does not help to slow down the collapse. The pressure is determined by the number of particles in a system and their average energy. The pressure by the electron is much bigger than of the nuclei. When the core is heated, some iron nuclei are broken into smaller nuclei, increasing the number of nuclear particles and raising the nuclear component of the pressure. This dissipation absorbs energy, which is taken from electrons, decreasing their pressures. The net result is that the collapse accelerates.

The entropy of the core, which has groups of 56 nucleons bonds, is lower than when it was composed of hydrogen. The high density in the collapsing core favors the reaction know as *electron capture*, which liberates a neutrino which carries entropy and energy out of the star. The loss of the electron also diminishes the electron pressure.

The first stage of the collapse comes to an end when the density of the stellar core reaches about $\rho = 4 \times 10^{11}$ grams per cubic centimeter. At this point, matter becomes opaque to neutrinos.

The role of the Chandrasekhar mass here is for the analysis of how the supernova changes: it is the largest mass that can collapse as a unit, the so called *homologous core and collapse.* Ares withing the core communicate by means of *sound waves* and *pressure waves.*

Within the homologously collapsing part of the core, the velocity of infalling material is directly proportional to the distance from the center, where the density and speed of sound decreases with distance from the center.

The radius at which the speed of the sound and the infall velocity coincide is called *sonic point* and is the the boundary of the homologous core. For a fraction of millisecond the sound waves at the sonic point build pressure there and slow the material falling through. This creates a discontinuity in velocity which forms the *shock waves.* The passages of the shock wave induces changes in density, pressure and entropy and it moves faster than the speed of the sound.

The shock wave will advance through the onionlike structure until erupts on the surface.

Core Collapse Supernova

The core collapse supernova begins with the collapse due to the force of gravity of the iron core of a massive star at the end of its thermonuclear evolution, a variant of the *shock reheating mechanism*

The rebound of the inner core is brought by a rapid increase of pressure with density when this rises above the nuclear matter density, generating a shock wave at a radius of ~ 20, driving into the outer core. The shock stalls and turns into an accretion shock at a radius of 100 to 200 km [2].

The energy transferred between neutrinos and matter before the shock is by charged currents that producing cooling are

$$e^- + p \to n + \nu_e$$

$$e^+ + n \to p + \bar{\nu}_e$$

which are proportional to the matter's temperature

$$\frac{cooling}{nucleons} \propto T^6_{matter}$$

where heating is produced by the inverse:

$$n + \nu_e \to e^- + p$$

$$p + \bar{\nu}_e \to e^+ n$$

proportional to the luminosity $L_{\nu_e,\bar{\nu}_e}$, energy $\epsilon_{\nu_e,\bar{\nu}_e}$, and the inverse of a geometric flux factor, of the neutrino and anti-neutrino

$$\frac{heating}{nucleons} \propto L_{\nu_e,\bar{\nu}_e} \cdot \left\langle \epsilon^2_{\nu_e,\bar{\nu}_e} \right\rangle \cdot \left\langle \frac{1}{\mathcal{F}_{\nu_e,\bar{\nu}_e}} \right\rangle. \tag{6.1}$$

1. Nuclear dissociation behind the shock lowers the ratio of pressure to energy, lowering the strength to the inner core.

2. There is a reduction in both the thermal and lepton numbers contribution in the shock pressure, resulting the burst of ν_e radiation.

3. As the matter continues to flow inward, neutrinos and heating increases the core temperature until the cooling rate, which goes to sixth power of the temperature, exceeding the heating rate. The radius at which the heating and cooling rates are equal is *gain radius.*

4. The inflowing matter cools and add onto the core.

5. If ν heating is sufficiently rapid in the region between the shock and the gain radius, the increased thermal pressure behind the shock allow it to overcome the accretion and propagate out, producing a supernova.

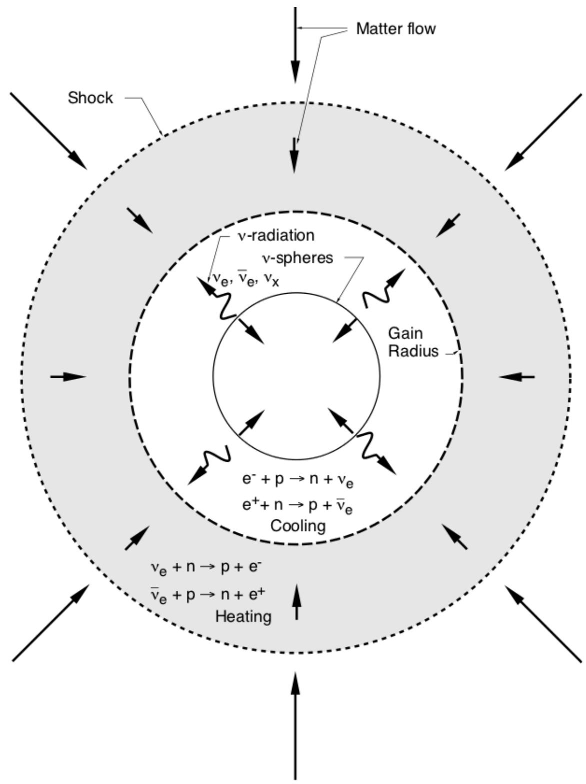

Figure 1: Stellar core after bounce from [2].

Convection

The flux-limited diffusion neutrino transport was investigate by [2] to study the role of the types of convection in core collapse supernovae. In special, two type of convention, *protoneutron star convection* and *neutrino-driven convention.*

In the presence of neutrino transport, protoneutron star convention velocities are too small to the bulk inflow to result to any significant convective transport of entropy and leptons. Moreover, neutrino-drive convection in stars with more than fifteen solar masses (compact iron core) still does not have a model that reproduces the explosions.

Convection Mechanism

Observations of supernovae indicates an extensive mixing thorough some of the ejected material with points to fluid instabilities from the explosion itself. The degree of mixings varies only from Rayleigh-Taylor instabilities in the expanding envelope so that these instabilities are preceded by former instabilities occurring during the explosion.

The postcollapse stellar core can be divided into two regions by the neutrinoshpere, where convection in the region below it can enhance the reheating mechanism by enhancing transporting of lepton from deep within the core, *protoneutron star convection.*

The factors L and ϵ from Eq. 6.1 depend on the conditions at the neutrinosphere and can be affected by the protoneutron star convection.

Neutrinosphere at a Negative Entropy Gradient

The entropy driven protoneutron star convection will advect high entropy material from deeper regions up to the vicinity of the neutrinosphere, raising the temperature, hence increasing L and ϵ, until the limit of zero electron degeneracy, at the ν_e-sphere.

The region between the neutrinosphere and the shock is divided into two regions: a region above the neutrinosphere of net neutrino cooling and a region above that and below the shock of net neutrino heating. these regions are divided by the gain radius.

The lepton fraction also will be increased by convection, from the fact that rapid e^- , e^+ capture and ν_e , $\bar{\nu}_e$ escapes at the neutrinoshpere ling at a minimum lepton fraction Y_l. The entropy and lepton drive convection will advect lepton matter to the border of the neutrinoshpere, increasing e^- degeneracy, hence increasing L and ϵ neutrinos decreasing antineutrino. This results heating rate of the material behind the shock.

Protoneutron Star Convention

Material in the vicinity and below the neutrinosphere are in nuclear statistical equilibrium and its thermodynamic state and composition can be specified by three variables: entropy per baryon s, lepton fraction Y_l and pressure P. Above the neutrinosphere, we use Y_e instead.

With gradients in these variables, a strong gravitational field and the neutrino transport of energy and leptons, the material become subject to many fluid instabilities. The simplistic case, with no neutrino transport, is the Raylength Taylor convective instability, where the criterion for convective instability, the Ledoux condition, is

$$\left(\frac{\partial\rho}{\partial\ln Y_l}\right)_{s,P}\left(\frac{\partial\ln Y_l}{\partial r}\right)_{s,P}+\left(\frac{\partial\rho}{\partial\ln s}\right)_{Y_l,P}\left(\frac{\partial\ln s}{\partial r}\right)_{Y_l,P}>0$$

where

- $\left(\frac{\partial\rho}{\partial\ln s}\right)_{Y_l,P}$ is negative for all thermodynamic states: entropy driven protoneutron star convection.

- $\left(\frac{\partial \rho}{\partial \ln Y_l}\right)_{s,P}$ is negative (positive) for large (small) Y_l: lepton driven protoneutron star convection.

Adding neutrino transport of energy and leptons, the above convection have reduced growth rates and two additional modes of instabilities are possible: neutron fingers and semi-convection, occurring on a diffusion scale (not dynamical), requiring that one of the gradients become destabilizing and the other stabilizing.

The fluid instabilities in the region below the neutrinoshpere play a roles in the shock reheating mechanism:

- instabilities tend to drive fluid motions that tend to circulate through the unstable region.
- fluid motions with entropy driven convection tend to advect high entropy material from deeper in the core to the neutrinosphere, increasing its temperature and ν_e , $\bar{\nu}_e$ emission rates.
- lepton drive convection tend to advect lepton material from the core to the neutrinosphere, increasing ν_e and decreasing $\bar{\nu}_e$ emission rates, producing the deleptozinzation of the core.

Neutrino Driven Convection

As infalling material encounter the shock, it is shock dissociated into free neutrons and protons if the shock is within a radius of 200 km. As the material continues to flow inward, it will be heated by the charged-current reaction until reaching the gain radius. Neutrino heating is strongest beyond the gain radius and decreases farther out as the neutrino flows becomes diluted. these factors create a negative entropy gradient between the gain radius and shock, unstable to entropy-drive convection/ this will persist until the explosion develops (neutrino heat)

If neutrino driven convection is able to develop, it plays a crucial role in the generating an explosion. efficient way of conveying low entropy matter from the shock to the gain radius and high entropy matter back to the shock.

Simulating Supernoave

Effects of gravity: the increased gravitational potential will pull the core into a deeper potential well the resulting redshift of the neutrino radiation will reduce the energy of the neutrinos in the heating region gravity will make successful explosions more difficult.

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