The Mathematics of Light

Math Club Topics

Overview

- Review of fractions
- Review of first degree equations
- Review of functions
- The sine function
- Electromagnetic waves, light, colors and the wavelength of light
- Geometrical optics: light rays, refraction and reflection of light
- Lenses, image formation through lenses and the thin lens equation
- Optical instruments with lenses.

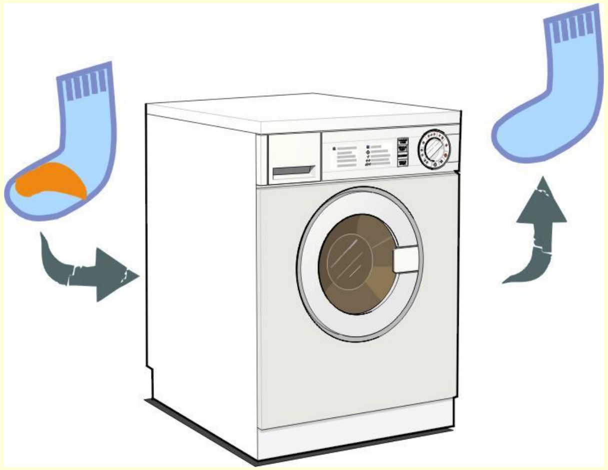
Review of functions

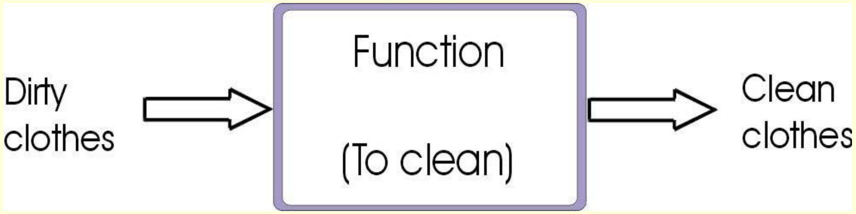
- In general, a function is a black box which is promised to do specific task(s), each and every time we ask it to do.

- You give something to the black box as *input*, and it returns the result as *output*, as shown by the schematic function diagram above

The "Clean" Function

- A washing machine can be regarded as a black box that performs the function "to clean"





Mathematical Functions

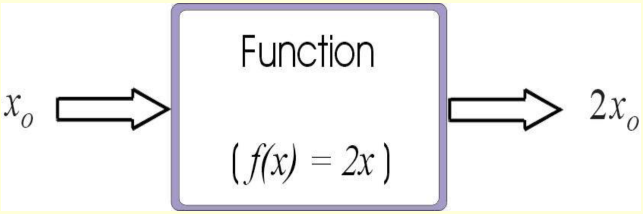
- Unlike the washing machine, ***a mathematical function*** is a rule (the promised “to do” list) that assigns to each real number *x* another real number *y.*

- The input to the function, *x*, is called the ***argument*** of the function. The output of the function, *y*, is called ***the value*** of the function.

- In other words, ***a mathematical function is a black box that transforms a set of numbers into another set of numbers, according to a specific rule***.

Example: The Doubling Function

- A simple example of a function: *the doubling function*

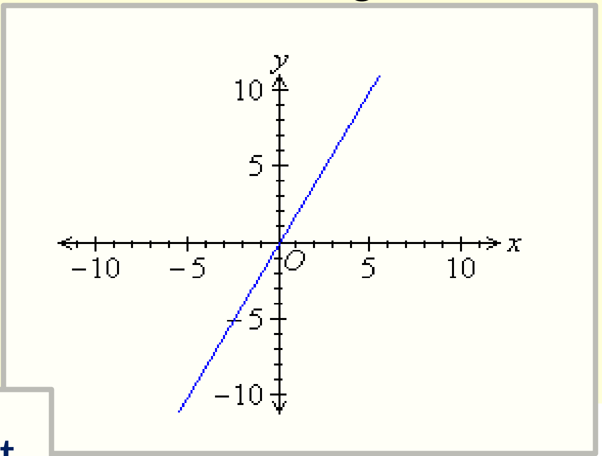


- For any value of the argument, x_0, the value of the function is twice the value of the argument, $y = 2x_0$.

- The input of the function can be any real number, and the output will be twice that number.

Graph of a Function

- The graph of a function is a powerful graphical way of representing what a mathematical function does.
- The graph of a function shows all the possible values of the input to (argument of) that function, and all the corresponding values of the output of that function
- Example: the graph of the doubling function



The Sine Function

- A particular class (or family) of functions of huge importance for many practical applications, including for Light and Optics, is the class of ***trigonometric functions***.

- The input of a trigonometric function is ***an angle*****.** Angles are measured in ***degrees*** (which have the symbol **°**).

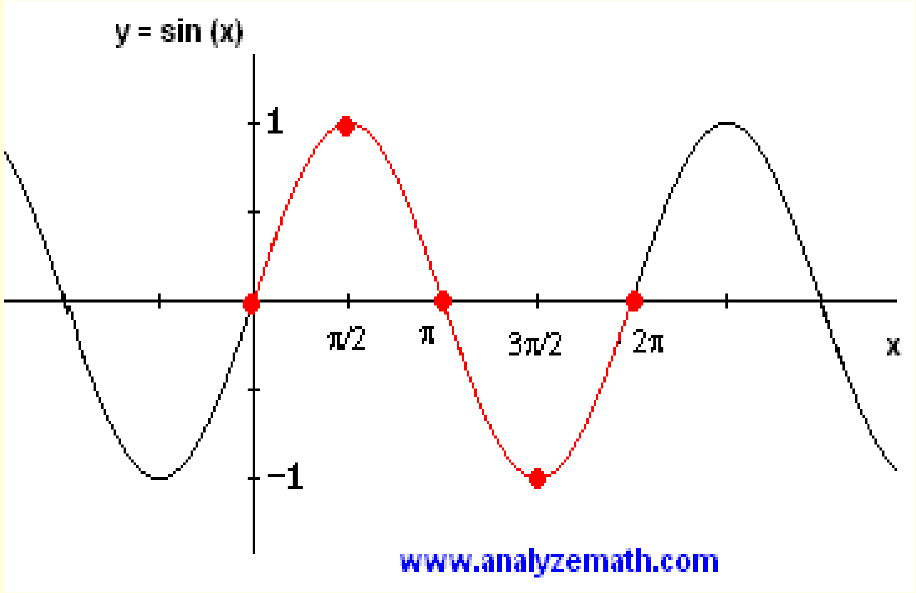
- Another unit used for angles in science and engineering is the ***radian*** (which has the symbol ***rad***). There is a simple relation between these two units:

$$180° = \pi \, radians$$

where $\pi = 3.14..$

The Action of the Sine Function

- The action of sine function can be understood from its graph.

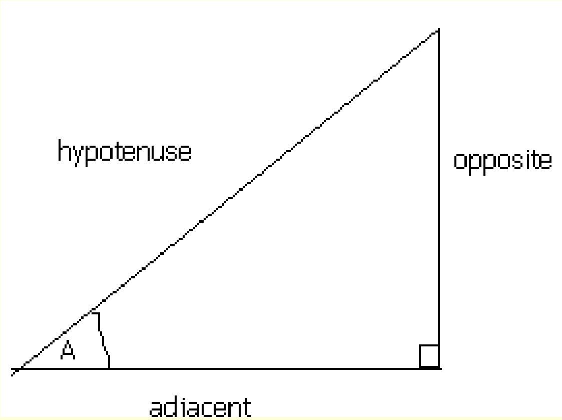


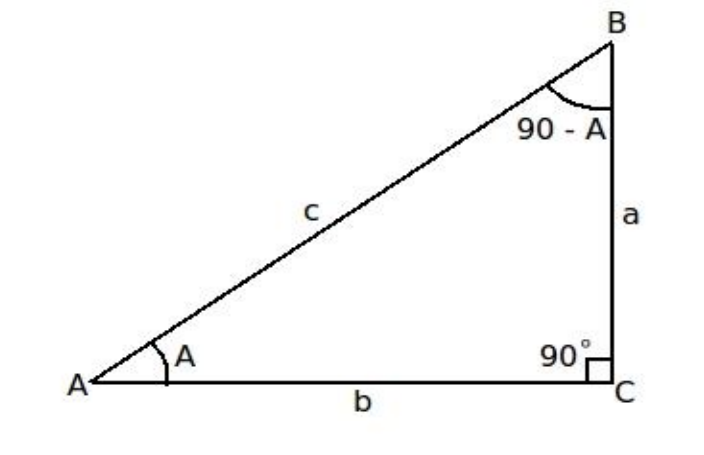
The Action of the Sine Function

- So how does a sine function actually do its black box magic?.

$$\boldsymbol{sinA = \frac{length\ of\ opposite\ side}{length\ of\ hypotenuse} = \frac{a}{c}}$$

- In a right angle triangle, the sine of an angle is equal to the ratio between the length of the side of the triangle opposite that angle and the length of the hypotenuse:





Class Activity

- Step 1: Please draw a right triangle with an acute angle A and measure the length of the opposite side of angle A and the length of the hypotenuse and evaluate the ratio
- and record it.
- Step 2: Now, use a rapporteur and measure angle A and compute sin A. Please compare this result with the one found in Step 1.
- Step 3: Please extend both the base and the hypotenuse of your triangle obtaining in this way a new triangle. Repeat Step 1 and 2 on this new triangle. Did you obtain a different value for sin A when using the measurements of this new triangle? Is the result consistent with your expectation?

Web Resources for Optics

- ***HyperPhysics – Light and Vision*** (Georgia State University)
  - http://hyperphysics.phy-astr.gsu.edu/hbase/hph.html#ahph

- ***Molecular Expressions – Physics of Light and Color*** (FSU and the National High Magnetic Fields Lab)
  - http://micro.magnet.fsu.edu/primer/lightandcolor/index.html

- ***General Physics Java Applets*** by Surendranath Reddy
  - http://surendranath.tripod.com/

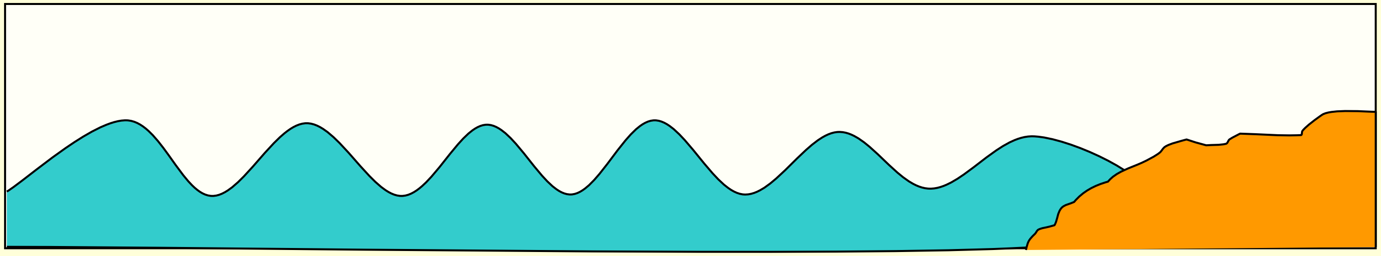
- Optics is by its nature a visual science. A wealth of Java Applets found on these sites help visualize optics phenomena.

The Broad Spectrum of Light

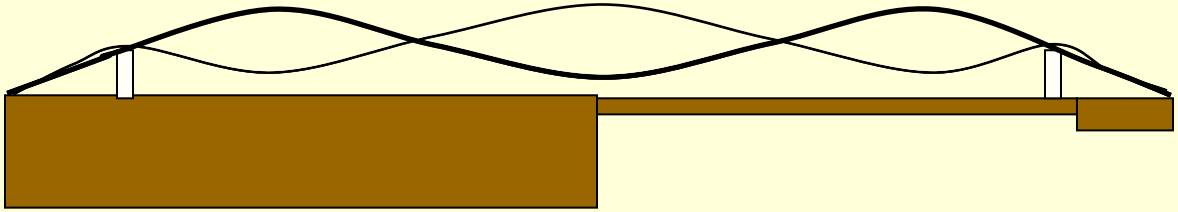
- ***Optics*** is the science that studies light.
- Light, in its broadest sense, is **more than only visible light**
- The ultraviolet (UV) and infrared (IR) regions of the spectrum facilitate some of the most important applications of optics.

Wave Properties of Light

- Light travels as a wave, similar to water waves



or to the waves on the vibrating strings of a guitar

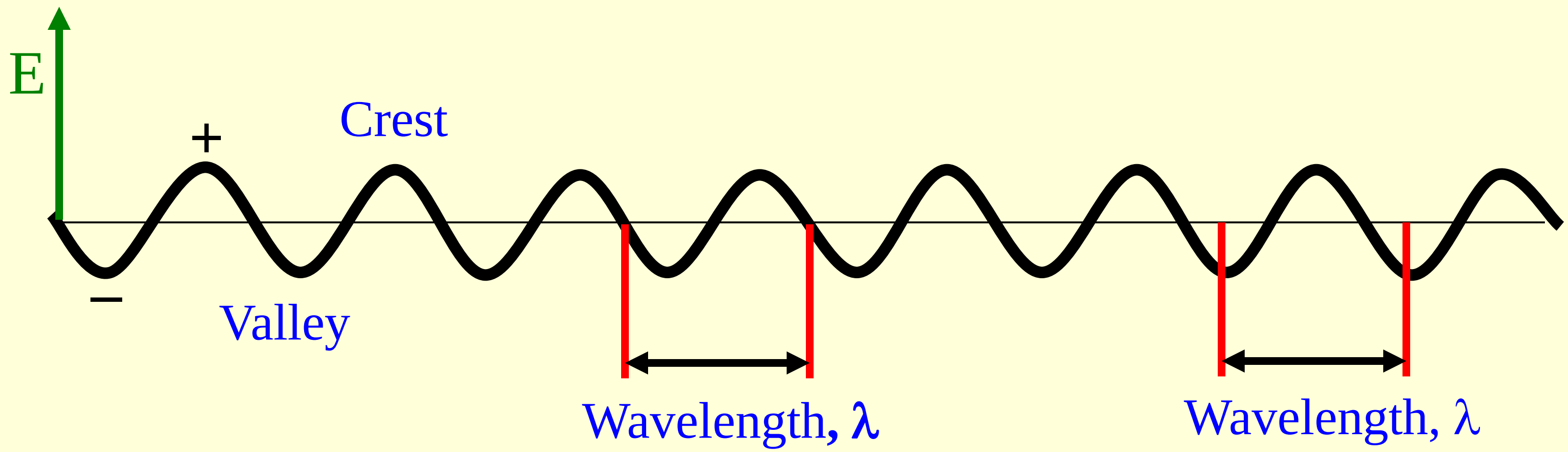


- But, unlike these waves, the oscillations of a light wave are in *electric field strength*, not in spatial displacement.

Wave Properties of Light

What oscillates and propagates in a light wave is an electric field

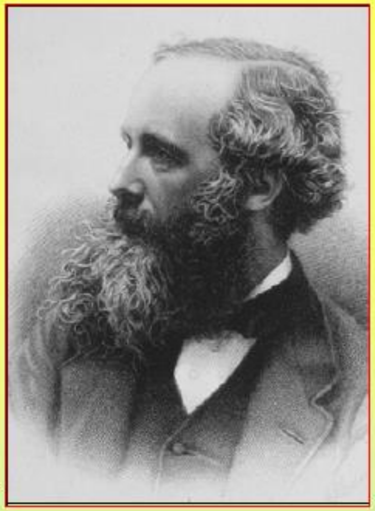
Snapshot (in time) of the electric field of the wave



A Built-in Length Scale – The Wavelength λ

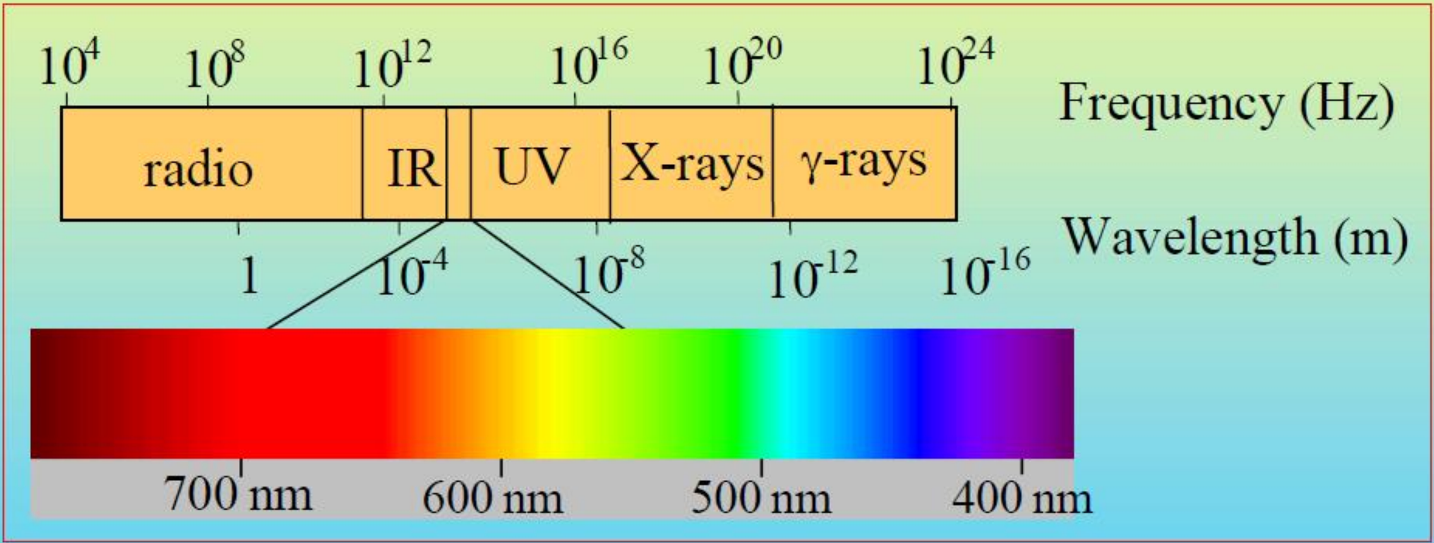
- Each "color" of light has a number associated to it – **the wavelength (λ)** of the light wave (measured in nm or μm).
- Visible light extends from **400 nm** (**violet**) to ~**800** (**Red**) **nm.**
- The spatial profile of the wave repeats itself identically after a distance equal to the wavelength – hence the name.
- The wavelength acts as **a length scale built into the light wave.**

The Electromagnetic Spectrum



James Clerk Maxwell

- Light is a tiny part of a huge spectrum of electromagnetic waves

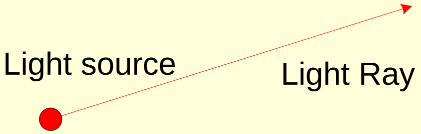


- Light in vacuum has **wavelengths** between ~400 nm and ~800 nm; 1 nm $\equiv 10^{-9}$ m (a nanometer)
- 1 μm $\equiv 10^{-6}$ m (a micron); 1 Å $\equiv 10^{-10}$ m $\equiv$ 0.1 nm

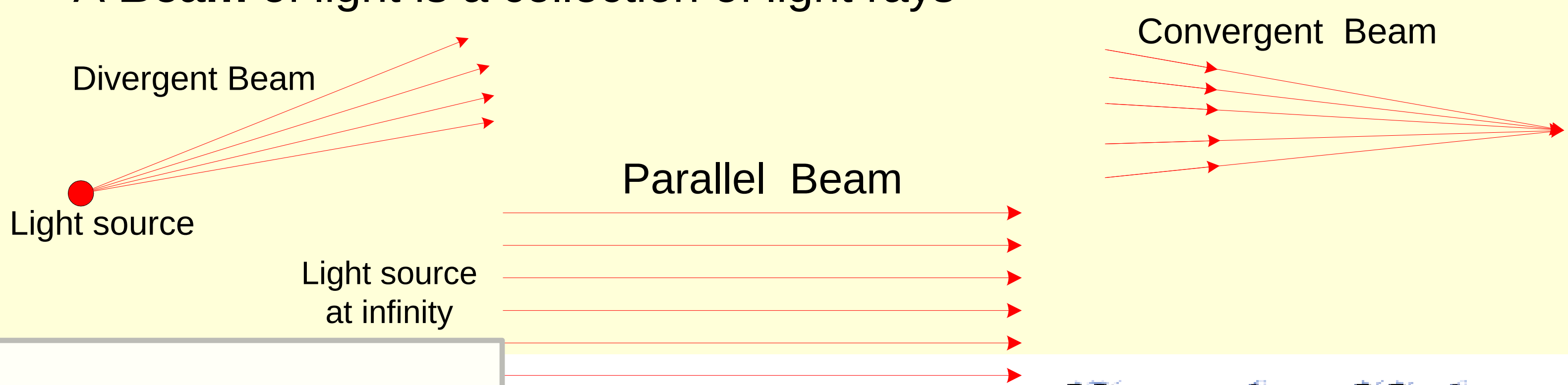
Propagation of Light

- In a ***homogeneous*** medium (a medium that has the same index of refraction everywhere), and ***only*** in a homogeneous medium, light propagates in a straight line.

  - A **Ray** of light is a line that shows the direction of propagation of a light wave



  - A **Beam** of light is a collection of light rays



The Index of Refraction

- In **vacuum** all electromagnetic waves - including visible light - travel at the same speed:

$$c = 2.99792458 \times 10^8 m/s \cong 3 \times 10^8 m/s$$

- The *index of refraction* of a medium is defined as the ratio of the speed of light in vacuum, c, and the speed of light in that medium, v

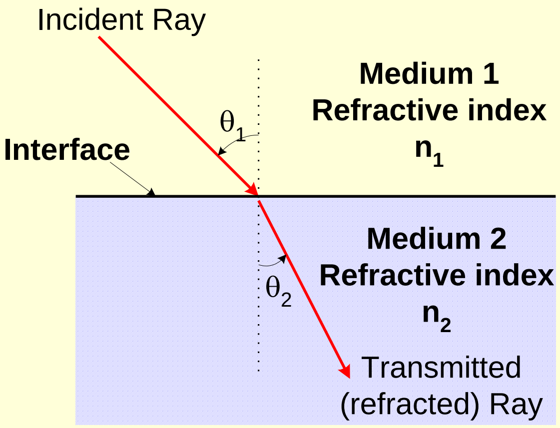
$$n = \frac{c}{v}$$

- Except for vacuum, the index of refraction of any medium has a (slight) dependence on the color (i.e. on the wavelength) of the light – this is called *dispersion of light:* different colors propagate through the medium at different speeds

$$n = n(\lambda) \rightarrow v = v(\lambda)$$

Refraction of Light

- Refraction of Light: “bending” of a light ray when it crosses the boundary (interface) between two different optical media.



- Snell’s Law
  - $n_1 \sin\theta_1 = n_2 \sin\theta_2$

- Snell’s Law Calculator

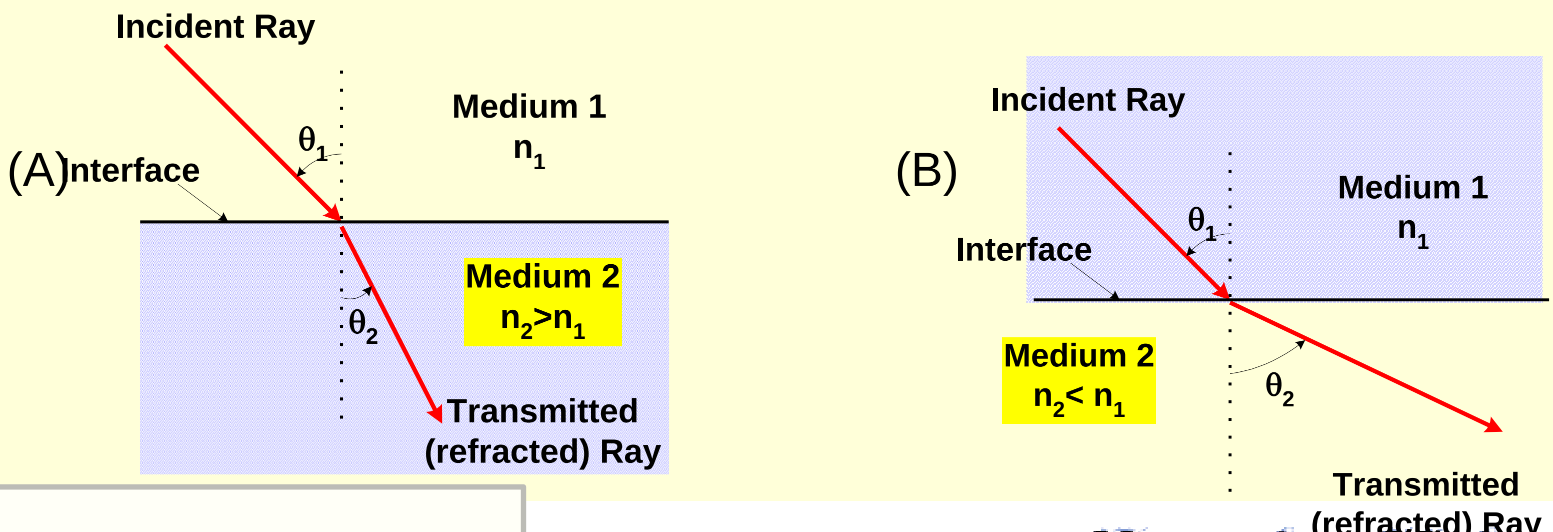
  http://hyperphysics.phy-astr.gsu.edu/hbase/geoopt/refr.html#c1

- Refraction JAVA applet:
  http://micro.magnet.fsu.edu/primer/java/refraction/refractionangles/index.html

Refraction of Light – "Take 2"

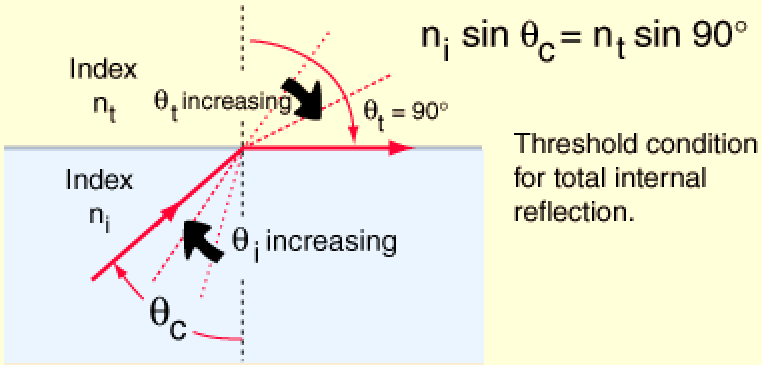
- Refraction of Light: "bending" of a light ray when it crosses the boundary (interface) between two different optical media.

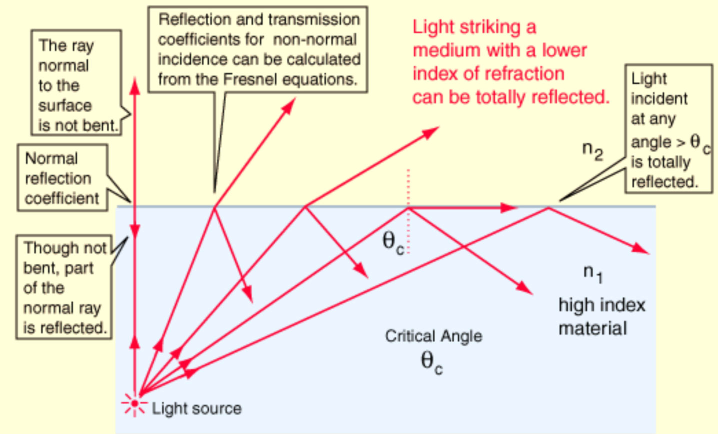
- Snell's Law: $n_1 \sin\theta_1 = n_2 \sin\theta_2$ (or $n_i \sin\theta_i = n_t \sin\theta_t$)
- Consequences:
  - (A) If $n_1 < n_2$, then $\theta_2 < \theta_1$, i.e. the light ray is bent towards the normal
  - (B) If $n_1 > n_2$, then $\theta_2 > \theta_1$, i.e. the light ray is bent away from the normal



Total Internal Reflection

- When light is incident upon a medium of lesser index of refraction, the exit (refraction) angle is greater than the incident angle.





- As the incident angle increases, the exit angle increases as well; when the incident angle reaches a value called ***the critical angle, θ_c*** the refraction angle will be equal to 90^o.

- For angles of incidence grater than the critical angle, there will be ***total internal reflection*** (i.e. no light will be transmitted into the 2nd medium)

Total Internal Reflection

- Snell's law at critical angle, $n_1 \sin\theta_c = n_2 \sin 90°$, leads to:

$$\sin\theta_c = \frac{n_2}{n_1} \quad \text{which means } \theta_c = arcsin\left(\frac{n_2}{n_1}\right)$$

- JAVA Applet: The critical angle of reflection

  http://micro.magnet.fsu.edu/primer/java/refraction/criticalangle/index.html

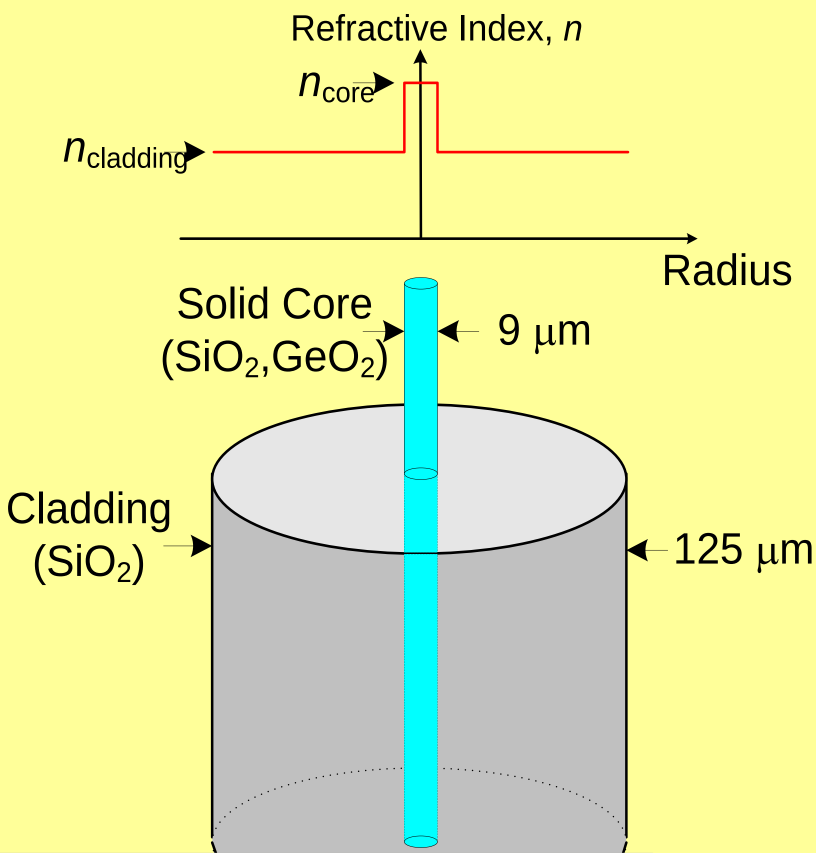
- JAVA Applet: Total internal reflection at water-air interface

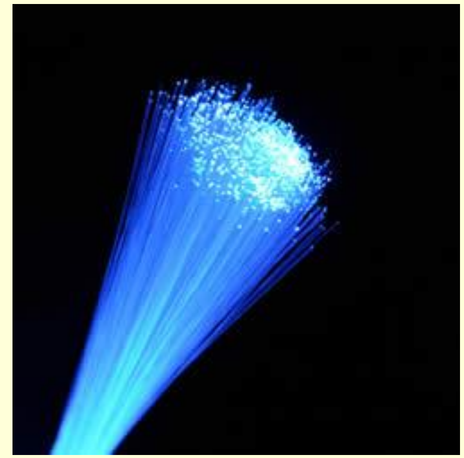
  http://www.phy.ntnu.edu.tw/java/light/flashLight.html

  http://www.physics.mclarenhigh.com/javapm/java/totintrefl/index.html

Application: Optical Fibers

- The light is trapped in the (solid) core of the fiber and propagates through it by **total internal reflection**.





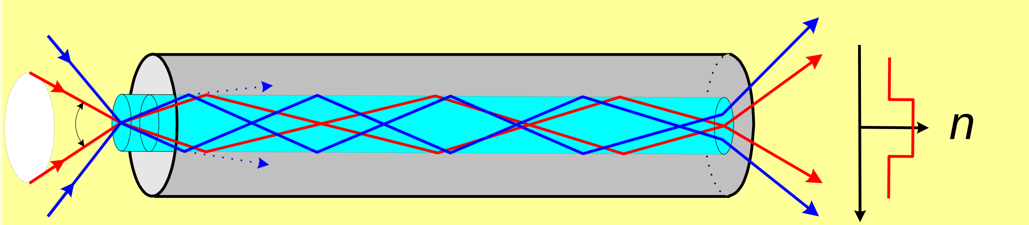
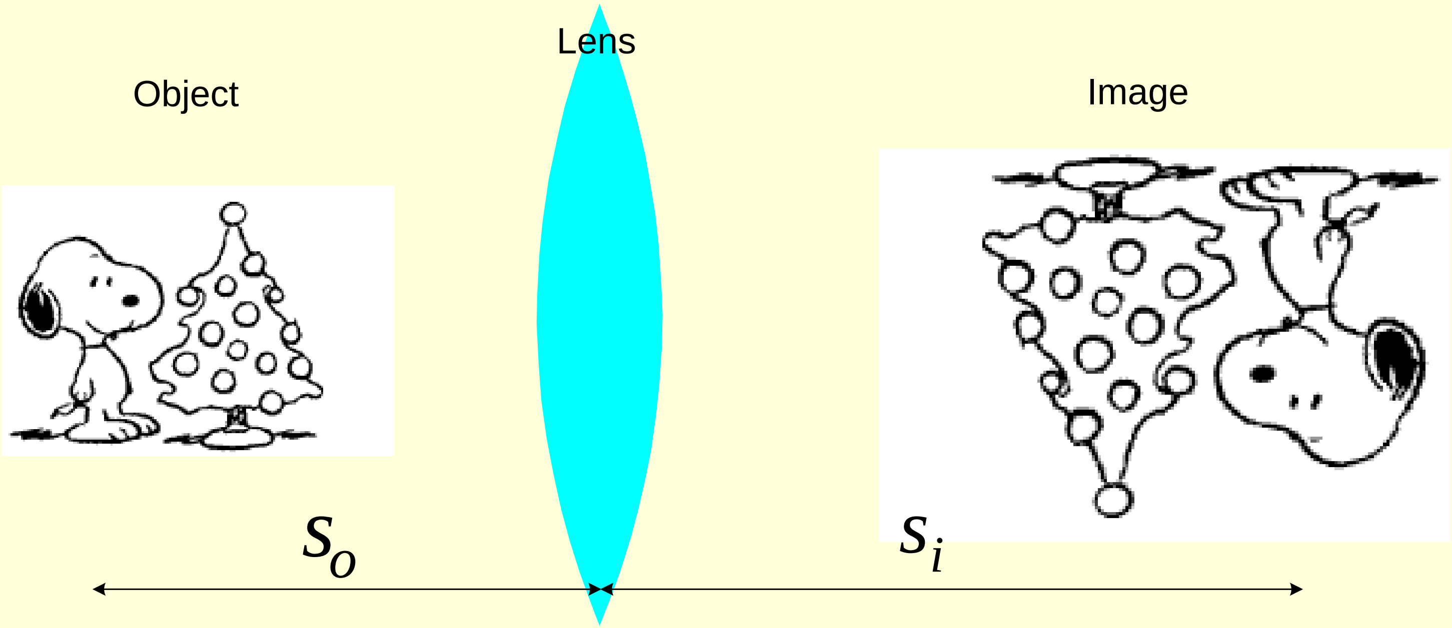


Image Formation by Lenses

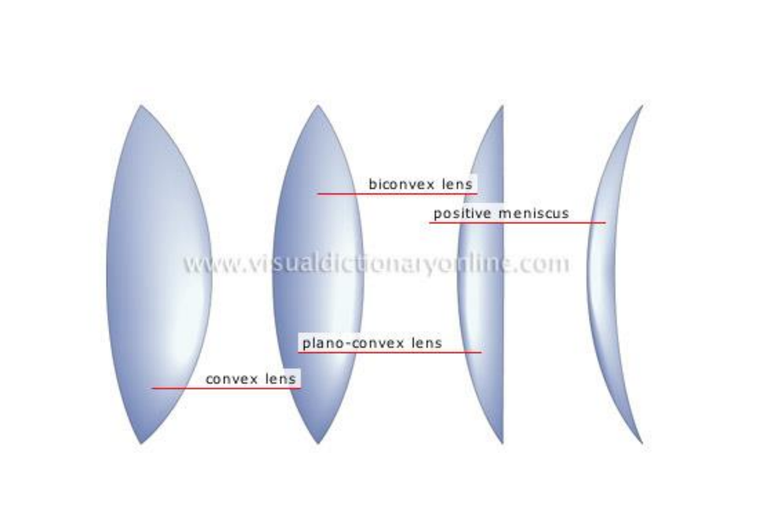
To create an image of an object, we use lenses and/or mirrors.

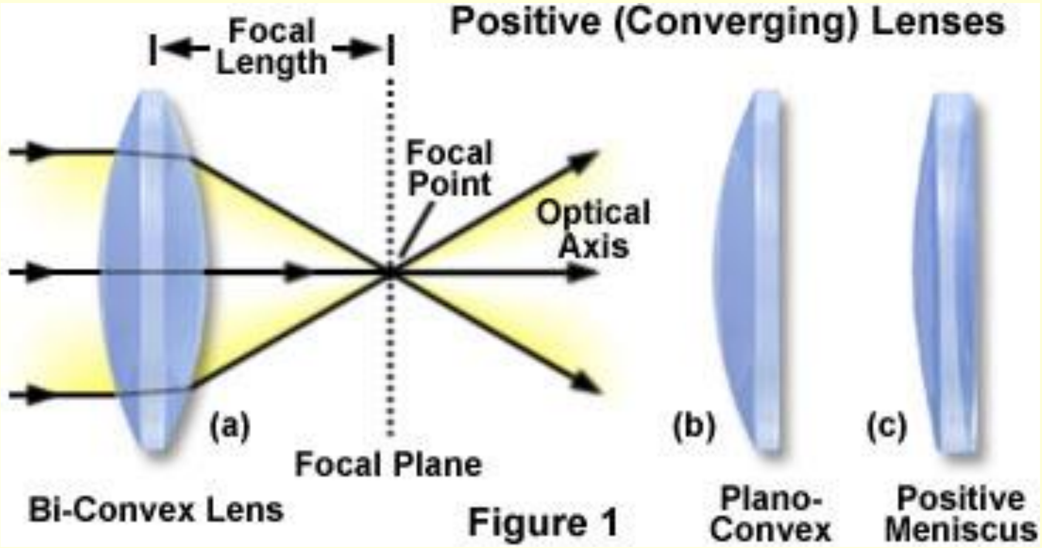


Lenses

- A *lens* is a transparent material (like glass or plastic) that (usually) has two spherical surfaces.

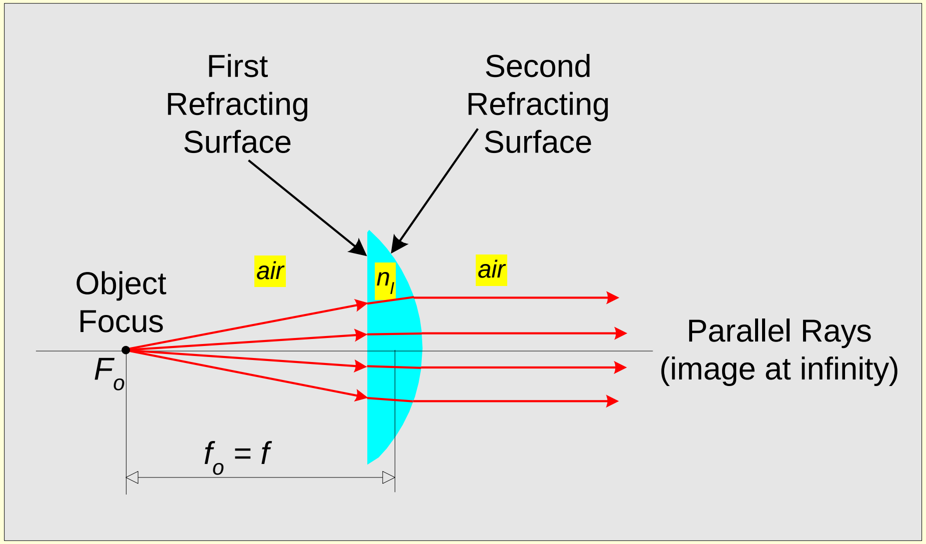
- *Converging lenses* are lenses that are thinner at the edges and thicker at the center. They transform a parallel beam of light into a converging beam of light (i.e. they “focus” together the light rays incident on the lens).

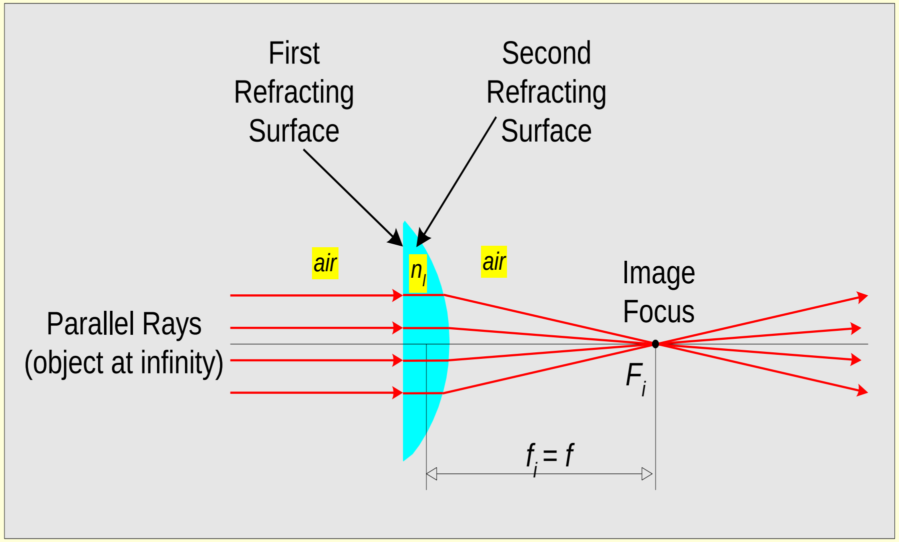




Focal Points and Focal Length

- ***Object focal point***: the point where one has to place a point light source for the emerging beam to be a parallel beam of light.





- ***Image Focal Point***: a parallel light beam parallel to the optical axis of the lens will be focused at a point called the image focal point.

Focal Points and Focal Length

- The two focal points are located at the same distance f from the lens, called ***the focal length of the lens***.

- *The thin lens equation* is an equation that relates (connects) the distance between the object and the lens (s_o) to the distance between its image and the lens (s_i)

$$\frac{1}{s_o} + \frac{1}{s_i} = \frac{1}{f}$$

- Our goal is to use this equation to find where the image is formed (to find s_i) for various positions of the object (various object distances s_o)

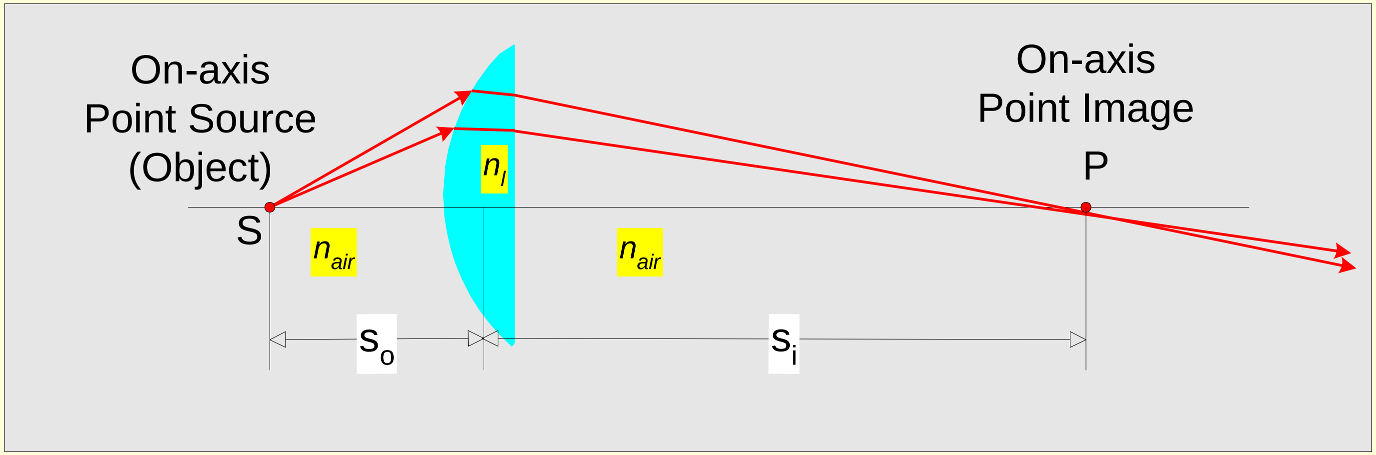
JAVA Applets for Lenses

- Converging Lens Focal Point
  http://micro.magnet.fsu.edu/primer/java/components/perfectlens/index.html
- Simple Bi-Convex Thin Lenses
  http://micro.magnet.fsu.edu/primer/java/lenses/simplethinlens/index.html
- Magnification with a Bi-Convex Lens
  http://micro.magnet.fsu.edu/primer/java/lenses/magnify/index.html
- Image Formation with Converging Lenses
  http://micro.magnet.fsu.edu/primer/java/lenses/magnify/index.html
- Image Formation with Diverging Lenses
  http://micro.magnet.fsu.edu/primer/java/lenses/diverginglenses/index.html
- Geometrical Construction of Ray Diagrams – Thick Lenses
  http://micro.magnet.fsu.edu/primer/java/components/characteristicrays/index.html

Worked –Out Example

- A point light source is placed on the optical axis at a distance of 60 cm in front (i.e. to the left) of a converging lens that has a focal length of 10 cm. Calculate the distance between the lens and the image.

$$f = 100mm$$



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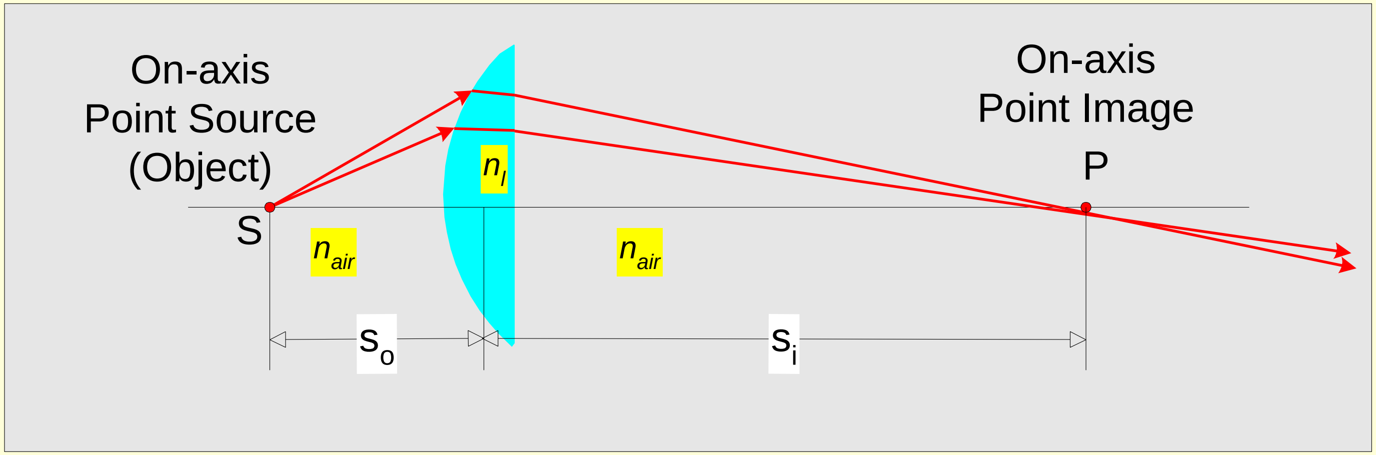


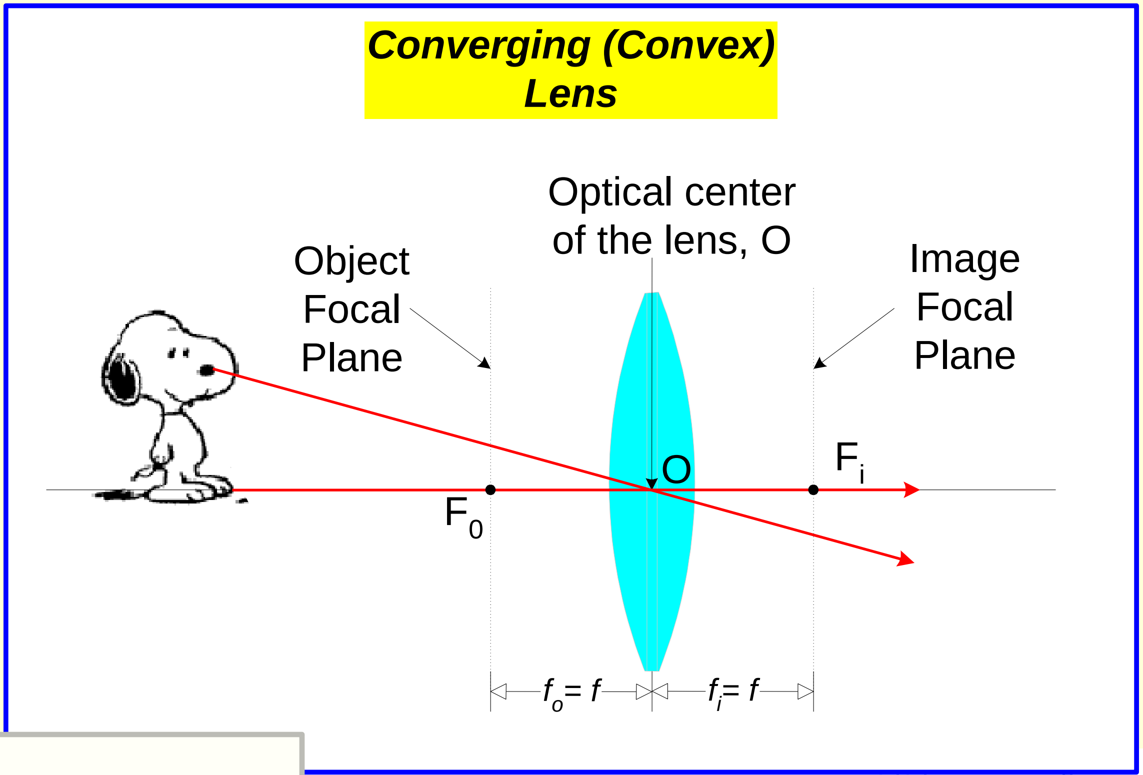
$$\frac{1}{s_o} + \frac{1}{s_i} = \frac{1}{f}$$

$$s_i = \frac{s_o \cdot f}{s_o - f} = \frac{(60)\cdot(10)}{60-10} = 12cm$$

Ray Tracing

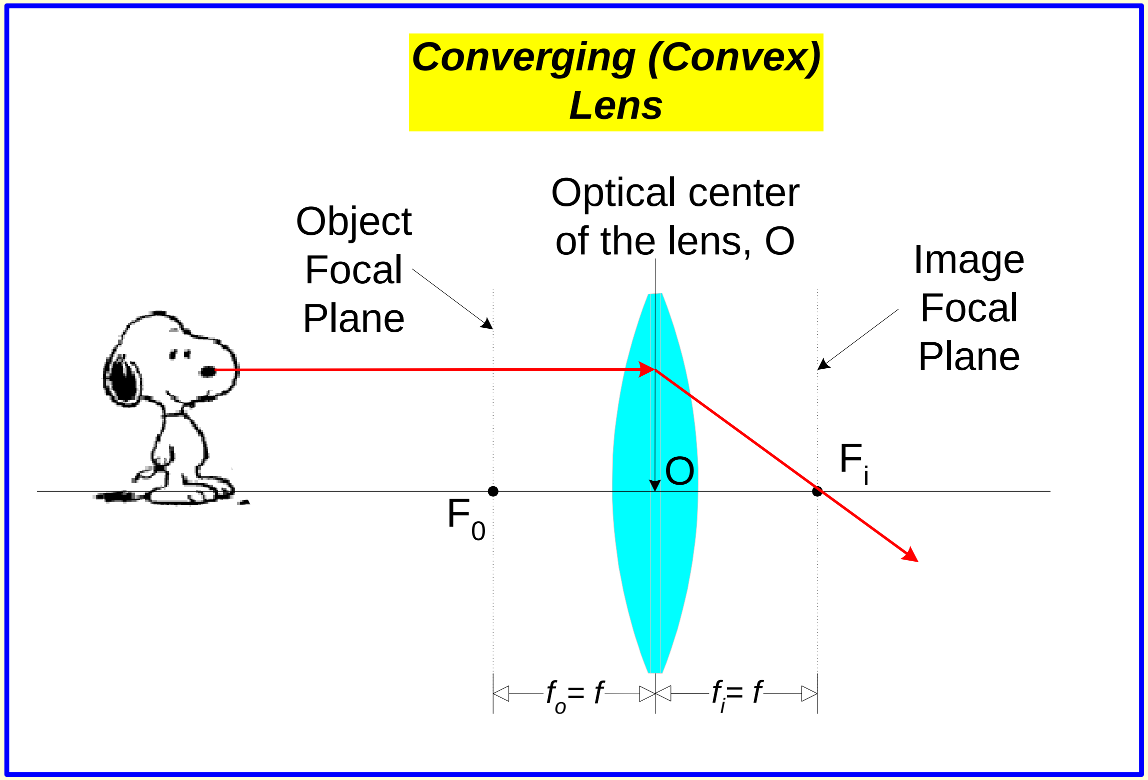
- To construct graphically the image of an object, we use a drawing technique called ***ray tracing***. Ray tracing makes use of three particular light rays:

1. A ray (any ray) passing through the *optical center* of the lens will pass undeviated (will continue in a straight line through the lens).



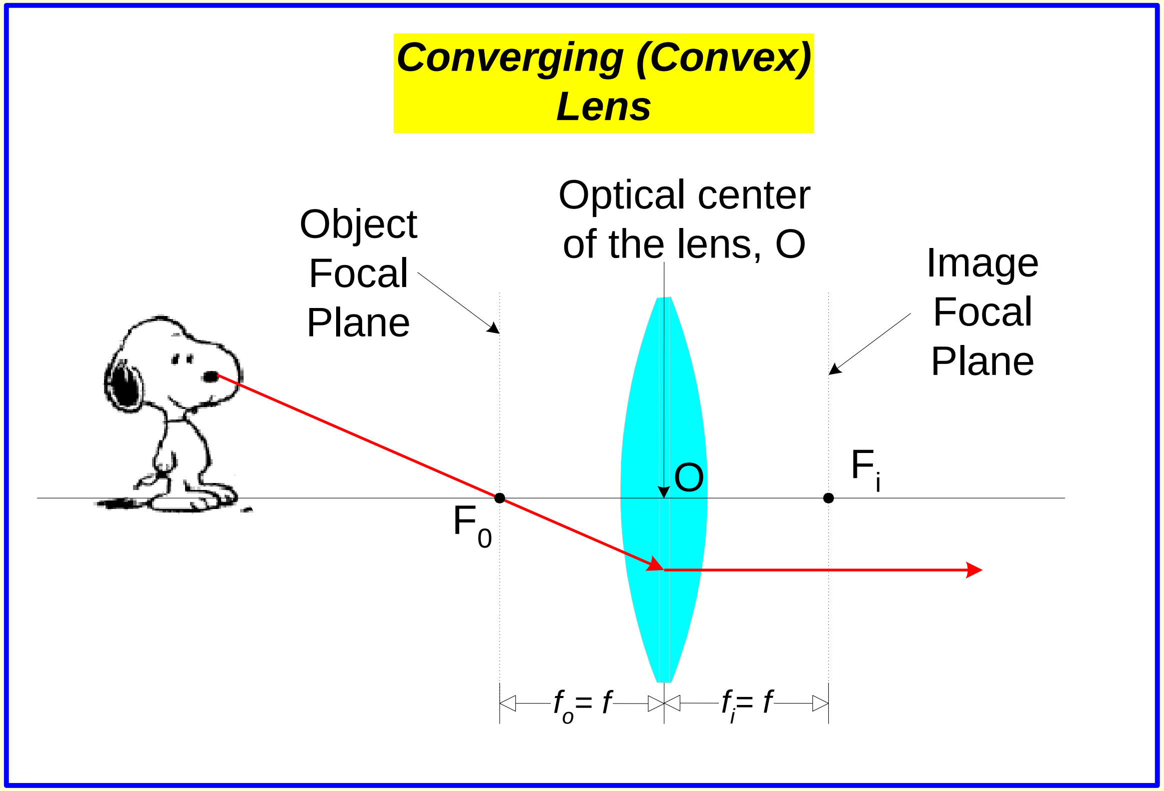
Ray Tracing

2. A ray (any ray) incident on the lens that is *parallel* to the optical axis will be refracted (bent) through the *image focal point* of the lens.



Ray Tracing

3. A ray (any ray) incident on the lens and passing through the *object focal point* of the lens will be refracted parallel to the optical axis.



Ray Tracing

- The image of a point of the object is found at the intersection of these three light rays. In fact, any two of the three rays are enough for constructing the image of a point of the object.
- The image of the entire object is the totality of images of the constituent object points.

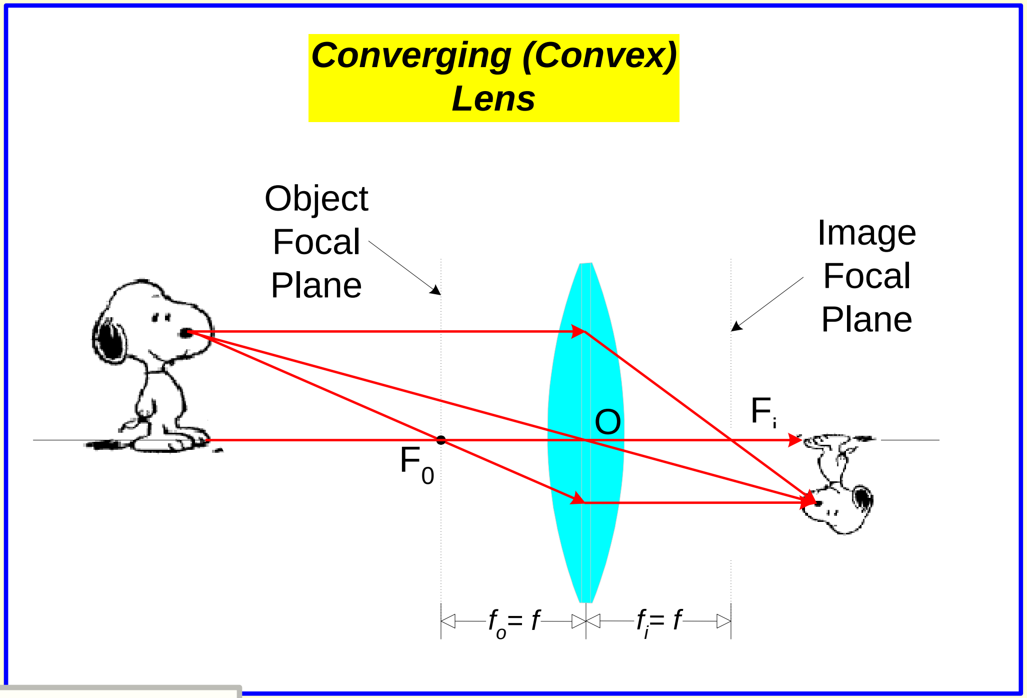
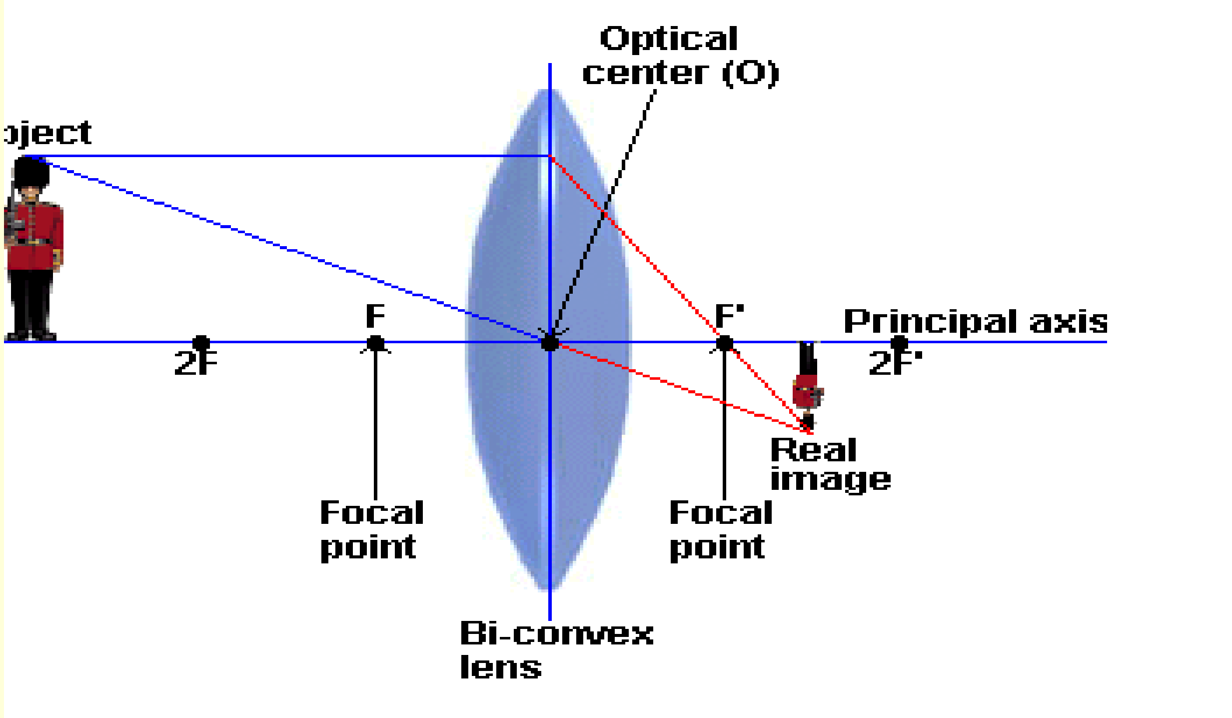


Image Formation Case Study 1



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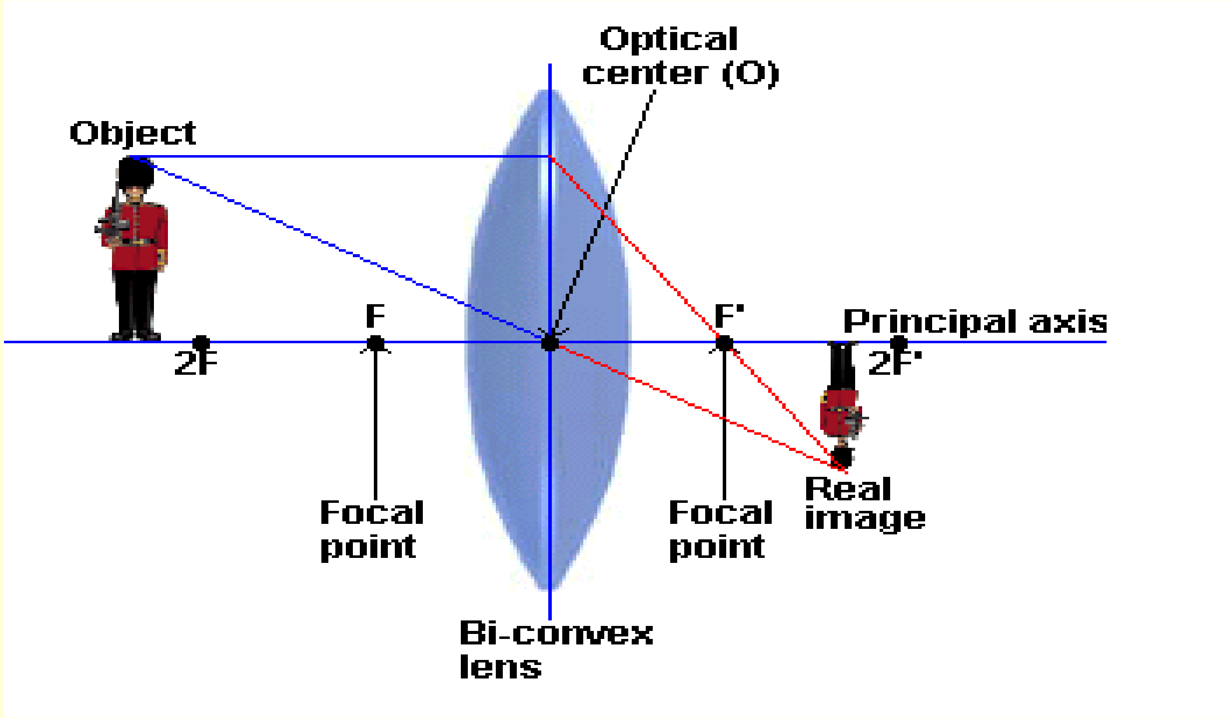

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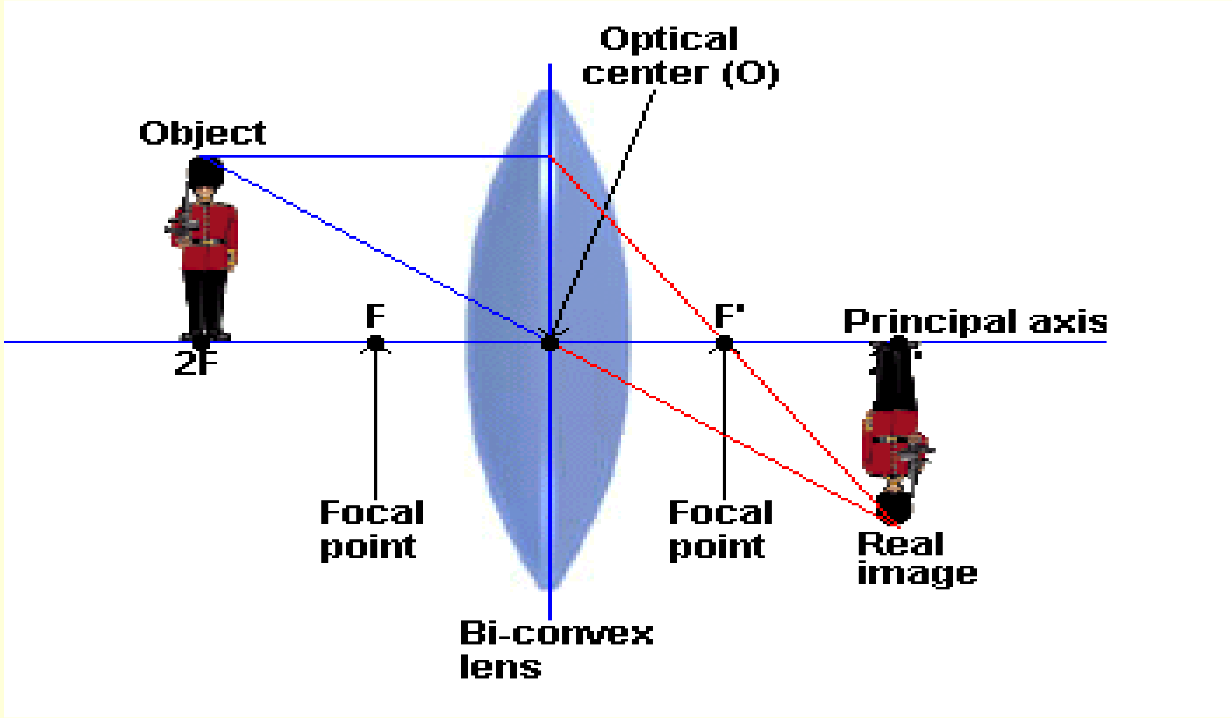

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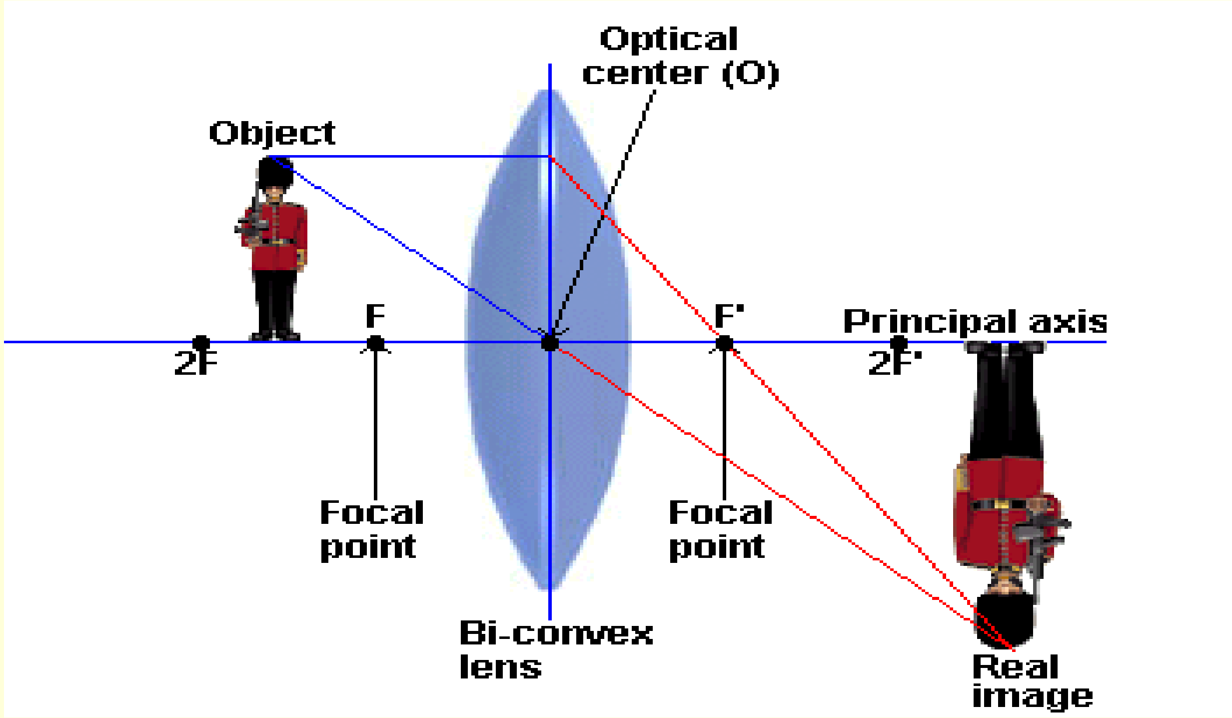

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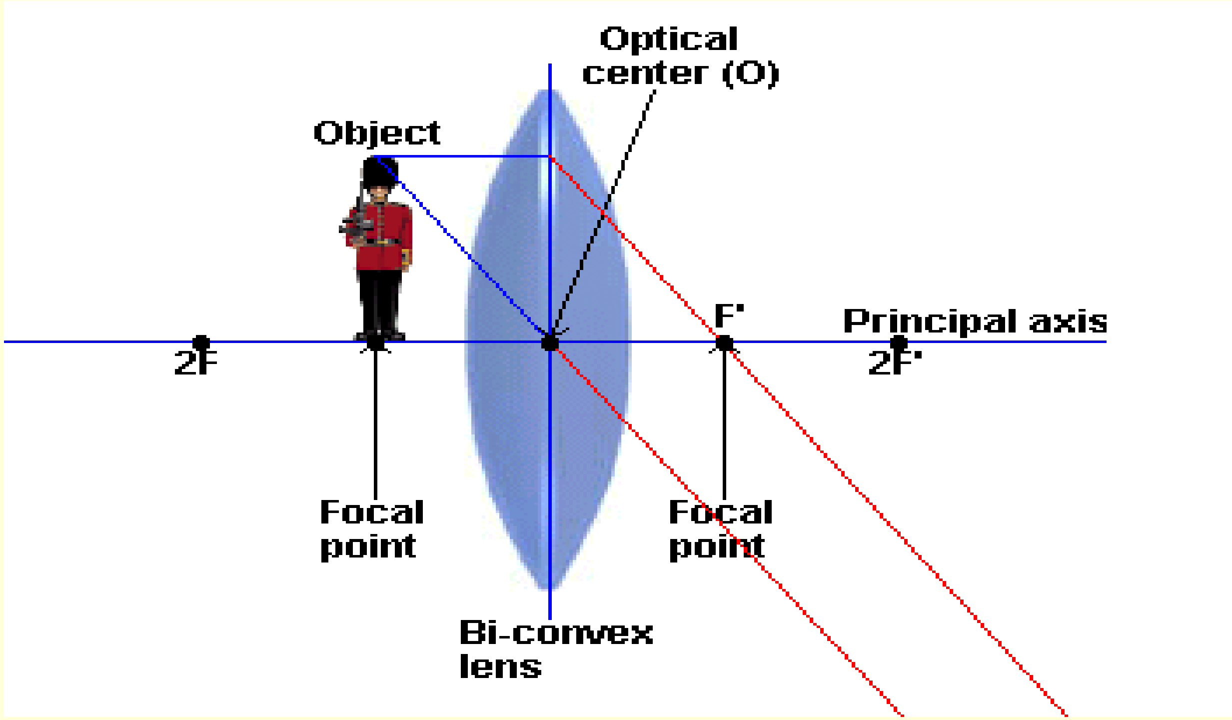

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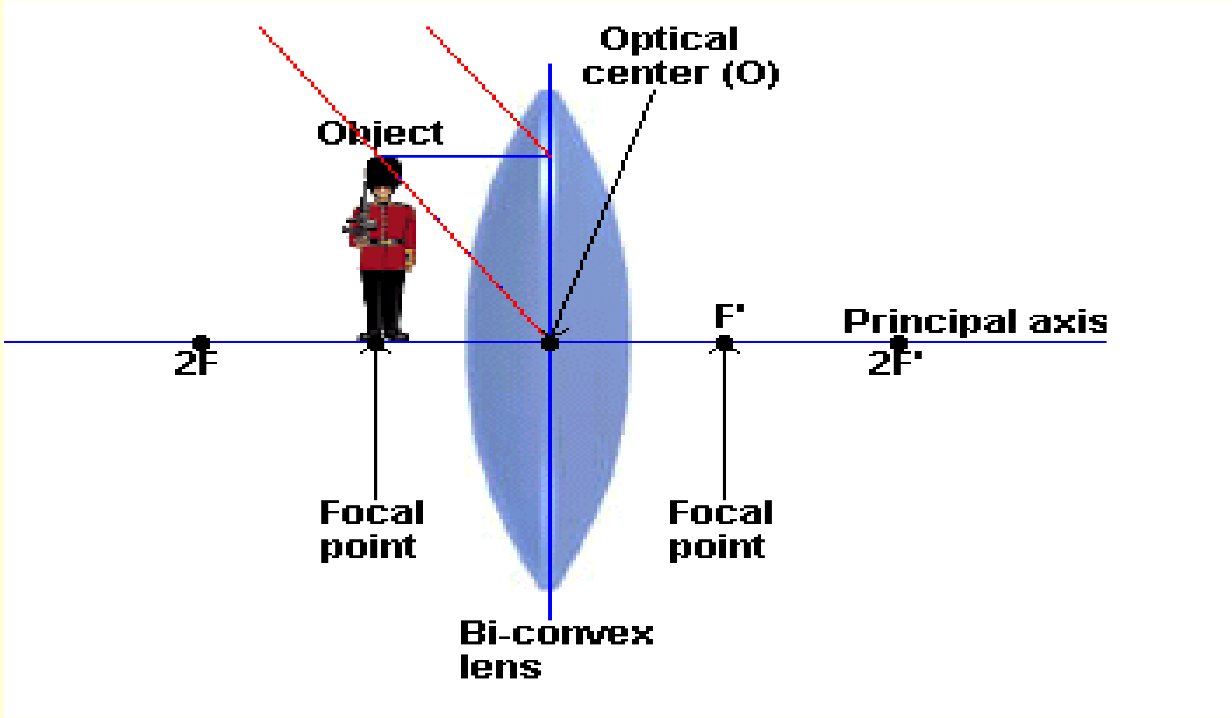

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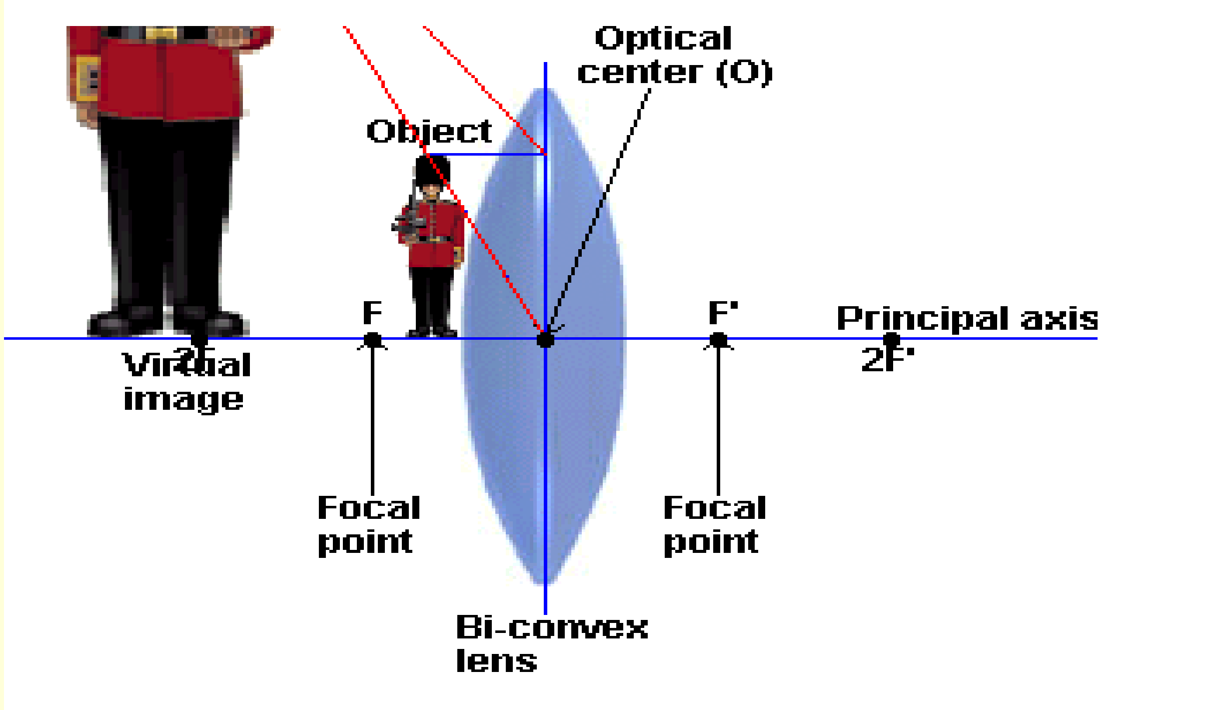

Image Formation

- JAVA Applets: Image Formation through Converging Lenses

  http://micro.magnet.fsu.edu/primer/java/lens/bi-convex.html

  http://www.phys.ksu.edu/perg/vqm/laserweb/Java/Javapm/java/Clens/index.html

  http://micro.magnet.fsu.edu/primer/java/lenses/simplethinlens/index.html

- JAVA Applet: Image Formation through Diverging Lenses

  http://micro.magnet.fsu.edu/primer/java/lens/bi-concave.html

  http://www.phys.ksu.edu/perg/vqm/laserweb/Java/Javapm/java/Dlens/index.html