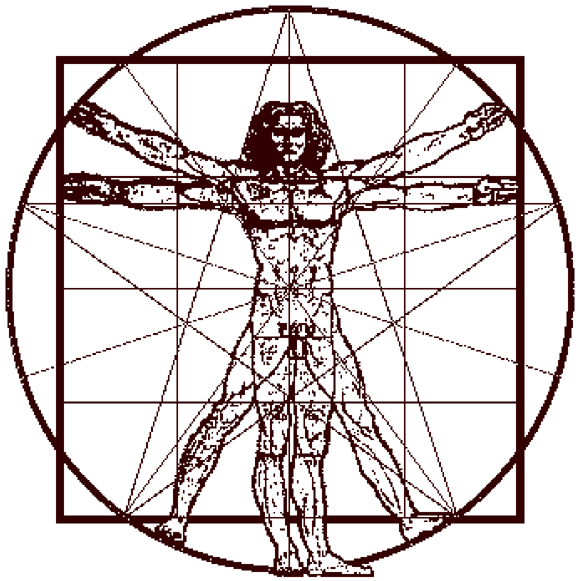


THE MATH & GEOMETRY OF MUSIC

Linking Sight and Sound

Daniel Arthur

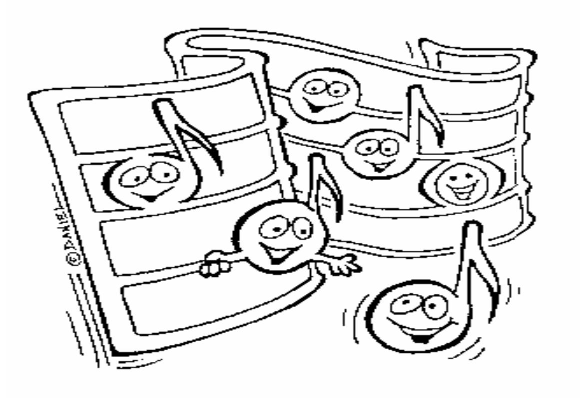
[Abstract] The academic community has historically exalted the Performing Arts while simultaneously denigrating their academic relevance and removing them from the classroom. The favored Music has primarily been operatic, symphonic and big band. Other genres, more appealing to the general public, have been minimized in importance and excluded from curricula. Further, the study of Music has focused on the specific tonality of the Western chromatic scale, taught linguistically rather than mathematically. This approach hides, mystifies and “complexifies” the true nature of the mathematic, geometric and harmonic structure of music, preventing this knowledge from being naturally linked to many other subjects.

[Keywords] Art, Architecture, Biology, Computer Animation, Educational Change, Geometry, Instructional Design, Language, Music, Physics.

Introduction

A three-year study at seven major universities, *Learning, Arts and the Brain*, by the Dana Foundation, finds strong links between arts education and cognitive development. With that in mind, this presentation demonstrates the cognitive connections of the Arts and Sciences in the visualization of Music, the design of musical instruments, and their applications to classroom activities and work environments. These multi-modal learning connections ultimately allow us to create such things as Art for the Blind and Music for the Deaf, beyond the realm of whatever languages are used to describe the phenomena.

What follows is a description of the unusual events that led to my personal quest, an overview of my twenty years of research, regarding the history and measurable facts of the subject matter, and elements that may benefit your own learning and teaching.



The Quest
One day I decided to design animated characters to represent musical notes. The shapes of my first drawings were arbitrary and I was concerned that, when they became popular, they would misrepresent the mathematic and geometric reality of musical tones. So I asked the question of a guitarist friend, Jaigh Lowder, "If a note was a shape, what shape would it be?" The thought had not crossed his mind, but he was curious. For the next few months, he tutored me in the study of music. I made a list of my favorite songs, and he casually remarked that most were in a minor key. At that moment I became stunningly aware of the depths of my musical ignorance. I had no idea what that meant.

One night, following several hours of comparing Bob Dylan's version of *All Along the Watchtower* with the Jimi Hendrix and Dave Mason versions, I had an unusual dream. It was so astonishing that I immediately wrote down a description of what I had seen, and discovered that it was a poem.

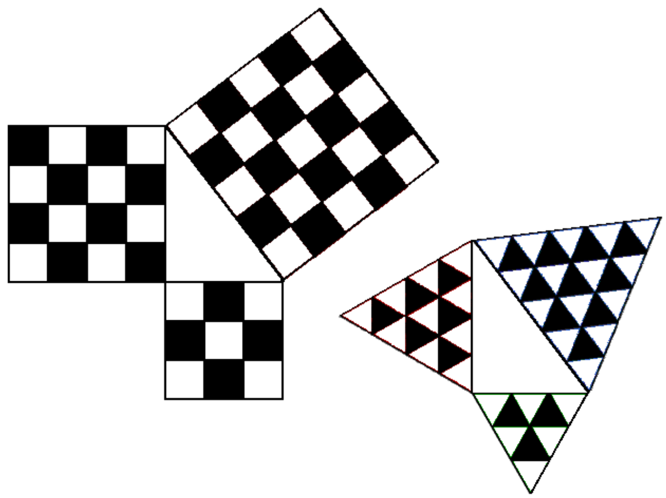
The Three Kittens - Dream Imagery
"There is a gray three-legged kitten, lying happy on a cloud.
Up to its right, soft and bright, is a kitten of white.
Down to the left, a dark one still wet, calls out in the night.
The white kitten sighs as the newborn tries to open its eyes,
And together they purr, as the dark kitten's fur turns white in the light."

A few months later, having been unable to make any sense of the kittens, I had a musical dream, the first I ever had. There was no imagery other than a starry sky, which slowly filled, note by note, with a melody. Although I heard no singing, I somehow knew the words and wrote them down, which, once again, formed a poem.

Children of the Sky - Lyrics
"I hear them. I hear them singing, out in the night.
I see them. I see them laughing, inside the light.
And I know they're coming. I can hear them humming this song that's in my mind.
They are the Children, Eternal Children.
They are the Children of the Sky."

This reminded me of the little aliens in the movie, *Close Encounters of the Third Kind*. The notes played over and over in my head but I could not replicate them. The closest I could get was playing the black notes on a keyboard, which weren't quite right. Jaigh gave me a guitar and showed me the imperfect structure of the "tempered" or chromatic scale. I couldn't find a book or a luthier that could tell me why frets were placed where they were. This began a four year quest to find the notes in my tune.

To make a very long story very short, after studying ancient tuning ratios and their relationship to the modern western musical scale, I discovered that the cycles per second (hertz) of the key note of any major chord is always divisible by 4, and the next two notes are divisible by 5 and 6, with the unheard interval being one. Applying these ratios to the haunting five note sequence from that famous film, it became apparent that ratios of the sequence are: 9, 10, 8, 4, 6. In the key of F, the notes are: G A F^2 F^1 C.



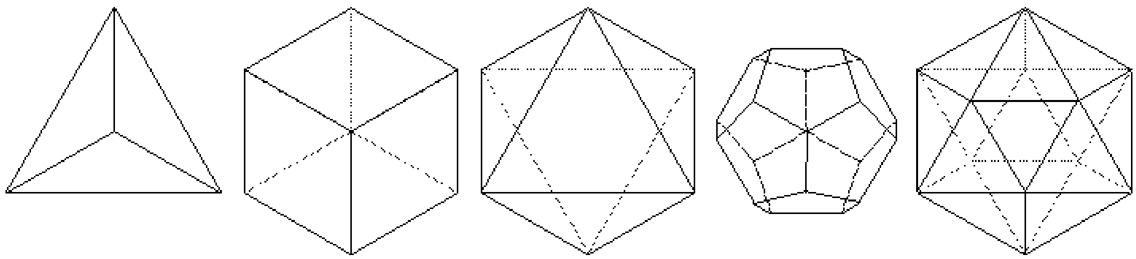
Mathematics

"A squared plus B squared equals C squared," is an ancient formula that has become known to us as the Pythagorean Theorem. The word, "squared," actually refers to the image of a square and the resulting information obtained from viewing it.

"A triangled plus B triangled equals C triangled," is also true and this knowledge allowed the Pythagorean Theorem to be folded into geodesic domes and spheres by Buckminster Fuller (1895-1983).

It is generally not mentioned that Pythagoras (569-480 BC) was also a musician and that he used his mathematical knowledge to build and play his own instruments, tuned in 3:4:5 ratios. This is the very nature of Harmony and is directly related to the measure of Time: 3 x 4 = 12, and 3 x 4 x 5 = 60, used to represent hours, minutes and seconds.

Our calendars are organized into weeks of seven days, representing seven key notes moving through twelve solar months of 30 days (or 13 lunar months of 28 days). Our clocks are based on twelve hours, divided into 60 minutes each, subdivided into 60 seconds each. These divisions can be plotted on grids of 360 degrees, (3x4x5x6) which are also divided and subdivided into minutes and seconds. Pythagorean ratios, therefore, allow music, and the time through which it flows, to be visualized in two-dimensional space using triangles, squares and pentagons, and in three-dimensional space using their related polyhedrons, known as “nested Platonic solids.”



Geometry

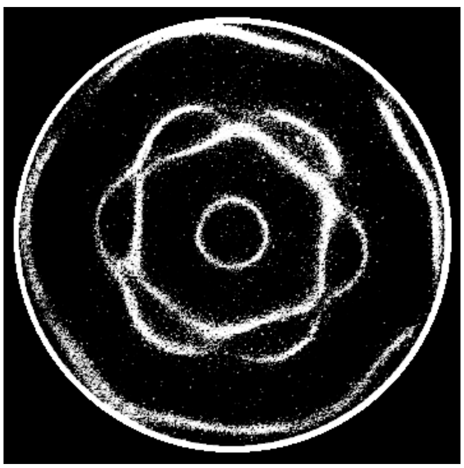
The motto for Plato’s Academy was: “ΑΓΕΩΜΕΤΡΗΤΟΣ ΜΗΔΕΙΣ ΕΙΣΙΤΩ (AGEOMETRETOS MEDEIS EISITO)” which translates into English as, “Let None Ignorant of Geometry Enter Here.” This is not surprising, considering that Plato (427-347 BC) studied Pythagoras, another in the Philosopher-Scientist fraternity that understood the old adage, “Architecture is Frozen Music.”

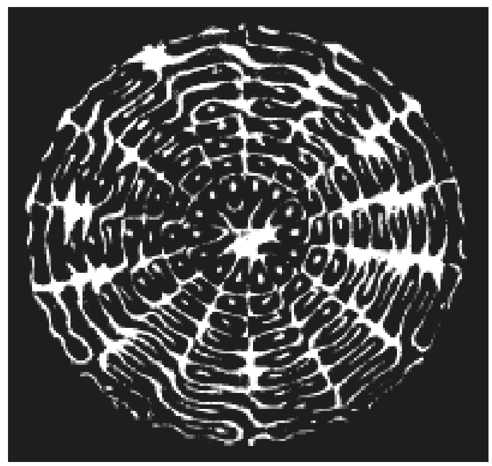
Ancient alphabets were constructed by using visual symbols to represent sounds. Ancient sacred architecture was constructed to send and receive those sounds to and from the gods. Because of that, ancient languages, like Egyptian, Hebrew and Greek were more poetic and musical than our own, with each letter symbol actually representing a specific tone, not just a generic phoneme. If Plato (427-347 BC) faithfully encoded the tones in his homage to the dialogues of Socrates (470-399 BC), the Greek text can be translated not only into other languages but into musical notes and their related shapes. Likewise, the text of the Hebrew Psalms can be translated directly into the languages of Music and Geometry, which are lost during translation into non-musical alphabets, such as English.

Hidden in the drawing of the Vitruvian Man, by Leonardo DaVinci (1452-1519 AD), are ratios upon which music can be visualized, taken from the book, On Architecture, by Marcus Vitruvius Pollio, (90 - 25 BC) which was used as a textbook for 1500 years and meticulously studied by DaVinci. The drawing transforms the geometry of the pentacle, used by Pythagoras, into a more acceptable Christian symbolism of the Squared Circle. This design clearly shows that the circumference of a circle is equal to the perimeter of a square, whenever the diameter is 14 and the side of the square is 11. So, we can easily calculate the circumference of the circle as 44. (Circumference equals Pi times the Diameter or 44 = 22/7 x 14). Pi is represented by the ratio of 22/7 and the ratio of 14/11 is the square root of Phi, with Phi being 196/121. The resulting dimensions of the perimeter and circumference visualize the cycles per second of musical tones.

The use of the decimal value of Pi, 3.14159 destroys the intent of the ratio, which is to compare the diameter of a circle with its circumference. By “decimalizing” these relationships, the shapes are removed and not referenced. As a result, multi-modal connections are hidden and meanings are lost. Likewise, the decimal value of Phi, 1.618034, is a symbolic abstraction which attempts to provide an ideal unattainable number around which Fibonacci sequences fluctuate. Using decimals removes the intended visual references associated with the ratios and their undulations. Since the Phi ratio exists in a multitude of geometries, it is important to delineate which visual relationships are present. For example, in a rectangle, Phi is not immediately obvious. In a chambered nautilus, it reveals itself in a spiral. From a pentagram, it generates stars.

Ernst Chladni (1756-1827) visualized sound with a violin bow and metal plates, which revealed patterns in sand, sprinkled over their surfaces, gathering at the nodal lines of the vibrations. These patterns have become known as "Chladni Figures." Although the method is still used today, a sinusoidal apparatus provides more accurate control. Other methods were developed by Hans Jenny, (1904-1972) who made use of crystal oscillators and his own invention, the Tonoscope, which visualizes the shapes of sounds being sung.





Music

The visual works of M. C. Escher, (1898-1972) contain implausible perspectives and three-dimensional illusions, impossible to construct. In other images he experiments with equalizing positive and negative space, by merging the subject and background through each other, forming checkerboard patterns. This technique can be used to visualize the mathematical and geometric relationships of two intersecting musical keys, as well as turning games of chess and checkers into musical sequences.

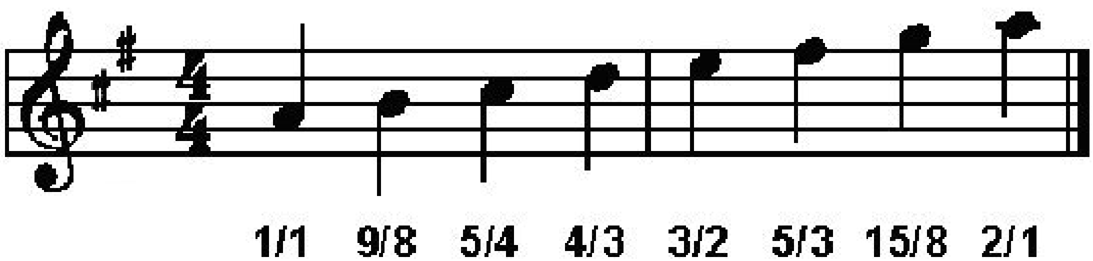
As an example, Escher's *Metamorphosis* can be used to visualize two tonal matrices of ancient harp tuning, as practiced by the last Sacred Irish Harper, Denis Hempson (1695-1807). The Harpers tuned differently for peasant music than for sacred music, in which they did not repeat the same notes every octave. A sacred harmonic tuning sequence goes through 24 strings and 35 pentatonic notes or 49 diatonic notes across seven octaves before repeating the sequence. Chromatic scales attempt to name these notes with seven letters and compress the resultant sharps and flats into a single octave. Advanced study of music goes through the memorization of Greek modes and other cultural variations around the world. The subject is intellectualized and philosophized, with little or no references to the practical applications of tuning, or the placement of frets and sound holes. By so doing, the sounds, the mathematics, and the very understanding of sacred tuning are removed from human consciousness.

Hempson's rule was to never tune a harp with both an E and an F. In the key of F, the E must be flat. In the key of E, the F must be sharp. He had an advantage over other harpists because he could fret the strings with his long fingernails, giving him a greater tonal range, if needed. In other words, he played the harp like a guitar. So, it seemed appropriate for me to design a guitar to play like his harp, in just intonation, E-flat minor pentatonic. I would have to figure out how to do that.

HEMPSON

Tuning two sacred harps with 25 strings, mirroring Escher's interlocking geometric grids:
Key of E minor: E G B D F# A C# E G# B D# F# A# C# F G# C D# G A# D F A C E
Key of F minor: F G# C D# G A# D F A C E G B D F# A C# E G# B D# F# A# C# F

Fractional Tone Ratios compared to Just Intonation and the Chromatic Scale.



Tone Ratios	**A440**	**B495**	**C#550**	**D586**	**E660**	**F#733**	**G#825**	**A880**
Just Intonation	**A440**	**B495**	**C#550**	**D594**	**E660**	**F#742**	**G792**	**A880**
Chromatic Scale	**A440**	**B494**	**C#554**	**D587**	**E659**	**F#740**	**G784**	**A880**

(Notes with decimal remainders are rounded to lower whole numbers)

J. S. Bach, (1685-1750) is renowned for the mathematical precision of his compositions, giving rise to the realization that "Music is Math You Can Hear." Unfortunately, the precision of his tone ratios is not heard in a chromatic scale, because the approximated sounds are similar to the effect of smudging colors of chalk into shades of brown.

The chromatic scale became popular only a hundred years ago, with concert musicians agreeing on a tuning that could be shared worldwide. Unfortunately, it has led to the manufacture of instruments that are locked in a chromatic box, with "A" at 440 cycles per second as the key, with every note but A and E tempered. In the case of the guitar, the Grand Staff is replaced by "Tablature," which shows where to put fingers on a fret board, using one accepted tuning, EADGBE. Lost is the fact that the Staff is an iconic representation of a series of harmonic strings, which can be retuned in various sequences for different instruments.

There is an approach to music called “Just Intonation.” The word, “just,” does not mean “merely,” but refers to “justice,” hence the “Law of Intonation.” The artistic advantage of Just Intonation over Tone Ratios and Chromatic Scales is the ability to visualize every ratio with circle circumferences and relative polygon perimeters, because each note ratio can be multiplied by eleven, providing a common denominator: A:11x40 = 440, B:11x45=495, C:11x48=528, D:11x54=594, E:11x60=660, F:11x64=704, G:11x72=792, A:11x80=880.

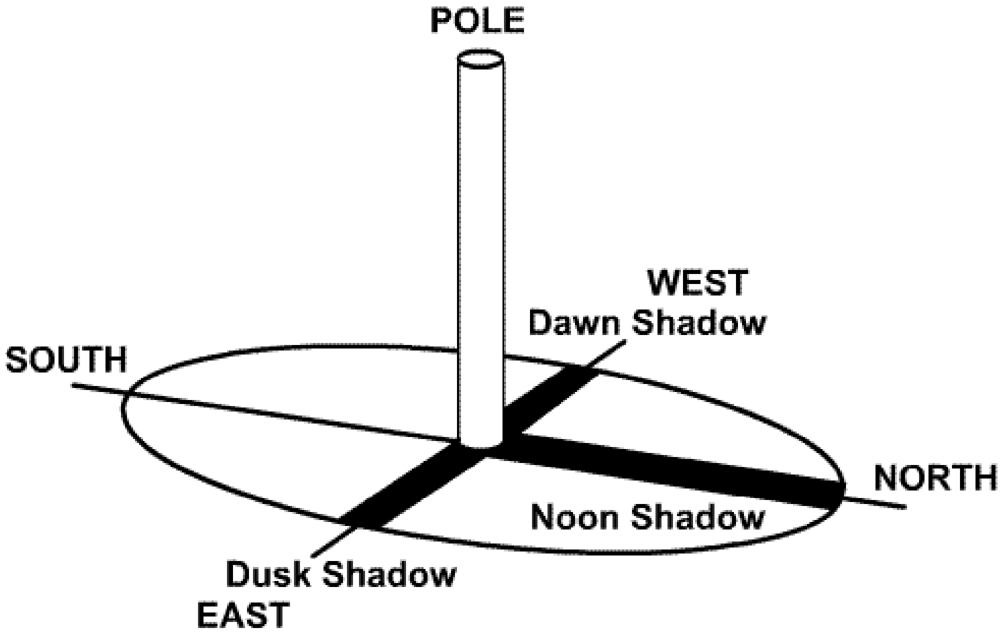
Additional harmonic musical scales can be constructed with the same shapes merely by changing the multiplier, a process that is similar to what occurs in the generation of fractals. This approach can be applied to architectural designs, astronomical alignments, and geographical positioning. For example: A:12x40 = 480, B:12x45 = 540, C:12x48 = 576, D:12x54 = 648, E:12x60 = 720, F:12x64 = 768, G:12x72 = 864, A:12x80=960.

Architecture

If “Architecture is Frozen Music,” it only occurs if the Architect is also a Musician. The beauty of a guitar is not in its physical appearance but in the sound it makes. Likewise, to understand the building, or to appreciate its beauty, the observer must have adequately studied the subject, learned to see beyond its skin, and be prepared to listen.

A number of ancient buildings have been constructed to display harmonic proportions in stone. These include the Great Pyramid of Giza, (2500BC) which connects Phi with Pi, aligning it to the sun, moon and stars. The Egyptian Temple of Luxor (1360 BC) is built as a gigantic musical instrument, with advanced knowledge of acoustics, to mix Earth vibrations with the singing of temple worshippers. The Greek Parthenon, (440BC) was designed on the “Golden Ratio,” which was renamed “Phi” after its architect, Phidias. The Cathedral at Chartres, France, (1200 AD) also containing the Phi ratio, is one of eighty gothic cathedrals built during the 12th century. Like Luxor, it was designed with harmonic proportions to resonate with the singing of the choir.

The design of each section of these buildings can be marked out on the ground, using a cord with 12 equal sections separated by 13 knots, to form Pythagorean triangles. This was the way the foundations of Gothic Cathedrals were measured and aligned to the heavens, a common practice for centuries, because “God Geometrizes.”



Before Phi got its name, it was known as the “Divine Proportion,” important because its five-fold symmetry is implicit in living things. Studying geometry in nature, it becomes obvious that it manifests itself in many ways. It can be seen in the chambered spiral shell of the nautilus, the Fibonacci patterns of leaves, the pentagonal distribution of flower petals, in human anatomy (as in the Vetruvian Man) and more recently, in DNA. Knowledge of Phi was seen by some as the proof of God. By others it was seen as a pathway to power. In either case, it began to be seen in man-made Architecture, hidden in the calculation of the square root of five. One method is to add one to the square root of five (2.236068) then divide by two (3.236068/2 = 1.618034). But it is more easily seen, measured and drawn using fractions, not decimals, as the ratio of 9/4, or 2 1/4. (3.25/2 = 1.625, a difference of 14/1000 of the square root of five and 7/1000 of Phi).

Instruments

Many years ago, I asked a luthier (professional instrument maker) how he calculated the placement of the frets on a guitar. He replied, “A fret is 17.8% of the distance from the bridge to the previous fret,” but he didn’t know why. The answer is that frets are placed according to a trigonometric function of the twelfth root of two, the understanding of which takes us into higher mathematics, relative to the study of acoustics, spectral analysis and string theory. This knowledge was not necessary for a monk to string a harp, for a peasant to fret a lute, for King David to build a psaltery, or for me to design a just-intoned guitar.

Having precise musical ratios and specific notes, in cycles per second, provides the information necessary to construct an instrument. The fret placements are calculated by inverting the fractions and multiplying the length of the string. So, if the string is 25 inches, tuned to A at 440 cycles per second, the interval is 110 and the next note in the chord is C# at 550 cps, so C# has a note ratio of 5/4. The fret is then placed at 4/5 of 25 or 20 inches from the bridge and 5 inches from the nut. This follows another ancient saying, “As Above, So Below, but Vice-Versa.”

The application of ratios in the creation of music and the making of instruments was so prevalent that way back in 325 AD at the Council of Nicaea, the Church banned a specific ratio, known as the “Devil’s Interval.” Also known as the “Tritone,” it is the ratio of 32/45 or 45/64, in modern notation F/B or B/F. Playing the notes together results in a discordant tone that “sounds like the devil.”

Nevertheless, using Just-Intoned ratios I designed a fret board and hired a luthier to build a guitar with fourteen frets per octave, not knowing if it would work or what sounds it would make. If it didn’t work, I would at least be able to hear the ratios of Pi and the square root of Phi. So, after four years, I held a guitar in my hand, translated the ratios into musical notes, found the tones of my musical dream and played the song.

A year later I revised the design for an E-flat just-intoned guitar with 12 frets per octave.

This tuning of the guitar follows precisely the tuning indicated by the standard musical staff. This means that the staff was at one time an iconic representation of a stringed instrument, in sacred harp tuning. String 6 is the bottom line of the treble clef, which is E, string 5 is the next line G, string 4 is the next line B, string 3 is the next line D, string 2 is the top line F#, string 1 is the added line above, which is A. The notes on the spaces between the lines can be played in either of two positions, fretted toward the bridge on the string beneath it or fretted toward the nut on the string above it.

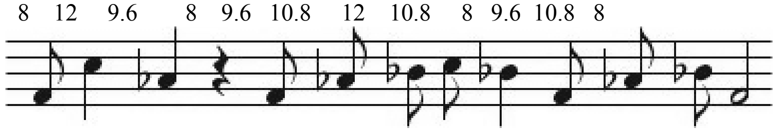
Imagery Interpretation
To play my dream, my left hand fingering on the guitar fret board matched the position of the legs of the three kittens. This fingering is the same on the other two pairs of strings and carries these ratios to any four fret grouping in any key:

R front leg = index	string 5 fret 13	ratio 9 G
L front leg = middle	string 6 fret 14	ratio 8 F
R back leg = ring	string 5 fret 15	ratio 10 A
L back leg = little	string 6 fret 16	ratio 9 G

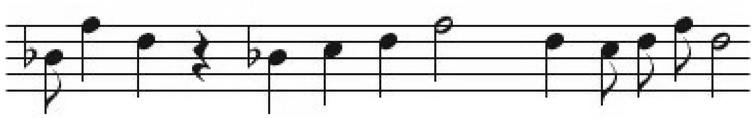
The left back leg plays the same note as the Right front, so one of them is not necessary.

Children of the Sky – Lyrics, with Notes and Ratios

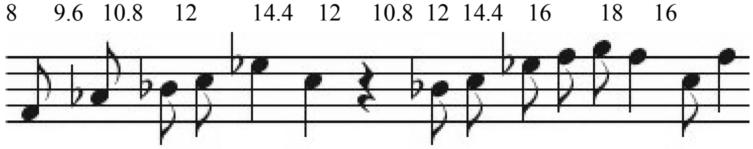
"I hear them. I hear them singing, out in the night.
F^1 C Ab F^1 Ab Bb C Bb F^1 Ab Bb F^1
8 12 9.6 8 9.6 10.8 12 10.8 8 9.6 10.8 8



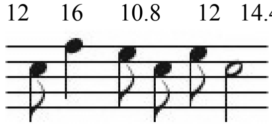
I see them. I see them laughing, inside the light.
Bb F^2 D Bb C D F^2 D C D F^2 D
10.8 16 13.5 10.8 12 13.5 16 13.5 12 13.5 16 13.5



And I know they're coming. I can hear them humming
F^1 Ab Bb C Eb C Bb C Eb F^2 G F^2
8 9.6 10.8 12 14.4 12 10.8 12 14.4 16 18 16



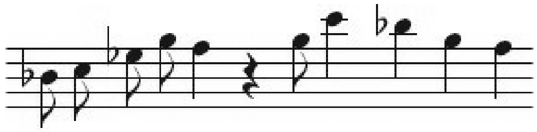
this song that's in my mind.
C F^2 Eb C Eb C
12 16 10.8 12 14.4 12



They are the Children, Eternal Children.

Bb C Eb G F^2 G C^2 Bb G F^2

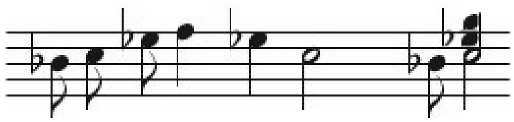
10.8 12 14.4 18 16 18 24 20 18 16



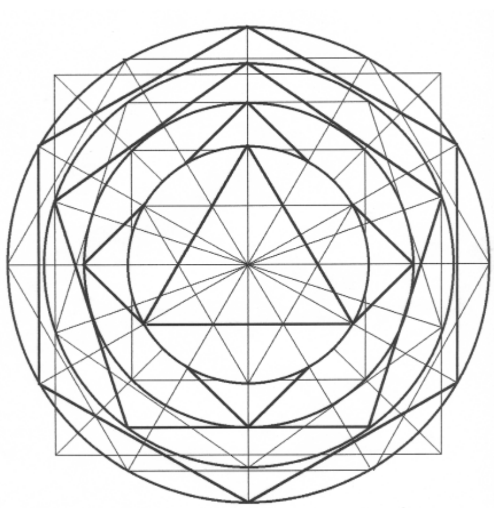
They are the Children of the Sky."

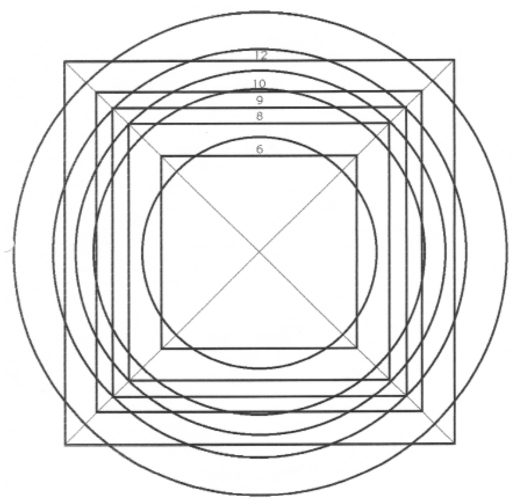
Bb C Eb F^2 Eb C Bb Cm chord

10.8 12 14.4 16 14.4 12 10.8 12 +14.4+18



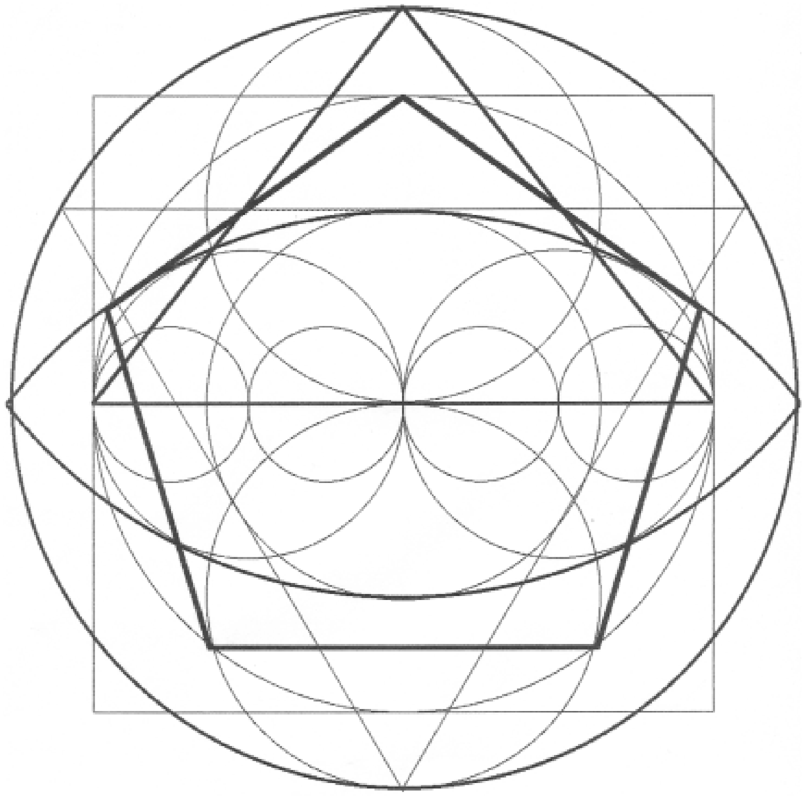
With the guitar finally figured out, I could get back to my original task of determining the shapes of the notes, then their personalities. I experimented with triangles, squares and circles, trying to find a logical structure. Then I began searching for representative colors. The process eventually evolved into the discovery of iconic representations of notes and chords in both two and three dimensions, with the geometry of nested Platonic solids.



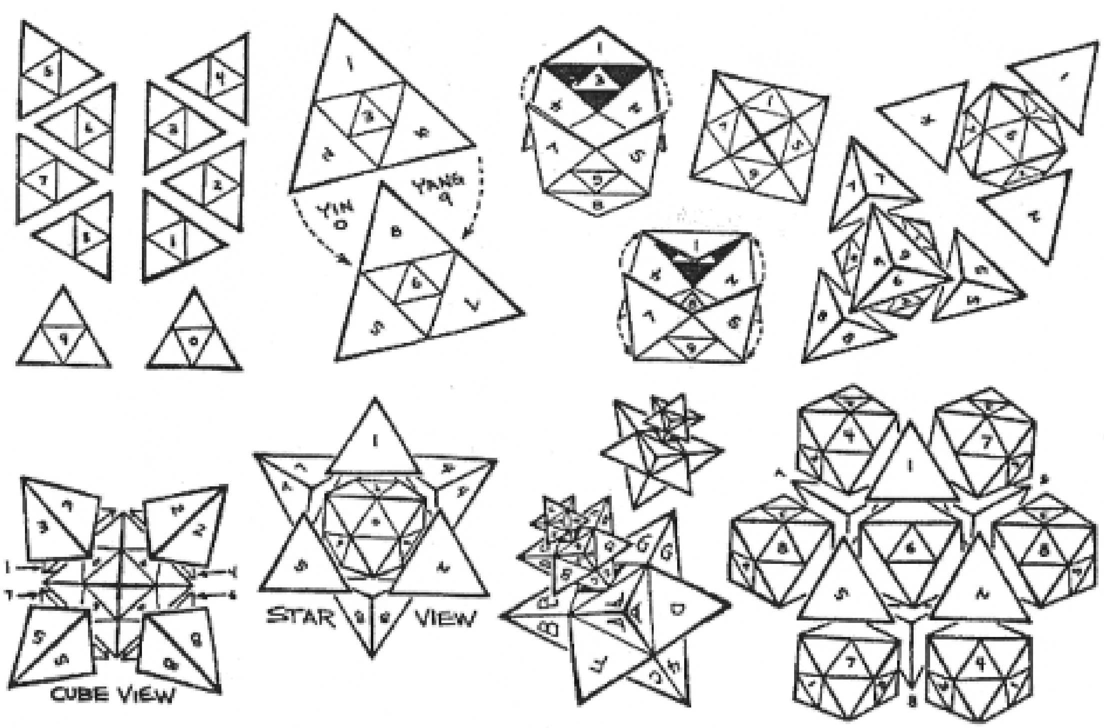


On the upper left are nested Platonic solids. On the right are squared circles representing a musical grid with a sequence of notes in the ratio of 6, 8, 9, 10, 12.

Below is the relative geometry of the Circle Squared, the Pyramid, Pentagon, and the octave circles inside what the Egyptians called the "Mouth of Re."

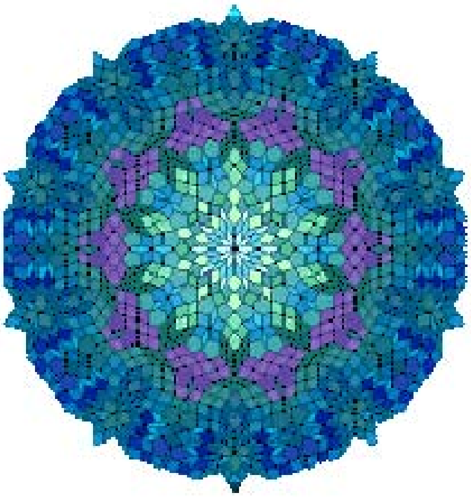


Below: Folding Triangles into Tetrahedra and constructing a Trisoctahedron.



Below: The harmonic sequence of nested Platonic polyhedra in a major chord.

Both the sounds and shapes are purely geometric and, I discovered, have historically been used for meditation and healing. For over ten years I suffered chronic pain in my right side. After numerous tests, no cause or cure was found. I discovered one evening while sitting in a chair playing my guitar that I could feel the vibrations of certain chords traveling from the strings through the wood body into my hipbone and then to the source of the pain in the soft tissues. After strumming a particular six-note chord for about 40 minutes, the pain subsided. Whenever it returned over the next week or two I strummed again until it completely went away. I would have never believed that such a thing could occur had I not experienced it myself. Since then I've discovered several other people who have witnessed a similar phenomenon, including musicologist Don Campbell, who has connected with Marianjoy Rehabilitation Center, in Wheaton, Illinois, to explore the potential of musical healing therapies.



Ancient Meditation Mandalas.

A couple of years ago, I heard that one of our family friends was undergoing chemotherapy and that he was in a lot of pain. I suggested that we get together and play guitar and that perhaps certain chords might diminish the pain. I tried to get two different luthiers to build a duplicate of my guitar, but they were not very interested. It would take awhile and be expensive. So I bought some equipment and re-fretted one of my standard guitars myself. In the meantime, our friend died.

I gave the guitar to his son and realized how terribly inadequate I was in explaining how and why I created the guitar and what difference it could possibly make in anybody's life. But I shouldn't have waited so long. I've been sitting here for years, playing this instrument every evening to myself. I could have taught him long ago, before he became ill, and we could have shared the joy of music even if it didn't take away his pain. I could have shown him how to play "All Along the Watchtower," because, regardless of how Dylan or Hendrix or Mason did it, I can play the chords with one finger, in Hempson's tuning. And so can you.

Conclusions

I have played my dream tune almost every night for twenty years and been inspired to create several others, but I'm still not a musician or a mathematician. That surprised me, because I had thought that my experiences would accelerate my learning and my skill level. That didn't happen. I designed my own guitar out of sheer frustration and am grateful that I was able to do it. And I can now build a "songscape" in 3D software.

The phenomena that I experienced were both inspirational and obsessive. Those of us that go through such events are often hesitant to share the information because we are simply unable to adequately describe them and often have no idea what they mean. I have come to the conclusion that this experience requires me to share it, because I know that it did not come from me. It was a gift. For me, it was a song, cartoon characters and a guitar. For others, the gift will be something else, specific for each of you. Take it and give it.

The Future

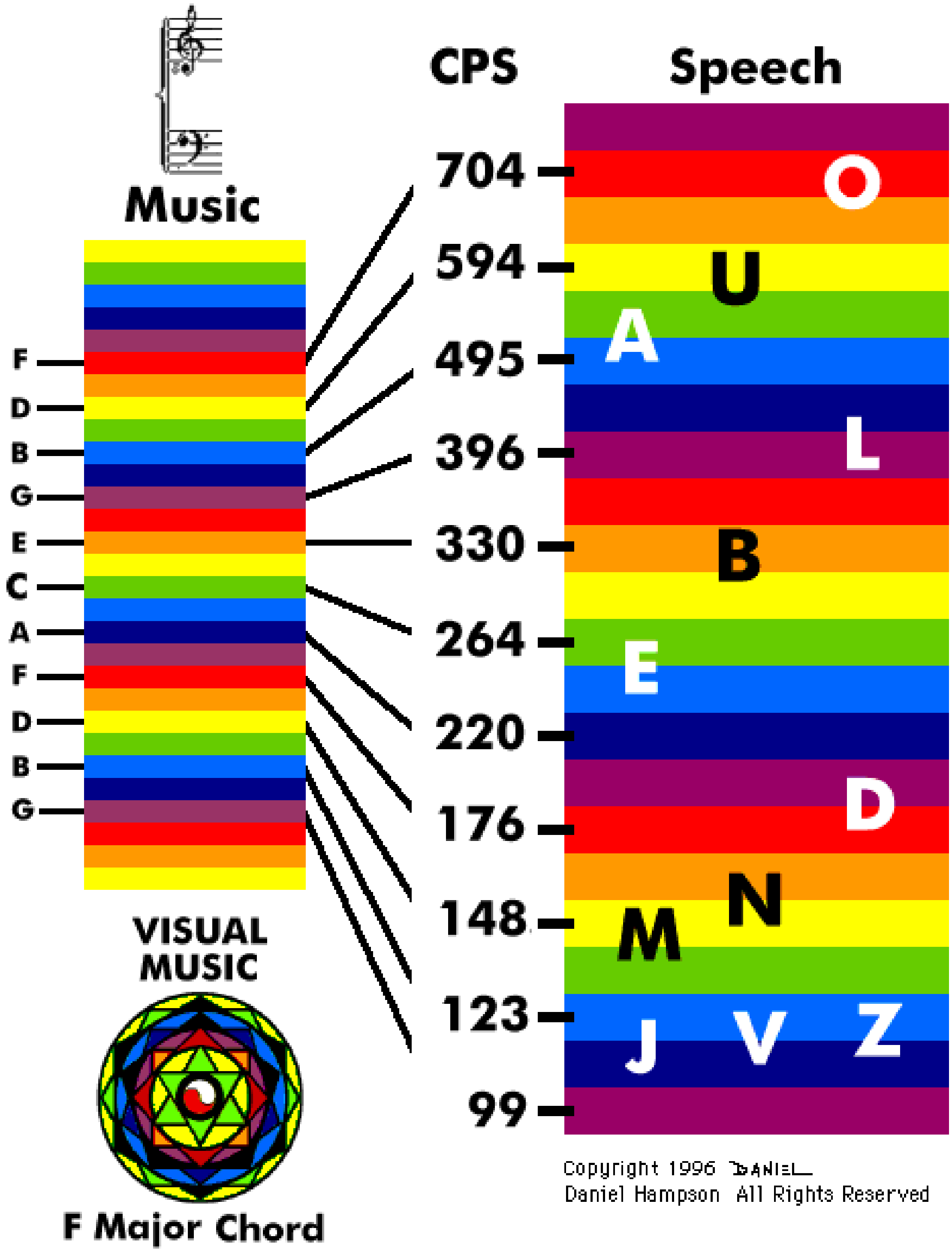
In the larger scheme of things, the meaning is that we do not need to teach art and music, and the design of musical instruments, in the same manner that we have been taught, or in a manner that has prevented us from learning. Ultimately, these tools quickly and effectively connect Math and Science to the Arts, and there should be no excuse for disconnecting them again.

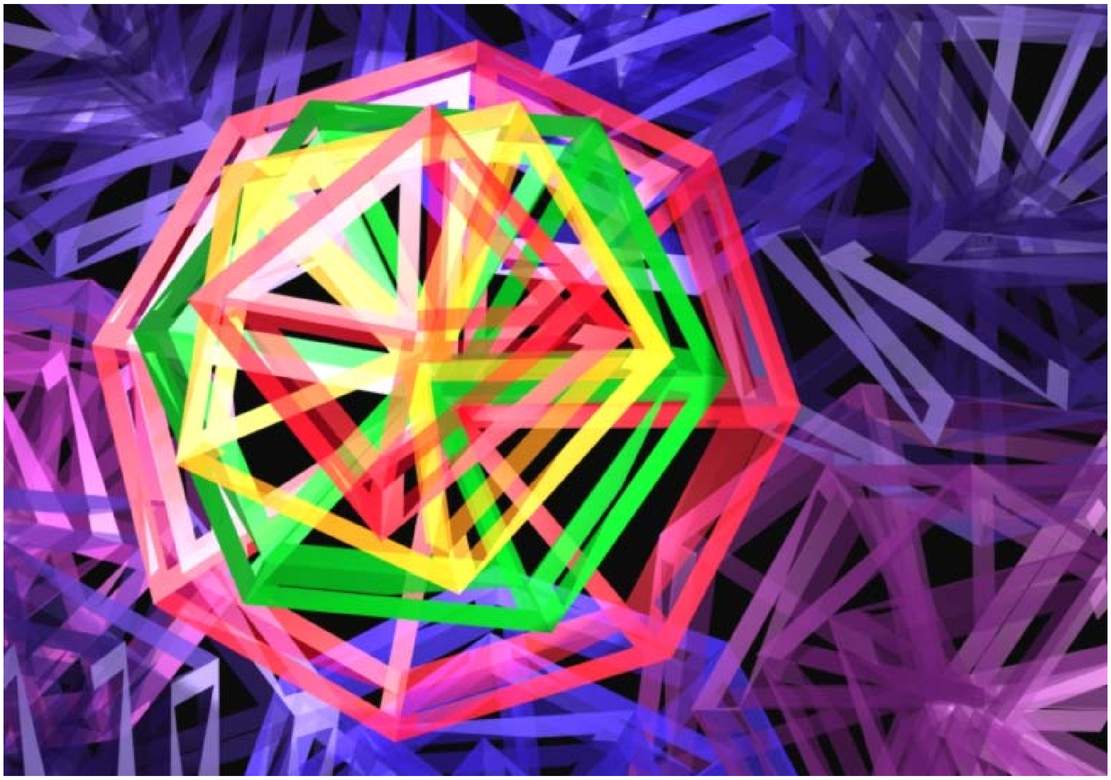
Several professionally made, just-intoned guitars have been created, with 12, 14 and 15 frets per octave. I have re-fretted several other necks myself and continue to do so, because it's cheaper and I would like to share them with friends. The designs, plans and fret spreadsheets have been put in the public domain because everyone should have the opportunity to create these themselves, especially in a classroom.

The advantages of playing a just-intoned guitar are, first, due to the EGBDF#A tuning and fret placements, relative major and minor chords can be played straight across the fret board with one finger. Second, there is no need to learn complex fingering positions for the basic chords. Third, you read music directly from the Grand Staff, placing your fingers on the strings, with no need for tablatures. Fourth, the notes are truly harmonic and the chords ring longer. Fifth, it’s easier to tune and change from major to minor keys.

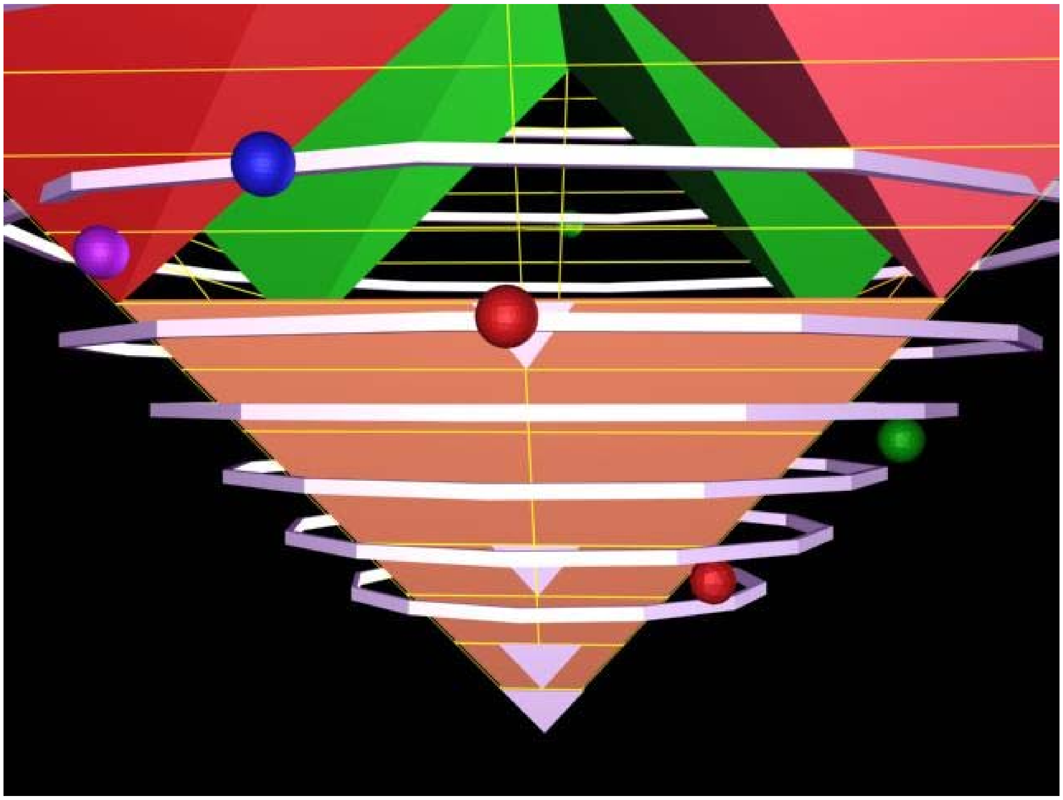
The advantages of building a just-intoned guitar are, first, the bridge can be one piece. It does not need moveable saddles for each string, to compensate for the discordant tuning of the EADGBE strings. Second, learning note ratios allows for easy calculation and placement of frets. Third, custom tunings and fret placements are possible, providing the opportunity for the creation of music that we have never heard. Fourth, by applying the principles to the construction of other instruments, like pipes and drums, the activity further merges Music with Visual Arts, Math and Science. These learn-by-doing activities provide visual and aural learning, pattern recognition, the development of fine motor skills, self esteem, and connections to other learning modalities, at any age. After all that, there is the Music and the Knowledge to make more.

SEE WHAT YOU HEAR

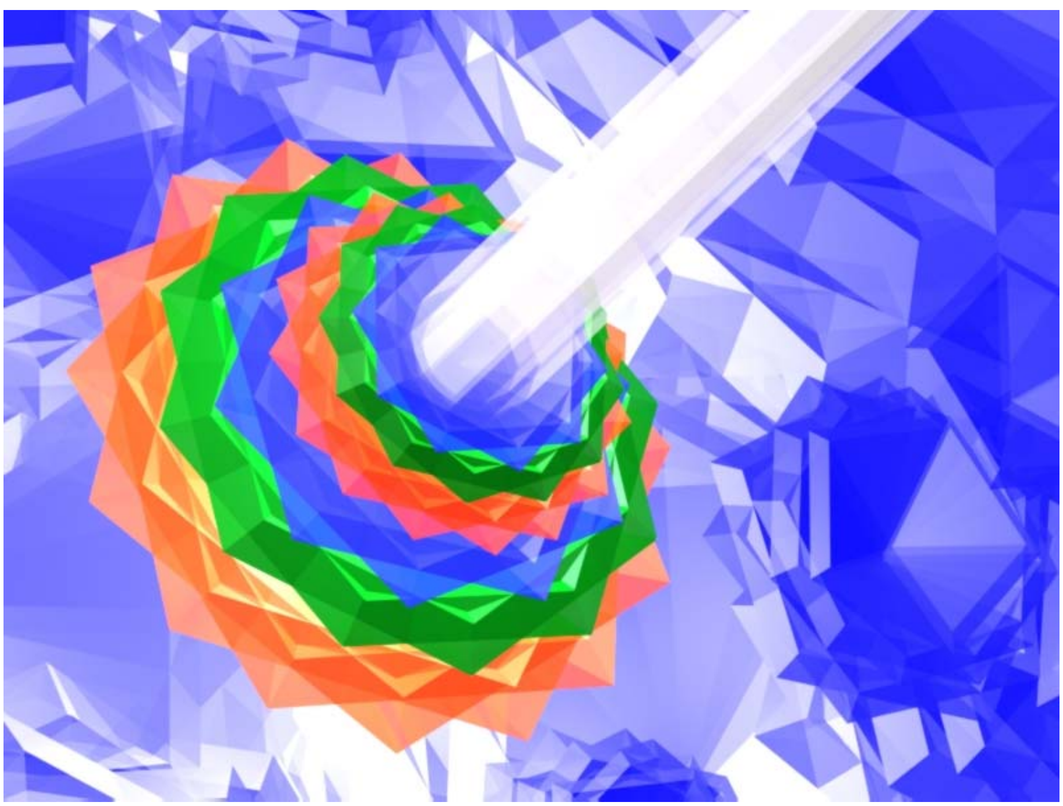




Platonic visual music in the key of F major.



3D Hempson chord models for instructional music games.



Author Bio

Daniel Arthur received his Bachelor of Arts degree from the University of Illinois, Springfield and his Master of Arts from the DePaul University School for New Learning. He has worked as a commercial artist since the age of nineteen. He is a professional cartoonist, illustrator, videographer, and media consultant. For two years he was an art instructor in a special education district. For the past ten years he has taught drawing and computer animation at an accredited new media arts college in suburban Chicago. As a result of these experiences and subsequent research, he invented the Jeometrik Guitar and designed the Artademic Curriculum. Contact: DanielArt@aol.com

References

Andrews, Ted. Sacred Sounds. Llewellyn: St. Paul, MN, 1993
Buerger, Martin. Elementary Crystallography. John Wiley: New York, NY, 1963
Campbell, Don The Mozart Effect. Harper, 2001.
Campbell, Don. The Roar of Silence. Quest Books, Wheaton,IL,1989.
Cole, Emily, Ed. The Grammar of Architecture. Ivy Press: East Sussex, UK, 2002
Cornell, Judith. Mandala. Quest Books, Wheaton, IL,1994.
Cornford, Francis. The Republic of Plato. Oxford University Press, Oxford, UK 1945.
Danielou, Alain. Music and the Power of Sound. Inner Traditions: Rochester, VT, 1995
Denyer, Ralph. The Guitar Handbook. Random House, 1992.
Fuller, R. Buckminster. Synergetics. Macmillan. New York, NY, 1982.
Gardner, Howard. The Arts and Human Development: A Psychological Study of the Artistic Process. Basic Books: New York, NY, 1973. www.pz.harvard.edu
Gardner, Howard. The Unschooled Mind: How Children Think and How Schools Should Teach. Basic Books. New York, NY, 1993
Gerstner, Louis, V. Jr. Reinventing Education. Penguin. New York, NY, 1995
Graham, F. Lanier. The Rainbow Book. Random House: New York, NY, 1979
Haik-Vantoura, Suzanne. The Music of the Bible Revealed. Bibal: Berkeley CA, 1991.
Hall, Manly P. The Secret Teachings of All Ages. Philosophical Research Society: New York, NY, 1978.
Hill, Paul T. Reinventing Public Education. University of Chicago. Chicago, IL, 1997
Hofstadter, Douglas R. Godel, Escher, Bach. Vintage: New York, NY, 1980.
Jones, Tom D. Art of Light & Color. VanNostrand Rheinhold: New York, NY, 1972
Lemesurier, Peter. The Great Pyramid Decoded. Avon: New York, NY, 1979.
May, Rollo. The Courage to Create W.W. Norton, New York, NY 1975
McClain Ernest G. The Myth of Invariance Nicolas-Hays, Berwick, ME 1985.
Perelman, Lewis J. School's Out. Avon. New York, NY, 1993
Plato. The Republic, Timaeus, Critius. 360 BCE. Jowett, Benjamin. (translator) Internet Classics Archive, Web Atomics. 1994. Copyright Daniel C. Stevenson, online: http://etext.library.adelaide.edu.au/mirror/classics.mit.edu/Plato/republic.html
Schank, Roger C. Engines for Education. Lawrence Erlbaum. Hillsdale, NJ, 1995
Tufte, Edward. Visual Explanations. Graphics Press: Cheshire, CT, 1997
West, John Anthony. Serpent In The Sky. Julian Press: New York, NY, 1987

Online Resources

Audio Visualization, http://www.audiovisualizers.com/
Center for Visual Music, http://www.centerforvisualmusic.org/
Charles Johnson's Research, http://www.gizapyramid.com/charles4.htm
Custom Guitars & Necks, http://www.baileycustomguitar.com/
Dana Foundation, http://www.dana.org/
Geometric Art, http://agutie.homestead.com/files/.geometric_art/index.html
Geometry Center, http://www.geom.uiuc.edu/
GeoMusic, http://members.aol.com/jamesfuria/secret.html
Just Intonation Network, http://www.justintonation.net/
Languages of the World, http://www.ethnologue.com/
M. C. Escher, http://www.mcescher.com/
Pythagoras & Music, http://www.dartmouth.edu/~matc/math5.geometry/unit3/unit3.html
Science in Ancient Artwork, http://www.earthmatrix.com/munck/extrac16.htm
Visual Music History http://homepage.eircom.net/~musima/visualmusic/visualmusic.htm
http://www.ronpellegrinoselectronicartsproductions.org/Pages/MusicVisualizers.html

Ranjer Jeometrik Guitar, http://www.jaighlowder.com/