The Mathematics of Sharing

The Mathematics of Sharing (Tannenbaum, chapter 3)

- Suppose that we have a set of **goods** (candy bars, diamond rings, first-round draft picks...)
- ...and a set of **players**, each of whom is entitled to a "fair share" of the goods.

How can we divide the goods to ensure that each player gets their fair share?

Here is a fair-division procedure we will not be studying.

Fair Division: An Example

Four kids (Arabella, Horace, Ludwig and Zenobia) are trying to divide a bag of jelly beans.

But, there are a few constraints.

Fair Division: An Example

Arabella and Horace both like the red ones best. Ludwig also likes red but he would rather have blue than purple. Zenobia likes red but also likes orange, which Arabella also likes. Horace wants at least three yellow ones. Arabella also likes yellow as long as she gets at least as many red ones also. Zenobia wants to make sure Horace doesn't get all the blue ones like he did last time. Horace insists he didn't hog all the blue ones, but Zenobia better not get all the pink ones. Zenobia doesn't really care about the pink ones but Zenobia knew that Horace knows that Arabella likes the pink ones and Zenobia was mad that Arabella got all the purple ones the time before that so Zenobia wanted to make sure Ludwig got more pink ones than Arabella. No one likes the weird green jelly beans.

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Your mission: Divide the jelly beans between the four kids so that no one has a temper tantrum.

Fair Division: Another Example

Having successfully divided the Halloween candy, you are now in charge of cutting the birthday cake.
Again, the problem is to divide the cake so that every kid receives a fair share.

Perhaps not like this.

Fair Division: Yet Another Example

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- and $197.28 in cash.

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- a case of rare Estonian whiskey;
- a marble statue of Charlie Chaplin;
- and $197.28 in cash.

Not only do the siblings disagree on who should receive what, they don't even agree on what the various items are worth.

As Olaf's executor, you must divide these goods fairly among the squabbling heirs.

Fair-Division Problems: Terminology and Notation

S: **set of goods to be divided** among N players

- S can be either **continuous** (if it can be divided in infinitely many ways: cake, pizza) or **discrete** (if it can't: dollars, marble busts of Charlie Chaplin).

Fair-Division Problems: Terminology and Notation

Each player is entitled to his/her own **value system** — a private way of evaluating how much each share of S is worth.

- Value systems have to be **rational:** one Hershey's Kiss can't be worth more than two Hershey's Kisses.
- But each player is free to decide that a Hershey's Kiss is worth twice (or half, or three times) as much as a Kit Kat.

Definition: A **fair share to player P** is a share of S that is worth at least $1/N$ of the value of S, **in P's opinion**.

What Constitutes A Fair Share?

Example: Arabella, Horace, Ludwig and Zenobia are to share a bag of 50 Peanut Eruptions, 20 Caramel Dreadnoughts, and 30 Jellyliciouses.

By the way, Arabella loves peanuts, but Horace is seriously allergic to them.

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Answer: It depends. For Arabella, yes. For Horace, no.

Fair Shares are Subjective

Very Important Point: Whether or not a share is fair depends on who receives it.

Fair Shares are Subjective

Goal of a Fair-Division Method: Ensure that each player receives a share that is fair **in his or her own opinion**.

- For example, if there are **four players**, then a division is fair if each player thinks s/he has received a share equal to **at least one-fourth (25%)** of the total value of the goods.

- It is OK if **some** players receive **more than** $1/N$ of the value (in their own estimation), as long as **every** player receives **at least** $1/N$.

Fair Shares are Subjective

	Value of Share (to that player)	
Player	**Case 1**	**Case 2**
Arabella	25%	25%
Horace	30%	40%
Ludwig	25%	20%
Zenobia	40%	45%

Which one is a fair division?

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Player	**Value of Share** (to that player) **Case 1**	**Case 2**
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- Case 1: **Fair.** Each of the four players receives a share worth at least 25% to him or her.

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- Case 1: **Fair.** Each of the four players receives a share worth at least 25% to him or her.
- Case 2: **Not fair.** Ludwig's share is worth **less than** 25% to him.

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- Case 1: **Fair.** Each of the four players receives a share worth at least 25% to him or her.
- Case 2: **Not fair.** Ludwig's share is worth **less than** 25% to him.
- Note: The columns do not have to add up to 100%.

Fair-Division Methods

A **fair-division method** is a systematic way to divide a set S of goods among N players so that each player receives a fair share — that is, at least $1/N$ of the value of S, in his or her own estimation.

The good news: In many cases, there are fair-division methods that are **mathematically guaranteed** to work.

Better yet, some or all players frequently wind up with **more** than a fair share!

Fair-Division Methods: Assumptions

In order to study fair-division methods, we have to make certain assumptions.

1. Cooperation: Each player accepts the rules of the fair-division method as binding.

2. Rationality: Each player is entitled to whatever value system he/she likes, but it must make mathematical sense.

For example, one Hershey's Kiss might be worth more than a bicycle, but it can't be worth more than two Hershey's Kisses.

Fair-Division Methods: Assumptions

In order to study fair-division methods, we have to make certain assumptions.

3. Privacy: No player knows anything about the other player's value system.

Without the privacy assumption, some players may be able to enrich themselves and deprive other players of a fair share. (This is analogous to strategic/insincere voting.)

The Divider-Chooser Method

The **Divider-Chooser Method** is as follows:

Step 1: Player P_1 divides the booty S into two shares.
Step 2: Player P_2 chooses one of the two shares; P_1 gets the other share.

- This is the "classic" fair-division method
- Applies to **two-player**, **continuous** fair-division games.

The Divider-Chooser Method

- Player P_1 (Divider) can guarantee himself a fair share by making sure the shares are of equal value in his opinion, so that either one will be a fair share.
- Player P_2 (Chooser) can guarantee herself a fair share by simply picking whichever she likes better, so that it is worth at least half the value of S in her opinion.
- Therefore, the Divider-Chooser method is **guaranteed** to yield a fair division, **regardless of the players' value systems**.

The Divider-Chooser Method

- The Divider-Chooser Method actually still works even without the privacy assumption.
- The method is a bit asymmetrical because it's typically better to be Chooser than Divider.
- On the other hand, if the players know each other's preferences, then it may be better to be Divider than Chooser.

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Big Question: What if there are more than two players?