The Maths Workbook

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The *Maths Workbook* has been developed for use by students preparing for the Preliminary Examinations in PPE, Economics and Management, and History and Economics. It covers the mathematical techniques that students will need for *Introductory Economics*, and part of what is required for the *Mathematics and Statistics* course for Economics and Management students.

The *Maths Workbook* was written by Margaret Stevens, with the help of: Alan Beggs, David Foster, Mary Gregory, Ben Irons, Godfrey Keller, Sujoy Mukerji, Mathan Satchi, Patrick Wallace, Tania Wilson.

Complementary Textbooks

As far as possible the *Workbook* is self-contained, but it should be used in conjunction with standard textbooks for a fuller coverage:

- Ian Jacques *Mathematics for Economics and Business* 3rd Edition 1999 or 4th Edition 2003. (The most elementary.)
- Martin Anthony and Norman Biggs *Mathematics for Economics and Finance*, 1996. (Useful and concise, but less suitable for students who have not previously studied mathematics to A-level.)
- Carl P. Simon and Lawrence Blume *Mathematics for Economists*, 1994. (A good but more advanced textbook, that goes well beyond the *Workbook*.)
- Hal R. Varian *Intermediate Microeconomics: A Modern Approach* covers many of the economic applications, particularly in the Appendices to individual chapters, where calculus is used.
- In addition, students who have not studied A-level maths, or feel that their maths is weak, may find it helpful to use one of the many excellent textbooks available for A-level Pure Mathematics (particularly the first three modules).

How to Use the Workbook

There are ten chapters, each of which can be used as the basis for a class. It is intended that students should be able to work through each chapter alone, doing the exercises and checking their own answers. References to the textbooks listed above are given at the end of each section.

At the end of each chapter is a worksheet, the answers for which are available for the use of tutors only.

The first two chapters are intended mainly for students who have not done A-level maths: they assume GCSE maths only. In subsequent chapters, students who have done A-level will find both familiar and new material.

Contents

(1) *Review of Algebra*
Simplifying and factorising algebraic expressions; indices and logarithms; solving equations (linear equations, equations involving parameters, changing the subject of a formula, quadratic equations, equations involving indices and logs); simultaneous equations; inequalities and absolute value.

(2) *Lines and Graphs*
The gradient of a line, drawing and sketching graphs, linear graphs ($y = mx+c$), quadratic graphs, solving equations and inequalities using graphs, budget constraints.

(3) *Sequences, Series and Limits; the Economics of Finance*
Arithmetic and geometric sequences and series; interest rates, savings and loans; present value; limit of a sequence, perpetuities; the number e, continuous compounding of interest.

(4) *Functions*
Common functions, limits of functions; composite and inverse functions; supply and demand functions; exponential and log functions with economic applications; functions of several variables, isoquants; homogeneous functions, returns to scale.

(5) *Differentiation*
Derivative as gradient; differentiating $y = x^n$; notation and interpretation of derivatives; basic rules and differentiation of polynomials; economic applications: MC, MPL, MPC; stationary points; the second derivative, concavity and convexity.

(6) *More Differentiation, and Optimisation*
Sketching graphs; cost functions; profit maximisation; product, quotient and chain rule; elasticities; differentiating exponential and log functions; growth; the optimum time to sell an asset.

(7) *Partial Differentiation*
First- and second-order partial derivatives; marginal products, Euler's theorem; differentials; the gradient of an isoquant; indifference curves, MRS and MRTS; the chain rule and implicit differentiation; comparative statics.

(8) *Unconstrained Optimisation Problems with One or More Variables*
First- and second-order conditions for optimisation, Perfect competition and monopoly; strategic optimisation problems: oligopoly, externalities; optimising functions of two or more variables.

(9) *Constrained Optimisation*
Methods for solving consumer choice problems: tangency condition and Lagrangian; cost minimisation; the method of Lagrange multipliers; other economic applications; demand functions.

(10) *Integration*
Integration as the reverse of differentiation; rules for integration; areas and definite integrals; producer and consumer surplus; integration by substitution and by parts; integrals and sums; the present value of an income flow.

CHAPTER 1

Review of Algebra

Much of the material in this chapter is revision from GCSE maths (although some of the exercises are harder). Some of it – particularly the work on logarithms – may be new if you have not done A-level maths. If you have done A-level, and are confident, you can skip most of the exercises and just do the worksheet, using the chapter for reference where necessary.

1. Algebraic Expressions

1.1. Evaluating Algebraic Expressions

EXAMPLES 1.1:

(i) A firm that manufactures widgets has m machines and employs n workers. The number of widgets it produces each day is given by the expression $m^2(n-3)$. How many widgets does it produce when $m = 5$ and $n = 6$?

$$\text{Number of widgets} = 5^2 \times (6-3) = 25 \times 3 = 75$$

(ii) In another firm, the cost of producing x widgets is given by $3x^2 + 5x + 4$. What is the cost of producing (a) 10 widgets (b) 1 widget?

$$\begin{aligned} \text{When } x = 10, \text{ cost} &= (3 \times 10^2) + (5 \times 10) + 4 = 300 + 50 + 4 = 354 \\ \text{When } x = 1, \text{ cost} &= 3 \times 1^2 + 5 \times 1 + 4 = 3 + 5 + 4 = 12 \end{aligned}$$

It might be clearer to use brackets here, but they are not essential: the rule is that $\times$ and $\div$ are evaluated before $+$ and $-$.

(iii) Evaluate the expression $8y^4 - \frac{12}{6-y}$ when $y = -2$.
(Remember that y^4 means $y \times y \times y \times y$.)

$$8y^4 - \frac{12}{6-y} = 8 \times (-2)^4 - \frac{12}{6-(-2)} = 8 \times 16 - \frac{12}{8} = 128 - 1.5 = 126.5$$

(If you are uncertain about using negative numbers, work through *Jacques* pp.7–9.)

EXERCISES 1.1: Evaluate the following expressions when $x = 1$, $y = 3$, $z = -2$ and $t = 0$:
(a) $3y^2 - z$ (b) $xt + z^3$ (c) $(x + 3z)y$ (d) $\frac{y}{z} + \frac{2}{x}$ (e) $(x + y)^3$ (f) $5 - \frac{x+3}{2t-z}$

1.2. Manipulating and Simplifying Algebraic Expressions

EXAMPLES 1.2:

(i) Simplify $1+3x-4y+3xy+5y^2+y-y^2+4xy-8$.
This is done by *collecting like terms*, and adding them together:

$$\begin{aligned} & 1+3x-4y+3xy+5y^2+y-y^2+4xy-8 \\ = & 5y^2-y^2+3xy+4xy+3x-4y+y+1-8 \\ = & 4y^2+7xy+3x-3y-7 \end{aligned}$$

The order of the terms in the answer doesn't matter, but we often put a positive term first, and/or write "higher-order" terms such as y^2 before "lower-order" ones such as y or a number.

(ii) Simplify $5(x-3)-2x(x+y-1)$.
Here we need to *multiply out the brackets* first, and then collect terms:

$$\begin{aligned} 5(x-3)-2x(x+y-1) &= 5x-15-2x^2-2xy+2x \\ &= 7x-2x^2-2xy-15 \end{aligned}$$

(iii) Multiply x^3 by x^2.

$$x^3 \times x^2 = x \times x \times x \times x \times x = x^5$$

(iv) Divide x^3 by x^2.
We can write this as a fraction, and cancel:

$$x^3 \div x^2 = \frac{x \times x \times x}{x \times x} = \frac{x}{1} = x$$

(v) Multiply $5x^2y^4$ by $4yx^6$.

$$\begin{aligned} 5x^2y^4 \times 4yx^6 &= 5 \times x^2 \times y^4 \times 4 \times y \times x^6 \\ &= 20 \times x^8 \times y^5 \\ &= 20x^8y^5 \end{aligned}$$

Note that you can always change the order of multiplication.

(vi) Divide $6x^2y^3$ by $2yx^5$.

$$\begin{aligned} 6x^2y^3 \div 2yx^5 &= \frac{6x^2y^3}{2yx^5} = \frac{3x^2y^3}{yx^5} = \frac{3y^3}{yx^3} \\ &= \frac{3y^2}{x^3} \end{aligned}$$

(vii) Add $\frac{3x}{y}$ and $\frac{y}{2}$.
The rules for algebraic fractions are just the same as for numbers, so here we find a *common denominator*:

$$\begin{aligned} \frac{3x}{y}+\frac{y}{2} &= \frac{6x}{2y}+\frac{y^2}{2y} \\ &= \frac{6x+y^2}{2y} \end{aligned}$$

(viii) Divide $\frac{3x^2}{y}$ by $\frac{xy^3}{2}$.

$$\begin{aligned} \frac{3x^2}{y} \div \frac{xy^3}{2} &= \frac{3x^2}{y} \times \frac{2}{xy^3} = \frac{3x^2 \times 2}{y \times xy^3} = \frac{6x^2}{xy^4} \\ &= \frac{6x}{y^4} \end{aligned}$$

Exercises 1.2: Simplify the following as much as possible:

(1) (a) $3x - 17 + x^3 + 10x - 8$ (b) $2(x+3y) - 2(x+7y-x^2)$

(2) (a) $z^2x - (z+1) + z(2xz+3)$ (b) $(x+2)(x+4) + (3-x)(x+2)$

(3) (a) $\dfrac{3x^2y}{6x}$ (b) $\dfrac{12xy^3}{2x^2y^2}$

(4) (a) $2x^2 \div 8xy$ (b) $4xy \times 5x^2y^3$

(5) (a) $\dfrac{2x}{y} \times \dfrac{y^2}{2x}$ (b) $\dfrac{2x}{y} \div \dfrac{y^2}{2x}$

(6) (a) $\dfrac{2x+1}{4} + \dfrac{x}{3}$ (b) $\dfrac{1}{x-1} - \dfrac{1}{x+1}$ (giving the answers as a single fraction)

1.3. Factorising

A number can be written as the product of its factors. For example: $30 = 5 \times 6 = 5 \times 3 \times 2$. Similarly "factorise" an algebraic expression means "write the expression as the product of two (or more) expressions." Of course, some numbers (primes) don't have any proper factors, and similarly, some algebraic expressions can't be factorised.

Examples 1.3:

(i) Factorise $6x^2 + 15x$.
Here, $3x$ is a common factor of each term in the expression so:

$$6x^2 + 15x = 3x(2x+5)$$

The factors are $3x$ and $(2x+5)$. You can check the answer by multiplying out the brackets.

(ii) Factorise $x^2 + 2xy + 3x + 6y$.
There is no common factor of all the terms but the first pair have a common factor, and so do the second pair, and this leads us to the factors of the whole expression:

$$\begin{aligned} x^2 + 2xy + 3x + 6y &= x(x+2y) + 3(x+2y) \\ &= (x+3)(x+2y) \end{aligned}$$

Again, check by multiplying out the brackets.

(iii) Factorise $x^2 + 2xy + 3x + 3y$.
We can try the method of the previous example, but it doesn't work. The expression can't be factorised.

(iv) Simplify $5(x^2+6x+3)-3(x^2+4x+5)$.
Here we can first multiply out the brackets, then collect like terms, then factorise:

$$\begin{aligned} 5(x^2+6x+3)-3(x^2+4x+5) &= 5x^2+30x+15-3x^2-12x-15 \\ &= 2x^2+18x \\ &= 2x(x+9) \end{aligned}$$

EXERCISES 1.3: **Factorising**

(1) Factorise: (a) $3x+6xy$ (b) $2y^2+7y$ (c) $6a+3b+9c$

(2) Simplify and factorise: (a) $x(x^2+8)+2x^2(x-5)-8x$ (b) $a(b+c)-b(a+c)$

(3) Factorise: $xy+2y+2xz+4z$

(4) Simplify and factorise: $3x(x+\frac{4}{x})-4(x^2+3)+2x$

1.4. Polynomials

Expressions such as

$$5x^2-9x^4-20x+7 \text{ and } 2y^5+y^3-100y^2+1$$

are called *polynomials*. A polynomial in x is a sum of terms, and each term is either a power of x (multiplied by a number called a *coefficient*), or just a number known as a *constant*. All the powers must be positive integers. (Remember: an *integer* is a positive or negative whole number.) The *degree* of the polynomial is the highest power. A polynomial of degree 2 is called a *quadratic* polynomial.

EXAMPLES 1.4: *Polynomials*

(i) $5x^2-9x^4-20x+7$ is a polynomial of degree 4. In this polynomial, the coefficient of x^2 is 5 and the coefficient of x is -20. The constant term is 7.

(ii) x^2+5x+6 is a quadratic polynomial. Here the coefficient of x^2 is 1.

1.5. Factorising Quadratics

In section 1.3 we factorised a quadratic polynomial by finding a common factor of each term: $6x^2+15x=3x(2x+5)$. But this only works because there is no constant term. Otherwise, we can try a different method:

EXAMPLES 1.5: *Factorising Quadratics*

(i) x^2+5x+6

- Look for two numbers that multiply to give 6, and add to give 5:

$$2\times 3=6 \text{ and } 2+3=5$$

- Split the "x"-term into two:

$$x^2+2x+3x+6$$

- Factorise the first pair of terms, and the second pair:

$$x(x+2)+3(x+2)$$

- $(x+2)$ is a factor of both terms so we can rewrite this as:

$$(x+3)(x+2)$$

- So we have:

$$x^2+5x+6=(x+3)(x+2)$$

(ii) y^2-y-12
In this example the two numbers we need are 3 and -4, because $3\times(-4)=-12$ and $3+(-4)=-1$. Hence:

$$\begin{aligned} y^2-y-12 &= y^2+3y-4y-12 \\ &= y(y+3)-4(y+3) \\ &= (y-4)(y+3) \end{aligned}$$

(iii) $2x^2-5x-12$
This example is slightly different because the coefficient of x^2 is not 1.

- Start by multiplying together the coefficient of x^2 and the constant:

$$2\times(-12)=-24$$

- Find two numbers that multiply to give -24, and add to give -5.

$$3\times(-8)=-24 \text{ and } 3+(-8)=-5$$

- Proceed as before:

$$\begin{aligned} 2x^2-5x-12 &= 2x^2+3x-8x-12 \\ &= x(2x+3)-4(2x+3) \\ &= (x-4)(2x+3) \end{aligned}$$

(iv) x^2+x-1
The method doesn't work for this example, because we can't see any numbers that multiply to give -1, but add to give 1. (In fact there is a pair of numbers that does so, but they are not integers so we are unlikely to find them.)

(v) x^2-49
The two numbers must multiply to give -49 and add to give zero. So they are 7 and -7:

$$\begin{aligned} x^2-49 &= x^2+7x-7x-49 \\ &= x(x+7)-7(x+7) \\ &= (x-7)(x+7) \end{aligned}$$

The last example is a special case of the result known as "the difference of two squares". If a and b are any two numbers:

$$a^2-b^2=(a-b)(a+b)$$

Exercises 1.4: Use the method above (if possible) to factorise the following quadratics:

(1) x^2+4x+3
(2) $y^2+10-7y$
(3) $2x^2+7x+3$
(4) $z^2+2z-15$
(5) $4x^2-9$
(6) $y^2-10y+25$
(7) x^2+3x+1

1.6. Rational Numbers, Irrational Numbers, and Square Roots

A *rational* number is a number that can be written in the form $\frac{p}{q}$ where p and q are integers. An *irrational* number is a number that is not rational. It can be shown that if a number can be written as a terminating decimal (such as 1.32) or a recurring decimal (such as 3.7425252525...) then it is rational. Any decimal that does not terminate or recur is irrational.

EXAMPLES 1.6: *Rational and Irrational Numbers*

(i) 3.25 is rational because $3.25 = 3\frac{1}{4} = \frac{13}{4}$.

(ii) -8 is rational because $-8 = \frac{-8}{1}$. Obviously, all integers are rational.

(iii) To show that 0.12121212... is rational check on a calculator that it is equal to $\frac{4}{33}$.

(iv) $\sqrt{2} = 1.41421356237...$ is irrational.

Most, but not all, square roots are irrational:

EXAMPLES 1.7: *Square Roots*

(i) (Using a calculator) $\sqrt{5} = 2.2360679774...$ and $\sqrt{12} = 3.4641016151...$

(ii) $5^2 = 25$, so $\sqrt{25} = 5$

(iii) $\frac{2}{3} \times \frac{2}{3} = \frac{4}{9}$, so $\sqrt{\frac{4}{9}} = \frac{2}{3}$

Rules for Square Roots:

$$\sqrt{ab} = \sqrt{a}\sqrt{b} \qquad \text{and} \qquad \sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$$

EXAMPLES 1.8: *Using the rules to manipulate expressions involving square roots*

(i) $\sqrt{2} \times \sqrt{50} = \sqrt{2 \times 50} = \sqrt{100} = 10$

(ii) $\sqrt{48} = \sqrt{16}\sqrt{3} = 4\sqrt{3}$

(iii) $\frac{\sqrt{98}}{\sqrt{8}} = \sqrt{\frac{98}{8}} = \sqrt{\frac{49}{4}} = \frac{\sqrt{49}}{\sqrt{4}} = \frac{7}{2}$

(iv) $\frac{-2+\sqrt{20}}{2} = -1 + \frac{\sqrt{20}}{2} = -1 + \frac{\sqrt{5}\sqrt{4}}{2} = -1 + \sqrt{5}$

(v) $\frac{8}{\sqrt{2}} = \frac{8\times\sqrt{2}}{\sqrt{2}\times\sqrt{2}} = \frac{8\sqrt{2}}{2} = 4\sqrt{2}$

(vi) $\frac{\sqrt{27y}}{\sqrt{3y}} = \sqrt{\frac{27y}{3y}} = \sqrt{9} = 3$

(vii) $\sqrt{x^3y}\sqrt{4xy} = \sqrt{x^3y \times 4xy} = \sqrt{4x^4y^2} = \sqrt{4}\sqrt{x^4}\sqrt{y^2} = 2x^2y$

EXERCISES 1.5: **Square Roots**

(1) Show that: (a) $\sqrt{2} \times \sqrt{18} = 6$ (b) $\sqrt{245} = 7\sqrt{5}$ (c) $\frac{15}{\sqrt{3}} = 5\sqrt{3}$

(2) Simplify: (a) $\frac{\sqrt{45}}{3}$ (b) $\sqrt{2x^3} \times \sqrt{8x}$ (c) $\sqrt{2x^3} \div \sqrt{8x}$ (d) $\frac{1}{3}\sqrt{18y^2}$

Further reading and exercises

- *Jacques* §1.4 has lots more practice of algebra. If you have had any difficulty with the work so far, you should work through it before proceeding.

2. Indices and Logarithms

2.1. Indices

We know that x^3 means $x \times x \times x$. More generally, if n is a positive integer, x^n means "x multiplied by itself n times". We say that x is *raised to the power* n. Alternatively, n may be described as the *index* of x in the expression x^n.

EXAMPLES 2.1:

(i) $5^4 \times 5^3 = 5 \times 5 \times 5 \times 5 \times 5 \times 5 \times 5 = 5^7$.

(ii) $\dfrac{x^5}{x^2} = \dfrac{x \times x \times x \times x \times x}{x \times x} = x \times x \times x = x^3$.

(iii) $\left(y^3\right)^2 = y^3 \times y^3 = y^6$.

Each of the above examples is a special case of the general rules:

- $a^m \times a^n = a^{m+n}$
- $\dfrac{a^m}{a^n} = a^{m-n}$
- $(a^m)^n = a^{m \times n}$

Now, a^n also has a meaning when n is zero, or negative, or a fraction. Think about the second rule above. If $m = n$, this rule says:

$$a^0 = \frac{a^n}{a^n} = 1$$

If $m = 0$ the rule says:

$$a^{-n} = \frac{1}{a^n}$$

Then think about the third rule. If, for example, $m = \frac{1}{2}$ and $n = 2$, this rule says:

$$\left(a^{\frac{1}{2}}\right)^2 = a$$

which means that

$$a^{\frac{1}{2}} = \sqrt{a}$$

Similarly $a^{\frac{1}{3}}$ is the cube root of a, and more generally $a^{\frac{1}{n}}$ is the n^{th} root of a:

$$a^{\frac{1}{n}} = \sqrt[n]{a}$$

Applying the third rule above, we find for more general fractions:

$$a^{\frac{m}{n}} = \left(\sqrt[n]{a}\right)^m = \sqrt[n]{a^m}$$

We can summarize the rules for zero, negative, and fractional powers:

- $a^0 = 1$ (if $a \neq 0$)
- $a^{-n} = \dfrac{1}{a^n}$
- $a^{\frac{1}{n}} = \sqrt[n]{a}$
- $a^{\frac{m}{n}} = \left(\sqrt[n]{a}\right)^m = \sqrt[n]{a^m}$

There are two other useful rules, which may be obvious to you. If not, check them using some particular examples:

- $a^n b^n = (ab)^n$ and
- $\dfrac{a^n}{b^n} = \left(\dfrac{a}{b}\right)^n$

EXAMPLES 2.2: *Using the Rules for Indices*

(i) $3^2 \times 3^3 = 3^5 = 243$

(ii) $\left(5^2\right)^{\frac{1}{2}} = 5^{2\times\frac{1}{2}} = 5$

(iii) $4^{\frac{3}{2}} = \left(4^{\frac{1}{2}}\right)^3 = 2^3 = 8$

(iv) $36^{-\frac{3}{2}} = \left(36^{\frac{1}{2}}\right)^{-3} = 6^{-3} = \dfrac{1}{6^3} = \dfrac{1}{216}$

(v) $\left(3\frac{3}{8}\right)^{\frac{2}{3}} = \left(\dfrac{27}{8}\right)^{\frac{2}{3}} = \dfrac{27^{\frac{2}{3}}}{8^{\frac{2}{3}}} = \dfrac{\left(27^{\frac{1}{3}}\right)^2}{\left(8^{\frac{1}{3}}\right)^2} = \dfrac{3^2}{2^2} = \dfrac{9}{4}$

2.2. Logarithms

You can think of *logarithm* as another word for index or power. To define a logarithm we first choose a particular *base*. Your calculator probably uses base 10, but we can take any positive integer, a. Now take any positive number, x.

The logarithm of x to the base a is:
the power to which the base must be raised to obtain x.
If $x = a^n$ then $\log_a x = n$

In fact the statement: $\log_a x = n$ is simply another way of saying: $x = a^n$. Note that, since a^n is positive for all values of n, there is no such thing as the log of zero or a negative number.

EXAMPLES 2.3:

(i) Since we know $2^5 = 32$, we can say that the log of 32 to the base 2 is 5: $\log_2 32 = 5$

(ii) From $3^4 = 81$ we can say $\log_3 81 = 4$

(iii) From $10^{-2} = 0.01$ we can say $\log_{10} 0.01 = -2$

(iv) From $9^{\frac{1}{2}} = 3$ we can say $\log_9 3 = 0.5$

(v) From $a^0 = 1$, we can say that the log of 1 to *any* base is zero: $\log_a 1 = 0$

(vi) From $a^1 = a$, we can say that for any base a, the log of a is 1: $\log_a a = 1$

Except for easy examples like these, you cannot calculate logarithms of particular numbers in your head. For example, if you wanted to know the logarithm to base 10 of 3.4, you would need to find out what power of 10 is equal to 3.4, which is not easy. So instead, you can use your calculator. Check the following examples of logs to base 10:

EXAMPLES 2.4: Using a calculator we find that (correct to 5 decimal places):

(i) $\log_{10} 3.4 = 0.53148$ (ii) $\log_{10} 125 = 2.09691$ (iii) $\log_{10} 0.07 = -1.15490$

There is a way of calculating logs to other bases, using logs to base 10. But the only other base that you really need is the special base e, which we will meet later.

2.3. Rules for Logarithms

Since logarithms are powers, or indices, there are rules for logarithms which are derived from the rules for indices in section 2.1:

- $\log_a xy = \log_a x + \log_a y$
- $\log_a \dfrac{x}{y} = \log_a x - \log_a y$
- $\log_a x^b = b \log_a x$
- $\log_a a = 1$
- $\log_a 1 = 0$

To see where the first rule comes from, suppose: $m = \log_a x$ and $n = \log_a y$
This is equivalent to: $x = a^m$ and $y = a^n$
Using the first rule for indices: $xy = a^m a^n = a^{m+n}$
But this means that: $\log_a xy = m + n = \log_a x + \log_a y$
which is the first rule for logs.
You could try proving the other rules similarly.

Before electronic calculators were available, printed tables of logs were used calculate, for example, $14.58 \div 0.3456$. You could find the log of each number in the tables, then (applying the second rule) subtract them, and use the tables to find the "anti-log" of the answer.

EXAMPLES 2.5: *Using the Rules for Logarithms*

(i) Express $2\log_a 5 + \frac{1}{3}\log_a 8$ as a single logarithm.

$$\begin{aligned} 2\log_a 5 + \tfrac{1}{3}\log_a 8 &= \log_a 5^2 + \log_a 8^{\frac{1}{3}} = \log_a 25 + \log_a 2 \\ &= \log_a 50 \end{aligned}$$

(ii) Express $\log_a \left(\frac{x^2}{y^3}\right)$ in terms of $\log x$ and $\log y$.

$$\begin{aligned} \log_a \left(\tfrac{x^2}{y^3}\right) &= \log_a x^2 - \log_a y^3 \\ &= 2\log_a x - 3\log_a y \end{aligned}$$

EXERCISES 1.6: **Indices and Logarithms**

(1) Evaluate (without a calculator):
(a) $64^{\frac{2}{3}}$ (b) $\log_2 64$ (c) $\log_{10} 1000$ (d) $4^{130} \div 4^{131}$

(2) Simplify: (a) $2x^5 \times x^6$ (b) $\dfrac{(xy)^2}{x^3y^2}$ (c) $\log_{10}(xy) - \log_{10} x$ (d) $\log_{10}(x^3) \div \log_{10} x$

(3) Simplify: (a) $\left(3\sqrt{ab}\right)^6$ (b) $\log_{10} a^2 + \frac{1}{3}\log_{10} b - 2\log_{10} ab$

Further reading and exercises

- *Jacques* §2.3 covers all the material in section 2, and provides more exercises.

3. Solving Equations

3.1. Linear Equations

Suppose we have an equation:

$$5(x-6) = x+2$$

Solving this equation means finding the value of x that makes the equation true. (Some equations have several, or many, solutions; this one has only one.)

To solve this sort of equation, we manipulate it by "doing the same thing to both sides." The aim is to get the variable x on one side, and everything else on the other.

Examples 3.1: Solve the following equations:

(i) $5(x-6) = x+2$

$$\begin{array}{lrcl} \text{Remove brackets:} & 5x-30 & = & x+2 \\ -x \text{ from both sides:} & 5x-x-30 & = & x-x+2 \\ \text{Collect terms:} & 4x-30 & = & 2 \\ +30 \text{ to both sides:} & 4x & = & 32 \\ \div \text{ both sides by 4:} & x & = & 8 \end{array}$$

(ii) $\dfrac{5-x}{3} + 1 = 2x+4$

Here it is a good idea to remove the fraction first:

$$\begin{array}{lrcl} \times \text{ all terms by 3:} & 5-x+3 & = & 6x+12 \\ \text{Collect terms:} & 8-x & = & 6x+12 \\ -6x \text{ from both sides:} & 8-7x & = & 12 \\ -8 \text{ from both sides:} & -7x & = & 4 \\ \div \text{ both sides by } -7\text{:} & x & = & -\frac{4}{7} \end{array}$$

(iii) $\dfrac{5x}{2x-9} = 1$

Again, remove the fraction first:

$$\begin{array}{lrcl} \times \text{ by } (2x-9)\text{:} & 5x & = & 2x-9 \\ -2x \text{ from both sides:} & 3x & = & -9 \\ \div \text{ both sides by 3:} & x & = & -3 \end{array}$$

All of these are linear equations: once we have removed the brackets and fractions, each term is either an x-term or a constant.

Exercises 1.7: Solve the following equations:

(1) $5x+4 = 19$

(2) $2(4-y) = y+17$

(3) $\frac{2x+1}{5} + x - 3 = 0$

(4) $2 - \frac{4-z}{z} = 7$

(5) $\frac{1}{4}(3a+5) = \frac{3}{2}(a+1)$

3.2. Equations involving Parameters

Suppose x satisfies the equation: $5(x-a) = 3x + 1$

Here a is a *parameter*: a letter representing an unspecified number. The solution of the equation will depend on the value of a. For example, you can check that if $a = 1$, the solution is $x = 3$, and if $a = 2$ the solution is $x = 5.5$.

Without knowing the value of a, we can still solve the equation for x, to find out exactly how x depends on a. As before, we manipulate the equation to get x on one side and everything else on the other:

$$\begin{aligned} 5x - 5a &= 3x + 1 \\ 2x - 5a &= 1 \\ 2x &= 5a + 1 \\ x &= \frac{5a+1}{2} \end{aligned}$$

We have obtained the solution for x in terms of the parameter a.

Exercises 1.8: **Equations involving parameters**

(1) Solve the equation $ax + 4 = 10$ for x.

(2) Solve the equation $\frac{1}{2}y + 5b = 3b$ for y.

(3) Solve the equation $2z - a = b$ for z.

3.3. Changing the Subject of a Formula

$V = \pi r^2 h$ is the formula for the volume of a cylinder with radius r and height h - so if you know r and h, you can calculate V. We could rearrange the formula to *make r the subject*:

Write the equation as:	$\pi r^2 h$	$=$	V
Divide by πh:	r^2	$=$	$\frac{V}{\pi h}$
Square root both sides:	r	$=$	$\sqrt{\frac{V}{\pi h}}$

This gives us a formula for r in terms of V and h. The procedure is exactly the same as solving the equation for r.

Exercises 1.9: **Formulae and Equations**

(1) Make t the subject of the formula $v = u + at$

(2) Make a the subject of the formula $c = \sqrt{a^2 + b^2}$

(3) When the price of an umbrella is p, and daily rainfall is r, the number of umbrellas sold is given by the formula: $n = 200r - \frac{p}{6}$. Find the formula for the price in terms of the rainfall and the number sold.

(4) If a firm that manufuctures widgets has m machines and employs n workers, the number of widgets it produces each day is given by the formula $W = m^2(n-3)$. Find a formula for the number of workers it needs, if it has m machines and wants to produce W widgets.

3.4. Quadratic Equations

A quadratic equation is one that, once brackets and fractions have removed, contains terms in x^2, as well as (possibly) x-terms and constants. A quadratic equation can be rearranged to have the form:

$$ax^2 + bx + c = 0$$

where a, b and c are numbers and $a \neq 0$.

A simple quadratic equation is:

$$x^2 = 25$$

You can see immediately that $x = 5$ is a solution, but note that $x = -5$ satisfies the equation too. There are two solutions:

$$x = 5 \quad \text{and} \quad x = -5$$

Quadratic equations have either two solutions, or one solution, or no solutions. The solutions are also known as the *roots* of the equation. There are two general methods for solving quadratics; we will apply them to the example:

$$x^2 + 5x + 6 = 0$$

Method 1: Quadratic Factorisation
We saw in section 1.5 that the quadratic polynomial $x^2 + 5x + 6$ can be factorised, so we can write the equation as:

$$(x + 3)(x + 2) = 0$$

But if the product of two expressions is zero, this means that one of them must be zero, so we can say:

$$\begin{aligned} \text{either } x + 3 = 0 \quad &\Rightarrow \quad x = -3 \\ \text{or } x + 2 = 0 \quad &\Rightarrow \quad x = -2 \end{aligned}$$

The equation has two solutions, -3 and -2. You can check that these are solutions by substituting them back into the original equation.

Method 2: The Quadratic Formula
If the equation $ax^2 + bx + c = 0$ can't be factorised (or if it can, but you can't see how) you can use[1]:

$$\boxed{\text{The Quadratic Formula:} \qquad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}}$$

The notation $\pm$ indicates that an expression may take either a positive or negative value. So this is a formula for the two solutions $x = \frac{-b+\sqrt{b^2-4ac}}{2a}$ and $x = \frac{-b-\sqrt{b^2-4ac}}{2a}$.

In the equation $x^2 + 5x + 6 = 0$, $a = 1$, $b = 5$ and $c = 6$. The quadratic formula gives us:

$$\begin{aligned} x &= \frac{-5 \pm \sqrt{5^2 - 4 \times 1 \times 6}}{2 \times 1} \\ &= \frac{-5 \pm \sqrt{25 - 24}}{2} \\ &= \frac{-5 \pm 1}{2} \end{aligned}$$

[1] *Antony & Biggs* §2.4 explains where the formula comes from.

So the two solutions are:

$$x = \frac{-5+1}{2} = -2 \text{ and } x = \frac{-5-1}{2} = -3$$

Note that in the quadratic formula $x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$, $b^2 - 4ac$ could turn out to be zero, in which case there is only one solution. Or it could be negative, in which case there are no solutions since we can't take the square root of a negative number.

EXAMPLES 3.2: Solve, if possible, the following quadratic equations.

(i) $x^2 + 3x - 10 = 0$
Factorise:

$$(x+5)(x-2) = 0$$

$$\Rightarrow x = -5 \text{ or } x = 2$$

(ii) $x(7-2x) = 6$
First, rearrange the equation to get it into the usual form:

$$\begin{aligned} 7x - 2x^2 &= 6 \\ -2x^2 + 7x - 6 &= 0 \\ 2x^2 - 7x + 6 &= 0 \end{aligned}$$

Now, we can factorise, to obtain:

$$(2x-3)(x-2) = 0$$

$$\begin{aligned} \text{either } 2x - 3 = 0 \quad &\Rightarrow \quad x = \tfrac{3}{2} \\ \text{or } x - 2 = 0 \quad &\Rightarrow \quad x = 2 \end{aligned}$$

The solutions are $x = \frac{3}{2}$ and $x = 2$.

(iii) $y^2 + 4y + 4 = 0$
Factorise:

$$\begin{aligned} (y+2)(y+2) &= 0 \\ \Longrightarrow y + 2 &= 0 \quad \Rightarrow y = -2 \end{aligned}$$

Therefore $y = -2$ is the only solution. (Or we sometimes say that the equation has a *repeated root* – the two solutions are the same.)

(iv) $x^2 + x - 1 = 0$
In section 1.5 we couldn't find the factors for this example. So apply the formula, putting $a = 1$, $b = 1$, $c = -1$:

$$x = \frac{-1 \pm \sqrt{1-(-4)}}{2} = \frac{-1 \pm \sqrt{5}}{2}$$

The two solutions, correct to 3 decimal places, are:

$$x = \tfrac{-1+\sqrt{5}}{2} = 0.618 \quad \text{and} \quad x = \tfrac{-1-\sqrt{5}}{2} = -1.618$$

Note that this means that the factors are, approximately, $(x - 0.618)$ and $(x + 1.618)$.

(v) $2z^2 + 2z + 5 = 0$

Applying the formula gives: $z = \dfrac{-2 \pm \sqrt{-36}}{4}$

So there are no solutions, because this contains the square root of a negative number.

(vi) $6x^2 + 2kx = 0$ (solve for x, treating k as a parameter)

Factorising:

$$2x(3x + k) = 0$$

$$\begin{aligned} \text{either } 2x = 0 \quad &\Rightarrow \quad x = 0 \\ \text{or } 3x + k = 0 \quad &\Rightarrow \quad x = -\frac{k}{3} \end{aligned}$$

Exercises 1.10: Solve the following quadratic equations, where possible:

(1) $x^2 + 3x - 13 = 0$

(2) $4y^2 + 9 = 12y$

(3) $3z^2 - 2z - 8 = 0$

(4) $7x - 2 = 2x^2$

(5) $y^2 + 3y + 8 = 0$

(6) $x(2x - 1) = 2(3x - 2)$

(7) $x^2 - 6kx + 9k^2 = 0$ (where k is a parameter)

(8) $y^2 - 2my + 1 = 0$ (where m is a parameter)
Are there any values of m for which this equation has no solution?

3.5. Equations involving Indices

Examples 3.3:

(i) $7^{2x+1} = 8$

Here the variable we want to find, x, appears in a power.

This type of equation can by solved by *taking logs of both sides*:

$$\begin{aligned} \log_{10}\left(7^{2x+1}\right) &= \log_{10}(8) \\ (2x+1)\log_{10} 7 &= \log_{10} 8 \\ 2x + 1 &= \frac{\log_{10} 8}{\log_{10} 7} = 1.0686 \\ 2x &= 0.0686 \\ x &= 0.0343 \end{aligned}$$

(ii) $(2x)^{0.65} + 1 = 6$

We can use the rules for indices to manipulate this equation:

$$
\begin{aligned}
&\text{Subtract 1 from both sides:} & (2x)^{0.65} &= 5 \\
&\text{Raise both sides to the power } \tfrac{1}{0.65}\text{:} & \left((2x)^{0.65}\right)^{\frac{1}{0.65}} &= 5^{\frac{1}{0.65}} \\
& & 2x &= 5^{\frac{1}{0.65}} = 11.894 \\
&\text{Divide by 2:} & x &= 5.947
\end{aligned}
$$

3.6. Equations involving Logarithms

EXAMPLES 3.4: Solve the following equations:

(i) $\log_5(3x-2) = 2$
From the definition of a logarithm, this equation is equivalent to:

$$3x - 2 = 5^2$$

which can be solved easily:

$$3x - 2 = 25 \Rightarrow x = 9$$

(ii) $10\log_{10}(5x+1) = 17$

$$
\begin{aligned}
\Rightarrow \log_{10}(5x+1) &= 1.7 \\
5x + 1 &= 10^{1.7} = 50.1187 \text{ (correct to 4 decimal places)} \\
x &= 9.8237
\end{aligned}
$$

EXERCISES 1.11: Solve the following equations:

(1) $\log_4(2+x) = 2$

(2) $16 = 5^{3t}$

(3) $2 + x^{0.4} = 8$

The remaining questions are a bit harder – skip them if you found this section difficult.

(4) $4.1 + 5x^{0.42} = 7.8$

(5) $6^{x^2-7} = 36$

(6) $\log_2(y^2+4) = 3$

(7) $3^{n+1} = 2^n$

(8) $2\log_{10}(x-2) = \log_{10}(x)$

Further reading and exercises

- For more practice on solving all the types of equation in this section, you could use an A-level pure maths textbook.
- *Jacques* §1.5 gives more detail on *Changing the Subject of a Formula*
- *Jacques* §2.1 and *Anthony & Biggs* §2.4 both cover the Quadratic Formula for *Solving Quadratic Equations*
- *Jacques* §2.3 has more *Equations involving Indices*

4. Simultaneous Equations

So far we have looked at equations involving one variable (such as x). An equation involving two variables, x and y, such as $x + y = 20$, has lots of solutions – there are lots of pairs of numbers x and y that satisfy it (for example $x = 3$ and $y = 17$, or $x = -0.5$ and $y = 20.5$).

But suppose we have two equations and two variables:

$$\begin{aligned} &(1) \qquad & x + y &= 20 \\ &(2) \qquad & 3x &= 2y - 5 \end{aligned}$$

There is just one pair of numbers x and y that satisfy both equations.

Solving a pair of simultaneous equations means finding the pair(s) of values that satisfy both equations. There are two approaches; in both the aim is to eliminate one of the variables, so that you can solve an equation involving one variable only.

Method 1: Substitution

Make one variable the subject of one of the equations (it doesn't matter which), and substitute it in the other equation.

From equation (1): $\quad x = 20 - y$

Substitute for x in equation (2): $\quad 3(20 - y) = 2y - 5$

Solve for y:

$$\begin{aligned} 60 - 3y &= 2y - 5 \\ -5y &= -65 \\ y &= 13 \end{aligned}$$

From the equation in the first step: $\quad x = 20 - 13 = 7$

The solution is $x = 7, y = 13$.

Method 2: Elimination

Rearrange the equations so that you can add or subtract them to eliminate one of the variables.

Write the equations as:

$$\begin{aligned} x + y &= 20 \\ 3x - 2y &= -5 \end{aligned}$$

Multiply the first one by 2:

$$\begin{aligned} 2x + 2y &= 40 \\ 3x - 2y &= -5 \end{aligned}$$

Add the equations together: $\quad 5x = 35 \quad \Rightarrow x = 7$

Substitute back in equation (1): $\quad 7 + y = 20 \quad \Rightarrow y = 13$

EXAMPLES 4.1: *Simultaneous Equations*

(i) Solve the equations $3x + 5y = 12$ and $2x - 6y = -20$

Multiply the first equation by 2 and the second one by 3:

$$\begin{array}{lrcll} & 6x+10y & = & 24 & \\ & 6x-18y & = & \text{-60} & \\ \text{Subtract:} & 28y & = & 84 & \Rightarrow y=3 \\ \text{Substitute back in the } 2^{nd} \text{ equation:} & 2x-18 & = & -20 & \Rightarrow x=-1 \end{array}$$

(ii) Solve the equations $x+y=3$ and $x^2+2y^2=18$

Here the first equation is linear but the second is quadratic.
Use the linear equation for a substitution:

$$\begin{aligned} x &= 3-y \\ \Rightarrow (3-y)^2+2y^2 &= 18 \\ 9-6y+y^2+2y^2 &= 18 \\ 3y^2-6y-9 &= 0 \\ y^2-2y-3 &= 0 \end{aligned}$$

Solving this quadratic equation gives two solutions for y:

$$y=3 \text{ or } y=-1$$

Now find the corresponding values of x using the linear equation: when $y=3, x=0$ and when $y=-1, x=4$. So there are two solutions:

$$x=0, y=3 \quad \text{and} \quad x=4, y=-1$$

(iii) Solve the equations $x+y+z=6$, $y=2x$, and $2y+z=7$

Here we have three equations, and three variables. We use the same methods, to eliminate first one variable, then another.
Use the second equation to eliminate y from both of the others:

$$\begin{aligned} x+2x+z=6 \Rightarrow 3x+z &= 6 \\ 4x+z &= 7 \end{aligned}$$

Eliminate z by subtracting: $x=1$
Work out z from $4x+z=7$: $z=3$
Work out y from $y=2x$: $y=2$
The solution is $x=1, y=2, z=3$.

Exercises 1.12: Solve the following sets of simultaneous equations:

(1) $2x=1-y$ and $3x+4y+6=0$

(2) $2z+3t=-0.5$ and $2t-3z=10.5$

(3) $x+y=a$ and $x=2y$ for x and y, in terms of the parameter a.

(4) $a=2b$, $a+b+c=12$ and $2b-c=13$

(5) $x-y=2$ and $x^2=4-3y^2$

Further reading and exercises

- *Jacques* §1.2 covers *Simultaneous Linear Equations* thoroughly.

5. Inequalities and Absolute Value

5.1. Inequalities

$$2x + 1 \leq 6$$

is an example of an inequality. Solving the inequality means "finding the set of values of x that make the inequality true." This can be done very similarly to solving an equation:

$$\begin{aligned} 2x + 1 &\leq 6 \\ 2x &\leq 5 \\ x &\leq 2.5 \end{aligned}$$

Thus, all values of x less than or equal to 2.5 satisfy the inequality.

> When manipulating inequalities you can add anything to both sides, or subtract anything, and you can multiply or divide both sides by a positive number. But if you multiply or divide both sides by a negative number you must reverse the inequality sign.

To see why you have to reverse the inequality sign, think about the inequality:

	5	$<$	8	(which is true)
If you multiply both sides by 2, you get:	10	$<$	16	(also true)
But if you just multiplied both sides by -2, you would get:	-10	$<$	-16	(NOT true)
Instead we reverse the sign when multiplying by -2, to obtain:	-10	$>$	-16	(true)

Examples 5.1: Solve the following inequalities:

(i) $3(x + 2) > x - 4$

$$\begin{aligned} 3x + 6 &> x - 4 \\ 2x &> -10 \\ x &> -5 \end{aligned}$$

(ii) $1 - 5y \leq -9$

$$\begin{aligned} -5y &\leq -10 \\ y &\geq 2 \end{aligned}$$

5.2. Absolute Value

The absolute value, or *modulus*, of x is the positive number which has the same "magnitude" as x. It is denoted by $|x|$. For example, if $x = -6$, $|x| = 6$ and if $y = 7$, $|y| = 7$.

> $|x| = x$ if $x \geq 0$
> $|x| = -x$ if $x < 0$

Examples 5.2: *Solving equations and inequalities involving absolute values*

(i) Find the values of x satisfying $|x + 3| = 5$.

$$|x + 3| = 5 \Rightarrow x + 3 = \pm 5$$

$$\begin{array}{rrcll} \text{Either:} & x+3 & = & 5 & \Rightarrow x=2 \\ \text{or:} & x+3 & = & -5 & \Rightarrow x=-8 \end{array}$$

So there are two solutions: $x = 2$ and $x = -8$

(ii) Find the values of y for which $|y| \leq 6$.

$$\begin{array}{rrcccc} \text{Either:} & y & \leq & 6 & & \\ \text{or:} & -y & \leq & 6 & \Rightarrow & y \geq -6 \end{array}$$

So the solution is: $-6 \leq y \leq 6$

(iii) Find the values of z for which $|z - 2| > 4$.

$$\begin{array}{rrclll} \text{Either:} & z-2 & > & 4 & \Rightarrow z > 6 & \\ \text{or:} & -(z-2) & > & 4 & \Rightarrow z-2 < -4 & \Rightarrow z < -2 \end{array}$$

So the solution is: $z < -2$ or $z > 6$

5.3. Quadratic Inequalities

EXAMPLES 5.3: Solve the inequalities:

(i) $x^2 - 2x - 15 \leq 0$
Factorise:

$$(x-5)(x+3) \leq 0$$

If the product of two factors is negative, one must be negative and the other positive:

$$\begin{array}{rl} \text{either:} & x-5 \leq 0 \text{ and } x+3 \geq 0 \quad \Rightarrow -3 \leq x \leq 5 \\ \text{or:} & x-5 \geq 0 \text{ and } x+3 \leq 0 \text{ which is impossible.} \end{array}$$

So the solution is: $-3 \leq x \leq 5$

(ii) $x^2 - 7x + 6 > 0$

$$\Rightarrow (x-6)(x-1) > 0$$

If the product of two factors is positive, both must be positive, or both negative:

$$\begin{array}{rll} \text{either:} & x-6 > 0 \text{ and } x-1 > 0 & \text{which is true if: } x > 6 \\ \text{or:} & x-6 < 0 \text{ and } x-1 < 0 & \text{which is true if: } x < 1 \end{array}$$

So the solution is: $x < 1$ or $x > 6$

EXERCISES 1.13: Solve the following equations and inequalities:

(1) (a) $2x + 1 \geq 7$ (b) $5(3 - y) < 2y + 3$

(2) (a) $|9 - 2x| = 11$ (b) $|1 - 2z| > 2$

(3) $|x + a| < 2$ where a is a parameter, and we know that $0 < a < 2$.

(4) (a) $x^2 - 8x + 12 < 0$ (b) $5x - 2x^2 \leq -3$

Further reading and exercises

- *Jacques* §1.4.1 has a little more on *Inequalities*.
- Refer to an A-level pure maths textbook for more detail and practice.

Solutions to Exercises in Chapter 1

EXERCISES 1.1:

(1) (a) 29
(b) -8
(c) -15
(d) $\frac{1}{2}$
(e) 64
(f) 3

EXERCISES 1.2:

(1) (a) $x^3 + 13x - 25$
(b) $2x^2 - 8y$ or $2(x^2 - 4y)$
(2) (a) $3z^2x + 2z - 1$
(b) $7x + 14$ or $7(x + 2)$
(3) (a) $\frac{xy}{2}$
(b) $\frac{6y}{x}$
(4) (a) $\frac{x}{4y}$
(b) $20x^3y^4$
(5) (a) y
(b) $\frac{4x^2}{y^3}$
(6) (a) $\frac{10x+3}{12}$
(b) $\frac{2}{x^2-1}$

EXERCISES 1.3:

(1) (a) $3x(1 + 2y)$
(b) $y(2y + 7)$
(c) $3(2a + b + 3c)$
(2) (a) $x^2(3x - 10)$
(b) $c(a - b)$
(3) $(x + 2)(y + 2z)$
(4) $x(2 - x)$

EXERCISES 1.4:

(1) $(x + 1)(x + 3)$
(2) $(y - 5)(y - 2)$
(3) $(2x + 1)(x + 3)$
(4) $(z + 5)(z - 3)$
(5) $(2x + 3)(2x - 3)$
(6) $(y - 5)^2$
(7) Not possible to split into integer factors.

EXERCISES 1.5:

(1) (a) $= \sqrt{2 \times 18} = \sqrt{36} = 6$
(b) $= \sqrt{49 \times 5} = 7\sqrt{5}$
(c) $\frac{15}{\sqrt{3}} = \frac{15\sqrt{3}}{3} = 5\sqrt{3}$
(2) (a) $\sqrt{5}$
(b) $4x^2$
(c) $\frac{x}{2}$
(d) $\sqrt{2}y$

EXERCISES 1.6:

(1) (a) 16
(b) 6
(c) 3
(d) $\frac{1}{4}$
(2) (a) $2x^{11}$
(b) $\frac{1}{x}$
(c) $\log_{10} y$
(d) 3
(3) (a) $(9ab)^3$
(b) $-\frac{5}{3}\log_{10} b$

EXERCISES 1.7:

(1) $x = 3$
(2) $y = -3$
(3) $x = 2$
(4) $z = -1$
(5) $a = -\frac{1}{3}$

EXERCISES 1.8:

(1) $x = \frac{6}{a}$
(2) $y = -4b$
(3) $z = \frac{a+b}{2}$

EXERCISES 1.9:

(1) $t = \frac{v-u}{a}$
(2) $a = \sqrt{c^2 - b^2}$
(3) $p = 1200r - 6n$
(4) $n = \frac{W}{m^2} + 3$

EXERCISES 1.10:

(1) $x = \frac{-3\pm\sqrt{61}}{2}$
(2) $y = 1.5$
(3) $z = -\frac{4}{3}, 2$
(4) $x = \frac{7\pm\sqrt{33}}{4}$
(5) No solutions.
(6) $x = \frac{7\pm\sqrt{17}}{4}$
(7) $x = 3k$
(8) $y = (m \pm \sqrt{m^2 - 1})$ No solution if $-1 < m < 1$

EXERCISES 1.11:

(1) $x = 14$
(2) $t = \frac{\log(16)}{3\log(5)} = 0.5742$
(3) $x = 6^{\frac{1}{0.4}} = 88.18$
(4) $x = (0.74)^{\frac{1}{0.42}} = 0.4883$
(5) $x = \pm 3$
(6) $y = \pm 2$
(7) $n = \frac{\log_2 3}{\log_2 \frac{2}{3}} = -2.7095$
(8) $x = 4, x = 1$

EXERCISES 1.12:

(1) $x = 2$, $y = -3$
(2) $t = 1.5$, $z = -2.5$
(3) $x = \frac{2a}{3}$, $y = \frac{a}{3}$
(4) $a = 10$, $b = 5$, $c = -3$
(5) $(x, y) = (2, 0)$ $(x, y) = (1, -1)$

EXERCISES 1.13:

(1) (a) $x \geq 3$
(b) $\frac{12}{7} < y$
(2) (a) $x = -1, 10$
(b) $z < -0.5$ or $z > 1.5$
(3) $-2 - a < x < 2 - a$
(4) (a) $2 < x < 6$
(b) $x \geq 3$, $x \leq -0.5$

[2]This Version of Workbook Chapter 1: September 25, 2014

Worksheet 1: Review of Algebra

(1) For a firm, the cost of producing q units of output is $C = 4 + 2q + 0.5q^2$. What is the cost of producing (a) 4 units (b) 1 unit (c) no units?

(2) Evaluate the expression $x^3(y + 7)$ when $x = -2$ and $y = -10$.

(3) Simplify the following algebraic expressions, factorising the answer where possible:
(a) $x(2y+3x-12)-3(2-5xy)-(3x+8xy-6)$ (b) $z(2-3z+5z^2)+3(z^2-z^3-4)$

(4) Simplify: (a) $6a^4b \times 4b \div 8ab^3c$ (b) $\sqrt{3x^3y} \div \sqrt{27xy}$ (c) $(2x^3)^3 \times (xz^2)^4$

(5) Write as a single fraction: (a) $\dfrac{2y}{3x} + \dfrac{4y}{5x}$ (b) $\dfrac{x+1}{4} - \dfrac{2x-1}{3}$

(6) Factorise the following quadratic expressions:
(a) $x^2 - 7x + 12$ (b) $16y^2 - 25$ (c) $3z^2 - 10z - 8$

(7) Evaluate (without using a calculator): (a) $4^{\frac{3}{2}}$ (b) $\log_{10} 100$ (c) $\log_5 125$

(8) Write as a single logarithm: (a) $2\log_a(3x) + \log_a x^2$ (b) $\log_a y - 3\log_a z$

(9) Solve the following equations:
(a) $5(2x-9) = 2(5-3x)$ (b) $1 + \dfrac{6}{y-8} = -1$ (c) $z^{0.4} = 7$ (d) $3^{2t-1} = 4$

(10) Solve these equations for x, in terms of the parameter a:
(a) $ax - 7a = 1$ (b) $5x - a = \dfrac{x}{a}$ (c) $\log_a(2x+5) = 2$

(11) Make Q the subject of: $P = \sqrt{\dfrac{a}{Q^2+b}}$

(12) Solve the equations: (a) $7 - 2x^2 = 5x$ (b) $y^2 + 3y - 0.5 = 0$ (c) $|1 - z| = 5$

(13) Solve the simultaneous equations:
(a) $2x - y = 4$ and $5x = 4y + 13$
(b) $y = x^2 + 1$ and $2y = 3x + 4$

(14) Solve the inequalities: (a) $2y - 7 \leq 3$ (b) $3 - z > 4 + 2z$ (c) $3x^2 < 5x + 2$

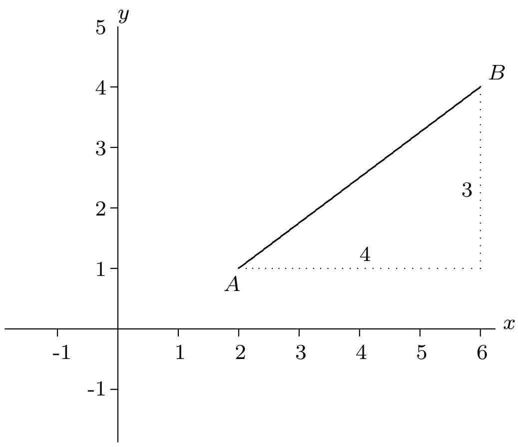
CHAPTER 2

Lines and Graphs

Almost everything in this chapter is revision from GCSE maths. It reminds you how to draw graphs, and focuses in particular on **straight line graphs** and their **gradients**. We also look at graphs of quadratic functions, and use graphs to solve equations and inequalities. An important economic application of straight line graphs is **budget constraints**.

1. The Gradient of a Line

A is a point with co-ordinates $(2, 1)$; B has co-ordinates $(6, 4)$.



When you move from A to B, the change in the x-coordinate is

$$\Delta x = 6 - 2 = 4$$

and the change in the y-coordinate is

$$\Delta y = 4 - 1 = 3$$

The gradient (or slope) of AB is Δy divided by Δx:

$$\text{Gradient} = \frac{\Delta y}{\Delta x} = \frac{3}{4} = 0.75$$

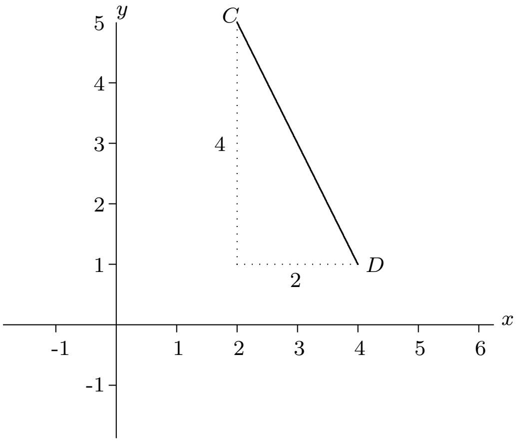
(The symbol Δ, pronounced "delta", denotes "change in".)

It doesn't matter which end of the line you start. If you move from B to A, the changes are negative, but the gradient is the same: $\Delta x = 2 - 6 = -4$ and $\Delta y = 1 - 4 = -3$, so the gradient is $(-3)/(-4) = 0.75$.

There is a general formula:

The gradient of the line joining (x_1, y_1) and (x_2, y_2) is:

$$\frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$



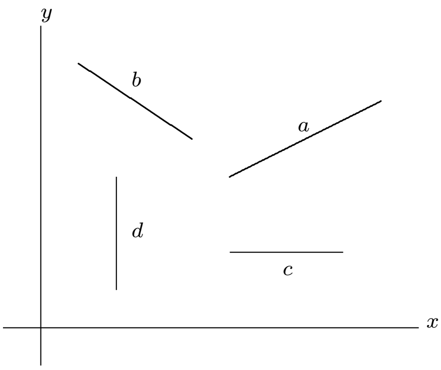
Here the gradient is negative. When you move from C to D:

$$\Delta x = 4 - 2 = 2$$

$$\Delta y = 1 - 5 = -4$$

The gradient of CD is:

$$\frac{\Delta y}{\Delta x} = \frac{-4}{2} = -2$$



In this diagram the gradient of line a is positive, and the gradient of b is negative: as you move in the x-direction, a goes uphill, but b goes downhill.

The gradient of c is zero. As you move along the line the change in the y-coordinate is zero: $\Delta y = 0$

The gradient of d is infinite. As you move along the line the change in the x-coordinate is zero (so if you tried to calculate the gradient you would be dividing by zero).

EXERCISES 2.1: **Gradients**

(1) Plot the points $A(1, 2), B(7, 10), C(-4, 14), D(9, 2)$ and $E(-4, -1)$ on a diagram.

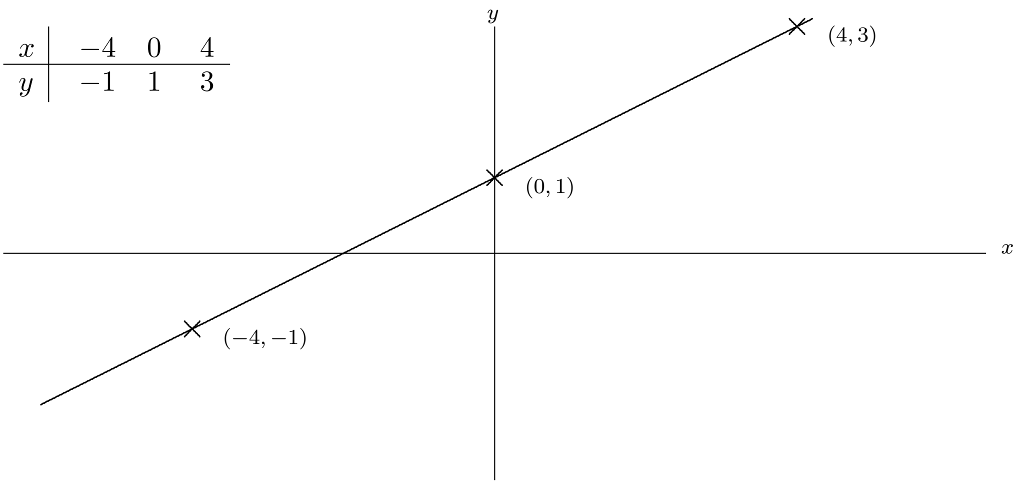
(2) Find the gradients of the lines AB, AC, CE, AD.

2. Drawing Graphs

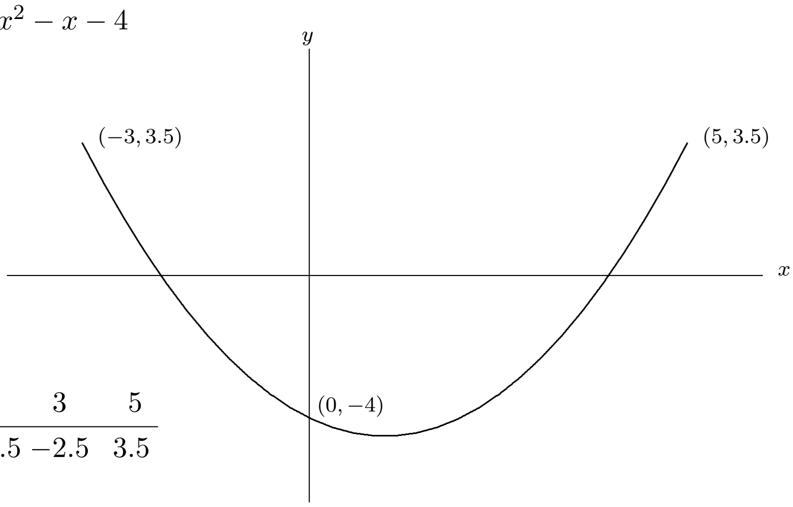
The equation $y = 0.5x + 1$ expresses a relationship between 2 quantities x and y (or a formula for y in terms of x) that can be represented as a graph in x-y space. To draw the graph, calculate y for a range of values of x, then plot the points and join them with a curve or line.

EXAMPLES 2.1: $y = 0.5x + 1$

x	-4	0	4
y	-1	1	3



EXAMPLES 2.2: $y = 0.5x^2 - x - 4$



x	-3	-1	0	1	3	5
y	3.5	-2.5	-4	-4.5	-2.5	3.5

EXERCISES 2.2: Draw the graphs of the following relationships:

(1) $y = 3x - 2$ for values of x between -4 and $+4$.

(2) $P = 10 - 2Q$, for values of Q between 0 and 5. (This represents a demand function: the relationship between the market price P and the total quantity sold Q.)

(3) $y = 4/x$, for values of x between -4 and $+4$.

(4) $C = 3 + 2q^2$, for values of q between 0 and 4. (This represents a firm's cost function: its total costs are C if it produces a quantity q of goods.)

3. Straight Line (Linear) Graphs

EXERCISES 2.3: **Straight Line Graphs**

(1) Using a diagram with x and y axes from -4 to $+4$, draw the graphs of:

(a) $y = 2x$
(b) $2x + 3y = 6$
(c) $y = 1 - 0.5x$
(d) $y = -3$
(e) $x = 4$

(2) For each graph find (i) the gradient, and (ii) the *vertical intercept* (that is, the value of y where the line crosses the y-axis, also known as the y-intercept).

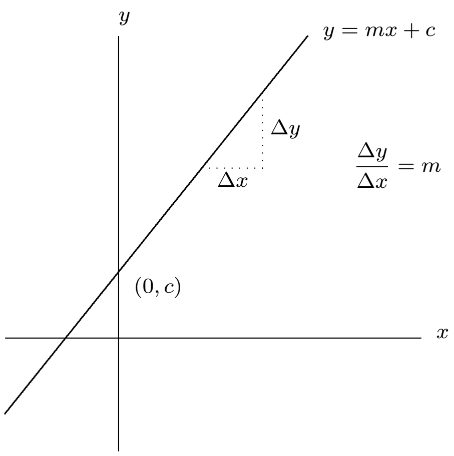
Each of the first four equations in this exercise can be rearranged to have the form $y = mx+c$:

(a) $y = 2x$	$\Rightarrow y = 2x + 0$	$\Rightarrow m = 2$	$c = 0$
(b) $2x + 3y = 6$	$\Rightarrow y = -\frac{2}{3}x + 2$	$\Rightarrow m = -\frac{2}{3}$	$c = 2$
(c) $y = 1 - 0.5x$	$\Rightarrow y = -0.5x + 1$	$\Rightarrow m = -0.5$	$c = 1$
(d) $y = -3$	$\Rightarrow y = 0x - 3$	$\Rightarrow m = 0$	$c = -3$

(e) is a special case. It cannot be written in the form $y = mx + c$, its gradient is infinite, and it has no vertical intercept.

Check these values of m and c against your answers. You should find that m is the gradient and c is the y-intercept.

Note that in an equation of the form $y = mx + c$, y is equal to a polynomial of degree 1 in x (see Chapter 1).



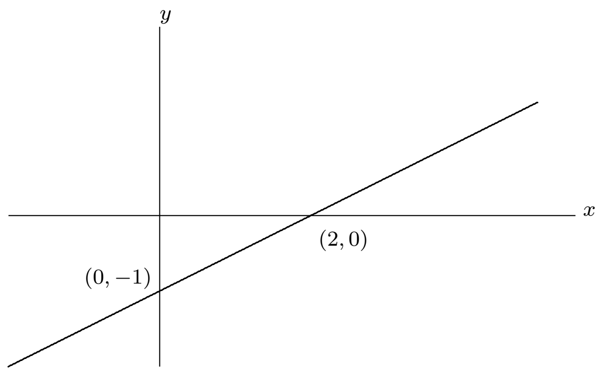
If an equation can be written in the form $y = mx + c$, then the graph is a straight line, with gradient m and vertical intercept c. We say "y is a linear function of x."

Examples 3.1: Sketch the line $x - 2y = 2$

"Sketching" a graph means drawing a picture to indicate its general shape and position, rather than plotting it accurately. First rearrange the equation:

$$y = 0.5x - 1$$

So the gradient is 0.5 and the y-intercept is -1. We can use this to sketch the graph.

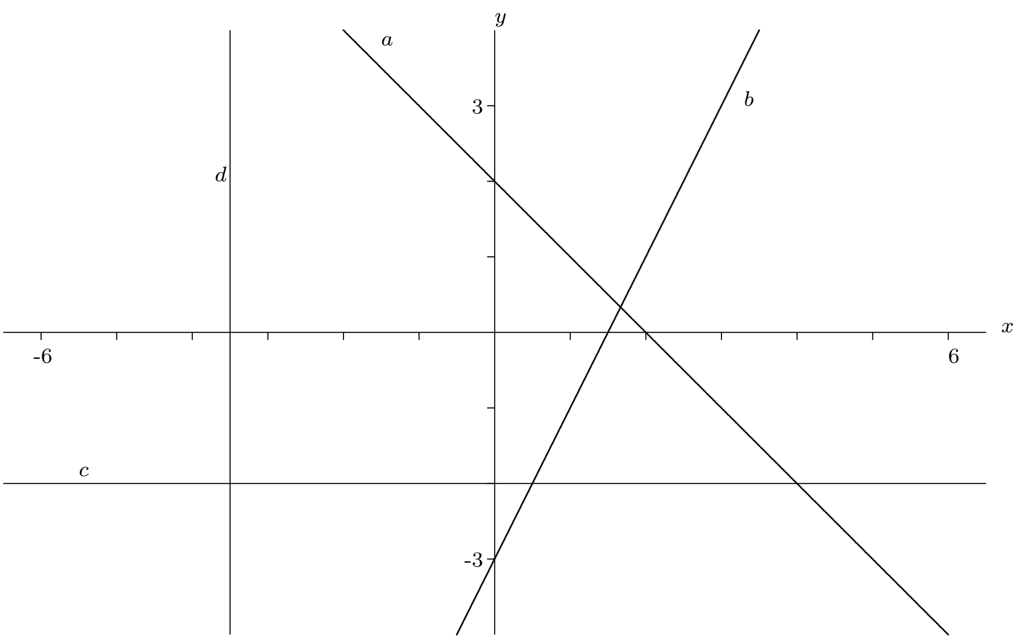


Exercises 2.4: $y = mx + c$

(1) For each of the lines in the diagram below, work out the gradient and hence write down the equation of the line.

(2) By writing each of the following lines in the form $y = mx + c$, find its gradient:
(a) $y = 4 - 3x$ (b) $3x + 5y = 8$ (c) $x + 5 = 2y$ (d) $y = 7$ (e) $2x = 7y$

(3) By finding the gradient and y-intercept, sketch each of the following straight lines:
$y = 3x + 5$
$y + x = 6$
$3y + 9x = 8$
$x = 4y + 3$

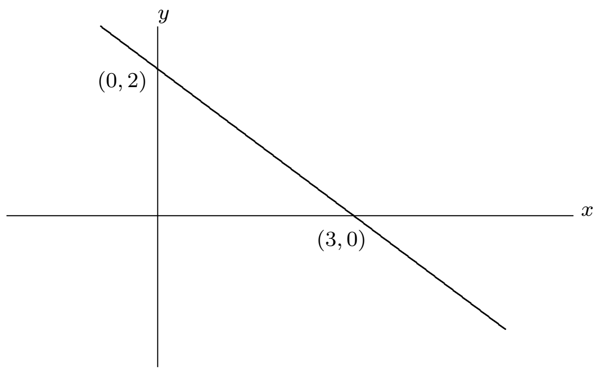


3.1. Lines of the Form $ax + by = c$

Lines such as $2x+3y = 6$ can be rearranged to have the form $y = mx+c$, and hence sketched, as in the previous exercise. But it is easier in this case to work out what the line is like by finding the points where it crosses both axes.

EXAMPLES 3.2:
Sketch the line $2x + 3y = 6$.

When $x = 0, y = 2$
When $y = 0, x = 3$



EXERCISES 2.5: Sketch the following lines: (1) $4x + 5y = 100$ (2) $2y + 6x = 7$

3.2. Working out the Equation of a Line

EXAMPLES 3.3: What is the equation of the line

(i) with gradient 3, passing through $(2, 1)$?
gradient $= 3$ $\Rightarrow y = 3x + c$
$y = 1$ when $x = 2$ $\Rightarrow 1 = 6 + c$ $\Rightarrow c = -5$
The line is $y = 3x - 5$.

(ii) passing through $(-1, -1)$ and $(5, 14)$?
First work out the gradient: $\frac{14-(-1)}{5-(-1)} = \frac{15}{6} = 2.5$
gradient $= 2.5$ $\Rightarrow y = 2.5x + c$
$y = 14$ when $x = 5$ $\Rightarrow 14 = 12.5 + c \Rightarrow c = 1.5$
The line is $y = 2.5x + 1.5$ (or equivalently $2y = 5x + 3$).

There is a formula that you can use (although the method above is just as good):

The equation of a line with gradient m, passing through the point (x_1, y_1) is: $y = m(x - x_1) + y_1$

EXERCISES 2.6: Find the equations of the following lines:
(1) passing through $(4, 2)$ with gradient 7
(2) passing through $(0, 0)$ with gradient 1
(3) passing through $(-1, 0)$ with gradient -3
(4) passing through $(-3, 4)$ and parallel to the line $y + 2x = 5$
(5) passing through $(0, 0)$ and $(5, 10)$
(6) passing through $(2, 0)$ and $(8, -1)$

4. Quadratic Graphs

If we can write a relationship between x and y so that y is equal to a quadratic polynomial in x (see Chapter 1):

$$y = ax^2 + bx + c$$

where a, b and c are numbers, then we say "y is a quadratic function of x", and the graph is a parabola (a U-shape) like the one in Example 2.2.

EXERCISES 2.7: **Quadratic Graphs**

(1) Draw the graphs of (i) $y = 2x^2 - 5$ (ii) $y = -x^2 + 2x$, for values of x between -3 and $+3$.

(2) For each graph note that: if a (the coefficient of x^2) is positive, the graph is a U-shape; if a is negative then it is an inverted U-shape; the vertical intercept is given by c; and the graph is symmetric.

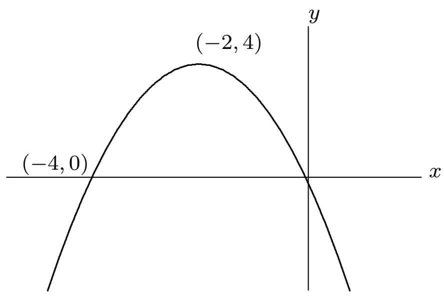
So, to sketch the graph of a quadratic you can:
- decide whether it is a U-shape or an inverted U;
- find the y-intercept;
- find the points where it crosses the x-axis (if any), by solving $ax^2 + bx + c = 0$;
- find its maximum or minimum point using symmetry: find two points with the same y-value, then the max or min is at the x-value halfway between them.

EXAMPLES 4.1: Sketch the graph of $y = -x^2 - 4x$

- $a = -1$, so it is an inverted U-shape.
- The y-intercept is 0.
- Solving $-x^2 - 4x = 0$ to find where it crosses the x-axis:

$$\begin{aligned} x^2 + 4x = 0 \quad &\Rightarrow \quad x(x+4) = 0 \\ &\Rightarrow \quad x = 0 \text{ or } x = -4 \end{aligned}$$

- Its maximum point is halfway between these two points, at $x = -2$, and at this point $y = -4 - 4(-2) = 4$.



EXERCISES 2.8: Sketch the graphs of the following quadratic functions:

(1) $y = 2x^2 - 18$

(2) $y = 4x - x^2 + 5$

5. Solving Equations and Inequalities using Graphs

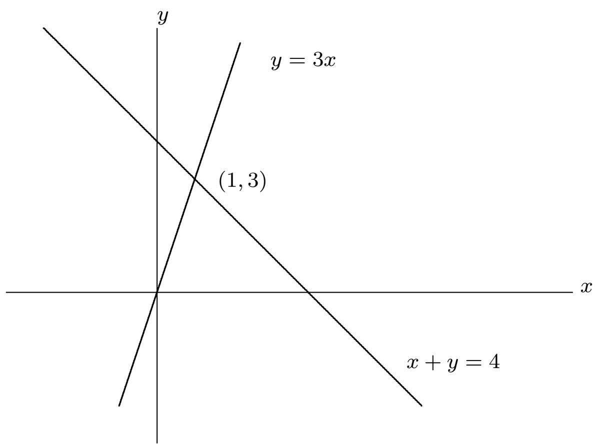
5.1. Solving Simultaneous Equations

The equations:

$$\begin{aligned} x + y &= 4 \\ y &= 3x \end{aligned}$$

could be solved algebraically (see Chapter 1). Alternatively we could draw their graphs, and find the point where they intersect.

The solution is $x = 1, y = 3$.



5.2. Solving Quadratic Equations

We could solve the solve the quadratic equation $x^2 - 5x + 2 = 0$ using the quadratic formula (see Chapter 1). Alternatively we could find an approximate solution by drawing the graph of $y = x^2 - 5x + 2$ (as accurately as possible), and finding where it crosses the x-axis (that is, finding the points where $y = 0$).

Exercises 2.9: **Solving Equations using Graphs**

(1) Solve the simultaneous equations $y = 4x-7$ and $y = x-1$ by drawing (accurately) the graphs of the two lines.

(2) By sketching their graphs, explain why you cannot solve either of the following pairs of simultaneous equations:

$$\begin{aligned} y = 3x - 5 &\quad \text{and} \quad 2y - 6x = 7; \\ x - 5y = 4 &\quad \text{and} \quad y = 0.2x - 0.8 \end{aligned}$$

(3) By sketching their graphs, show that the simultaneous equations $y = x^2$ and $y = 3x + 4$ have two solutions. Find the solutions algebraically.

(4) Show algebraically that the simultaneous equations $y = x^2 + 1$ and $y = 2x$ have only one solution. Draw the graph of $y = x^2 + 1$ and use it to show that the simultaneous equations:

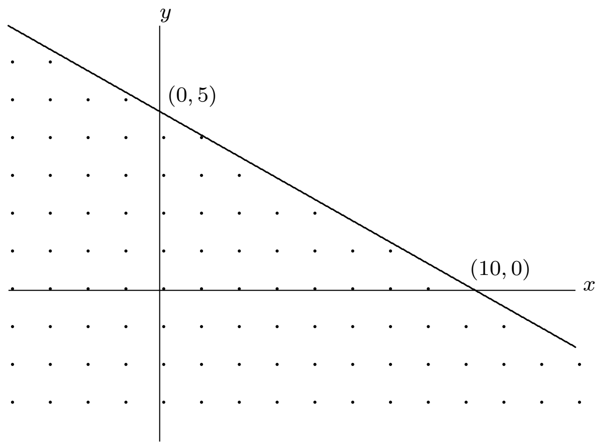
$$y = x^2 + 1 \text{ and } y = mx$$

have: no solutions if $0 < m < 2$; one solution if $m = 2$; and two solutions if $m > 2$.

(5) Sketch the graph of the quadratic function $y = x^2 - 8x$. From your sketch, find the approximate solutions to the equation $x^2 - 8x = -4$.

5.3. Representing Inequalities using Graphs

If we draw the graph of the line $2y+x=10$, all the points satisfying $2y+x<10$ lie on one side of the line. The dotted region shows the inequality $2y+x<10$.



> EXERCISES 2.10: **Representing Inequalities**
> (1) Draw a sketch showing all the points where $x+y<1$ and $y<x+1$ and $y>-3$.
> (2) Sketch the graph of $y=3-2x^2$, and show the region where $y<3-2x^2$.

5.4. Using Graphs to Help Solve Quadratic Inequalities

In Chapter 1, we solved quadratic inequalities such as: $x^2-2x-15\leq 0$. To help do this quickly, you can sketch the graph of the quadratic polynomial $y=x^2-2x-15$.

EXAMPLES 5.1: *Quadratic Inequalities*

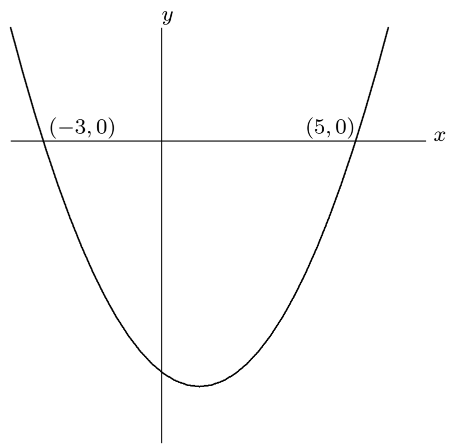
(i) Solve the inequality $x^2-2x-15\leq 0$.
As before, the first step is to factorise:

$$(x-5)(x+3)\leq 0$$

Now sketch the graph of $y=x^2-2x-15$.
We can see from the graph that $x^2-2x-15\leq 0$ when:

$$-3\leq x\leq 5$$

This is the solution of the inequality.



(ii) Solve the inequality $x^2-2x-15>0$.
From the same graph the solution is: $x>5$ or $x<-3$.

> EXERCISES 2.11: Solve the inequalities:
> (1) $x^2-5x>0$
> (2) $3x+5-2x^2\geq 0$
> (3) $x^2-3x+1\leq 0$ (Hint: you will need to use the quadratic formula for the last one.)

6. Economic Application: Budget Constraints

6.1. An Example

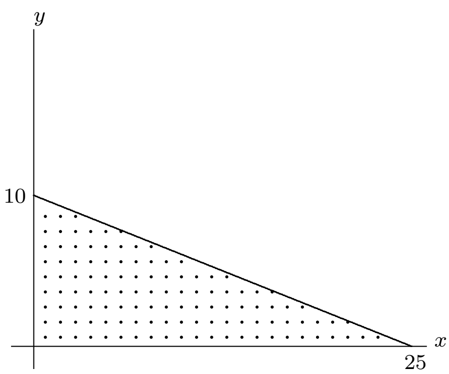
Suppose pencils cost 20p, and pens cost 50p. If a student buys x pencils and y pens, the total amount spent is:

$$20x + 50y$$

If the maximum amount he has to spend on writing implements is £5 his budget constraint is:

$$20x + 50y \leq 500$$

His budget set (the choices of pens and pencils that he can afford) can be shown as the shaded area on a diagram:



From the equation of the budget line:

$$20x + 50y = 500$$

we can see that the gradient of the budget line is:

$$-\frac{20}{50}$$

(You could rewrite the equation in the form $y = mx + c$.)

6.2. The General Case

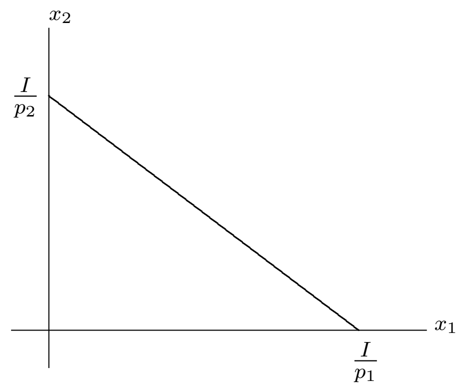
Suppose that a consumer has a choice of two goods, good 1 and good 2. The price of good 1 is p_1 and the price of good 2 is p_2. If he buys x_1 units of good 1, and x_2 units of good 2, the total amount spent is:

$$p_1x_1 + p_2x_2$$

If the maximum amount he has to spend is his income I his budget constraint is:

$$p_1x_1 + p_2x_2 \leq I$$

To draw the budget line:



$$p_1x_1 + p_2x_2 = I$$

note that it crosses the x_1-axis where:

$$x_2 = 0 \Rightarrow p_1x_1 = I \Rightarrow x_1 = \frac{I}{p_1}$$

and similarly for the x_2 axis.

The gradient of the budget line is:

$$-\frac{p_1}{p_2}$$

EXERCISES 2.12: **Budget Constraints**

(1) Suppose that the price of coffee is 30p and the price of tea is 25p. If a consumer has a daily budget of £1.50 for drinks, draw his budget set and find the slope of the budget line. What happens to his budget set and the slope of the budget line if his drinks budget increases to £2?

(2) A consumer has budget constraint $p_1x_1 + p_2x_2 \leq I$. Show diagrammatically what happens to the budget set if

   (a) income I increases

   (b) the price of good 1, p_1, increases

   (c) the price of good 2, p_2, decreases.

Further reading and exercises

- *Jacques* §1.1 has more on co-ordinates and straight-line graphs.
- *Jacques* §2.1 includes graphs of quadratic functions.
- *Varian* discusses budget constraints in detail.

Solutions to Exercises in Chapter 2

EXERCISES 2.1:

(1)

(2) gradient (AB) $= \frac{4}{3}$,
(AC) $= -\frac{12}{5}$,
(CE) $= \infty$,
(AD) $= 0$

EXERCISES 2.2:

EXERCISES 2.3:

(1)

(2) (a) (i) 2 (ii) $y = 0$
(b) (i) $-\frac{2}{3}$ (ii) $y = 2$
(c) (i) $-\frac{1}{2}$ (ii) $y = 1$
(d) (i) 0 (ii) $y = -3$
(e) (i) ∞ (ii) None.

EXERCISES 2.4:

(1) (a) $y = 2 - x$
(b) $y = 2x - 3$
(c) $y = -2$
(d) $x = -3\frac{1}{2}$

(2) (a) -3
(b) $-\frac{3}{5}$
(c) $\frac{1}{2}$
(d) 0
(e) $\frac{2}{7}$

EXERCISES 2.5:

EXERCISES 2.6:

(1) $y = 7x - 26$
(2) $y = x$
(3) $y + 3x = -3$
(4) $y + 2x = -2$
(5) $y = 2x$
(6) $6y + x = 2$

EXERCISES 2.7:

EXERCISES 2.8:

EXERCISES 2.9:

(1) $(x, y) = (2, 1)$
(2) The lines are parallel.
(3) $(x, y) = (-1, 1)$ and $(4, 16)$
(4) $(x, y) = (1, 2)$
(5) $(x, y) \approx (0.536, -4)$ and $(7.464, -4)$ Actual values for $x = 4 \pm \sqrt{12}$

EXERCISES 2.10:

EXERCISES 2.11:

(1) $x > 5$ or $x < 0$
(2) $-1 \leq x \leq \frac{5}{2}$
(3) $\frac{3-\sqrt{5}}{2} \leq x \leq \frac{3+\sqrt{5}}{2}$

EXERCISES 2.12:

(1) 6 Coffee + 5 Tea $\leq$ 30. Gradient (with Coffee on horiz. axis) $= -\frac{6}{5}$ Budget line shifts outwards; budget set larger; slope the same.

(2)
(a) Budget line shifts outwards; budget set larger; slope the same.
(b) Budget line pivots around intercept with x_2-axis; budget set smaller; (absolute value of) gradient increases.
(c) Budget line pivots around intercept with x_1-axis; budget set larger; (absolute value of) gradient increases.

[1]This Version of Workbook Chapter 2: August 22, 2003

Worksheet 2: Lines and Graphs

(1) Find the gradients of the lines AB, BC, and CA where A is the point $(5, 7)$, B is $(-4, 1)$ and C is $(5, -17)$.

(2) Draw (accurately), for values of x between -5 and 5, the graphs of:
(a) $y - 2.5x = -5$ (b) $z = \frac{1}{4}x^2 + \frac{1}{2}x - 1$
Use (b) to solve the equation $\frac{1}{4}x^2 + \frac{1}{2}x = 1$

(3) Find the gradient and y-intercept of the following lines:
(a) $3y = 7x - 2$ (b) $2x + 3y = 12$ (c) $y = -x$

(4) Sketch the graphs of $2P = Q+5$, $3Q+4P = 12$, and $P = 4$, with Q on the horizontal axis.

(5) What is the equation of the line through $(1, 1)$ and $(4, -5)$?

(6) Sketch the graph of $y = 3x - x^2 + 4$, and hence solve the inequality $3x - x^2 < -4$.

(7) Draw a diagram to represent the inequality $3x - 2y < 6$.

(8) Electricity costs 8p per unit during the daytime and 2p per unit if used at night. The quarterly charge is £10. A consumer has £50 to spend on electricity for the quarter.
   (a) What is his budget constraint?
   (b) Draw his budget set (with daytime units as "good 1" on the horizontal axis).
   (c) Is the bundle (440, 250) in his budget set?
   (d) What is the gradient of the budget line?

(9) A consumer has a choice of two goods, good 1 and good 2. The price of good 2 is 1, and the price of good 1 is p. The consumer has income M.
   (a) What is the budget constraint?
   (b) Sketch the budget set, with good 1 on the horizontal axis, assuming that $p > 1$.
   (c) What is the gradient of the budget line?
   (d) If the consumer decides to spend all his income, and buy equal amounts of the two goods, how much of each will he buy?
   (e) Show on your diagram what happens to the budget set if the price of good 1 falls by 50%.

CHAPTER 3

Sequences, Series, and Limits; the Economics of Finance

If you have done A-level maths you will have studied Sequences and Series (in particular **Arithmetic** and **Geometric** ones) before; if not you will need to work carefully through the first two sections of this chapter. Sequences and series arise in many economic applications, such as the **economics of finance and investment**. Also, they help you to understand the concept of a **limit** and the significance of **the natural number**, e. You will need both of these later.

1. Sequences and Series

1.1. Sequences

A sequence is a set of terms (or numbers) arranged in a definite order.

Examples 1.1: *Sequences*

(i) $3, 7, 11, 15, \ldots$
In this sequence each term is obtained by adding 4 to the previous term. So the next term would be 19.

(ii) $4, 9, 16, 25, \ldots$
This sequence can be rewritten as $2^2, 3^2, 4^2, 5^2, \ldots$ The next term is 6^2, or 36.

The dots($\ldots$) indicate that the sequence continues indefinitely – it is an *infinite sequence*. A sequence such as $3, 6, 9, 12$ (stopping after a finite number of terms) is a *finite sequence*. Suppose we write u_1 for the first term of a sequence, u_2 for the second and so on. There may be a formula for u_n, the n^{th} term:

Examples 1.2: *The n^{th} term of a sequence*

(i) $4, 9, 16, 25, \ldots$ The formula for the n^{th} term is $u_n = (n+1)^2$.

(ii) $u_n = 2n + 3$. The sequence given by this formula is: $5, 7, 9, 11, \ldots$

(iii) $u_n = 2^n + n$. The sequence is: $3, 6, 11, 20, \ldots$

Or there may be a formula that enables you to work out the terms of a sequence from the preceding one(s), called a *recurrence relation*:

Examples 1.3: *Recurrence Relations*

(i) Suppose we know that: $u_n = u_{n-1} + 7n$ and $u_1 = 1$.
Then we can work out that $u_2 = 1 + 7 \times 2 = 15$, $u_3 = 15 + 7 \times 3 = 36$, and so on, to find the whole sequence : $1, 15, 36, 64, \ldots$

(ii) $u_n = u_{n-1} + u_{n-2}$, $u_1 = 1$, $u_2 = 1$
The sequence defined by this formula is: $1, 1, 2, 3, 5, 8, 13, \ldots$

1.2. Series

A series is formed when the terms of a sequence are added together. The Greek letter Σ (pronounced "sigma") is used to denote "the sum of":

$$\sum_{r=1}^{n} u_r \text{ means } u_1 + u_2 + \cdots + u_n$$

EXAMPLES 1.4: *Series*

(i) In the sequence $3, 6, 9, 12, \ldots$, the sum of the first five terms is the series: $3 + 6 + 9 + 12 + 15$.

(ii) $\displaystyle\sum_{r=1}^{6}(2r + 3) = 5 + 7 + 9 + 11 + 13 + 15$

(iii) $\displaystyle\sum_{r=5}^{k} \frac{1}{r^2} = \frac{1}{25} + \frac{1}{36} + \frac{1}{49} + \cdots + \frac{1}{k^2}$

EXERCISES 3.1: **Sequences and Series**

(1) Find the next term in each of the following sequences:

(a) $2, 5, 8, 11, \ldots$
(b) $0.25, 0.75, 1.25, 1.75, 2.25, \ldots$
(c) $5, -1, -7, \ldots$
(d) $36, 18, 9, 4.5, \ldots$
(e) $1, -2, 3, -4, 5, \ldots$

(2) Find the 2^{nd}, 4^{th} and 6^{th} terms in the sequence given by: $u_n = n^2 - 10$

(3) If $u_n = \frac{u_{n-1}}{2} + 2$ and $u_1 = 4$ write down the first five terms of the sequence.

(4) If $u_n = u_{n-1}^2 + 3u_{n-1}$ and $u_3 = -2$, find the value of u_4.

(5) Find the value of $\sum_{r=1}^{4} 3r$

(6) Write out the following sums without using sigma notation:

(a) $\sum_{r=1}^{5} \frac{1}{r^2}$ (b) $\sum_{i=0}^{3} 2^i$ (c) $\sum_{j=0}^{n}(2j + 1)$

(7) In the series $\sum_{i=0}^{n-1}(4i+1)$, (a) how many terms are there? (b) what is the formula for the last term?

(8) Express using the Σ notation:

(a) $1^2 + 2^2 + 3^2 + ... + 25^2$
(b) $6 + 9 + 12 + \cdots + 21$
(c) $16 + 25 + 36 + 49 + \cdots + n^2$

Further reading and exercises

- For more practice with simple sequences and series, you could use an A-level pure maths textbook.

2. Arithmetic and Geometric Sequences

2.1. Arithmetic Sequences

An *arithmetic sequence* is one in which each term can be obtained by adding a fixed number (called the *common difference*) to the previous term.

EXAMPLES 2.1: *Some Arithmetic Sequences*

(i) $1, 3, 5, 7, \ldots$ The common difference is 2.

(ii) $13, 7, 1, -5, \ldots$ The common difference is -6.

> In an arithmetic sequence with first term a and common difference d, the formula for the n^{th}term is:
> $u_n = a + (n-1)d$

EXAMPLES 2.2: *Arithmetic Sequences*

(i) What is the 10^{th} term of the arithmetic sequence $5, 12, 19, \ldots$?
In this sequence $a = 5$ and $d = 7$. So the 10^{th} term is: $5 + 9 \times 7 = 68$.

(ii) If an arithmetic sequence has $u_{10} = 24$ and $u_{11} = 27$, what is the first term?
The common difference, d, is 3. Using the formula for the 11th term:

$$27 = a + 10 \times 3$$

Hence the first term, a, is -3.

2.2. Arithmetic Series

When the terms in an arithmetic sequence are summed, we obtain an arithmetic series. Suppose we want to find the sum of the first 5 terms of the arithmetic sequence with first term 3 and common difference 4. We can calculate it directly:

$$S_5 = 3 + 7 + 11 + 15 + 19 = 55$$

But there is a general formula:

> If an arithmetic sequence has first term a and common difference d, the sum of the first n terms is:
> $$S_n = \frac{n}{2}\left(2a + (n-1)d\right)$$

We can check that the formula works: $S_5 = \frac{5}{2}\left(2 \times 3 + 4 \times 4\right) = 55$

2.3. To Prove the Formula for an Arithmetic Series

Write down the series in order and in reverse order, then add them together, pairing terms:

$$\begin{array}{rcccccccc}
S_n = & a & + & (a+d) & + & (a+2d) & +\cdots+ & (a+(n-1)d) \\
S_n = & (a+(n-1)d) & + & (a+(n-2)d) & + & (a+(n-3)d) & +\cdots+ & a \\
2S_n = & (2a+(n-1)d) & + & (2a+(n-1)d) & + & (2a+(n-1)d) & +\cdots+ & (2a+(n-1)d) \\
= & n\left(2a+(n-1)d\right) & & & & & &
\end{array}$$

Dividing by 2 gives the formula.

EXERCISES 3.2: **Arithmetic Sequences and Series**

(1) Using the notation above, find the values of a and d for the arithmetic sequences:
(a) $4, 7, 10, 13, \ldots$ (b) $-1, 2, 5, \ldots$ (c) $-7, -8.5, -10, -11.5, \ldots$

(2) Find the n^{th} term in the following arithmetic sequences:
(a) $44, 46, 48, \ldots$ (b) $-3, -7, -11, -15, \ldots$

(3) If an arithmetic sequence has $u_{20} = 100$, and $u_{22} = 108$, what is the first term?

(4) Use the formula for an arithmetic series to calculate the sum of the first 8 terms of the arithmetic sequence with first term 1 and common difference 10.

(5) (a) Find the values of a and d for the arithmetic sequence: $21, 19, 17, 15, 13, \ldots$
(b) Use the formula for an arithmetic series to calculate $21 + 19 + 17 + 15 + 13$.
(c) Now use the formula to calculate the sum of $21{+}19{+}17{+}\ldots{+}1$.

(6) Is the sequence $u_n = -4n + 2$ arithmetic? If so, what is the common difference?

2.4. Geometric Sequences

A *geometric sequence* is one in which each term can be obtained by multiplying the previous term by a fixed number, called the *common ratio*.

EXAMPLES 2.3: *Geometric Sequences*

(i) $\frac{1}{2}, 1, 2, 4, 8, \ldots$ Each term is double the previous one. The common ratio is 2.

(ii) $81, 27, 9, 3, 1, \ldots$ The common ratio is $\frac{1}{3}$.

In a geometric sequence with first term a and common ratio r, the formula for the n^{th} term is:
$u_n = ar^{n-1}$

EXAMPLES 2.4: Consider the geometric sequence with first term 2 and common ratio 1.1.

(i) What is the 10^{th} term?
Applying the formula, with $a = 2$ and $r = 1.1$, $u_{10} = 2 \times (1.1)^9 = 4.7159$

(ii) Which terms of the sequence are greater than 20?
The n^{th} term is given by $u_n = 2 \times (1.1)^{n-1}$. It exceeds 20 if:

$$\begin{aligned} 2 \times (1.1)^{n-1} &> 20 \\ (1.1)^{n-1} &> 10 \end{aligned}$$

Taking logs of both sides (see chapter 1, section 5):

$$\begin{aligned} \log_{10}(1.1)^{n-1} &> \log_{10} 10 \\ (n-1)\log_{10} 1.1 &> 1 \\ n &> \frac{1}{\log_{10} 1.1} + 1 = 25.2 \end{aligned}$$

So all terms from the 26^{th} onwards are greater than 20.

2.5. Geometric Series

Suppose we want to find the sum of the first 10 terms of the geometric sequence with first term 3 and common ratio 0.5:

$$S_{10} = 3 + 1.5 + 0.75 + \cdots + 3 \times (0.5)^9$$

There is a general formula:

For a geometric sequence with first term a and common ratio r, the sum of the first n terms is:

$$S_n = \frac{a(1-r^n)}{1-r}$$

So the answer is: $$S_{50} = \frac{3(1-(0.5)^{10})}{1-0.5} = 5.994$$

2.6. To Prove the Formula for a Geometric Series

Write down the series and then multiply it by r:

$$\begin{array}{rcccccccccccl} S_n = & a & + & ar & + & ar^2 & + & ar^3 & + & \ldots & + & ar^{n-1} & \\ rS_n = & & & ar & + & ar^2 & + & ar^3 & + & \ldots & + & ar^{n-1} & + ar^n \end{array}$$

Subtract the second equation from the first:

$$\begin{aligned} S_n - rS_n &= a - ar^n \\ \implies S_n &= \frac{a(1-r^n)}{1-r} \end{aligned}$$

EXERCISES 3.3: **Geometric Sequences and Series**

(1) Find the 8^{th} term and the n^{th} term in the geometric sequence: $5, 10, 20, 40, \ldots$

(2) Find the 15^{th} term and the n^{th} term in the geometric sequence: $-2, 4, -8, 16, \ldots$

(3) In the sequence $1, 3, 9, 27, \ldots$, which is the first term term greater than 1000?

(4) (a) Using the notation above, what are the values of a and r for the sequence: $4, 2, 1, 0.5, 0.25, \ldots$?

(b) Use the formula for a geometric series to calculate: $4 + 2 + 1 + 0.5 + 0.25$.

(5) Find the sum of the first 10 terms of the geometric sequence: $4, 16, 64, \ldots$

(6) Find the sum of the first n terms of the geometric sequence: $20, 4, 0.8, \ldots$ Simplify your answer as much as possible.

(7) Use the formula for a geometric series to show that: $1 + \frac{x}{2} + \frac{x^2}{4} + \frac{x^3}{8} = \frac{16 - x^4}{16 - 8x}$

Further reading and exercises

- For more practice with arithmetic and geometric sequences and series, you could use an A-level pure maths textbook.
- *Jacques* §3.3 Geometric Series.

3. Economic Application: Interest Rates, Savings and Loans

Suppose that you invest £500 at the bank, at a fixed interest rate of 6% (that is, 0.06) per annum, and the interest is paid at the end of each year. At the end of one year you receive an interest payment of $0.06 \times 500 = £30$, which is added to your account, so you have £530. After two years, you receive an interest payment of $0.06 \times 530 = £31.80$, so that you have £561.80 in total, and so on.[1]

More generally, if you invest an amount P (the "principal") and interest is paid annually at interest rate i, then after one year you have a total amount y_1:

$$y_1 = P(1+i)$$

after two years:

$$y_2 = (P(1+i))(1+i) = P(1+i)^2$$

and after t years:

$$y_t = P(1+i)^t$$

This is a geometric sequence with common ratio $(1+i)$.

Examples 3.1: If you save £500 at a fixed interest rate of 6% paid annually:

(i) How much will you have after 10 years?
Using the formula above, $y_{10} = 500 \times 1.06^{10} = £895.42$.

(ii) How long will you have to wait to double your initial investment?
The initial amount will have doubled when:

$$\begin{aligned} 500 \times (1.06)^t &= 1000 \\ \Longrightarrow (1.06)^t &= 2 \end{aligned}$$

Taking logs of both sides (see chapter 1, section 5):

$$\begin{aligned} t \log_{10} 1.06 &= \log_{10} 2 \\ t &= \frac{\log_{10} 2}{\log_{10} 1.06} = 11.8957 \end{aligned}$$

So you will have to wait 12 years.

3.1. Interval of Compounding

In the previous section we assumed that interest was paid annually. However, in practice, financial institutions often pay interest more frequently, perhaps quarterly or even monthly. We call the time period between interest payments the interval of compounding.

Suppose the bank has a *nominal* (that is, stated) interest rate i, but pays interest m times a year at a rate of $\frac{i}{m}$. After 1 year you would have:

$$P\left(1+\frac{i}{m}\right)^m$$

and after t years:

$$P\left(1+\frac{i}{m}\right)^{mt}$$

[1] If you are not confident with calculations involving percentages, work through *Jacques* Chapter 3.1

EXAMPLES 3.2: You invest £1000 for five years in the bank, which pays interest at a nominal rate of 8%.

(i) How much will you have at the end of one year if the bank pays interest annually? You will have: $1000 \times 1.08 =$ £1080.

(ii) How much will you have at the end of one year if the bank pays interest quarterly? Using the formula above with $m = 4$, you will have $1000 \times 1.02^4 =$ £1082.43. Note that you are better off (for a given nominal rate) if the interval of compounding is shorter.

(iii) How much will you have at the end of 5 years if the bank pays interest monthly? Using the formula with $m = 12$ and $t = 5$, you will have:

$$1000 \times \left(1 + \frac{0.08}{12}\right)^{5\times 12} = £1489.85$$

From this example, you can see that if the bank pays interest quarterly and the nominal rate is 8%, then your investment actually grows by 8.243% in one year. The *effective* annual interest rate is 8.243%. In the UK this rate is known as the *Annual Equivalent Rate (AER)* (or sometimes the *Annual Percentage Rate (APR)*). Banks often describe their savings accounts in terms of the AER, so that customers do not need to do calculations involving the interval of compounding.

If the nominal interest rate is i, and interest is paid m times a year, an investment P grows to $P(1 + i/m)^m$ in one year. So the formula for the Annual Equivalent Rate is:

$$AER = \left(1 + \frac{i}{m}\right)^m - 1$$

EXAMPLES 3.3: *Annual Equivalent Rate*
If the nominal interest rate is 6% and the bank pays interest monthly, what is the AER?

$$AER = \left(1 + \frac{0.06}{12}\right)^{12} - 1 = 0.0617$$

The Annual Equivalent Rate is 6.17%.

3.2. Regular Savings

Suppose that you invest an amount A at the beginning of every year, at a fixed interest rate i (compounded annually). At the end of t years, the amount you invested at the beginning of the first year will be worth $A(1+i)^t$, the amount you invested in the second year will be worth $A(1+i)^{t-1}$, and so on. The total amount that you will have at the end of t years is:

$$\begin{aligned} S_t &= A(1+i)^t + A(1+i)^{t-1} + A(1+i)^{t-2} + \cdots + A(1+i)^2 + A(1+i) \\ &= A(1+i) + A(1+i)^2 + A(1+i)^3 + \cdots + A(1+i)^{t-1} + A(1+i)^t \end{aligned}$$

This is the sum of the first t terms of a geometric sequence with first term $A(1+i)$, and common ratio $(1+i)$. We can use the formula from section 2.5. The sum is:

$$S_t = \frac{A(1+i)\left(1 - (1+i)^t\right)}{1 - (1+i)} = \frac{A(1+i)}{i}\left((1+i)^t - 1\right)$$

So, for example, if you saved £200 at the beginning of each year for 10 years, at 5% interest, then you would accumulate $200\frac{1.05}{0.05}((1.05)^{10} - 1) =$ £2641.36.

3.3. Paying Back a Loan

If you borrow an amount L, to be paid back in annual repayments over t years, and the interest rate is i, how much do you need to repay each year?

Let the annual repayment be y. At the end of the first year, interest will have been added to the loan. After repaying y you will owe:

$$X_1 = L(1+i) - y$$

At the end of two years you will owe:

$$\begin{aligned} X_2 &= (L(1+i) - y)(1+i) - y \\ &= L(1+i)^2 - y(1+i) - y \end{aligned}$$

At the end of three years: $X_3 = L(1+i)^3 - y(1+i)^2 - y(1+i) - y$

and at the end of t years: $X_t = L(1+i)^t - y(1+i)^{t-1} - y(1+i)^{t-2} - \ldots - y$

But if you are to pay off the loan in t years, X_t must be zero:

$$\implies L(1+i)^t = y(1+i)^{t-1} + y(1+i)^{t-2} + \ldots + y$$

The right-hand side of this equation is the sum of t terms of a geometric sequence with first term y and common ratio $(1+i)$ (in reverse order). Using the formula from section 2.5:

$$\begin{aligned} L(1+i)^t &= \frac{y\left((1+i)^t - 1\right)}{i} \\ \implies y &= \frac{Li(1+i)^t}{((1+i)^t - 1)} \end{aligned}$$

This is the amount that you need to repay each year.

Exercises 3.4: **Interest Rates, Savings, and Loans**
(Assume annual compounding unless otherwise specified.)

(1) Suppose that you save £300 at a fixed interest rate of 4% per annum.
   (a) How much would you have after 4 years if interest were paid annually?
   (b) How much would you have after 10 years if interest were compounded monthly?
(2) If you invest £20 at 15% interest, how long will it be before you have £100?
(3) If a bank pays interest daily and the nominal rate is 5%, what is the AER?
(4) If you save £10 at the beginning of each year for 20 years, at an interest rate of 9%, how much will you have at the end of 20 years?
(5) Suppose you take out a loan of £100000, to be repaid in regular annual repayments, and the annual interest rate is 5%.
   (a) What should the repayments be if the loan is to be repaid in 25 years?
   (b) Find a formula for the repayments if the repayment period is T years.

Further reading and exercises

- *Jacques* §3.1 and 3.2.
- *Anthony & Biggs* Chapter 4

4. Present Value and Investment

Would you prefer to receive (a) a gift of £1000 today, or (b) a gift of £1050 in one year's time?

Your decision (assuming you do not have a desperate need for some immediate cash) will depend on the interest rate. If you accepted the £1000 today, and saved it at interest rate i, you would have £$1000(1+i)$ in a year's time. We could say:

$$\begin{aligned} \text{Future value of (a)} &= 1000(1+i) \\ \text{Future value of (b)} &= 1050 \end{aligned}$$

You should accept the gift that has higher future value. For example, if the interest rate is 8%, the future value of (a) is £1080, so you should accept that. But if the interest rate is less than 5%, it would be better to take (b).

Another way of looking at this is to consider what cash sum *now* would be equivalent to a gift of a gift of £1050 in one year's time. An amount P received now would be equivalent to an amount £1050 in one year's time if:

$$\begin{aligned} P(1+i) &= 1050 \\ \Longrightarrow P &= \frac{1050}{1+i} \end{aligned}$$

We say that:

$$\text{The } \textit{Present Value} \text{ of "£1050 in one year's time" is } \frac{1050}{1+i}$$

More generally:

> If the annual interest rate is i, the *Present Value* of an amount A to be received in t years' time is:
> $$P = \frac{A}{(1+i)^t}$$

The present value is also known as the *Present Discounted Value*; payments received in the future are worth less – we "discount" them at the interest rate i.

Examples 4.1: *Present Value and Investment*

(i) The prize in a lottery is £5000, but the prize will be paid in two years' time. A friend of yours has the winning ticket. How much would you be prepared to pay to buy the ticket, if you are able to borrow and save at an interest rate of 5%?

The present value of the ticket is:

$$P = \frac{5000}{(1.05)^2} = 4535.15$$

This is the maximum amount you should pay. If you have £4535.15, you would be indifferent between (a) paying this for the ticket, and (b) saving your money at 5%. Or, if you don't have any money at the moment, you would be indifferent between (a) taking out a loan of £4535.15, buying the ticket, and repaying the loan after 2 years when you receive the prize, and (b) doing nothing. Either way, if your friend will sell the ticket for less than £4535.15, you should buy it.

We can see from this example that *Present Value* is a powerful concept: a single calculation of the PV enables you to answer the question, without thinking about exactly how the money to buy the ticket is to be obtained. This does rely, however, on the assumption that you can borrow and save at the same interest rate.

(ii) An investment opportunity promises you a payment of £1000 at the end of each of the next 10 years, and a capital sum of £5000 at the end of the 11th year, for an initial outlay of £10000. If the interest rate is 4%, should you take it?

We can calculate the present value of the investment opportunity by adding up the present values of all the amounts paid out and received:

$$P = -10000 + \frac{1000}{1.04} + \frac{1000}{1.04^2} + \frac{1000}{1.04^3} + \cdots + \frac{1000}{1.04^{10}} + \frac{5000}{1.04^{11}}$$

In the middle of this expression we have (again) a geometric series. The first term is $\frac{1000}{1.04}$ and the common ratio is $\frac{1}{1.04}$. Using the formula from section 2.5:

$$\begin{aligned}
P &= -10000 + \frac{\frac{1000}{1.04}\left(1 - \left(\frac{1}{1.04}\right)^{10}\right)}{1 - \frac{1}{1.04}} + \frac{5000}{1.04^{11}} \\
&= -10000 + \frac{1000\left(1 - \left(\frac{1}{1.04}\right)^{10}\right)}{0.04} + 3247.90 \\
&= -10000 + 25000\left(1 - \left(\frac{1}{1.04}\right)^{10}\right) + 3247.90 \\
&= -10000 + 8110.90 + 3247.90 = -10000 + 11358.80 = £1358.80
\end{aligned}$$

The present value of the opportunity is positive (or equivalently, the present value of the return is greater than the initial outlay): you should take it.

4.1. Annuities

An *annuity* is a financial asset which pays you an amount A each year for N years. Using the formula for a geometric series, we can calculate the present value of an annuity:

$$\begin{aligned}
PV &= \frac{A}{1+i} + \frac{A}{(1+i)^2} + \frac{A}{(1+i)^3} + \cdots + \frac{A}{(1+i)^N} \\
&= \frac{\frac{A}{1+i}\left(1 - \left(\frac{1}{1+i}\right)^N\right)}{1 - \left(\frac{1}{1+i}\right)} \\
&= \frac{A\left(1 - \left(\frac{1}{1+i}\right)^N\right)}{i}
\end{aligned}$$

The present value tells you the price you would be prepared to pay for the asset.

EXERCISES 3.5: **Present Value and Investment**

(1) On your 18^{th} birthday, your parents promise you a gift of £500 when you are 21. What is the present value of the gift (a) if the interest rate is 3% (b) if the interest rate is 10%?

(2) (a) How much would you pay for an annuity that pays £20 a year for 10 years, if the interest rate is 5%?

(b) You buy it, then after receiving the third payment, you consider selling the annuity. What price will you be prepared to accept?

(3) The useful life of a bus is five years. Operating the bus brings annual profits of £10000. What is the value of a new bus if the interest rate is 6%?

(4) An investment project requires an initial outlay of £2400, and can generate revenue of £2000 per year. In the first year, operating costs are £600; thereafter operating costs increase by £500 a year.

(a) What is the maximum length of time for which the project should operate?

(b) Should it be undertaken if the interest rate is 5%?

(c) Should it be undertaken if the interest rate is 10%?

Further reading and exercises

- *Jacques* §3.4
- *Anthony & Biggs* Chapter 4
- *Varian* also discusses Present Value and has more economic examples.

5. Limits

5.1. The Limit of a Sequence

If we write down some of the terms of the geometric sequence: $u_n = (\frac{1}{2})^n$:

$$\begin{aligned} u_1 = &\quad (\tfrac{1}{2})^1 &= 0.5 \\ u_{10} = &\quad (\tfrac{1}{2})^{10} &= 0.000977 \\ u_{20} = &\quad (\tfrac{1}{2})^{20} &= 0.000000954 \end{aligned}$$

we can see that as n gets larger, u_n gets closer and closer to zero. We say that "the limit of the sequence as n tends to infinity is zero" or "the sequence converges to zero" or:

$$\lim_{n\to\infty} u_n = 0$$

Examples 5.1: *Limits of Sequences*

(i) $u_n = 4 - (0.1)^n$
The sequence is:

$$3.9, 3.99, 3.999, 3.9999, \ldots$$

We can see that it converges:

$$\lim_{n\to\infty} u_n = 4$$

(ii) $u_n = (-1)^n$
This sequence is $-1, +1, -1, +1, -1, +1, \ldots$ It has no limit.

(iii) $u_n = \frac{1}{n}$
The terms of this sequence get smaller and smaller:

$$1, \tfrac{1}{2}, \tfrac{1}{3}, \tfrac{1}{4}, \tfrac{1}{5}, \ldots$$

It converges to zero:

$$\lim_{n\to\infty} \tfrac{1}{n} = 0$$

(iv) $2, 4, 8, 16, 32, \ldots$
This is a geometric sequence with common ratio 2. The terms get bigger and bigger. It *diverges*:

$$u_n \to \infty \quad \text{as } n \to \infty$$

(v) $u_n = \dfrac{2n^3 + n^2}{3n^3}$.

A useful trick is to divide the numerator and the denominator by the highest power of n; that is, by n^3. Then:

$$u_n = \left(\frac{2 + \frac{1}{n}}{3} \right)$$

and we know that $\frac{1}{n} \to 0$, so:

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \left(\frac{2 + \frac{1}{n}}{3} \right) = \frac{2}{3}$$

EXERCISES 3.6: Say whether each if the following sequences converges or diverges as $n \to \infty$. If it converges, find the limit.

(1) $u_n = (\frac{1}{3})^n$

(2) $u_n = -5 + (\frac{1}{4})^n$

(3) $u_n = (-\frac{1}{3})^n$

(4) $u_n = 7 - (\frac{2}{5})^n$

(5) $u_n = \frac{10}{n^3}$

(6) $u_n = (1.2)^n$

(7) $u_n = 25n + \frac{10}{n^3}$

(8) $u_n = \frac{7n^2+5n}{n^2}$

From examples like these we can deduce some general results that are worth remembering:

- $\lim_{n\to\infty} \dfrac{1}{n} = 0$
- Similarly $\lim_{n\to\infty} \dfrac{1}{n^2} = 0$ and $\lim_{n\to\infty} \dfrac{1}{n^3} = 0$ etc
- If $|r| < 1$, $\quad \lim_{n\to\infty} r^n = 0$
- If $r > 1$, $\quad r^n \to \infty$

5.2. Infinite Geometric Series

Consider the sequence: $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$ It is a geometric sequence with first term $a = 1$ and common ratio $r = \frac{1}{2}$. We can find the sum of the first n terms:

$$S_n = 1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \cdots + \left(\tfrac{1}{2}\right)^{n-1}$$

using the formula from section 2.5:

$$\begin{aligned} S_n &= \frac{a(1-r^n)}{1-r} = \frac{1-\left(\frac{1}{2}\right)^n}{1-\frac{1}{2}} = 2\left(1-\left(\tfrac{1}{2}\right)^n\right) \\ &= 2 - \left(\tfrac{1}{2}\right)^{n-1} \end{aligned}$$

As the number of terms gets larger and larger, their sum gets closer and closer to 2:

$$\lim_{n\to\infty} S_n = 2$$

Equivalently, we can write this as:

$$1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \cdots = 2$$

So we have found the sum of an *infinite* number of terms to be a finite number. Using the sigma notation this can be written:

$$\sum_{i=1}^{\infty} \left(\tfrac{1}{2}\right)^{i-1} = 2$$

or (a little more neatly):

$$\sum_{i=0}^{\infty} \left(\tfrac{1}{2}\right)^i = 2$$

The same procedure works for any geometric series with common ratio r, provided that $|r| < 1$. The sum of the first n terms is:

$$S_n = \frac{a(1 - r^n)}{1 - r}$$

As $n \to \infty$, $r^n \to 0$ so the series converges:

$$\lim_{n\to\infty} S_n = \frac{a}{1 - r}$$

or equivalently:

$$\sum_{i=1}^{\infty} ar^{i-1} = \frac{a}{1 - r}$$

But note that if $|r| > 1$ the terms of the series get bigger and bigger, so it *diverges*: the infinite sum does not exist.

EXAMPLES 5.2: Find the sum to infinity of the following series:

(i) $2 - 1 + \frac{1}{2} - \frac{1}{4} + \frac{1}{8} \dots$
This is a geometric series, with $a = 2$ and $r = -\frac{1}{2}$. It converges because $|r| < 1$.
Using the formula above:

$$S_\infty = \frac{a}{1 - r} = \frac{2}{\frac{3}{2}} = \frac{4}{3}$$

(ii) $x + 2x^2 + 4x^3 + 8x^4 + \dots$ (assuming $0 < x < 0.5$)
This is a geometric series with $a = x$ and $r = 2x$. We know $0 < r < 1$, so it converges.
The formula for the sum to infinity gives:

$$S_\infty = \frac{a}{1 - r} = \frac{x}{1 - 2x}$$

5.3. Economic Application: Perpetuities

In section 4.1 we calculated the present value of an annuity – an asset that pays you an amount A each year for a fixed number of years. A *perpetuity* is an asset that pays you an amount A each year *forever*.
If the interest rate is i, the present value of a perpetuity is:

$$PV = \frac{A}{1 + i} + \frac{A}{(1 + i)^2} + \frac{A}{(1 + i)^3} + \dots$$

This is an infinite geometric series. The common ratio is $\frac{1}{1+i}$. Using the formula for the sum of an infinite series:

$$\begin{aligned} PV &= \frac{\frac{A}{1+i}}{1 - \left(\frac{1}{1+i}\right)} \\ &= \frac{A}{i} \end{aligned}$$

Again, the present value tells you the price you would be prepared to pay for the asset. Even an asset that pays out forever has a finite price. (Another way to get this result is to let

$N \to \infty$ in the formula for the present value of an annuity that we obtained earlier.)

EXERCISES 3.7: **Infinite Series, and Perpetuities**

(1) Evaluate the following infinite sums:

(a) $\frac{1}{3} + \left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^3 + \left(\frac{1}{3}\right)^4 + \dots$

(b) $1 + 0.2 + 0.04 + 0.008 + 0.0016 + \dots$

(c) $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} - \dots$

(d) $\frac{2}{3} + \left(\frac{2}{3}\right)^4 + \left(\frac{2}{3}\right)^7 + \dots$

(e) $\sum_{r=3}^{\infty} \left(\frac{1}{2}\right)^r$

(f) $\sum_{r=0}^{\infty} x^r$ assuming $|x| < 1$. (Why is this assumption necessary?)

(g) $\frac{2y^2}{x} + \frac{4y^3}{x^2} + \frac{8y^4}{x^3} + \dots$ What assumption is needed here?

(2) If the interest rate is 4%, what is the present value of:

(a) an annuity that pays £100 each year for 20 years?

(b) a perpetuity that pays £100 each year forever?

How will the value of each asset have changed after 10 years?

(3) A firm's profits are expected to be £1000 this year, and then to rise by 2% each year after that (forever). If the interest rate is 5%, what is the present value of the firm?

Further reading and exercises

- *Anthony & Biggs*: §3.3 discusses limits briefly.
- *Varian* has more on financial assets including perpetuities, and works out the present value of a perpetuity in a different way.
- For more on limits of sequences, and infinite sums, refer to an A-level pure maths textbook.

6. The Number e

If we evaluate the numbers in the sequence:

$$u_n = \left(1 + \frac{1}{n}\right)^n$$

we get: $u_1 = 2$, $u_2 = (1 + \frac{1}{2})^2 = 2.25$, $u_3 = (1 + \frac{1}{3})^3 = 2.370, \ldots$
For some higher values of n we have, for example:

$$\begin{aligned}
u_{10} &= && (1.1)^{10} && = 2.594 \\
u_{100} &= && (1.01)^{100} && = 2.705 \\
u_{1000} &= && (1.001)^{1000} && = 2.717 \\
u_{10000} &= && (1.0001)^{10000} && = 2.71814 \\
u_{100000} &= && (1.00001)^{100000} && = 2.71826 \quad \ldots
\end{aligned}$$

As $n \to \infty$, u_n gets closer and closer to a limit of $2.718281828459\ldots$ This is an irrational number (see Chapter 1) known simply as e. So:

$$\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e \quad (\approx 2.71828)$$

e is important in calculus (as we will see later) and arises in many economic applications.
We can generalise this result to:

> For any value of r, $\displaystyle\lim_{n\to\infty} \left(1 + \frac{r}{n}\right)^n = e^r$

> Exercises 3.8: Verify (approximately, using a calculator) that $\displaystyle\lim_{n\to\infty} \left(1 + \frac{2}{n}\right)^n = e^2$.
> *Hint*: Most calculators have a button that evaluates e^x for any number x.

6.1. Economic Application: Continuous Compounding

Remember, from section 3.1, that if interest is paid m times a year and the nominal rate is i, then the return after t years from investing an inital amount P is:

$$P\left(1 + \frac{i}{m}\right)^{mt}$$

Interest might be paid quarterly ($m = 4$), monthly ($m = 12$), weekly ($m = 52$), or daily ($m = 365$). Or it could be paid even more frequently – every hour, every second ... As the interval of compounding get shorter, interest is compounded almost continuously.
As $m \to \infty$, we can apply our result above to say that:

$$\lim_{m\to\infty} \left(1 + \frac{i}{m}\right)^m = e^i$$

and so:

> If interest is compounded continuously at rate i, the return after t years on an initial amount P is:
> Pe^{it}

EXAMPLES 6.1: If interest is compounded continuously, what is the AER if:

(i) the interest rate is 5%?
Applying the formula, an amount P invested for one year yields:

$$Pe^{0.05} = 1.05127P = (1 + 0.05127)P$$

So the AER is 5.127%.

(ii) the interest rate is 8%?
Similarly, $e^{0.08} = 1.08329$, so the AER is 8.329%.

We can see from these examples that with continuous compounding the AER is little different from the interest rate. So, when solving economic problems we often simplify by assuming continuous compounding, because it avoids the messy calculations for the interval of compounding.

6.2. Present Value with Continuous Compounding

In section 4, when we showed that the present value of an amount A received in t years time is $\frac{A}{(1+i)^t}$, we were assuming annual compounding of interest.

> With continuous compounding, if the interest rate is i, the present value of an amount A received in t years is:
> $P = Ae^{-it}$

Continuous compounding is particularly useful because it allows us to calculate the present value when t is not a whole number of years.

To see where the formula comes from, note that if you have an amount Ae^{-it} now, and you save it for t years with continuous compounding, you will then have $Ae^{-it}e^{it} = A$. So "Ae^{-it} now" and "A after t years", are worth the same.

EXERCISES 3.9: e

(1) Express the following in terms of e:

(a) $\lim_{n\to\infty}(1 + \frac{1}{n})^n$ (b) $\lim_{n\to\infty}(1 + \frac{5}{n})^n$ (c) $\lim_{n\to\infty}(1 + \frac{1}{2n})^n$

(2) If you invest £100, the interest rate is 5%, and interest is compounded countinuously:

(a) How much will you have after 1 year?

(b) How much will you have after 5 years?

(c) What is the AER?

(3) You expect to receive a gift of £100 on your next birthday. If the interest rate is 5%, what is the present value of the gift (a) six months before your birthday (b) 2 days before your birthday?

Further reading and exercises

- *Anthony & Biggs*: §7.2 and §7.3.
- *Jacques* §2.4.

Solutions to Exercises in Chapter 3

EXERCISES 3.1:

(1) (a) 14
  (b) 2.75
  (c) -13
  (d) 2.25
  (e) -6
(2) $-6, 6, 26$
(3) $4, 4, 4, 4, 4$
(4) $u_4 = -2$
(5) 30
(6) (a) $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25}$
  (b) $1 + 2 + 4 + 8$
  (c) $1+3+5+\ldots+(2n+1)$
(7) (a) n
  (b) $4n - 3$
(8) There are several possibilities. e.g.
  (a) $\sum_{r=1}^{25} r^2$
  (b) $\sum_{r=2}^{7} 3r$
  (c) $\sum_{r=4}^{n} r^2$

EXERCISES 3.2:

(1) (a) $a = 4$ $d = 3$
  (b) $a = -1$ $d = 3$
  (c) $a = -7$ $d = -1.5$
(2) (a) $u_n = 44 + 2(n-1)$
    $= 42 + 2n$
  (b) $u_n = -3 - 4(n-1)$
    $= -4n + 1$
(3) $u_1 = 24$
(4) 288
(5) (a) $a = 21$ $d = -2$
  (b) $S_5 = 85$
  (c) $S_{11} = 121$
(6) Yes, $u_n = -2 - 4(n-1)$, so $d = -4$

EXERCISES 3.3:

(1) 640, $5 \times 2^{n-1}$
(2) -32768, $(-2)^n$
(3) 8^{th}
(4) (a) $a = 4$, $r = \frac{1}{2}$
  (b) $S_5 = 7.75$
(5) $S_{10} = 1398100$
(6) $S_n = 25(1 - \frac{1}{5}^n)$
  $= 25 - \left(\frac{1}{5}\right)^{n-2}$
(7) $a = 1; r = \left(\frac{x}{2}\right); n = 4$
  $\Rightarrow S_4 = \frac{16-x^4}{16-8x}$

EXERCISES 3.4:

(1) (a) 350.96
  (b) 447.25
(2) $t = 12$
(3) 5.13%
(4) $S_{20} = £557.65$
(5) (a) £7095.25
  (b) $y = \frac{5000(1.05)^T}{1.05^T - 1}$

EXERCISES 3.5:

(1) (a) £457.57
  (b) £375.66
(2) (a) £154.43
  (b) £115.73
(3) £42123
(4) (a) 3 years.
  (b) Yes PV=£95.19
  (c) No PV=$-$£82.95

EXERCISES 3.6:

(1) $u_n \to 0$ as $n \to \infty$
(2) $u_n \to -5$ as $n \to \infty$
(3) $u_n \to 0$ as $n \to \infty$
(4) $u_n \to 7$ as $n \to \infty$
(5) $u_n \to 0$ as $n \to \infty$
(6) $u_n \to \infty$ as $n \to \infty$
(7) $u_n \to \infty$ as $n \to \infty$
(8) $u_n \to 7$ as $n \to \infty$

EXERCISES 3.7:

(1) (a) $\frac{1}{2}$
  (b) $\frac{5}{4}$
  (c) $\frac{2}{3}$
  (d) $\frac{18}{19}$
  (e) $\frac{1}{4}$
  (f) $\frac{1}{1-x}$ If $|x| \geq 1$ sequence diverges
  (g) $\frac{2y^2}{x-2y}$ assuming $|2y| < |x|$
(2) (a) £1359.03. Is only worth £811.09 after ten years.
  (b) £2500. Same value in ten years.
(3) £33,333.33

EXERCISES 3.8:

(1) ($e^2 = 7.389$ to 3 decimal places)

EXERCISES 3.9:

(1) (a) e
  (b) e^5
  (c) $e^{\frac{1}{2}}$
(2) (a) £105.13
  (b) £128.40
  (c) 5.13%
(3) (a) $P = 100e^{-\frac{i}{2}}$
    $= £97.53$
  (b) $P = 100e^{-\frac{2i}{365}}$
    $= £99.97$

[2]This Version of Workbook Chapter 3: September 25, 2014

Worksheet 3: Sequences, Series, and Limits; the Economics of Finance

Quick Questions

(1) What is the n^{th} term of each of the following sequences:
(a) $20, 15, 10, 5, \ldots$ (b) $1, 8, 27, 64, \ldots$ (c) $0.2, 0.8, 3.2, 12.8, \ldots$

(2) Write out the series: $\sum_{r=0}^{n-1}(2r-1)^2$ without using sigma notation, showing the first four terms and the last two terms.

(3) For each of the following series, work out how many terms there are and hence find the sum:
(a) $3+4+5+\cdots+20$ (b) $1+0.5+0.25+\cdots+(0.5)^{n-1}$ (c) $5+10+20+\cdots+5\times2^n$

(4) Express the series $3+7+11+\cdots+(4n-1)+(4n+3)$ using sigma notation.

(5) If you invest £500 at a fixed interest rate of 3% per annum, how much will you have after 4 years:
 (a) if interest is paid annually?
 (b) if interest is paid monthly? What is the AER in this case?
 (c) if interest is compounded continuously?
If interest is paid annually, when will your savings exceed £600?

(6) If the interest rate is 5% per annum, what is the present value of:
 (a) An annuity that pays £100 a year for 20 years?
 (b) A perpetuity that pays £50 a year?

(7) Find the limit, as $n \to \infty$, of:
(a) $3(1+(0.2)^n)$ (b) $\dfrac{5n^2+4n+3}{2n^2+1}$ (c) $0.75+0.5625+\cdots+(0.75)^n$

Longer Questions

(1) Carol (an economics student) is considering two possible careers. As an acrobat, she will earn £30000 in the first year, and can expect her earnings to increase at 1% per annum thereafter. As a beekeeper, she will earn only £20000 in the first year, but the subsequent increase will be 5% per annum. She plans to work for 40 years.
 (a) If she decides to be an acrobat:
  (i) How much will she earn in the 3^{rd} year of her career?
  (ii) How much will she earn in the n^{th} year?
  (iii) What will be her total career earnings?
 (b) If she decides to be a beekeeper:
  (i) What will be her total career earnings?
  (ii) In which year will her annual earnings first exceed what she would have earned as an acrobat?
 (c) She knows that what matters for her choice of career is the present value of her earnings. The rate of interest is i. (Assume that earnings are received at the end of each year, and that her choice is made on graduation day.) If she decides to be an acrobat:

(i) What is the present value of her first year's earnings?
(ii) What is the present value of the n^{th} year's earnings?
(iii) What is the total present value of her career earnings?

(d) Which career should she choose if the interest rate is 3%?
(e) Which career should she choose if the interest rate is 15%?
(f) Explain these results.

(2) Bill is due to start a four year degree course financed by his employer and he feels the need to have his own computer. The computer he wants cost £1000. The insurance premium on the computer will start at £40 for the first year, and decline by £5 per year throughout the life of the computer, while repair bills start at £50 in the computer's first year, and increase by 50% per annum thereafter. (The grant and the insurance premium are paid at the beginning of each year, and repair bills at the end.)
The resale value for computers is given by the following table:

	Resale value at end of year
Year 1	75% of initial cost
Year 2	60% of initial cost
Year 3	20% of initial cost
Year 4 and onwards	£100

The interest rate is 10% per annum.

(a) Bill offers to forgo £360 per annum from his grant if his employer purchases a computer for him and meets all insurance and repair bills. The computer will be sold at the end of the degree course and the proceeds paid to the employer. Should the employer agree to the scheme? If not, what value of computer would the employer agree to purchase for Bill (assuming the insurance, repair and resale schedules remain unchanged)?

(b) Bill has another idea. He still wants the £1000 computer, but suggests that it be replaced after two years with a new one. Again, the employer would meet all bills and receive the proceeds from the sale of both computers, and Bill would forgo £360 per annum from his grant. How should the employer respond in this case?

CHAPTER 4

Functions

> Functions, and the language of functions, are widely used in economics. Linear and quadratic functions were discussed in Chapter 2. Now we introduce other useful functions (including **exponential** and **logarithmic** functions) and concepts (such as **inverse functions** and **functions of several variables**). Economic applications are **supply and demand functions**, **utility functions** and **production functions**.

1. Function Notation, and Some Common Functions

We have already encountered some functions in Chapter 2. For example:

$$y = 5x - 8$$

Here y is a function of x (or in other words, y depends on x).

$$C = 3 + 2q^2$$

Here, a firm's total cost, C, of producing output is a function of the quantity of output, q, that it produces.

To emphasize that y *is a function of* x, and that C *is a function of* q, we often write:

$$y(x) = 5x - 8 \quad \text{and} \quad C(q) = 3 + 2q^2$$

Also, for general functions it is common to use the letter f rather than y:

$$f(x) = 5x - 8$$

Here, x is called the *argument* of the function.

In general, we can think of a function $f(x)$ as a "black box" which takes x as an input, and produces an output $f(x)$:

$$x \longrightarrow \boxed{\quad f \quad} \longrightarrow f(x)$$

For each input value, there is a unique output value.

EXAMPLES 1.1: For the function $f(x) = 5x - 8$

(i) Evaluate $f(3)$

$$f(3) = 15 - 8 = 7$$

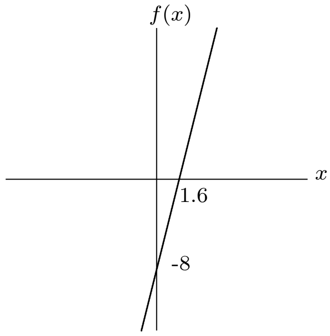
(ii) Evaluate $f(0)$

$$f(0) = 0 - 8 = -8$$

(iii) Solve the equation $f(x) = 0$

$$\begin{aligned} f(x) &= 0 \\ \implies 5x - 8 &= 0 \\ \implies x &= 1.6 \end{aligned}$$

(iv) Hence sketch the graph.



EXERCISES 4.1: **Using Function Notation**

(1) For the function $f(x) = 9 - 2x$, evaluate $f(2)$ and $f(-4)$.

(2) Solve the equation $g(x) = 0$, where $g(x) = 5 - \frac{10}{x+1}$.

(3) If a firm has cost function $C(q) = q^3 - 5q$, what are its costs of producing 4 units of output?

1.1. Polynomials

Polynomials were introduced in Chapter 1. They are functions of the form:

$$f(x) = a_n x^n + a_{n-1} x^{n+1} + ... + a_0$$

EXAMPLES 1.2:

(i) $g(x) = 5x^3 - 2x + 1$ is a polynomial of degree 3

(ii) $h(x) = 5 + x^4 - x^9 + x^2$ is a polynomial of degree 9

A linear function is a polynomial of degree 1, and a quadratic function is a polynomial of degree 2. We already know what their graphs look like (see Chapter 2); a linear function crosses the x-axis once (or not at all); a quadratic function crosses the x-axis twice, or touches it once, or not at all.

The graph of a polynomial of degree n crosses the x-axis up to n times.

EXERCISES 4.2: **Polynomials**

(1) For the polynomial function $f(x) = x^3 - 3x^2 + 2x$:

(a) Factorise the function and hence show that it crosses the x-axis at $x = 0$, $x = 1$ and $x = 2$.

(b) Check whether the function is positive or negative when $x < 0$, when $0 < x < 1$, when $1 < x < 2$, and when $x > 2$ and hence sketch a graph of the function.

(2) Consider the polynomial function $g(x) = 5x^2 - x^4 - 4$.

(a) What is the degree of the polynomial?

(b) Factorise the function. (*Hint*: it is a quadratic in x^2: $-(x^2)^2 + 5x^2 - 4$)

(c) Hence sketch the graph, using the same method as in the previous example.

1.2. The Function $f(x) = x^n$

When n is a positive integer, $f(x) = x^n$ is just a simple polynomial. But n could be negative:

$$f(x) = \frac{1}{x^2} \quad (n = -2)$$

and in economic applications, we often use fractional values of n, which we can do if x represents a positive variable (such as "output", or "employment"):

$$\begin{aligned} f(x) &= x^{0.3} \\ f(x) &= x^{\frac{3}{2}} \\ f(x) &= \sqrt{x} \quad (n = 0.5) \end{aligned}$$

EXERCISES 4.3: **The Function x^n**

On a single diagram, with x-axis from 0 to 4, and vertical axis from 0 to 6.5, plot carefully the graphs of $f(x) = x^n$, for n=1.3, n=1, n=0.7 and $n = -0.3$.

Hint: You cannot evaluate $x^{-0.3}$ when $x = 0$, but try values of x close to zero.

You should find that your graphs have the standard shapes below.

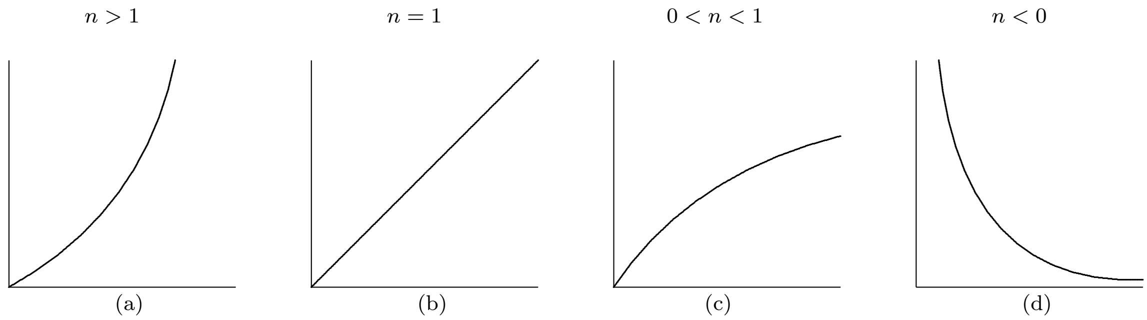


FIGURE 1. The function $f(x) = x^n$

1.3. Increasing and Decreasing Functions

If $f(x)$ increases whenever x increases:

- the graph is upward-sloping;
- we say that f is an *increasing function* of x
  (or sometimes that it is a *monotonic increasing function*).

Similarly, a function that decreases whenever x increases has a downward-sloping graph and is known as a (monotonic) *decreasing function*. *Monotonic* means simply that f moves in one direction (either up or down) as x increases.

EXAMPLES 1.3: $f(x) = x^n \quad (x \geq 0)$

We can see from Figure 1 that this function is monotonic, whatever the value of n.

(i) When $n > 0$ f is an increasing function of x (graphs (a) to (c))

(ii) When $n < 0$ f is a decreasing function of x (graph (d))

1.4. Limits of Functions

In Chapter 3, we looked at limits of sequences. We also use this idea for functions.

EXAMPLES 1.4: $f(x) = x^n$

Looking at Figure 1 again:

(i) If $n > 0$, $\lim_{x\to\infty} x^n = \infty$ and $\lim_{x\to 0} x^n = 0$

(ii) If $n < 0$, $\lim_{x\to\infty} x^n = 0$ and $\lim_{x\to 0} x^n = \infty$

EXERCISES 4.4: **Increasing and decreasing functions, and limits**

(1) If $f(x) = 2 - x^2$ find:

(a) $\lim_{x\to\infty} f(x)$

(b) $\lim_{x\to-\infty} f(x)$

(c) $\lim_{x\to 0} f(x)$

(2) If $y = \frac{2}{x} + 3$ for values of $x \geq 0$, is y an increasing or a decreasing function of x? What is the limit of y as x tends to infinity?

(3) If $g(x) = 1 - \frac{5}{x^2}$ for $x \geq 0$:

(a) Is g an increasing or a decreasing function?

(b) What is $\lim_{x\to\infty} g(x)$?

(c) What is $\lim_{x\to 0} g(x)$?

Further reading and exercises

- *Jacques* §1.3.
- *Anthony & Biggs* §2.2.
- Both of the above are brief. For more examples, use an A-level pure maths textbook.

2. Composite Functions

If we have two functions, $f(x)$ and $g(x)$, we can take the output of f and input it to g:

$$x \longrightarrow \boxed{\;f\;} \longrightarrow f(x) \longrightarrow \boxed{\;g\;} \longrightarrow g(f(x))$$

We can think of the final output as the output of a new function, $g(f(x))$, called a *composite function* or a "function of a function."

EXAMPLES 2.1: If $f(x) = 2x + 3$ and $g(x) = x^2$:

(i) $g(f(2)) = g(7) = 49$

(ii) $f(g(2)) = f(4) = 11$

(iii) $f(g(-4)) = f(16) = 35$

(iv) $g(f(-4)) = g(-5) = 25$
Note that $f(g(x))$ is not the same as $g(f(x))$!

EXAMPLES 2.2: If $f(x) = 2x + 3$ and $g(x) = x^2$, what are the functions:

(i) $f(g(x))$?

$$f(g(x)) = 2g(x) + 3 = 2x^2 + 3$$

(ii) $g(f(x))$?

$$g(f(x)) = (f(x))^2 = (2x + 3)^2 = 4x^2 + 12x + 9$$

Note that we can check these answers using the previous ones. For example:

$$f(g(x)) = 2x^2 + 3 \implies f(g(2)) = 11$$

EXERCISES 4.5: **Composite Functions**

(1) If $f(x) = 3 - 2x$ and $g(x) = 8x - 1$ evaluate $f(g(2))$ and $g(f(2))$.

(2) If $g(x) = \frac{4}{x+1}$ and $h(x) = x^2 + 1$, find $g(h(1))$ and $h(g(1))$.

(3) If $k(x) = \sqrt[3]{x}$ and $m(x) = x^3$, evaluate $k(m(3))$ and $m(k(3))$. Why are the answers the same in this case?

(4) If $f(x) = x + 1$, and $g(x) = 2x^2$, what are the functions $g(f(x))$, and $f(g(x))$?

(5) If $h(x) = \frac{5}{x+2}$ and $k(x) = \frac{1}{x}$ find the functions $h(k(x))$, and $k(h(x))$.

Further reading and exercises

- *Jacques* §1.3.
- *Anthony & Biggs* §2.3.
- Both of the above are brief. For more examples, use an A-level maths pure textbook.

3. Inverse Functions

Suppose we have a function $f(x)$, and we call the output y. If we can find another function that takes y as an input and produces the original value x as output, it is called the inverse function $f^{-1}(y)$.

$$x \longrightarrow \boxed{\;f\;} \longrightarrow y \longrightarrow \boxed{f^{-1}} \longrightarrow x$$

FIGURE 2. f^{-1} is the inverse of f: $f^{-1}(f(x)) = x$

If we can find such a function, and we take its output and input it to the original function, we find also that f is the inverse of f^{-1}:

$$y \longrightarrow \boxed{f^{-1}} \longrightarrow x \longrightarrow \boxed{\;f\;} \longrightarrow y$$

FIGURE 3. f is the inverse of f^{-1}: $f(f^{-1}(y)) = y$

EXAMPLES 3.1: What is the inverse of the function

(i) $f(x) = 3x + 1$?

Call the output of the function y:

$$y = 3x + 1$$

This equation tells you how to find y if you know x (that is, it gives y in terms of x). Now rearrange it, to find x in terms of y (that is, make x the subject):

$$\begin{aligned} 3x &= y - 1 \\ x &= \frac{y-1}{3} \end{aligned}$$

So the inverse function is:

$$f^{-1}(y) = \frac{y-1}{3}$$

(ii) $f(x) = \frac{2}{x-1}$?

$$\begin{aligned} y &= \frac{2}{x-1} \\ y(x-1) &= 2 \\ x &= 1 + \frac{2}{y} \end{aligned}$$

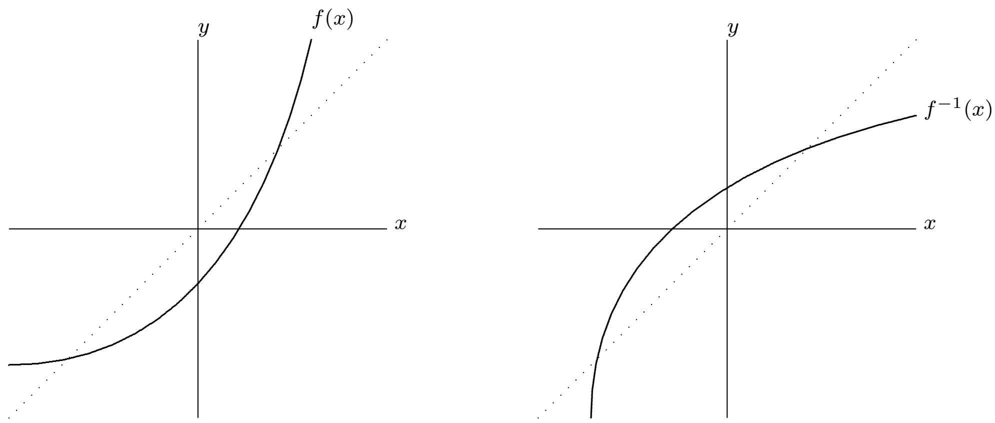
So the inverse function is:

$$f^{-1}(y) = 1 + \frac{2}{y}$$

- It doesn't matter what letter we use for the argument of a function. The function above would still be the same function if we wrote it as:
$$f^{-1}(z) = 1 + \frac{2}{z}$$
- We could say, for example, that
$$f^{-1}(x) = 1 + \frac{2}{x} \text{ is the inverse of } f(x) = \frac{2}{x-1}$$
using the same argument for both (although this can be be confusing).

- *Warning:* It is usually only possible to find the inverse if the function is monotonic (see section 1.3). Otherwise, if you know the output, you can't be sure what the input was. For example, think about the function $y = x^2$. If we know that the output is 9, say, we can't tell whether the input was 3 or -3. In such cases, we say that the inverse function "doesn't exist."

- There is an easy way to work out what the graph of the inverse function looks like: just reflect it in the line $y = x$.



EXERCISES 4.6: Find the inverse of each of the following functions:

(1) $f(x) = 8x + 7$

(2) $g(x) = 3 - 0.5x$

(3) $h(x) = \frac{1}{x+4}$

(4) $k(x) = x^3$

Further reading and exercises

- *Jacques* §1.3.
- *Anthony & Biggs* §2.2.
- A-level pure maths textbooks.

4. Economic Application: Supply and Demand Functions

4.1. Demand

The market demand function for a good tells us how the quantity that consumers want to buy depends on the price. Suppose the demand function in a market is:

$$Q^d(P) = 90 - 5P$$

This is a linear function of P. You can see that it is downward-sloping (it is a decreasing function). If the price increases, consumers will buy less.

The inverse demand function tells us how much consumers will pay if the quantity available is Q. To find the inverse demand function:

$$\begin{aligned} Q &= 90 - 5P \\ 5P &= 90 - Q \\ P &= \frac{90 - Q}{5} \end{aligned}$$

So the inverse demand function is:

$$P^d(Q) = \frac{90 - Q}{5}$$

4.2. Supply

Suppose the supply function (the quantity that firms are willing to supply if the market price is P) is:

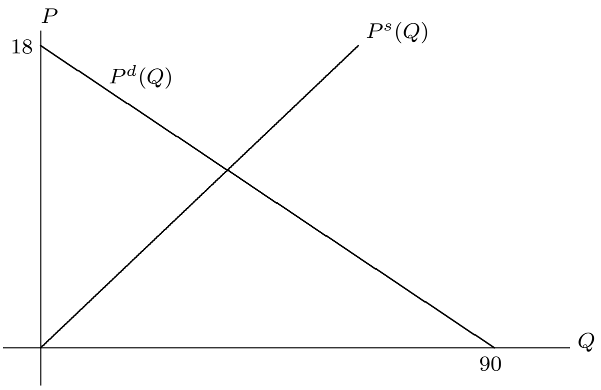
$$Q^s(P) = 4P$$

This is an increasing function (upward-sloping). Firms will supply more if the price is higher. The inverse supply function is:

$$P^s(Q) = \frac{Q}{4}$$

4.3. Market Equilibrium

We can draw the supply and demand functions to show the equilibrium in the market. It is conventional to draw the graph of P against Q – that is, to put P on the vertical axis, and draw the *inverse* supply and demand functions).



The equilibrium in the market is where the supply price equals the demand price:

$$\begin{aligned} P^s(Q) &= P^d(Q) \\ \Longrightarrow \frac{Q}{4} &= \frac{90 - Q}{5} \\ 5Q &= 360 - 4Q \\ Q &= 40 \end{aligned}$$

and so:

$$P = 10$$

EXERCISES 4.7: Suppose that the supply and demand functions in a market are:

$$\begin{aligned} Q^s(P) &= 6P - 10 \\ Q^d(P) &= \frac{100}{P} \end{aligned}$$

(1) Find the inverse supply and demand functions, and sketch them.

(2) Find the equilibrium price and quantity in the market.

4.4. Using Parameters to Specify Functions

In economic applications, we often want to specify the general shape of a function, without giving its exact formula. To do this, we can include *parameters* in the function.

For example, in the previous section, we used a particular demand function:

$$P^d(Q) = \frac{90 - Q}{5}$$

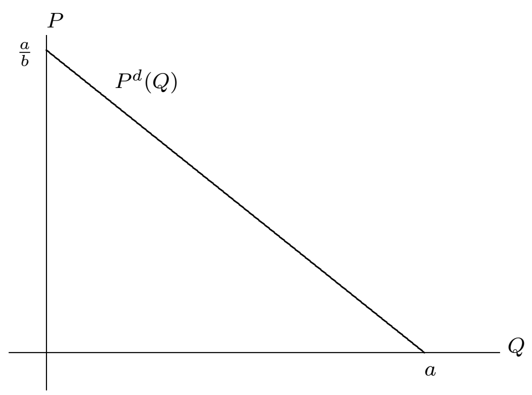
A more general specification would be to write it using two parameters a and b, instead of the numbers:

$$P^d(Q) = \frac{a - Q}{b} \text{ where } a > 0 \text{ and } b > 0$$

This gives us enough information to sketch the general shape of the function:

- it is a downward-sloping straight line (if we write it as
$$P = -\frac{1}{b}Q + \frac{a}{b}$$
we can see that the gradient is negative);

- and we can find the points where it crosses the axes.



EXERCISES 4.8: Suppose that the inverse supply and demand functions in a market are:

$$\begin{aligned} P^d(Q) &= a - Q \\ P^s(Q) &= cQ + d \quad \text{where } a, c, d > 0 \end{aligned}$$

(1) Sketch the functions.

(2) Find the equilibrium quantity in terms of the parameters a, c and d.

(3) What happens if $d > a$?

Further reading and exercises

- *Jacques* §1.3.
- *Anthony & Biggs* §1.

5. Exponential and Logarithmic Functions

5.1. Exponential Functions

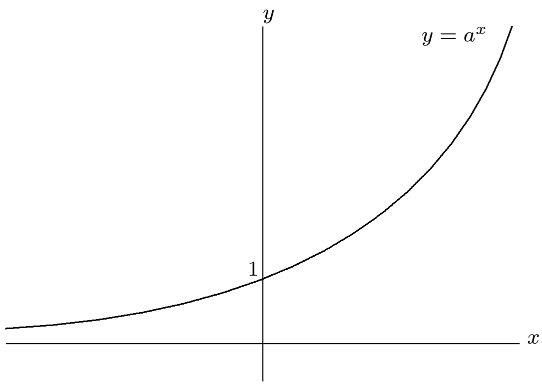
$$f(x) = a^x$$

where a is any positive number, is called an *exponential function*. For example, 3^x and 8.31^x are both exponential functions.

Exponential functions have the same general shape, whatever the value of a. If $a > 1$:

- the y-intercept is 1, because $a^0 = 1$;
- y is positive, and increasing, for all values of x;
- as x gets bigger, y increases very fast (exponentially);
- as x gets more negative, y gets closer to zero (but never actually gets there):

$$\lim_{x \to -\infty} a^x = 0$$



(If $a < 1$, a^x is a decreasing function.)

5.2. Logarithmic Functions

$$g(x) = \log_a x$$

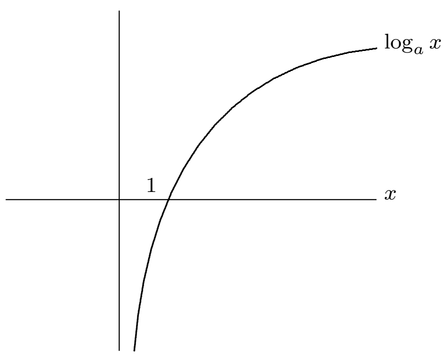
is a logarithmic function, with base a. Remember, from the definition of a logarithm in Chapter 1, that:

$$z = \log_a x \quad \text{is equivalent to} \quad x = a^z$$

In other words, a logarithmic function is the inverse of an exponential function.

Since a logarithmic function is the inverse of an exponential function, we can find the shape of the graph by reflecting the graph above in the line $y = x$. We can see that if $a > 1$:

- Since a^z is always positive, the logarithmic function is only defined for positive values of x.
- It is an increasing function
- Since $a^0 = 1$, $\log_a(1) = 0$
- $\log_a x < 0$ when $0 < x < 1$
- $\log_a x > 0$ when $x > 1$
- $\lim_{x \to 0} \log_a x = -\infty$



EXERCISES 4.9: **Exponential and Logarithmic Functions**

(1) Plot the graph of the exponential function $f(x) = 2^x$, for $-3 \le x \le 3$.

(2) What is (a) $\log_2 2$ (b) $\log_a a$?

(3) Plot the graph of $y = \log_{10} x$ for values of x between 0 and 10.
Hint: The "log" button on most calculators gives you $\log_{10}$.

5.3. *The* Exponential Function

In Chapter 3 we came across the number e: $e \approx 2.71828$. The function

$$e^x$$

is known as *the* exponential function.

5.4. Natural Logarithms

The inverse of the exponential function is the *natural logarithm function*. We could write it as:

$$\log_e x$$

but it is often written instead as:

$$\ln x$$

Note that since e^x and ln are inverse functions:

$$\boxed{\ln(e^x) = x \text{ and } e^{\ln x} = x}$$

5.5. Where the Exponential Function Comes From

Remember (or look again at Chapter 3) where e comes from: it is the limit of a sequence of numbers. For any number r:

$$e^r = \lim_{n\to\infty} \left(1 + \frac{r}{n}\right)^n$$

So, we can think of the exponential function as the limit of a sequence of functions:

$$e^x = \lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n$$

EXERCISES 4.10: **Base e**

(1) Plot the graphs of $1 + x$, $\left(1 + \frac{x}{2}\right)^2$, $\left(1 + \frac{x}{3}\right)^3$, and e^x, for $0 \le x \le 1$.

(2) What is (i) $\ln 1$ (ii) $\ln e$ (iii) $\ln\left(e^{5x}\right)$ (iv) $e^{\ln x^2}$ (v) $e^{\ln 3 + \ln x}$?

(3) Plot the graph of $y = \ln x$ for values of x between 0 and 3.

(4) Sketch the graph of $y = e^x$. From this sketch, work out how to sketch the graphs of $y = e^{-x}$ and $y = e^{3x+1}$.
Hint: There is an *ln* button, and an e^x button, on most calculators.

Further reading and exercises

- *Jacques* §2.4.
- *Anthony & Biggs* §7.1 to §7.4.

6. Economic Examples using Exponential and Logarithmic Functions

From Chapter 3 we know that an initial amount A_0 invested at interest rate i with continuous compounding of interest would grow to be worth

$$A(t) = A_0 e^{it}$$

after t years. We can think of this as an exponential function of time. Exponential functions are used to model the growth of other economic variables over time:

Examples 6.1: A company selling cars expects to increase its sales over future years. The number of cars sold per day after t years is expected to be:

$$S(t) = 5e^{0.08t}$$

(i) How many cars are sold per day now?
When $t = 0$, $S = 5$.

(ii) What is the expected sales rate after (a) 1 year; (b) 5 years (c) 10 years?
$S(1) = 5e^{0.08} = 5.4$.
Similarly, (b) $S(5) = 7.5$ and (c) $S(10) = 11.1$.

(iii) Sketch the graph of the sales rate against time.

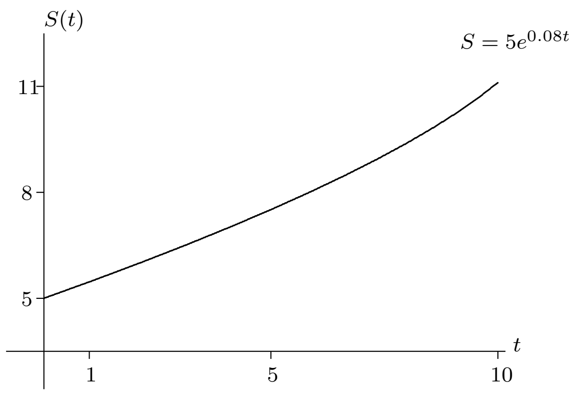
(iv) When will daily sales first exceed 14?

To find the time when $S = 14$, we must solve the equation:

$$14 = 5e^{0.08t}$$

Take (natural) logs of both sides:

$$\begin{aligned} \ln 14 &= \ln 5 + \ln(e^{0.08t}) \\ \ln 14 - \ln 5 &= 0.08t \\ 1.03 &= 0.08t \\ t &= 12.9 \end{aligned}$$



Examples 6.2: Y is the GDP of a country. GDP in year t satisfies (approximately) the equation:

$$Y = ae^{0.03t}$$

(i) What is GDP in year 0?
Putting $t = 0$ in the equation, we obtain: $Y(0) = a$. So the parameter a represents the initial value of GDP.

(ii) What is the percentage change in GDP between year 5 and year 6?
$Y(5) = ae^{0.15} = 1.1618a$ and $Y(6) = ae^{0.18} = 1.1972a$

So the percentage increase in GDP is:

$$100 \times \frac{Y(6) - Y(5)}{Y(5)} = 100\frac{.0354a}{1.1618a} = 3.05$$

The increase is approximately 3 percent.

(iii) What is the percentage change in GDP between year t and year $t+1$?

$$\begin{aligned} 100 \times \frac{Y(t+1) - Y(t)}{Y(t)} &= 100\frac{ae^{0.03(t+1)} - ae^{0.03t}}{ae^{0.03t}} \\ &= 100\frac{ae^{0.03t}(e^{0.03} - 1)}{ae^{0.03t}} \\ &= 100(e^{0.03} - 1) \\ &= 3.05 \end{aligned}$$

So the percentage increase is approximately 3 percent in *every* year.

(iv) Show that the graph of $\ln Y$ against time is a straight line.

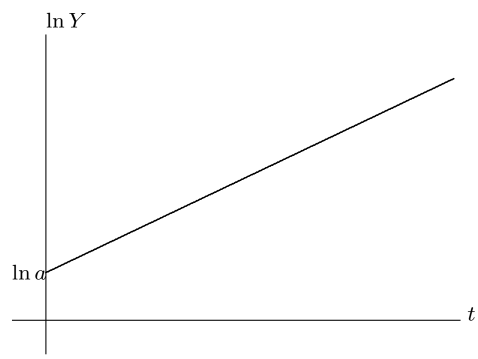
The equation relating Y and t is:

$$Y = ae^{0.03t}$$

Take logs of both sides of this equation:

$$\begin{aligned} \ln Y &= \ln(ae^{0.03t}) \\ &= \ln a + \ln(e^{0.03t}) \\ &= \ln a + 0.03t \end{aligned}$$

So if we plot a graph of $\ln Y$ against t, we will get a straight line, with gradient 0.03, and vertical intercept $\ln a$.



EXERCISES 4.11: **Economic Example using the Exponential Function**

The percentage of a firm's workforce who know how to use its computers increases over time according to:

$$P = 100\left(1 - e^{-0.5t}\right)$$

where t is the number of years after the computers are introduced.

(1) Calculate P for $t = 0, 1, 5$ and 10, and hence sketch the graph of P against t.

(2) What happens to P as $t \to \infty$?

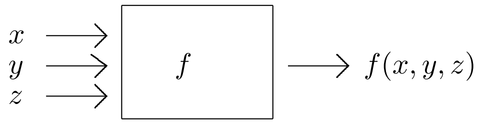
(3) How long is it before 95% of the workforce know how to use the computers?
Hint: You can use the method that we used in the first example, but you will need to rearrange your equation before you take logs of both sides.

Further reading and exercises

- *Jacques* §2.4.

7. Functions of Several Variables

A function can have more than one input:



Examples 7.1: *Functions of Several Variables*

(i) For the function $F(x,y) = x^2 + 2y$

(1) $F(3,2) = 3^2 + 4 = 13$

(2) $F(-1,0) = (-1)^2 + 0 = 1$

(ii) The production function of a firm is: $Y(K,L) = 3K^{0.4}L^{0.6}$
In this equation, Y is the number of units of output produced with K machines and L workers.

(1) How much output is produced with one machine and 4 workers?
$Y(1,4) = 3 \times 4^{0.6} = 6.9$

(2) If the firm has 3 machines, how many workers does it need to produce 10 units of output?
When the firm has 3 machines and L workers, it produces:

$$\begin{aligned} Y &= 3 \times 3^{0.4}L^{0.6} \\ &= 4.66L^{0.6} \end{aligned}$$

So if it wants to produce 10 units we need to find the value of L for which:

$$\begin{aligned} 10 &= 4.66L^{0.6} \\ \Longrightarrow L^{0.6} &= 2.148 \\ \Longrightarrow L &= 2.148^{\frac{1}{0.6}} \\ &= 3.58 \end{aligned}$$

It needs 3.58 (possibly 4) workers.[1]

Exercises 4.12: Functions of Several Variables

(1) For the function $f(x,y,z) = 2x + 6y - 7 + z^2$ evaluate (i) $f(0,0,0)$ (ii) $f(5,3,1)$

(2) A firm has production function: $Y(K,L) = 4K^2L^3$, where K is the number of units of capital, and L is the number of workers, that it employs. How much output does it produce with 2 workers and 3 units of capital? If it has 5 units of capital, how many workers does it need to produce 6400 units of output?

[1]If you are unsure about manipulating indices as we have done here, refer back to Chapter 1.

7.1. Drawing Functions of Two Variables: Isoquants

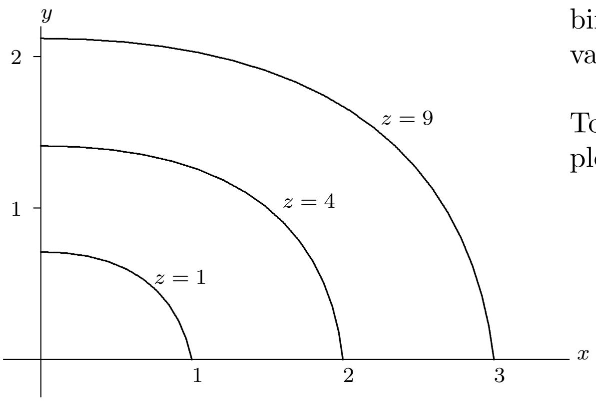
Suppose we have a function of two variables:

$$z = f(x, y)$$

To draw a graph of the function, you could think of (x, y) as a point on a horizontal plane, like the co-ordinates of a point on a map, and z as a distance above the horizontal plane, corresponding to the height of the land at the point (x, y). So the graph of the function will be a "surface" in 3-dimensions. See *Jacques* §5.1 for an example of a picture of a surface.

Since you really need 3 dimensions, graphs of functions of 2 variables are difficult to draw (and graphs of functions of 3 or more variables are impossible). But for functions of two variables we can draw a diagram like a contour map:

EXAMPLES 7.2: $z = x^2 + 2y^2$



The lines are *isoquants*: they show all the combinations of x and y that produce a particular value of the function, z.

To draw the isoquant for $z = 4$, for example:

- Write down the equation:
$$x^2 + 2y^2 = 4$$
- Make y the subject:
$$y = \sqrt{\frac{4 - x^2}{2}}$$
- Draw the graph of y against x.

7.2. Economic Application: Indifference curves

Suppose that a consumer's preferences over 2 goods are represented by the utility function:

$$U(x_1, x_2) = x_1^3 x_2^2$$

In this case an isoquant is called an *indifference curve*. It shows all the bundles (x_1, x_2) that give the same amount of utility: k, for example. So an indifference curve is represented by the equation:

$$x_1^3 x_2^2 = k$$

where k is a constant. To draw the indifference curves, make x_2 the subject:

$$x_2 = \left(\frac{k}{x_1^3}\right)^{\frac{1}{2}}$$

Then you can sketch them for different values of k (putting x_2 on the vertical axis).

EXERCISES 4.13: Sketch the indifference curves for a consumer whose preferences are represented by the utility function: $U(x_1, x_2) = x_1^{0.5} + x_2$

Further reading and exercises

- *Jacques* §5.1.
- *Anthony & Biggs* §11.1.

8. Homogeneous Functions, and Returns to Scale

EXERCISES 4.14: A firm has production function $Y(K, L) = 4K^{\frac{1}{3}}L^{\frac{2}{3}}$.

(1) How much output does it produce with

(1) 2 workers and 5 units of capital?

(2) 4 workers and 10 units of capital?

(3) 6 workers and 15 units of capital?

In this exercise, you should have found that when the firm doubles its inputs, it doubles its output, and when it trebles the inputs, it trebles the output. That is, the firm has constant return to scale. In fact, for the production function

$$Y(K, L) = 4K^{\frac{1}{3}}L^{\frac{2}{3}}$$

if the inputs are multiplied by any positive number λ:

$$\begin{aligned}
Y(\lambda K, \lambda L) &= 4(\lambda K)^{\frac{1}{3}}(\lambda L)^{\frac{2}{3}} \\
&= 4\lambda^{\frac{1}{3}}K^{\frac{1}{3}}\lambda^{\frac{2}{3}}L^{\frac{2}{3}} = \lambda^{\frac{1}{3}+\frac{2}{3}}4K^{\frac{1}{3}}L^{\frac{2}{3}} = \lambda 4K^{\frac{1}{3}}L^{\frac{2}{3}} \\
&= \lambda Y(K, L)
\end{aligned}$$

– the output is multiplied by λ. We say that the function is *homogeneous of degree* 1.

A function $f(x, y)$ is said to be homogeneous of degree n if:
$$f(\lambda x, \lambda y) = \lambda^n f(x, y)$$
for all positive numbers λ.

We can extend this for functions of more than 2 variables in the obvious way.

EXAMPLES 8.1: *Homogeneous Functions*

(i) $f(x, y, z) = 2x^{\frac{1}{2}} + 3y^{\frac{1}{2}} + z^{\frac{1}{2}}$
For this function:

$$\begin{aligned}
f(\lambda x, \lambda y, \lambda z) &= 2(\lambda x)^{\frac{1}{2}} + 3(\lambda y)^{\frac{1}{2}} + (\lambda z)^{\frac{1}{2}} \\
&= 2\lambda^{\frac{1}{2}}x^{\frac{1}{2}} + 3\lambda^{\frac{1}{2}}y^{\frac{1}{2}} + \lambda^{\frac{1}{2}}z^{\frac{1}{2}} \\
&= \lambda^{\frac{1}{2}}\left(2x^{\frac{1}{2}} + 3y^{\frac{1}{2}} + z^{\frac{1}{2}}\right) = \lambda^{\frac{1}{2}}f(x, y, z)
\end{aligned}$$

So it is homogeneous of degree $\frac{1}{2}$.

(ii) The function $g(x_1, x_2) = x_1 + x_2^3$ is not homogeneous because:

$$g(\lambda x_1, \lambda x_2) = \lambda x_1 + (\lambda x_2)^3 = \lambda x_1 + \lambda^3 x_2^3$$

which cannot be written in the required form.

(iii) The production function $F(K, L) = 3KL$ is homogeneous of degree 2:

$$F(\lambda K, \lambda L) = 3\lambda K \lambda L = \lambda^2 F(K, L)$$

This means that if $K$and L are *increased* by the same factor $\lambda > 1$, then output increases by more:

$$F(\lambda K, \lambda L) = \lambda^2 F(K, L) > \lambda F(K, L)$$

So this production function has *increasing returns to scale.*

From examples like this we can see that:

A production function that is homogeneous of degree n has:
constant returns to scale if $n = 1$
increasing returns to scale if $n > 1$
decreasing returns to scale if $n < 1$

Exercises 4.15: **Homogeneous Functions**

(1) Determine whether each of the following functions is homogeneous, and if so, of what degree:

(a) $f(x, y) = 5x^2 + 3y^2$

(b) $g(z, t) = t(z + 1)$

(c) $h(x_1, x_2) = x_1^2(2x_2^3 - x_1^3)$

(d) $F(x, y) = 8x^{0.7}y^{0.9}$

(2) Show that the production function $F(K, L) = aK^c + bL^c$ is homogeneous. For what parameter values does it have constant, increasing and decreasing returns to scale?

Further reading and exercises

- *Jacques* §2.3.
- *Anthony & Biggs* §12.4.

Solutions to Exercises in Chapter 4

EXERCISES 4.1:

(1) $f(2) = 5$, and $f(-4) = 17$
(2) $g(x) = 0 \Rightarrow x = 1$
(3) $C(4) = 44$

EXERCISES 4.2:

(1) (a) $f(x) = x(x-1)(x-2) \Rightarrow f(0) = 0, f(1) = 0, f(2) = 0$
 (b) $f(x) < 0,\ x < 0$
 $f(x) > 0,\ 0 < x < 1$
 $f(x) < 0,\ 1 < x < 2$
 $f(x) > 0,\ x > 2$
(2) (a) 4
 (b) $g(x) = (x^2 - 1)(4 - x^2) \Rightarrow$
 $g(-2) = 0$
 $g(-1) = 0$
 $g(1) = g(2) = 0$
 (c) $x < -2 : g(x) < 0$
 $-2 < x < -1 : g(x) > 0$
 $-1 < x < 1 : g(x) < 0$
 $1 < x < 2 : g(x) > 0$
 $x > 2 : g(x) < 0$

EXERCISES 4.3:

EXERCISES 4.4:

(1) (a) $\lim_{x\to\infty} f(x) = -\infty$
 (b) $\lim_{x\to-\infty} f(x) = -\infty$
 (c) $\lim_{x\to 0} f(x) = 2$
(2) Decreasing, $\lim_{x\to\infty} y = 3$
(3) (a) Increasing function.
 (b) $\lim_{x\to\infty} g(x) = 1$
 (c) $\lim_{x\to 0} g(x) = -\infty$

EXERCISES 4.5:

(1) $f(g(2)) = -27$, $g(f(2)) = -9$
(2) $g(h(1)) = \frac{4}{3}$, $h(g(1)) = 5$
(3) $k(m(3)) = 3$, $m(k(3)) = 3$. *Cube root* is the inverse of *cube*.
(4) $g(f(x)) = 2(x+1)^2$
 $f(g(x)) = 2x^2 + 1$
(5) $h(k(x)) = \frac{5x}{2x+1}$, $k(h(x)) = \frac{x+2}{5}$

EXERCISES 4.6:

(1) $f^{-1}(y) = \frac{y-7}{8}$
(2) $g^{-1}(y) = 6 - 2y$
(3) $h^{-1}(y) = \frac{1-4y}{y} = \frac{1}{y} - 4$
(4) $k^{-1}(y) = \sqrt[3]{y}$

EXERCISES 4.7:

(1) $P^S = \frac{Q+10}{6}$, $P^D = \frac{100}{Q}$
(2) $Q = 20$, $P = 5$

EXERCISES 4.8:

(1)
(2) $Q = \frac{a-d}{c+1}$, $P = \frac{ac+d}{c+1}$.
(3) $d > a \Rightarrow$ there is no equilibrium in the market.

EXERCISES 4.9:

(1)
(2) (a) 1
 (b) 1

EXERCISES 4.10:

(1)
(2) (a) 0
 (b) 1
 (c) $5x$
 (d) x^2
 (e) $3x$

EXERCISES 4.11:

(1) $t = 0$, $P = 0$
 $t = 1$, $P \approx 39.35$
 $t = 5$, $P \approx 91.79$
 $t = 10$, $P \approx 99.33$
(2) $\lim_{t\to\infty} P = 100$
(3) $t \approx 6$

EXERCISES 4.12:

(1) $f(0,0,0) = -7$, $f(5,3,1) = 22$
(2) $Y(3,2) = 288$, $L = 4$

EXERCISES 4.13:

EXERCISES 4.14:

(1) $Y(5,2) \approx 10.86$
(2) $Y(10,4) \approx 21.72$
(3) $Y(15,6) \approx 32.57$

EXERCISES 4.15:

(1) (a) 2
 (b) No
 (c) 5
 (d) 1.6
(2) $F(\lambda K, \lambda L) = a\lambda^c K^c + b\lambda^c L^c = \lambda^c F(K,L)$. Increasing $c > 1$, constant $c = 1$, decreasing $c < 1$

[2]This Version of Workbook Chapter 4: September 25, 2014

Worksheet 4: Functions

Quick Questions

(1) If $f(x) = 2x - 5$, $g(x) = 3x^2$ and $h(x) = \frac{1}{1+x}$:
  (a) Evaluate: $h\left(\frac{1}{3}\right)$ and $g(h(2))$
  (b) Solve the equation $h(x) = \frac{3}{4}$
  (c) Find the functions $h(f(x))$, $f^{-1}(x)$, $h^{-1}(x)$ and $f(g(x))$.

(2) A country's GDP grows according to the equation $Y(t) = Y_0 e^{gt}$.
(Y_0 and g are parameters; g is the growth rate.)
  (a) What is GDP when $t = 0$?
  (b) If $g = 0.05$, how long will it take for GDP to double?
  (c) Find a formula for the time taken for GDP to double, in terms of the growth rate g.

(3) Consider the function: $g(x) = 1 - e^{-x}$.
  (a) Evaluate $g(0)$, $g(1)$ and $g(2)$.
  (b) Is it an increasing or a decreasing function?
  (c) What is $\lim_{x\to\infty} g(x)$?
  (d) Use this information to sketch the function for $x \geq 0$.

(4) The inverse supply and demand functions for a good are: $P^s(Q) = 1 + Q$ and $P^d(Q) = a - bQ$, where a and b are parameters. Find the equilibrium quantity, in terms of a and b. What conditions must a and b satisfy if the equilibrium quantity is to be positive?

(5) Are the following functions homogeneous? If so, of what degree?
(a) $g(z, t) = 2t^2 z$ (b) $h(a, b) = \sqrt[3]{a^2 + b^2}$

Longer Questions

(1) The supply and demand functions for beer are given by:

$$q^s(p) = 50p \quad \text{and} \quad q^d(p) = 100\left(\frac{12}{p} - 1\right)$$

  (a) How many bottles of beer will consumers demand if the price is 5?
  (b) At what price will demand be zero?
  (c) Find the equilibrium price and quantity in the market.
  (d) Determine the inverse supply and demand functions $p^s(q)$ and $p^d(q)$.
  (e) What is $\lim_{q\to\infty} p^d(q)$?
  (f) Sketch the inverse supply and demand functions, showing the market equilibrium.

The technology for making beer changes, so that the unit cost of producing a bottle of beer is 1, whatever the scale of production. The government introduces a tax on the production of beer, of t per bottle. After these changes, the demand function remains the same, but the new inverse supply function is:

$$p^s(q) = 1 + t$$

(g) Show the new supply function on your diagram.
(h) Find the new equilibrium price and quantity in the market, in terms of the tax, t.
(i) Find a formula for the total amount of tax raised, in terms of t, and sketch it for $0 \leq t \leq 12$. Why does it have this shape?

(2) A firm has m machines and employs n workers, some of whom are employed to maintain the machines, and some to operate them. One worker can maintain 4 machines, so the number of production workers is $n - \frac{m}{4}$. The firm's produces:

$$Y(n, m) = \left(n - \frac{m}{4}\right)^{\frac{2}{3}} m^{\frac{1}{3}}$$

units of output provided that $n > \frac{m}{4}$; otherwise it produces nothing.

(a) Show that the production function is homogeneous. Does the firm have constant, increasing or decreasing returns to scale?
(b) In the short-run, the number of machines is fixed at 8.
  (i) What is the firm's short-run production function $y(n)$?
  (ii) Sketch this function. Does the firm have constant, increasing, or decreasing returns to labour?
  (iii) Find the inverse of this function and sketch it. Hence or otherwise determine how many workers the firm needs if it is to produce 16 units of output.
(c) In the long-run, the firm can vary the number of machines.
  (i) Draw the isoquant for production of 16 units of output.
  (ii) What happens to the isoquants when the number of machines gets very large?
  (iii) Explain why this happens. Why would it not be sensible for the firm to invest in a large number of machines?

CHAPTER 5

Differentiation

Differentiation is a technique that enables us to find out how a function changes when its argument changes. It is an essential tool in economics. If you have done A-level maths, much of the maths in this chapter will be revision, but the economic applications may be new. We use the **first and second derivatives** to work out the shapes of simple functions, and find **maximum and minimum points**. These techniques are applied to **cost, production and consumption functions**. **Concave and convex functions** are important in economics.

1. What is a Derivative?

1.1. Why We Are Interested in the Gradient of a Function

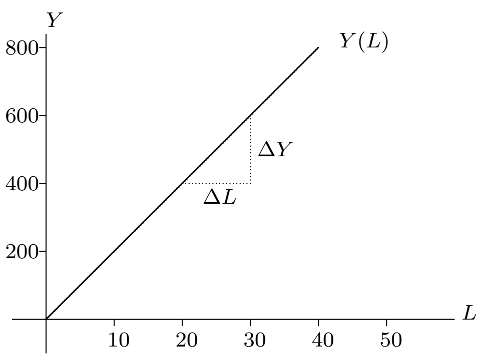
In many economic applications we want to know how a function changes when its argument changes – so we need to know its gradient.

This graph shows the *production function* $Y(L)$ of a firm: if it employs L workers it produces Y units of output.

$$\text{gradient} = \frac{\Delta Y}{\Delta L} = \frac{200}{10} = 20$$

The gradient represents the *marginal product of labour* - the firm produces 20 more units of output for each extra worker it employs.

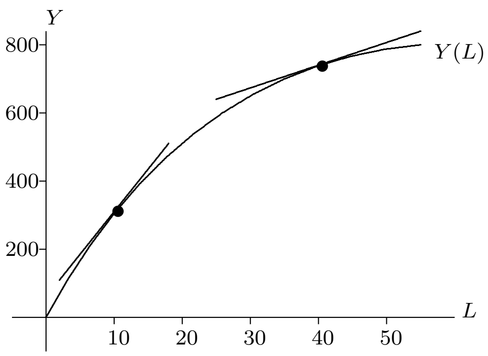
The steeper the production function, the greater is the marginal product of labour.



With this production function, the gradient changes as we move along the graph.

The gradient of a curve at a particular point is the gradient of the *tangent*.

The gradient, and hence the marginal product of labour, is higher when the firm employs 10 workers than when it employs 40 workers.



1.2. Finding the Gradient of a Function

If we have a linear function: $y(x) = mx + c$

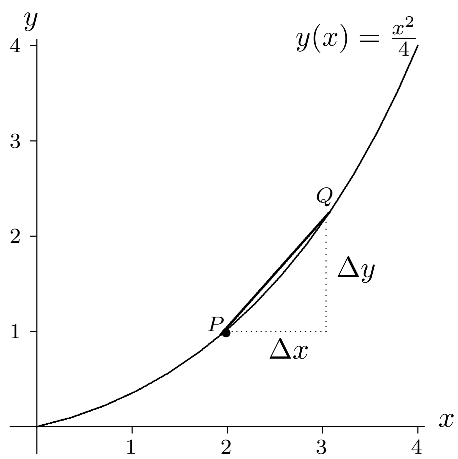
we know immediately that its graph is a straight line, with gradient equal to m (Chapter 2).

But the graph of the function: $y(x) = \dfrac{x^2}{4}$

is a curve, so its gradient changes. To find the gradient at a particular point, we could try to draw the graph accurately, draw a tangent, and measure its gradient. But we are unlikely to get an accurate answer. Alternatively:

EXAMPLES 1.1: For the function $y(x) = \frac{x^2}{4}$

(i) What is the gradient at the point where $x = 2$?



An approximation to the gradient at P is:

$$\text{gradient of } PQ = \frac{y(3) - y(2)}{3 - 2} = 1.25$$

For a better approximation take a point nearer to P:

$$\frac{y(2.5) - y(2)}{2.5 - 2} = 1.125$$

or much nearer:

$$\frac{y(2.001) - y(2)}{2.001 - 2} = 1.00025$$

So we can see that:

- In general the gradient at P is given by: $\displaystyle\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$
- In this case the gradient at P is 1.

(ii) What is the gradient at the point where at the point where $x = 3$?
Applying the same method:

$$\begin{aligned}
\frac{y(3.1) - y(3)}{3.1 - 3} &= 1.525 \\
\frac{y(3.01) - y(3)}{3.01 - 3} &= 1.5025 \\
\frac{y(3.001) - y(3)}{3.001 - 3} &= 1.50025 \\
&\cdots \\
\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} &= 1.5
\end{aligned}$$

1.3. The Derivative

In the example above we found the gradient of a function by working out $\lim\limits_{\Delta x \to 0} \dfrac{\Delta y}{\Delta x}$.

- The shorthand for: $\lim\limits_{\Delta x \to 0} \dfrac{\Delta y}{\Delta x}$ is $\dfrac{dy}{dx}$
- $\dfrac{dy}{dx}$ is pronounced "dee y by dee x"
- $\dfrac{dy}{dx}$ measures the gradient
- $\dfrac{dy}{dx}$ is called the *derivative* of y

We found the derivative of the function $y(x) = \dfrac{x^2}{4}$ at particular points:

$$\begin{aligned} \frac{dy}{dx} &= 1 \quad \text{at } x = 2 \\ \frac{dy}{dx} &= 1.5 \quad \text{at } x = 3 \end{aligned}$$

EXERCISES 5.1: Use the method above to find the derivative of the function $y(x) = \frac{3}{4}x^3$ at the point where $x = 2$

Further Reading

- *Jacques* §4.1 introduces derivatives slowly and carefully.
- *Anthony & Biggs* §6.2 is brief.

2. Finding the Derivative of the Function $y = x^n$

Although we could use the method in the previous section to find the derivative of any function at any particular point, it requires tedious calculations. Instead there is a simple formula, which we can use to find the derivative of any function of the form $y = x^n$, at any point.

If $y = x^n$, then $\dfrac{dy}{dx} = nx^{n-1}$

So the derivative of the function x^n is itself a function: nx^{n-1}
and we just have to evaluate this to find the gradient at any point.

EXAMPLES 2.1: For the function $y = x^3$:

(i) Find the derivative.

Applying the formula, with $n = 3$: $\dfrac{dy}{dx} = 3x^2$

(ii) Find the gradient at the points: $x = 1$, $x = -2$ and $x = 10$

When $x = 1$ $\dfrac{dy}{dx} = 3 \times 1^2 = 3.$ So the gradient is 3.

When $x = -2$ $\dfrac{dy}{dx} = 3 \times (-2)^2 = 12$

When $x = 10$ $\dfrac{dy}{dx} = 3 \times (10)^2 = 300$

Some Notation: We could write this as: $\left.\dfrac{dy}{dx}\right|_{x=10} = 300$

The process of finding the derivative function is called *differentiation*

EXAMPLES 2.2: Differentiate the following functions.

(i) $y(x) = x^9$

$$\frac{dy}{dx} = 9x^8$$

(ii) $Q(P) = Q^{\frac{1}{2}}$

$$\frac{dQ}{dP} = \frac{1}{2}Q^{-\frac{1}{2}} = \frac{1}{2Q^{\frac{1}{2}}}$$

(iii) $Y(t) = t^{1.2}$

$$\frac{dY}{dt} = 1.2t^{0.2}$$

(iv) $F(L) = L^{\frac{2}{3}}$

$$\frac{dF}{dL} = \frac{2}{3}L^{-\frac{1}{3}} = \frac{2}{3L^{\frac{1}{3}}}$$

(v) $y = \dfrac{1}{x^2}$

In this case $n = -2$:

$$y = x^{-2} \quad \Rightarrow \quad \frac{dy}{dx} = -2x^{-3} = -\frac{2}{x^3}$$

We have not proved that the formula is correct, but for two special cases we can verify it:

EXAMPLES 2.3: *The special cases $n = 0$ and $n = 1$*

(i) Consider the linear function: $y = x$
We know (from Chapter 2) that it is a straight line with gradient 1.

We could write this function as:	$y(x)$	$= x^1$
and apply the formula:	$\dfrac{dy}{dx}$	$= 1 \times x^0 = 1$

So the formula also tells us that the gradient of $y = x$ is 1 for *all* values of x.

(ii) Consider the constant function: $y(x) = 1$
We know (Chapter 2) that it is a straight line with gradient zero and y-intercept 1.

We could write this function as:	$y(x)$	$= x^0$
and apply the formula:	$\dfrac{dy}{dx}$	$= 0 \times x^{-1} = 0$

So the formula also tells us that the gradient is zero.

EXERCISES 5.2: **Using the formula to find derivatives**

(1) If $y = x^4$, find $\dfrac{dy}{dx}$ and hence find the gradient of the function when $x = -2$

(2) Find the derivative of $y = x^{\frac{3}{2}}$. What is the gradient at the point where $x = 16$?

(3) If $y = x^7$ what is $\left.\dfrac{dy}{dx}\right|_{x=1}$?

(4) Find the derivative of the function $z(t) = t^2$ and evaluate it when $t = 3$

(5) Differentiate the supply function $Q(P) = P^{1.7}$

(6) Differentiate the utility function $u(y) = y^{\frac{5}{6}}$ and hence find its gradient when $y = 1$

(7) Find the derivative of the function $y = \dfrac{1}{x^3}$, and the gradient when $x = 2$.

(8) Differentiate the demand function $Q = \dfrac{1}{P}$
What happens to the derivative (a) as $P \to 0$ (b) as $P \to \infty$?

Further Reading and exercises

- *Jacques* §4.1.
- *Anthony & Biggs* §6.2.
- For more examples, use an A-level pure maths textbook.

3. Optional Section: Where Does the Formula Come From?

Remember the method we used in section 1 to find the gradient of a function $y(x)$ at the point $x = 2$. We calculated:

$$\frac{y(2+h) - y(2)}{(2+h) - 2}$$

for values of h such as 0.1, 0.01, 0.001, getting closer and closer to zero.
In other words, we found:

$$\left.\frac{dy}{dx}\right|_{x=2} = \lim_{h \to 0} \frac{y(2+h) - y(2)}{(2+h) - 2} = \lim_{h \to 0} \frac{y(2+h) - y(2)}{h}$$

More generally, we could do this for any value of x (not just 2):

$$\boxed{\frac{dy}{dx} = \lim_{h \to 0} \frac{y(x+h) - y(x)}{h}}$$

So, for the function $y = x^2$:

$$\begin{aligned} \frac{dy}{dx} &= \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h} \\ &= \lim_{h \to 0} \frac{x^2 + 2xh + h^2 - x^2}{h} \\ &= \lim_{h \to 0} \frac{2xh + h^2}{h} \\ &= \lim_{h \to 0} (2x + h) \\ &= 2x \end{aligned}$$

We have *proved* that if $y = x^2$, $\dfrac{dy}{dx} = 2x$.

This method is called *Differentiation from First Principles.* You could do it for other integer values of n to check that the formula gives the correct answer. You could also apply it to some functions that are not of the simple form $y = x^n$ as in the third exercise below.

Exercises 5.3: Using the method of Differentiation from First Principles:

(1) Prove that if $y = x^3$, $\dfrac{dy}{dx} = 3x^2$

(2) Prove that if $y = \dfrac{1}{x}$, $\dfrac{dy}{dx} = -\dfrac{1}{x^2}$

(3) Find the derivative of $y = 3x^2 + 5x + 1$

Further Reading

- *Anthony & Biggs* §6.1.
- Most A-level pure maths textbooks give a general proof of the formula for the derivative of x^n, by *Differentiation from First Principles*, for integer values of n.
- The proof for fractional values is a bit harder. See, for example, *Simon & Blume.*

4. Differentiating More Complicated Functions

The simple rule for differentiating $y = x^n$ can be easily extended so that we can differentiate other functions, such as polynomials. First it is useful to have some alternative notation:

> The derivative of the function $y(x)$ can be written as $y'(x)$ instead of $\dfrac{dy}{dx}$

Examples 4.1: For the function $y(x) = x^2$, we can write:

Either:	$\dfrac{dy}{dx} = 2x$	and	$\left.\dfrac{dy}{dx}\right\|_{x=5} = 10$
Or:	$y'(x) = 2x$	and	$y'(5) = 10$

You can see that the notation $y'(x)$ (read "y prime x") emphasizes that the derivative is a function, and will often be neater, especially when we evaluate the derivative at particular points. It enables us to write the following rules neatly.

Some rules for differentiation

- If $f(x) = a$ $\quad f'(x) = 0$
- If $f(x) = ag(x)$ $\quad f'(x) = ag'(x)$
- If $f(x) = g(x) \pm h(x)$ $\quad f'(x) = g'(x) \pm h'(x)$

(a is a constant, and f, g and h are functions.)

The first rule is obvious: the graph of a constant function is a horizontal line with zero gradient, so its derivative must be zero for all values of x. You could prove all of these rules using the method of differentiation from first principles in the previous section.

Examples 4.2: Differentiate the following functions.

(i) $f(x) = 7$
By the first rule:

$$f'(x) = 0$$

(ii) $y(x) = 4x^2$
We know the derivative of x^2 is $2x$, so using the second rule:

$$y'(x) = 4 \times 2x = 8x$$

(iii) $z(x) = x^3 - x^4$
Here we can use the third rule:

$$z'(x) = 3x^2 - 4x^3$$

(iv) $g(x) = 6x^2 + 3x + 5$

$$g'(x) = 6 \times 2x + 3 \times 1 + 0 = 12x + 3$$

(v) $h(x) = \dfrac{7}{x^3} - 5x + 4x^3$

$$\begin{aligned} h(x) &= 7x^{-3} - 5x + 4x^3 \\ h'(x) &= 7 \times (-3x^{-4}) - 5 + 4 \times 3x^2 \\ &= -\frac{21}{x^4} - 5 + 12x^2 \end{aligned}$$

(vi) $P(Q) = 5Q^{1.2} - 4Q + 10$

$$P'(Q) = 6Q^{0.2} - 4$$

(vii) $Y(t) = k\sqrt{t}$ (where k is a parameter)

$$\begin{aligned} Y(t) &= kt^{\frac{1}{2}} \\ Y'(t) &= k \times \tfrac{1}{2}t^{-\frac{1}{2}} \\ &= \frac{k}{2t^{\frac{1}{2}}} \end{aligned}$$

(viii) $F(x) = (2x+1)(1-x^2)$

First multiply out the brackets:

$$\begin{aligned} F(x) &= 2x + 1 - 2x^3 - x^2 \\ F'(x) &= 2 - 6x^2 - 2x \end{aligned}$$

EXAMPLES 4.3:

(i) Differentiate the quadratic function $f(x) = 3x^2 - 6x + 1$, and hence find its gradient at the points $x = 0$, $x = 1$ and $x = 2$.

$f'(x) = 6x - 6$, and the gradients are $f'(0) = -6$, $f'(1) = 0$, and $f'(2) = 6$.

(ii) Find the gradient of the function $g(y) = y - \frac{1}{y}$ when $y = 1$.

$g'(y) = 1 + \dfrac{1}{y^2}$, so $g'(1) = 2$.

EXERCISES 5.4: **Differentiating more complicated functions**

(1) If $f(x) = 8x^2 - 7$, what is (a) $f'(x)$ (b) $f'(2)$ (c) $f'(0)$?

(2) Differentiate the functions (a) $u(y) = 10y^4 - y^2$ (b) $v(y) = 7y + 5 - 6y^2$

(3) If $Q(P) = \frac{20}{P} - \frac{10}{P^2}$, what is $Q'(P)$?

(4) Find the gradient of $h(z) = z^3(z+4)$ at the point where $z = 2$.

(5) If $y = t - 8t^{\frac{3}{2}}$, what is $\frac{dy}{dt}$?

(6) Find the derivative of $g(x) = 3x^2 + 2 - 8x^{-\frac{1}{4}}$ and evaluate it when $x = 1$.

(7) Differentiate $y(x) = 12x^4 + 7x^3 - 4x^2 - 2x + 8$

(8) If $F(Y) = (Y-1)^2 + 2Y(1+Y)$, what is $F'(1)$?

Further Reading and Exercises

- *Jacques* §4.2.
- A-level pure maths textbooks have lots of practice exercises.

5. Economic Applications

The derivative tells you the gradient of a function $y(x)$. That means it tells you the *rate of change* of the function: how much y changes if x increases a little. So, it has lots of applications in economics:

- How much would a firm's output increase if the firm increased employment?
- How much do a firm's costs increase if it increases production?
- How much do consumers increase their consumption if their income increases?

Each of these questions can be answered by finding a derivative.

5.1. Production Functions

If a firm produces $Y(L)$ units of output when the number of units of labour employed is L, the derivative of the production function is the marginal product of labour:

$$\text{MPL} = \frac{dY}{dL}$$

(Units of labour could be workers, or worker-hours, for example.)

EXAMPLES 5.1: Suppose that a firm has production function $Y(L) = 60L^{\frac{1}{2}}$

The marginal product of labour is: $\dfrac{dY}{dL} = 60\times\ \frac{1}{2}L^{-\frac{1}{2}} = \dfrac{30}{L^{\frac{1}{2}}}$

If we calculate the marginal product of labour when $L = 1$, 4, and 9 we get:

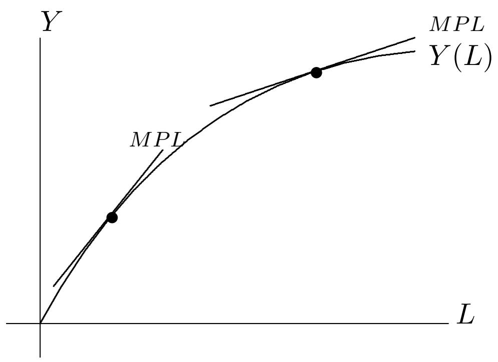
$$\left.\frac{dY}{dL}\right|_{L=1} = 30, \quad \left.\frac{dY}{dL}\right|_{L=4} = 15, \quad \left.\frac{dY}{dL}\right|_{L=9} = 10$$

These figures suggest that the firm has diminishing returns to labour.

If we look again at

$$\frac{dY}{dL} = \frac{30}{L^{\frac{1}{2}}}$$

we can see that the marginal product of labour falls as the labour input increases.



5.2. Cost Functions

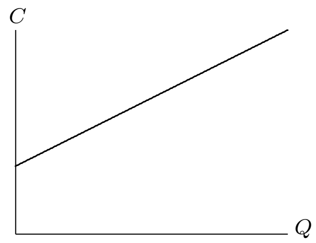
A cost function specifies the total cost $C(Q)$ for a firm of producing a quantity Q of output, so the derivative tells you how much an additional unit of output adds to costs:

> The derivative of the total cost function $C(Q)$ is the marginal cost: $\text{MC} = \dfrac{dC}{dQ}$

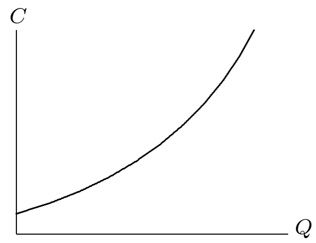
EXAMPLES 5.2: Consider the following total cost functions:

(i) $C(Q) = 4Q + 7$
Differentiating the cost function: MC=4. So marginal cost is constant for all levels of output Q. The cost function is an upward-sloping straight line.

(ii) $C(Q) = 3Q^2 + 4Q + 1$
In this case, MC=$6Q + 4$. Looking at this expression for MC, we can see that the firm has increasing marginal cost – the higher the level of output, the more it costs to make an additional unit.



(i) Constant marginal cost



(ii) Increasing marginal cost

5.3. Consumption Functions

In macroeconomics the consumption function $C(Y)$ specifies how aggregate consumption C depends on national income Y.

> The derivative of the consumption function $C(Y)$ is the marginal propensity to consume:
> $$\text{MPC} = \frac{dC}{dY}$$

The marginal propensity to consume tell us how responsive consumer spending would be if, for example, the government were increase income by reducing taxes.

EXERCISES 5.5: **Economic Applications**

(1) Find the marginal product of labour for a firm with production function $Y(L) = 300L^{\frac{2}{3}}$. What is the MPL when it employs 8 units of labour?

(2) A firm has cost function $C(Q) = a + 2Q^k$, where a and k are positive parameters. Find the marginal cost function. For what values of k is the firm's marginal cost (a) increasing (b) constant (c) decreasing?

(3) If the aggregate consumption function is $C(Y) = 10 + Y^{0.9}$, what is (a) aggregate consumption, and (b) the marginal propensity to consume, when national income is 50?

Further Reading and Exercises

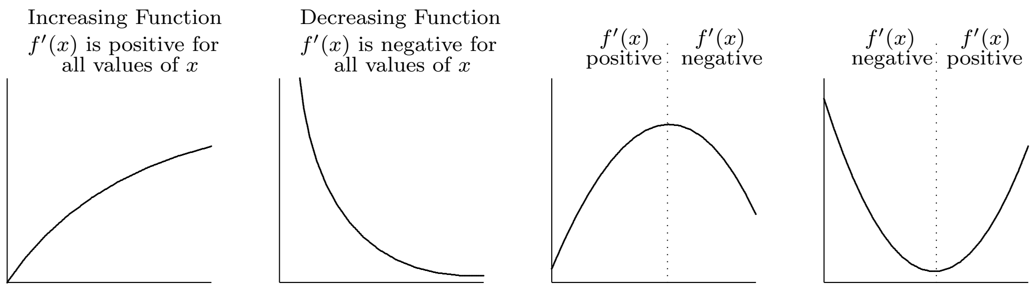
- *Jacques* §4.3.

6. Finding Stationary Points

6.1. The Sign of the Gradient

Since the derivative $f'(x)$ of a function is the gradient, we can look at the derivative to find out which way the function slopes - that is, whether the function is increasing or decreasing for different values of x.

$f'(x) > 0$	f is increasing at x
$f'(x) < 0$	f is decreasing at x



(i) (ii) (iii) (iv)

6.2. Stationary Points

In diagram (iii) there is one point where the gradient is zero. This is where the function reaches a *maximum*. Similarly in diagram (iv) the gradient is zero where the function reaches a *minimum*.

A point where $f'(x) = 0$ is called a *stationary point*, or a *turning point*, or a *critical point*, of the function.

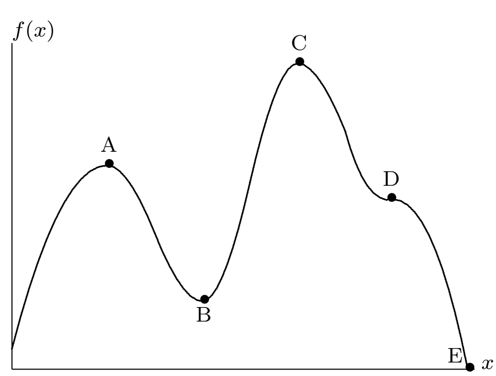
If we want to know where a function reaches a maximum or minimum, we can try to do it by looking for stationary points. But we have to be careful: not all stationary points are maxima or minima, and some maxima and minima are not stationary points. For example:

This function has four stationary points:

- a *local maximum* at A and C
- a *local minimum* at B
- and a stationary point which is neither a max or a min at D. This is called a *point of inflection*.

The *global maximum* of the function (over the range of x-values we are looking at here) is at C.

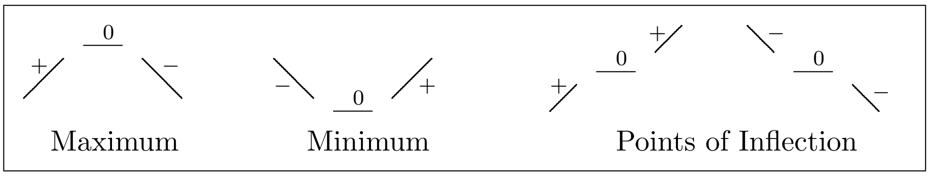
The *global minimum* of the function is at E, which is not a stationary point.



Note that there may not be a global maximum and/or minimum; we know that some functions go towards $\pm\infty$. We will see some examples below.

6.3. Identifying Maxima and Minima

- Find the stationary points of the function
- Look at the sign of the gradient on either side of the stationary points:



Examples 6.1: Find the stationary points, and the global maximum and minimum if they exist, of the following functions:

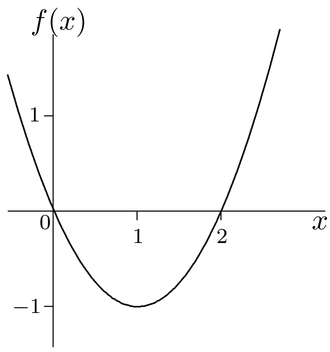
(i) $f(x) = x^2 - 2x$

The derivative is: $f'(x) = 2x - 2$
Stationary points occur where $f'(x) = 0$:

$$2x - 2 = 0 \quad \Rightarrow \quad x = 1$$

So there is one stationary point, at $x = 1$.

(You can sketch the function to see exactly what it looks like)

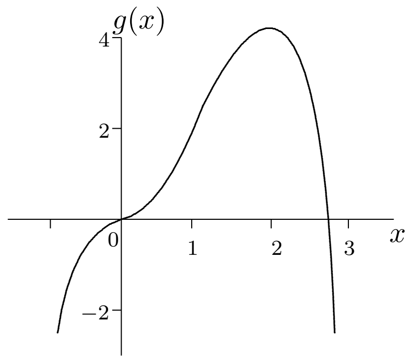


When $x < 1$, $f'(x) = 2x - 2 < 0$
When $x > 1$, $f'(x) = 2x - 2 > 0$

So the shape of the function at $x = 1$ is $\smile$
It is a minimum point.
The value of the function there is $f(1) = -1$.

The function has no maximum points, and no global maximum.

(ii) $g(x) = 2x^3 - \frac{3}{4}x^4$



Derivative is: $g'(x) = 6x^2 - 3x^3 = 3x^2(2 - x)$

Stationary points occur where $g'(x) = 0$:

$$3x^2(2 - x) = 0 \quad \Rightarrow \quad x = 0 \quad \text{or} \quad x = 2$$

So two stationary points, at $x = 0$ and $x = 2$.

$$\begin{aligned} x < 0: &\quad g'(x) &= 3x^2(2-x) > 0 \\ 0 < x < 2: &\quad g'(x) &> 0 \\ x > 2: &\quad g'(x) &< 0 \end{aligned}$$

At $x = 0$ the gradient doesn't change sign. Point of inflection. $f(0) = 0$
At $x = 2$ the shape of the function is $\frown$. Maximum point. $f(2) = 4$
There are no minimum points, and the global minimum does not exist.

(iii) $h(x) = \dfrac{2}{x}$ for $0 < x \leq 1$

In this example we are only considering a limited range of values for x.

The derivative is: $h'(x) = -\dfrac{2}{x^2}$

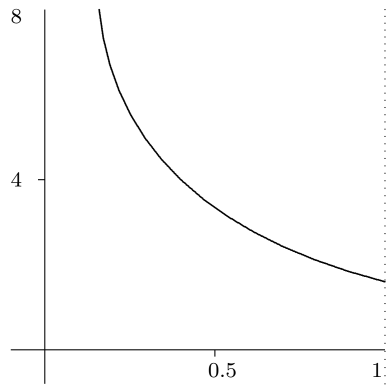
There are *no* stationary points because:

$-\dfrac{2}{x^2} < 0$ for all values of x between 0 and 1

The gradient is always negative: h is a decreasing function of x.

So the minimum value of the function is at $x = 1$: $h(1) = 2$.

The function doesn't have a maximum value because:
$h(x) \to \infty$ as $x \to 0$.



Exercises 5.6: **Stationary Points**

For each of the following functions (i) find and identify all of the stationary points, and (ii) find the global maximum and minimum values of the function, if they exist.

(1) $a(x) = x^2 + 4x + 2$

(2) $b(x) = x^3 - 3x + 1$

(3) $c(x) = 1 + 4x - x^2$ for $0 \leq x \leq 3$

(4) $d(x) = \frac{\alpha}{2}x^2 + \alpha x + 1$ where α is a parameter.
Hint: consider the two cases $\alpha \geq 0$ and $\alpha < 0$.

Further Reading and Exercises

- *Jacques* §4.6
- *Anthony & Biggs* §8.2 and §8.3

Both of these books use the second derivative to identify stationary points, so you may want to read section 7.2 before looking at them.

7. The Second Derivative

- The derivative $y'(x)$ tells us the gradient, or in other words how y changes as x increases.
- But $y'(x)$ is a function.
- So we can differentiate $y'(x)$, to find out how $y'(x)$ changes as x increases.

The derivative of the derivative is called the *second derivative*. It can be denoted $y''(x)$, but there are alternatives:

For the function $y(x)$ the derivatives can be written:

$$\begin{array}{lcccccc} 1^{st} \text{ derivative:} & y'(x) & \text{or} & \dfrac{d}{dx}(y(x)) & \text{or} & \dfrac{dy}{dx} \\ 2^{nd} \text{ derivative:} & y''(x) & \text{or} & \dfrac{d}{dx}\left(\dfrac{dy}{dx}\right) & \text{or} & \dfrac{d^2y}{dx^2} \end{array}$$

EXAMPLES 7.1: *Second Derivatives*

(i) $f(x) = ax^2 + bx + c$

$$\begin{array}{lrcl} \text{The } 1^{st} \text{ derivative is:} & f'(x) & = & 2ax + b \\ \text{Differentiating again:} & f''(x) & = & 2a \end{array}$$

(ii) $y = \dfrac{5}{x^2}$

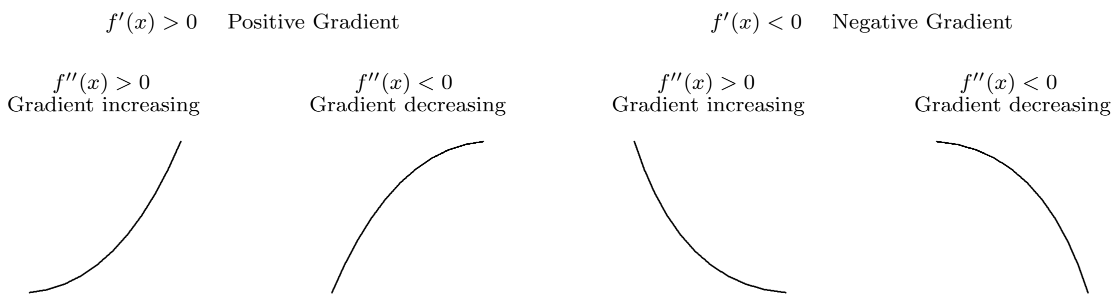
This function can be written as $y = 5x^{-2}$ so:

$$\begin{array}{lccccc} \text{First derivative:} & \dfrac{dy}{dx} & = & -10x^{-3} & = & -\dfrac{10}{x^3} \\ \text{Second derivative:} & \dfrac{d^2y}{dx^2} & = & 30x^{-4} & = & \dfrac{30}{x^4} \end{array}$$

EXERCISES 5.7: Find the first and second derivatives of:
(1) $f(x) = x^4$ (2) $g(x) = x + 1$ (3) $h(x) = 4x^{\frac{1}{2}}$ (4) $k(x) = x^3(5x^2 + 6)$

7.1. Using the Second Derivative to Find the Shape of a Function

- The 1^{st} derivative tells you whether the gradient is positive or negative.
- The 2^{nd} derivative tells you whether the gradient is increasing or decreasing.



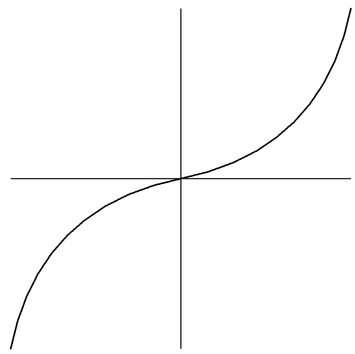
EXAMPLES 7.2: $y = x^3$

From the derivatives of this function:

$$\frac{dy}{dx} = 3x^2 \quad \text{and} \quad \frac{d^2y}{dx^2} = 6x$$

we can see:

- There is one stationary point, at $x = 0$ and at this point $y = 0$
- When $x < 0$:
  $$\frac{dy}{dx} > 0 \text{ and } \frac{d^2y}{dx^2} < 0$$
  The gradient is positive, and decreasing.
- When $x > 0$:
  $$\frac{dy}{dx} > 0 \text{ and } \frac{d^2y}{dx^2} > 0$$
  The gradient is positive, and increasing.



7.2. Using the 2^{nd} Derivative to Classify Stationary Points

In section 6.3 we looked at the gradient on either side of a stationary point to determine its type. But instead we can look at the sign of the second derivative, to see whether the gradient is increasing (as it will be at a minimum point, from negative to positive) or decreasing (as it will be at a maximum point, from positive to negative).

> If $f(x)$ has a stationary point at $x = x_0$
> (that is, if $f'(x_0) = 0$)
>
> then if $f''(x_0) > 0$ it is a minimum point
> and if $f''(x_0) < 0$ it is a maximum point

If the second derivative is zero it doesn't help you to classify the point, and you have to revert to the previous method.

EXAMPLES 7.3: Applying this method to the examples in section 6.3:

(i) $f(x) = x^2 - 2x$

$$f'(x) = 2x - 2$$

There is one stationary point at $x = 1$.

$$f''(x) = 2$$

This is positive, for all values of x. So the stationary point is a minimum.

(ii) $g(x) = 2x^3 - \frac{3}{4}x^4$

$$g'(x) = 6x^2 - 3x^3 = 3x^2(x - 2)$$

There are two stationary points, at $x = 0$ and $x = 2$.

$$g''(x) = 12x - 9x^2 = 3x(4 - 3x)$$

Look at each stationary point:

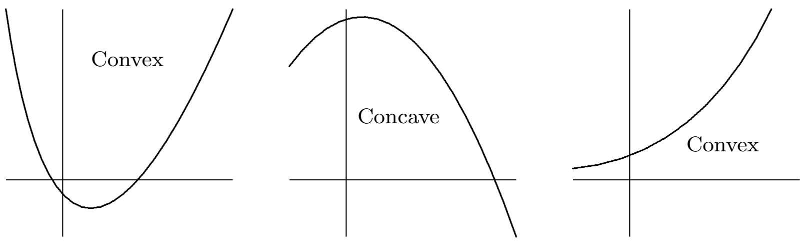
$x = 0$: $g''(0) = 0$ – we can't tell its type. (We found before that it is a point of inflection.)
$x = 2$: $g''(2) = -12$ – it is a maximum point.

EXERCISES 5.8: Find and classify all the stationary points of the following functions:
(1) $a(x) = x^3 - 3x$
(2) $b(x) = x + \dfrac{9}{x} + 7$
(3) $c(x) = x^4 - 4x^3 - 8x^2 + 12$

7.3. Concave and Convex Functions

If the gradient of a function is *increasing* for *all* values of x, the function is called *convex.* If the gradient of a function is *decreasing* for *all* values of x, the function is called *concave.* Since the second derivative tells us whether the gradient is increasing or decreasing:

$$\text{A function } f \text{ is called } \left\{ \begin{array}{c} \text{concave} \\ \text{convex} \end{array} \right\} \text{ if}$$
$$f''(x) \left\{ \begin{array}{c} \leq \\ \geq \end{array} \right\} 0 \quad \text{for all values of } x$$



EXAMPLES 7.4: Are the following functions concave, convex, or neither?

(i) The quadratic function $f(x) = ax^2 + bx + c$.

$f''(x) = 2a$, so if $a > 0$ it is convex and if $a < 0$ it is concave.

(ii) $P(q) = \dfrac{1}{q^2}$

$P'(q) = -2q^{-3}$, and $P''(q) = 6q^{-4} = \dfrac{6}{q^4}$

The second derivative is positive for all values of q, so the function is convex.

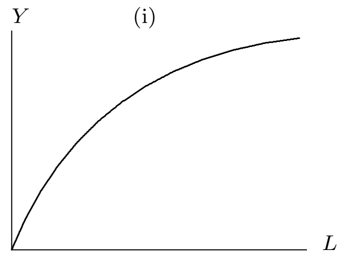
(iii) $y = x^3$

We looked at this function in section 7.1. The second derivative changes sign, so the function is neither concave nor convex.

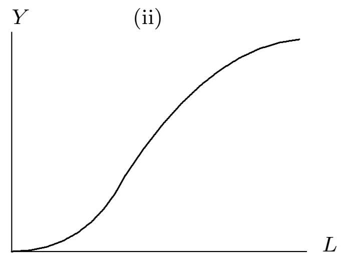
7.4. Economic Application: Production Functions

The economic characteristics of a production function $Y(L)$ depend on its shape:

- It is (usually) an increasing function: $Y'(L) > 0$ for all values of x;
- but the second derivative could be positive or negative: it tells you whether the marginal product of labour is increasing or decreasing.



- $Y''(L) < 0$ for all values of L
- The production function is concave
- MPL is decreasing
- Diminishing returns to labour



This function is neither concave nor convex.
When employment is low:

- $Y''(L) > 0$
- Increasing returns to labour

But when emplyment is higher:

- $Y''(L) < 0$
- Decreasing returns to labour

Exercises 5.9: **Concave and Convex Functions in Economics**

(1) Consider the cost function $C(q) = 3q^2 + 2q + 5$

(a) What is the marginal cost?

(b) Use the second derivative to find out whether the firm has increasing or decreasing marginal cost.

(c) Is the cost function convex or concave?

(2) For the aggregate consumption function is $C(Y) = 10 + Y^{0.9}$

(a) Find the first and second derivatives.

(b) How does the marginal propensity to consume change as income increases?

(c) Is the function concave or convex?

(d) Sketch the function.

Further Reading and Exercises

- *Jacques* §4.3 on the derivatives of economic functions
and §4.6 on maxima and minima
- *Anthony & Biggs* §8.2 and §8.3 for stationary points, maxima and minima

Solutions to Exercises in Chapter 5

EXERCISES 5.1:

(1) $\left.\frac{dy}{dx}\right|_{x=2} = 9$

EXERCISES 5.2:

(1) $\frac{dy}{dx} = 4x^3$,
$\left.\frac{dy}{dx}\right|_{x=-2} = -32$
(2) $\frac{dy}{dx} = \frac{3}{2}x^{\frac{1}{2}}$,
$\left.\frac{dy}{dx}\right|_{x=16} = 6$
(3) $\left.\frac{dy}{dx}\right|_{x=1} = 7$
(4) $z'(t) = 2t$, $z'(3) = 6$
(5) $Q'(P) = 1.7P^{0.7}$
(6) $u'(y) = \frac{5}{6}y^{-\frac{1}{6}}$, $u'(1) = \frac{5}{6}$
(7) $\frac{dy}{dx} = \frac{-3}{x^4}$, $\left.\frac{dy}{dx}\right|_{x=2} = -\frac{3}{16}$
(8) $\frac{dQ}{dP} = -\frac{1}{P^2}$
  (a) $\lim_{P\to 0} \frac{dQ}{dP} = -\infty$
  (b) $\lim_{P\to\infty} \frac{dQ}{dP} = 0$

EXERCISES 5.3:

(1) $\lim_{h\to 0} \frac{(x+h)^3-x^3}{h} =$
$\lim_{h\to 0} \frac{3x^2h+3xh^2+h^3}{h} =$
$\lim_{h\to 0} 3x^2 + 3xh + h^2 =$
$3x^2$
(2) $\lim_{h\to 0} \frac{\frac{1}{x+h}-\frac{1}{x}}{h} =$
$\lim_{h\to 0} \frac{x-x-h}{x(x+h)} \cdot \frac{1}{h} =$
$\lim_{h\to 0} \frac{-1}{x^2+xh} = \frac{-1}{x^2}$
(3) $\frac{dy}{dx} = 6x + 5$

EXERCISES 5.4:

(1) (a) $f'(x) = 16x$
  (b) $f'(2) = 32$
  (c) $f'(0) = 0$
(2) (a) $u'(y) = 40y^3 - 2y$
  (b) $v'(y) = 7 - 12y$
(3) $Q'(P) = \frac{20}{P^3} - \frac{20}{P^2}$
$= \frac{20}{P^3}(1 - P)$
(4) $h'(2) = 80$
(5) $\frac{dy}{dt} = 1 - 12\sqrt{t}$
(6) $g'(x) = 6x + 2x^{-\frac{5}{4}}$,
$g'(1) = 8$
(7) $y'(x) =$
$48x^3 + 21x^2 - 8x - 2$
(8) $F'(1) = 6$

EXERCISES 5.5:

(1) $\text{MPL} = 200L^{-\frac{1}{3}}$,
$\text{MPL}(8) = 100$
(2) $C'(Q) = 2kQ^{k-1}$
  (a) $k > 1$
  (b) $k = 1$
  (c) $k < 1$
(3) (a) $C(50) \approx 43.81$
  (b) $C'(Q) \approx 0.61$

EXERCISES 5.6:

(1) $x = -2, a(-2) = -2$:
global minimum.
(2) $x = -1, b(-1) = 3$:
maximum point.
$x = 1, b(1) = -1$:
minimum point.
No global max or min.
(3) $x = 2, c(2) = 5$:
global maximum.
No other stationary points; global minimum (in range $0 \leq x \leq 3$) is at $x = 0, c(0) = 1$.
(4) Stationary point at
$x = 1$, $d(1) = \frac{3}{2}\alpha + 1$.
It is a global max if $\alpha < 0$ and a global min if $\alpha > 0$

EXERCISES 5.7:

(1) $f'(x) = 4x^3$
$f''(x) = 12x^2$
(2) $g'(x) = 1$
$g''(x) = 0$
(3) $h'(x) = 2x^{-\frac{1}{2}}$
$h''(x) = -x^{-\frac{3}{2}}$
(4) $k'(x) = 25x^4 + 18x^2$
$k''(x) = 100x^3 + 36x$

EXERCISES 5.8:

(1) $(-1, 2)$ max
$(1, -2)$ min
(2) $(-3, 1)$ max
$(3, 13)$ min
(3) $(-1, 9)$ min
$(0, 12)$ max
$(4, -116)$ min

EXERCISES 5.9:

(1) (a) $C'(q) = 6q + 2$
  (b) $C''(q) = 6 \Rightarrow$ increasing MC
  (c) convex
(2) (a) $C'(Y) = 0.9Y^{-0.1}$,
$C''(Y) =$
$-0.09Y^{-1.1}$
  (b) It decreases
  (c) Concave

[1]This Version of Workbook Chapter 5: September 24, 2010

Worksheet 5: Differentiation

Quick Questions

(1) Differentiate:
  (a) $y = 9x^3 - 7x^2 + 15$
  (b) $f(x) = \dfrac{3}{4x^2}$
  (c) $Y(t) = 100t^{1.3}$
  (d) $P(Q) = Q^2 - 4Q^{\frac{1}{2}}$

(2) Determine whether each of the following functions is concave, convex, or neither:
  (a) $y = 5x^2 - 8x + 7$
  (b) $C(y) = \sqrt{4y}$ (for $y \geq 0$)
  (c) $P(q) = q^2 - 4q^{\frac{1}{2}}$ (for $q \geq 0$)
  (d) $k(x) = x^2 - x^3$

(3) Find the first and second derivatives of the production function $F(L) = 100L + 200L^{\frac{2}{3}}$ (for $L \geq 0$) and hence determine whether the firm has decreasing, constant, or increasing returns to labour.

(4) Find and classify all the the stationary points of the following functions. Find the global maxima and minima, if they exist, and sketch the functions.
  (a) $y = 3x - x^2 + 4$ for values of x between 0 and 4.
  (b) $g(x) = 6x^{\frac{1}{2}} - x$ for $x \geq 0$.
  (c) $f(x) = x^4 - 8x^3 + 18x^2 - 5$ (for all values of x)

(5) If a firm has cost function $C(Q) = aQ(b + Q^{1.5}) + c$, where a, b and c are positive parameters, find and sketch the marginal cost function. Does the firm have concave or convex costs?

Longer Questions

(1) The number of meals, y, produced in a hotel kitchen depends on the number of cooks, n, according to the production function $y(n) = 60n - \dfrac{n^3}{5}$
  (a) What is the marginal product of labour?
  (b) Does the kitchen have increasing, constant, or decreasing returns to labour?
  (c) Find:
    (i) the output of the kitchen
    (ii) the average output per cook
    (iii) the marginal product of labour
  when the number of cooks is 1, 5, 10 and 15.
  (d) Is this a realistic model of a kitchen? Suggest a possible explanation for the figures you have obtained.
  (e) Draw a careful sketch of the production function of the kitchen.
  (f) What is the maximum number of meals that can be produced?

(2) A manufacturer can produce economics textbooks at a cost of £5 each. The textbook currently sells for £10, and at this price 100 books are sold each day. The

manufacturer figures out that each pound decrease in price will sell ten additional copies each day.

(a) What is the demand function $q(p)$ for the textbooks?
(b) What is the cost function?
(c) What is the inverse demand function?
(d) Write down the manufacturer's daily profits as function of the quantity of textbooks sold, $\Pi(q)$.
(e) If the manufacturer now maximises profits, find the new price, and the profits per day.
(f) Sketch the profit function.
(g) If the government introduces a law that at least 80 economics textbooks must be sold per day, and the (law-abiding) manufacturer maximises profits, find the new price.

CHAPTER 6

More Differentation, and Optimisation

In the previous chapter we differentiated simple functions: **powers of** x, and **polynomials**. Now we introduce further techniques of differentiation, such as the **product, quotient and chain rules**, and the rules for differentiating **exponential and logarithmic functions**. We use differentiation in a variety of economic applications: for example, to find the **profit-maximising output** for a firm, the **growth rate of GDP**, the **elasticity of demand** and the **optimum time to sell an asset**.

1. Graph Sketching

The first step towards understanding an economic model is often to draw pictures: graphs of supply and demand functions, utility functions, production functions, cost functions, and so on. In previous chapters we have used a variety of methods, including differentiation, to work out the shape of a function. Here it is useful to summarise them, so that you can use them in the economic applications in this chapter.

1.1. Guidelines for Sketching the Graph of a Function $y(x)$

- Look at the function $y(x)$ itself:
  - to find where it crosses the x- and y-axes (if at all), and whether it is positive or negative elsewhere;
  - to see what happens to y when x goes towards $\pm\infty$;
  - to see whether there are any values of x for which y goes to $\pm\infty$.

- Look at the first derivative $y'(x)$ to find the gradient:
  - It may be positive for all values of x, so the function is increasing; or negative for all values of x, so the function is decreasing.
  - Otherwise, find the stationary points where $y'(x) = 0$, and check whether the graph slopes up or down elsewhere.

- Look at the second derivative $y''(x)$ to see which way the function curves:
  - Use it to check the types of the stationary points.
  - It may be positive for all values of x, so the function is convex; or negative for all values of x, so the function is concave.
  - Otherwise, it tells you whether the gradient is increasing or decreasing in different parts of the graph.

Depending on the function, you may not need to do all of these things. Sometimes some of them are too difficult – for example, you may try to find where the function crosses the

x-axis, but end up with an equation that you can't solve. But if you miss out some steps, the others may still give you enough information to sketch the function. The following example illustrates the procedure.

EXAMPLES 1.1: Sketch the graph of the function $y(x) = 2x^3 - 3x^2 - 12x$

(i) $y(0) = 0$, so it crosses the y-axis at $y = 0$

(ii) To find where it crosses the x-axis, we need to solve the equation $y(x) = 0$:

$$\begin{aligned} 2x^3 - 3x^2 - 12x &= 0 \\ x(2x^2 - 3x - 12) &= 0 \end{aligned}$$

One solution of this equation is $x = 0$. To find the other solutions we need to solve the equation $2x^2 - 3x - 12 = 0$. There are no obvious factors, so we use the quadratic formula:

$$\begin{aligned} 2x^2 - 3x - 12 &= 0 \\ x &= \frac{3 \pm \sqrt{105}}{4} \\ x &\approx 3.31 \text{ or } x \approx -1.81 \end{aligned}$$

So the function crosses the x-axis at -1.81, 0, and 3.31.

(iii) What happens to $y(x) = 2x^3 - 3x^2 - 12x$ as x becomes infinite?
For a polynomial, it is the term with the highest power of x that determines what happens when x becomes infinite – it dominates all the other terms. In this case:

$$\begin{aligned} 2x^3 \to \infty \text{ as } x \to \infty, &\quad \text{so} \quad y(x) \to \infty \text{ as } x \to \infty \\ 2x^3 \to -\infty \text{ as } x \to -\infty, &\quad \text{so} \quad y(x) \to -\infty \text{ as } x \to -\infty \end{aligned}$$

(iv) Are there any finite values of x that make $y(x)$ infinite?
No; we can evaluate $y(x)$ for all finite values of x.

(v) The first derivative of the function is:

$$y'(x) = 6x^2 - 6x - 12$$

Stationary points occur where $y'(x) = 0$:

$$\begin{aligned} 6x^2 - 6x - 12 &= 0 \\ x^2 - x - 2 &= 0 \\ (x+1)(x-2) &= 0 \\ x &= -1 \text{ or } x = 2 \end{aligned}$$

So there are stationary points at $x = -1$, $y = 7$ and $x = 2$, $y = -20$

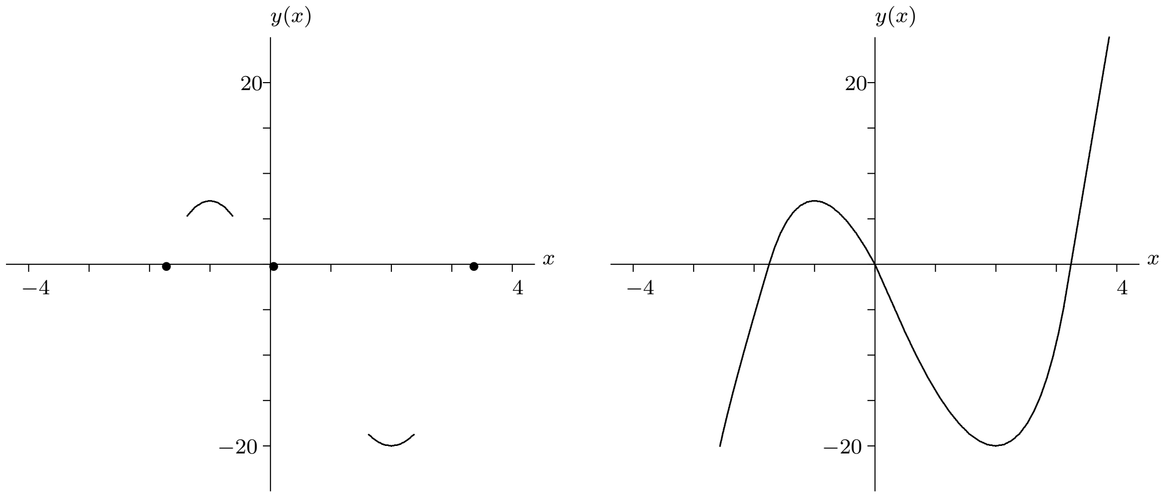
(vi) The second derivative of the function is:

$$y''(x) = 12x - 6$$

At $(-1, 7)$, $y''(x) = -18$ so this is a maximum point.
At $(2, -20)$, $y''(x) = 18$ so this is a minimum point.

(vii) The second derivative is zero when $x = \frac{1}{2}$.
When $x < \frac{1}{2}$, $y''(x) < 0$, so the *gradient* is decreasing.
When $x > \frac{1}{2}$, $y''(x) > 0$, so the *gradient* is increasing.

To draw the graph, first put in the points where the function crosses the axes, and the stationary points. Then fill in the curve, using what you know about what happens as $x \to \pm\infty$, and about where the gradient is increasing and decreasing:



1.2. Economic Application: Cost Functions

EXAMPLES 1.2: A firm's total cost of producing output y is $C(y) = 2y^2 + 3y + 8$.

(i) What are its fixed and variable costs?
Fixed costs are costs that must be paid whatever the level of output; variable costs depend on the level of output. If the firm produces *no* output, its costs are: $C(0) = 8$. So fixed costs are $F = 8$, and variable costs are $VC(y) = 2y^2 + 3y$

(ii) What is the marginal cost function?
The marginal cost function is the derivative of the cost function:

$$MC(y) = C'(y) = 4y + 3$$

(iii) What is the average cost function?

$$AC(y) = \frac{C(y)}{y} = 2y + 3 + \frac{8}{y}$$

(iv) Show that the marginal cost curve passes through the point of minimum average cost.

The derivative of the average cost function is:

$$\frac{dAC}{dy} = 2 - \frac{8}{y^2}$$

The average cost function has stationary point(s) where its derivative is zero:

$$2 - \frac{8}{y^2} = 0 \quad \Rightarrow \quad y = 2 \quad \text{(ignoring negative values of } y\text{)}$$

Look at the second derivative to see if this is a miminum:

$$\frac{d^2AC}{dy^2} = \frac{16}{y^3}$$

Since this is positive when $y = 2$ we have found a minimum point. At this point:

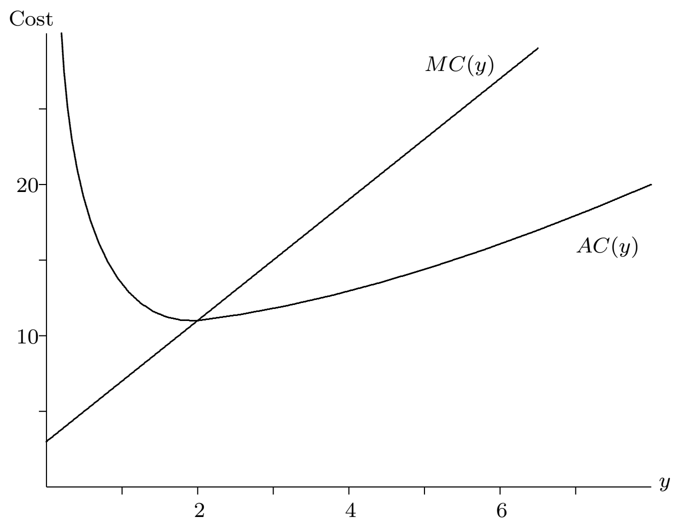
$$y = 2, \quad AC(y) = 11 \text{ and } MC(y) = 11$$

So the marginal cost curve passes through the point of minimum average cost.

(v) Sketch the marginal and average cost functions.
We will sketch the functions for positive values of y only, since negative values are of no economic interest. Sketching $MC(y)$ is easy: it is a straight line, of gradient 4, crossing the vertical axis at 3, and (from above) passing throught the point (2,11).
For the average cost function:

$$AC(y) = \frac{C(y)}{y} = 2y + 3 + \frac{8}{y}$$

we already know that it has one stationary point, which is a minimum. Also, we have found the second derivative, and it is positive for all positive values of y, so the gradient is always increasing.
We can see by looking at $AC(y)$ that as $y \to 0$, $AC(y) \to \infty$, and that as $y \to \infty$, $AC(y) \to \infty$. The function does not cross either of the axes.



Exercises 6.1: **Sketching Graphs**

(1) Sketch the graphs of the functions:
(a) $a(x) = (1 - x)(x - 2)(x - 3)$ (b) $b(x) = x^{\frac{1}{2}}(x - 6)$ for $x \geq 0$.

(2) A firm has cost function $C(y) = y^3 + 54$. Show that the marginal cost function passes through the point of minimum average cost, and sketch the marginal and average cost functions.

(3) A firm has cost function $C(y) = 2y + 7$. Sketch the marginal and average cost functions.

2. Introduction to Optimisation

In economic applications we often want to find the (global) maximum or minimum of a function. For example, we may want to find:

- the level of output where a firm's profit is maximised
- the level of output where a firm's average cost of production is minimised
- the number of hours per week that an individual should work, if he wants to maximise his utility.

Usually in economic applications the relevant function just has one stationary point, and this is the maximum or minimum that we want. But, having found a stationary point it is always wise to check that it is a maximum or minimum point (whichever is expected) and that it is a global optimum.

2.1. Profit Maximisation

Suppose that a firm producing y units of output has cost function $C(y)$, and faces inverse demand function $P(y)$. (In other words, if the firm makes y units, the total cost is $C(y)$, and each unit can be sold at a price $P(y)$.)

- The firm's *Revenue Function* is: $R(y) = yP(y)$
- The firm's *Profit Function* is: $\Pi(y) = R(y) - C(y) = yP(y) - C(y)$
- The firm's *optimal choice of output* is the value of y that maximises its profit $\Pi(y)$.

So we can find the optimal output by finding the maximum point of the profit function.

EXAMPLES 2.1: A monopolist has inverse demand function $P(y) = 16-2y$ and cost function $C(y) = 4y + 1$.

(i) What is the optimal level of output?
The profit function is:

$$\begin{aligned}\Pi(y) &= y(16-2y) - (4y+1) \\ &= 12y - 2y^2 - 1\end{aligned}$$

Look for a maximum point where the derivative is zero:

$$\frac{d\Pi}{dy} = 12 - 4y = 0 \quad \Rightarrow \quad y = 3$$

To check that this *is* a maximum look at the second derivative:

$$\frac{d^2\Pi}{dy^2} = -4 \quad \text{so it is a maximum}$$

If you think about the shape of the function (or sketch it), you can see that this the *global* maximum. So the optimal choice of output is $y = 3$.

(ii) How much profit does it make?
When $y = 3$:

$$\Pi(3) = 12 \times 3 - 2 \times 9 - 1 = 17$$

Profit is 17.

2.2. Marginal Cost, Marginal Revenue, and Profit Maximisation

We know that *Marginal Cost* is the cost of an additional unit of output, which is the derivative of the cost function:

$$MC(y) = \frac{dC}{dy}$$

Similarly *Marginal Revenue* is the revenue obtained when one more unit of output is sold, and it is the derivative of the revenue function:

$$MR(y) = \frac{dR}{dy}$$

To find the optimal level of output, we look for the point where $\frac{d\Pi}{dy} = 0$.
But we can write the profit function as:

$$\begin{aligned} \Pi(y) &= R(y) - C(y) \\ \Rightarrow \frac{d\Pi}{dy} &= \frac{dR}{dy} - \frac{dC}{dy} \end{aligned}$$

So the optimal level of output is where:

$$\begin{aligned} \frac{dR}{dy} &= \frac{dC}{dy} \\ \Rightarrow \text{Marginal Revenue} &= \text{Marginal Cost} \end{aligned}$$

> The optimal level of output for a profit-maximising firm is where $\frac{d\Pi}{dy} = 0$, or (equivalently) where $MR = MC$

If you want to find the profit-maximising output you can *either* look for a stationary point of the profit function, *or* solve the equation $MR = MC$.

EXERCISES 6.2: **Introduction to Optimisation**

(1) A competitive firm has cost function $C(q) = 3q^2 + 1$. The market price is $p = 24$. Write down the firm's profit as a function of output, $\Pi(q)$. Find the profit-maximising level of output. How much profit does the firm make if it produces the optimal amount of output?

(2) A monopolist faces demand curve $Q^d(P) = 10 - 0.5P$. Find the inverse demand function $P^d(Q)$, and hence the revenue function $R(Q)$, and marginal revenue. Suppose the monopoly wants to maximise its *revenue*. What level of output should it choose?

(3) A competitive firm produces output using labour as its only input. It has no fixed costs. The production function is $y = 6L^{0.5}$; the market price is $p = 4$, and the wage (the cost of labour per unit) is $w = 2$. Write down the firm's profit, as a function of the number of units of labour it employs, $\Pi(L)$. How many units of labour should the firm employ if it wants to maximise its profit?

Further Reading and Exercises

- *Jacques* §4.6.
- *Anthony & Biggs* §8.1, 8.2.

3. More Rules for Differentiation

3.1. Product Rule

To differentiate the function:

$$y(x) = (5x+3)(x^2-x+1)$$

we could multiply out the brackets and differentiate as before. Alternatively we could use:

The Product Rule: If $y(x) = u(x)v(x)$ then

$$y'(x) = u(x)v'(x) + u'(x)v(x)$$

or equivalently

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

Examples 3.1: *Product Rule*

(i) Differentiate $y(x) = (5x+3)(x^2-x+1)$

Put $u(x) = 5x+3$, and $v(x) = x^2-x+1$:

$$\begin{aligned} \frac{dy}{dx} &= (5x+3)(2x-1) + 5(x^2-x+1) \\ &= 15x^2 - 4x + 2 \end{aligned}$$

(ii) Differentiate $2x^{\frac{3}{2}}(x^3-2)$

$$f(x) = 2x^{\frac{3}{2}}(x^3-2)$$

Put $u(x) = 2x^{\frac{3}{2}}$ and $v(x) = x^3-2$:

$$\begin{aligned} f'(x) &= 2x^{\frac{3}{2}} \times 3x^2 + 3x^{\frac{1}{2}}(x^3-2) \\ &= 9x^{\frac{7}{2}} - 6x^{\frac{1}{2}} \end{aligned}$$

3.2. Quotient Rule

So far we have no method for differentiating a function such as:

$$y(x) = \frac{3x-2}{2x+1}$$

but there is a rule for differentiating the quotient of two functions:

The Quotient Rule: If $y(x) = \dfrac{u(x)}{v(x)}$ then

$$y'(x) = \frac{u'(x)v(x) - u(x)v'(x)}{v(x)^2}$$

or equivalently: $$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

EXAMPLES 3.2: *Quotient Rule*

(i) Differentiate $g(x) = \dfrac{3x-2}{2x+1}$
Putting $u = 3x - 2$ and $v = 2x + 1$:

$$\begin{aligned}\frac{dg}{dx} &= \frac{(2x+1)\times 3 - (3x-2)\times 2}{(2x+1)^2}\\ &= \frac{6x+3-6x+4}{(2x+1)^2} = \frac{7}{(2x+1)^2}\end{aligned}$$

(ii) Differentiate $h(x) = \dfrac{3x}{x^2+2}$

$$h'(x) = \frac{(x^2+2)\times 3 - 3x\times 2x}{(x^2+2)^2} = \frac{3x^2+6-6x^2}{(x^2+2)^2} = \frac{6-3x^2}{(x^2+2)^2}$$

3.3. Chain Rule

To differentiate the function:

$$y(x) = 4(2x^3-1)^7$$

we could multiply out the brackets to obtain a very long polynomial (of degree 21) and then differentiate. But this would take a very long time. Instead we can do it by writing y as a *function of a function*:

$$y = 4u^7 \quad \text{where} \quad u = 2x^3 - 1$$

and then using the Chain Rule:

The Chain Rule
If y is a function of u, and u is a function of x:

$$\frac{dy}{dx} = \frac{dy}{du}\times\frac{du}{dx}$$

Applying this to the example above:

$$\begin{aligned}\frac{dy}{dx} &= 28u^6\times 6x^2\\ &= 28(2x^3-1)^6\times 6x^2\\ &= 168x^2(2x^3-1)^6\end{aligned}$$

The Chain Rule is the rule for differentiating a composite function (see Chapter 4). Another way of writing it is:

Alternative Form of the Chain Rule
If $y(x) = v(u(x))$

$$y'(x) = v'(u(x))u'(x)$$

EXAMPLES 3.3: Differentiate the following functions using the Chain Rule:

(i) $y(x) = (1 - 8x)^5$

$$y = u^5 \quad \text{where} \quad u = 1 - 8x$$
$$\frac{dy}{dx} = 5u^4 \times (-8) = 5(1 - 8x)^4 \times (-8) = -40(1 - 8x)^4$$

(ii) $f(x) = \sqrt{(2x - 3)}$

First write this as: $f(x) = (2x - 3)^{\frac{1}{2}}$. Then:

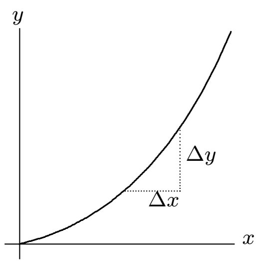
$$\begin{aligned} f &= u^{\frac{1}{2}} \quad \text{where} \quad u = 2x - 3 \\ f'(x) &= \frac{1}{2}u^{-\frac{1}{2}} \times 2 \\ &= \frac{1}{2}(2x - 3)^{-\frac{1}{2}} \times 2 \\ &= \frac{1}{\sqrt{2x - 3}} \end{aligned}$$

(iii) $g(x) = \dfrac{1}{1 + x^2}$

$$\begin{aligned} g &= \frac{1}{u} \quad \text{where} \quad u = 1 + x^2 \\ g'(x) &= -\frac{1}{u^2} \times 2x \\ &= -\frac{2x}{(1 + x^2)^2} \end{aligned}$$

(Note that we could have done this example using the quotient rule.)

3.4. Differentiating the Inverse Function



Remember from Chapter 5 that the derivative of a function is approximately equal to the ratio of the change in y to the change in x as you move along the function:

$$\frac{dy}{dx} \approx \frac{\Delta y}{\Delta x}$$

If a function has an inverse:

$$y = f(x) \quad \text{and} \quad x = f^{-1}(y)$$

then the gradients of these two functions are:

$$\frac{dy}{dx} \approx \frac{\Delta y}{\Delta x} \quad \text{and} \quad \frac{dx}{dy} \approx \frac{\Delta x}{\Delta y}$$

$$\begin{aligned} \text{So the gradient of the inverse function } f^{-1} &= \frac{1}{\text{the gradient of the function } f} \\ \frac{dx}{dy} &= \frac{1}{\frac{dy}{dx}} \end{aligned}$$

We will use this rule in later sections of this chapter.

EXERCISES 6.3: **More Rules for Differentiation**

(1) Use the product rule to differentiate:
(a) $y(x) = (x^2 + 3x + 2)(x - 1)$ (b) $f(z) = z(1 - \sqrt{z})$

(2) Use the quotient rule to differentiate: $k(x) = \dfrac{x+1}{2x^3+1}$

(3) Use the chain rule to differentiate: (a) $a(x) = 2(2x-7)^9$ (b) $b(y) = \sqrt{y^2+5}$

(4) For a firm facing inverse demand function $P(y) = \dfrac{1}{2y+3}$, find the revenue function $R(y)$, and the marginal revenue.

3.5. More Complicated Examples

EXAMPLES 3.4: Sometimes we need more than one of the product, quotient and chain rules:

(i) Differentiate the function $y(x) = x^3(x^2+1)^4$.

By the Product Rule: $$\frac{dy}{dx} = x^3 \times \frac{d}{dx}\left\{(x^2+1)^4\right\} \quad + \quad 3x^2 \times (x^2+1)^4$$

Then use the Chain Rule to differentiate the term in curly brackets:

$$\begin{aligned} \frac{dy}{dx} &= x^3 \times 4(x^2+1)^3 \times 2x \quad + \quad 3x^2(x^2+1)^4 \\ &= 8x^4(x^2+1)^3 \quad + \quad 3x^2(x^2+1)^4 \\ \text{Factorising: } \frac{dy}{dx} &= x^2(x^2+1)^3(11x^2+3) \end{aligned}$$

(ii) Differentiate: $f(x) = \left(\dfrac{x}{x+\alpha}\right)^\beta$

First use the Chain Rule, with $u = \frac{x}{x+\alpha}$:

$$f'(x) = \beta\left(\frac{x}{x+\alpha}\right)^{\beta-1} \frac{d}{dx}\left\{\frac{x}{x+\alpha}\right\}$$

Then use the Quotient Rule to differentiate the term in curly brackets:

$$f'(x) = \beta\left(\frac{x}{x+\alpha}\right)^{\beta-1}\left(\frac{\alpha}{(x+\alpha)^2}\right) = \alpha\beta\frac{x^{\beta-1}}{(x+\alpha)^{\beta+1}}$$

EXERCISES 6.4:

(1) Find the derivative of $y(x) = \sqrt{\dfrac{x}{1-x}}$

(2) Differentiate $z(t) = \dfrac{t^a}{(2t+1)^b}$ where a and b are parameters.

(3) If $f(q) = q^5(1-q)^3$, for what values of q is $f(q)$ increasing?

(4) (*Harder*) Using the chain rule and the product rule, derive the quotient rule.

Further Reading and Exercises

- *Jacques* §4.4.
- *Anthony & Biggs* §6.2, §6.4 and §6.5.

4. Economic Applications

4.1. Minimum Average Cost

If a firm has total cost function $C(q)$, where q is its output, its average cost function is:

$$AC(q) = \frac{C(q)}{q}$$

The derivative of the average cost function can be found using the Quotient Rule:

$$\frac{dAC}{dq} = \frac{qC'(q) - C(q)}{q^2}$$

At a point where average cost is minimised, $\frac{dAC}{dq} = 0$, so:

$$\begin{aligned} qC'(q) - C(q) &= 0 \\ C'(q) &= \frac{C(q)}{q} \\ MC &= AC \end{aligned}$$

We have proved that for any cost function, $MC = AC$ when AC is minimised: the marginal cost function passes through the minimum point of the average cost function.

4.2. The Marginal Revenue Product of Labour

If a firm uses labour to produces its output y, with production function $f(L)$, revenue is a function of output, and output is a function of labour:

$$Revenue = R(y) \quad \text{where} \quad y = f(L)$$

The *marginal revenue product of labour* is the additional revenue generated by employing one more unit of labour:

$$MRPL = \frac{dR}{dL}$$

Using the chain rule:

$$\begin{aligned} \frac{dR}{dL} &= \frac{dR}{dy} \times \frac{dy}{dL} \\ &= MR \times MPL \end{aligned}$$

The *marginal revenue product of labour* is the *marginal revenue* multiplied by the *marginal product of labour.*

EXERCISES 6.5:

(1) A firm has cost function $C(q) = (q+2)^{\beta}$ where β is a parameter greater than one. Find the level of output at which average cost is minimised.

(2) A firm produces output Q using labour L. Its production function is $Q = 3L^{\frac{1}{3}}$. If it produces Q units of output, the market price will be $P(Q) = \dfrac{54}{Q+9}$.

   (a) What is the firm's revenue function $R(Q)$?

   (b) If the firm employs 27 units of labour, find: (i) output (ii) the marginal product of labour (iii) the market price (iv) marginal revenue (v) the marginal revenue product of labour

5. Economic Application: Elasticity

5.1. The Price Elasticity of Demand

The gradient $\frac{dq}{dp}$ of a demand curve $q(p)$ gives a measure of how "responsive" demand is to price: how much demand falls (it is usually negative) when price rises a little:

$$\frac{dq}{dp} \approx \frac{\Delta q}{\Delta p}$$

But the gradient depends on the units in which price and quantity are measured. So, for example, the demand for cars in the UK would appear by this measure to be more responsive to price than in the US, simply because of the difference in currencies.

A better measure, which avoids the problem of units, is the *elasticity*, which measures the proportional change in quantity in response to a small proportional change in price. The *price elasticity of demand* is the answer to the question "By want percentage would demand change, if the price rose by one percent?"

$$\text{Price elasticity of demand, } \epsilon = \frac{p}{q}\frac{dq}{dp}$$

You can see where this definition comes from if we write:

$$\frac{p}{q}\frac{dq}{dp} \approx \frac{p}{q}\frac{\Delta q}{\Delta p} = \frac{\Delta q/q}{\Delta p/p} = \frac{\text{proportional change in } q}{\text{proportional change in } p}$$

EXAMPLES 5.1: For the demand function $q(p) = 10 - 2p$:

(i) Find the price elasticity of demand.

$$\begin{aligned} \frac{dq}{dp} &= -2 \\ \Rightarrow \epsilon &= \frac{p}{q} \times (-2) \end{aligned}$$

Substituting for q we obtain the elasticity as a function of price only:

$$\epsilon = \frac{-2p}{10 - 2p}$$

(ii) Evaluate the elasticity when the price is 1, and when the price is 4.
Substituting into the expression for the elasticity in (i):

$$\text{when } p = 1, \epsilon = -\frac{1}{4} \quad \text{and when } p = 4, \epsilon = -4$$

In this example, demand is inelastic (unresponsive to price changes) when the price is low, but elastic when price is high.

We say that demand is:

- inelastic if $|\epsilon| < 1$
- unit-elastic if $|\epsilon| = 1$
- elastic if $|\epsilon| > 1$

Warning: The price elasticity of demand is negative, but economists are often careless about the minus sign, saying, for example "The elasticity is greater than one" when we mean that the absolute value of the elasticity, $|\epsilon|$, is greater than one. Also, some books define the elasticity as $-\frac{p}{q}\frac{dq}{dp}$, which is positive.

5.2. Manipulating Elasticities

In the example above we started from the demand function $q(p) = 10 - 2p$, and obtained the elasticity as a function of price. But if we can find the inverse demand function, we could express the elasticity as a function of quantity instead:

$$q = 10 - 2p \quad \Rightarrow \quad p = 5 - \frac{q}{2}$$

Then:

$$\begin{aligned} \Rightarrow \epsilon &= \frac{p}{q} \times (-2) \quad \text{(as before)} \\ &= \frac{q - 10}{q} \end{aligned}$$

We can also find the price elasticity of demand if we happen to know the inverse demand function $p(q)$, rather than the demand function. We can differentiate to find its gradient $\frac{dp}{dq}$. Then the gradient of the demand function is:

$$\frac{dq}{dp} = \frac{1}{\frac{dp}{dq}}$$

(from section 3.4) and we can use this to evaluate the elasticity.

Examples 5.2: If the inverse demand function is $p = q^2 - 20q + 100$ (for $0 \leq q \leq 10$), find the elasticity of demand when $q = 6$.

Differentiatiating the inverse demand function and using the result above:

$$\frac{dp}{dq} = 2q - 20 \quad \Rightarrow \quad \frac{dq}{dp} = \frac{1}{2q - 20}$$

$$\Rightarrow \epsilon = \frac{p}{q}\frac{1}{2q - 20}$$

Here, it is natural to express the elasticity as a function of quantity:

$$\epsilon = \frac{q^2 - 20q + 100}{q(2q - 20)}$$

Evaluating it when $q = 6$:

$$\epsilon = \frac{16}{6 \times (12 - 20)} = -\frac{1}{3}$$

5.3. Elasticity and Revenue

If a firm faces inverse demand function $p(q)$, its revenue function is:

$$R(q) = p(q)q$$

As quantity increases, more items are sold, but at a lower price. What happens to revenue?

Differentiating, using the Product Rule, we obtain Marginal Revenue:

$$\begin{aligned}\frac{dR}{dq} &= p + q\frac{dp}{dq} \\ &= p\left(1 + \frac{q}{p}\frac{dp}{dq}\right) \\ &= p\left(1 + \frac{1}{\frac{p}{q}\frac{dq}{dp}}\right) \\ &= p\left(1 + \frac{1}{\epsilon}\right) \\ &= p\left(1 - \frac{1}{|\epsilon|}\right)\end{aligned}$$

From this expression we can see that revenue is increasing with quantity if and only if $|\epsilon| > 1$: that is, if and only if demand is elastic.

5.4. Other Elasticities

We can define other elasticities in just the same way as the price elasticity of demand. For example:

- If $q^s(p)$ is a supply function, the elasticity of supply is $\dfrac{p}{q^s}\dfrac{dq^s}{dp}$
- If $Y(L)$ is a production function, $\dfrac{L}{Y}\dfrac{dY}{dL}$ is the elasticity of output with respect to labour.

Exercises 6.6: **Elasticity**

(1) Find the price elasticity of demand for the demand function $Q^d(P) = 100 - 5P$, as a function of the price. What is the elasticity when $P = 4$?

(2) If the inverse demand function is $p^d(q) = 10 + \frac{20}{q}$, find the price elasticity of demand when $q = 4$.

(3) Find the elasticity of supply, as a function of price, when the supply function is: $q^s(p) = \sqrt{3p+4}$.

(4) Show that demand function $q^d(p) = \alpha p^{-\beta}$, where α and β are positive parameters, has constant elasticity of demand.

Further Reading and Exercises

- *Jacques* §4.5.
- *Anthony & Biggs* §9.1.
- *Varian* "Intermediate Microeconomics" Chapter 15 discusses elasticity fully, without using calculus in the main text, and using calculus in the Appendix.

6. Differentiation of Exponential and Logarithmic Functions

6.1. The Derivative of the Exponential Function

The reason why the exponential function e^x is so important in mathematics and economics is that it has a surprising property:[1] the derivative of e^x is e^x.

$$\text{If } y = \mathrm{e}^x \text{ then } \frac{dy}{dx} = e^x$$

Using the chain rule we can show easily that if a is a constant:

$$\text{If } y = e^{ax} \text{ then } \frac{dy}{dx} = ae^{ax}$$

EXAMPLES 6.1: Differentiate:

(i) $2e^{3x}$

$$y = 2e^{3x} \quad \Rightarrow \quad \frac{dy}{dx} = 2 \times 3e^{3x} = 6e^{3x}$$

(ii) $Y(t) = ae^{-bt}$

$$Y'(t) = a \times (-b) \times e^{-bt} = -abe^{-bt}$$

(iii) $k(x) = 5e^{x^4+2x}$

$$\begin{aligned} \text{Using the chain rule:} \quad \Rightarrow \quad k'(x) &= 5e^{x^4+2x} \times (4x^3+2) \\ &= 5(4x^3+2)e^{x^4+2x} \end{aligned}$$

6.2. The Derivative of the Logarithmic Function

The logarithmic function $\ln x$ is the inverse of the exponential function, and we can use this to find its derivative:

$$\begin{aligned} y = \ln x \quad \Rightarrow \quad x &= e^y \\ \frac{dx}{dy} &= e^y = x \\ \frac{dy}{dx} &= \frac{1}{\frac{dx}{dy}} = \frac{1}{x} \end{aligned}$$

So we have obtained a rule for differentiating $\ln x$:

$$\text{If } y = \ln x \text{ then } \frac{dy}{dx} = \frac{1}{x}$$

[1]You could try to verify this by differentiating $e^x = \lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n$.

Examples 6.2: Differentiate the following functions:

(i) $f(x) = \ln(5x)$

Again we can use the chain rule: $f'(x) = \dfrac{1}{5x} \times 5 = \dfrac{1}{x}$

(ii) $p(q) = 3\ln(q^2 - 5q)$

$$p'(q) = 3 \times \frac{1}{q^2 - 5q} \times (2q - 5) = \frac{3(2q-5)}{q^2 - 5q}$$

Exercises 6.7: **Differentiation of Exponential and Logarithmic Functions**

(1) Differentiate the function: $f(x) = \frac{1}{2}e^{8x}$.

(2) What is the derivative of $4\ln(3x+1)$?

(3) Find the derivative of $C(q) = 3qe^{q^2} + 4e^{-2q}$.

(4) Differentiate the function: $g(y) = y^2\ln(2y)+2$. For what values of y is g increasing?

(5) Find and classify the stationary points of the function: $F(t) = e^{-t}(t^2 + 2t - 2)$.

(6) If $f(x)$ is any function, find the derivative of (a) $y(x) = e^{f(x)}$ (b) $z(x) = \ln f(x)$.

6.3. Economic Application: Growth

In Chapter 4, we modelled the GDP of a country as depending on time, according to the equation:

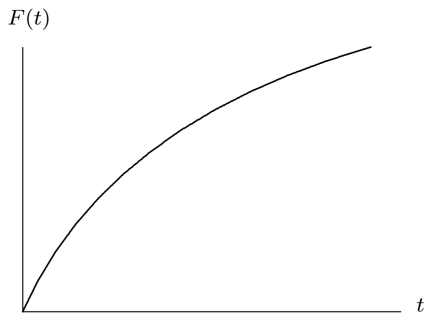
$$Y(t) = Y_0 e^{0.03t}$$

(Remember that the parameter Y_0 represents GDP at time $t = 0$.)
Differentiating this equation we get:

$$\begin{aligned} Y'(t) &= 0.03Y_0e^{0.03t} \\ &= 0.03Y(t) \end{aligned}$$

So we can now see that the rate of increase of GDP, $Y'(t)$, is a constant proportion, 0.03, of GDP. In other words, GDP is growing at a rate of 3%.

6.4. Economic Application: The Optimum Time to Sell an Asset



Suppose that, as a forest grows over time, the value of the timber is given by $F(t)$, where F is an increasing, concave, function of time. Then, if the interest rate is r, the present value of the forest at time t is:

$$V(t) = F(t)e^{-rt}$$

(See Chapter 3.)

The best time to cut down the forest and sell the timber is when the present value is maximised. Differentiating (using the product rule):

$$\begin{aligned} \frac{dV}{dt} &= F'(t)e^{-rt} - rF(t)e^{-rt} \\ &= (F'(t) - rF(t))e^{-rt} \end{aligned}$$

The optimum moment is when $\frac{dV}{dt} = 0$:

$$F'(t) = rF(t) \quad \Rightarrow \quad \frac{F'(t)}{F(t)} = r$$

This means that the optimum time to sell is when the proportional growth rate of the forest is equal to the interest rate. (You can verify, by differentiating again, that this is a maximum point.)

To see why, call the optimum moment t^*. After t^*, the value of the forest is growing more slowly than the interest rate, so it would be better to sell it and put the money in the bank. Before t^*, the forest grows fast: it offers a better rate of return than the bank.

Exercises 6.8:

(1) A painting is bought as an investment for £3000. The artist is increasing in popularity, so the market price is rising by £300 each year: $P(t) = 3000 + 300t$. If the interest rate is 4%, when should the painting be sold?

(2) The output Y of a country is growing at a proportional rate g and the population N is growing at a proportional rate n:

$$Y(t) = Y_0 e^{gt} \quad \text{and} \quad N(t) = N_0 e^{nt}$$

Output per person is $y(t) = \frac{Y(t)}{N(t)}$. Using the quotient rule, show that the proportional rate of growth of output per person is $g - n$.

Further Reading and Exercises

- *Jacques* §4.8.
- *Anthony & Biggs* §7.2 and §7.4.
- For more discussion of "When to cut a forest" see *Varian*, "Intermediate Microeconomics", Chapter 11 and its Appendix.

Solutions to Exercises in Chapter 6

Exercises 6.1:

(1) (a) Graph crosses y-axis at $y = 6$ and x-axis at 1,2 and 3. Max at $x = 2.577$; min at $x = 1.423$. $a \to -\infty$ as $x \to \infty$ and $a \to \infty$ as $x \to -\infty$.
(b) Graph passes through (0,0) and crosses x-axis at $x = 6$. Min at $x = 2, b = -5.66$. Convex: $b''(x) > 0$ for all x. $b \to \infty$ as $x \to \infty$.

(2) $MC = 3y^2$; $AC = y^2 + \frac{54}{y}$. AC is minimised when $y = 3$. At this point $MC = AC = 27$.

(3) MC is a horizontal line $MC = 2$. AC is downward-sloping and convex. As $y \to 0$, $AC \to \infty$, and as $y \to \infty$, $AC \to 2$.

Exercises 6.2:

(1) $\Pi(q) = 24q - 3q^2 - 1$. Profit is maximised when $q = 4$. $\Pi(4) = 47$.

(2) $P^d(Q) = 20 - 2Q$
$R(Q) = (20 - 2Q)Q$
$MR = 20 - 4Q$
Revenue is maximised when $Q = 5$.

(3) $\Pi(L) = 24L^{0.5} - 2L$. Profit is maximised when $L = 36$.

Exercises 6.3:

(1) (a) $y'(x) = 3x^2 + 4x - 1$
(b) $f'(z) = 1 - \frac{3}{2}\sqrt{z}$

(2) $k'(x) = \frac{1-4x^3-6x^2}{(2x^3+1)^2}$

(3) (a) $a'(x) = 36(2x - 7)^8$
(b) $b'(y) = \frac{y}{\sqrt{y^2+5}}$

(4) $R(y) = \frac{y}{2y+3}$
$MR = \frac{3}{(2y+3)^2}$

Exercises 6.4:

(1) $y'(x) = \frac{1}{2x^{\frac{1}{2}}(1-x)^{\frac{3}{2}}}$

(2) $z'(t) = \frac{t^{a-1}(a+2at-2bt)}{(2t+1)^{b+1}}$

(3) $f'(q)$
$= 5q^4(1-q)^3 - 3q^5(1-q)^2$
$= q^4(1-q)^2(5-8q)$
So f is increasing when $q < \frac{5}{8}$.

(4) Let $y(x) = \frac{u(x)}{v(x)}$. Write it as:
$y(x) = u(x)\,(v(x))^{-1}$
$\frac{dy}{dx}$
$= \frac{du}{dx}v^{-1} + u\frac{d}{dx}\{v^{-1}\}$
$= \frac{du}{dx}v^{-1} - uv^{-2}\frac{dv}{dx}$
$= v^{-2}\left(v\frac{du}{dx} - u\frac{dv}{dx}\right)$
$= \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

Exercises 6.5:

(1) $q = \frac{2}{\beta-1}$

(2) (a) $R(Q) = \frac{54Q}{Q+9}$
(b) (i) $Q = 9$
(ii) $MPL = \frac{1}{9}$
(iii) $P = 3$
(iv) $MR = \frac{3}{2}$
(v) $MRPL = \frac{1}{6}$

Exercises 6.6:

(1) $\epsilon = -\frac{5P}{100-5P}$; $\epsilon = -\frac{1}{4}$

(2) -3

(3) $\frac{3p}{2(3p+4)}$

(4) $\epsilon = -\alpha\beta p^{-\beta-1} \times \frac{p}{q}$
$= -\alpha\beta p^{-\beta} \times \frac{1}{q} = -\beta$

Exercises 6.7:

(1) $f'(x) = 4e^{8x}$

(2) $\frac{12}{3x+1}$

(3) $3(1 + 2q^2)e^{q^2} - 8e^{-2q}$

(4) $g'(y) = y(1 + 2\ln(2y))$
Increasing when
$1 + 2\ln(2y) > 0$
$\Rightarrow y > \frac{1}{2e^{\frac{1}{2}}}$

(5) Max at $(2, 6e^{-2})$
Min at $(-2, -2e^2)$

(6) (a) $y'(x) = f'(x)e^{f(x)}$
(b) $z'(x) = \frac{f'(x)}{f(x)}$

Exercises 6.8:

(1) The present value is $(3000 + 300t)e^{-0.04t}$. This is maximised when $t = 15$.

(2) $y'(t) = \frac{N(t)Y'(t)-Y(t)N'(t)}{N^2}$
But:
$Y' = gY$ and $N' = nN$
so:
$y' = \frac{gNY-nNY}{N^2}$
$= (g-n)\frac{Y}{N} = (g-n)y$

[2]This Version of Workbook Chapter 6: August 22, 2003

Worksheet 6: More Differentiation and Optimisation

Quick Questions

(1) Find the derivatives of the following functions:

(a) $f(x) = (2x+1)(x+3)$ (b) $g(y) = \dfrac{y^2+1}{2-y}$ (c) $h(z) = (6z+1)^5$

(2) For what values of the parameter a is $y(x) = \dfrac{x+1}{x+a}$ an increasing function of x?

(3) Differentiate: (a) e^{3x} (b) $\ln(3x^2+1)$ (c) e^{x^3} (d) $e^{2x}(x+\frac{1}{x})$

(4) If the cost function of a firm is $C(q) = q^2\sqrt{1+2q}$, what is the marginal cost function?

(5) Does the production function $F(L) = \ln(2L+5)$ have increasing or decreasing returns to labour?

(6) Consider the function:

$$y(x) = \left(2x^2 - 2x - 1\right)e^{2x}$$

(a) Find and classify the stationary points.
(b) For what values of x is y positive?
(c) Sketch the graph of the function.

(7) Find the elasticity of demand when the demand function is $q(p) = 120 - 5p$. For what values of p is demand inelastic?

(8) Suppose that a monopoly has cost function $C(q) = q^2$, and a demand function $q(p) = 10 - \frac{p}{2}$. Find (i) the inverse demand curve; (ii) the profit function; and (iii) the profit-maximizing level of output, and corresponding price.

(9) The supply and demand functions in a market are

$$q^s(p) = 16(p-t) \qquad q^d(p) = 24 - 8p$$

where t is a per unit sales tax. Find the equilibrium price and quantity in the market, and the total tax revenue for the government, in terms of the tax, t. What value of t should the government choose if it wants to maximise tax revenue? How much tax will then be raised?

Longer Questions

(1) Suppose that a monopolist faces the demand function: $q^D(p) = \dfrac{a}{p^2+1}$
where a is a positive parameter.

(a) Find the elasticity of demand (i) as a function of the price, and (ii) as a function of the quantity sold.
(b) At what value of q is the elasticity of demand equal to 1?
(c) Sketch the demand curve, showing where demand is inelastic, and where it is elastic.
(d) Find the firm's revenue function, and its marginal revenue function.

(e) Sketch the revenue function, showing where revenue is maximised.
(f) Explain intuitively the relationship between the shape of the revenue function and the elasticity of demand.
(g) Prove that, in general for a downward-sloping demand function, marginal revenue is zero when the elasticity is one. Is it necessarily true that revenue is maximised at this point? Explain.

(2) For a firm with cost function $C(y)$, the average cost function is defined by:
$$AC = \frac{C(y)}{y}$$
(a) If $C(y) = \dfrac{2y^3}{3} + y + 36$:
(i) Find the fixed cost, the average cost function, and the marginal cost function.
(ii) At what level of output is average cost minimised? What is the average cost at this point?
(iii) Sketch carefully the average and marginal cost functions.

(b) If $C(y) = cy + F$ where c and F are positive parameters:
(i) Show that the average cost function is decreasing at all levels of output.
(ii) Show that marginal cost is less than average cost for all values of y.
(iii) Sketch the marginal and average cost functions.

(c) Prove that, for any cost function $C(y)$, AC is decreasing whenever $MC < AC$, and AC is increasing whenever $MC > AC$ (for $y > 0$). Explain intuitively why this is so.

(3) The ageing bursar of St. Giles College and its young economist have disagreed about the disposal of college wine. The Bursar would like to sell up at once, whereas she is of the view that the college should wait 25 years. Both agree, however, that if the wine is sold t years from now, the revenue received will be:
$$W(t) = A\exp(t^{\frac{1}{2}})$$
and that the interest rate, r, will be stable.
(a) Differentiate W, and hence show that the proportional growth rate of the potential revenue is
$$\frac{1}{W}\frac{dW}{dt} = \frac{1}{2\sqrt{t}}$$
(b) Explain why the present value of the revenue received if the wine is sold in t years is given by:
$$V(t) = A\exp\left(t^{\frac{1}{2}} - rt\right)$$
(c) Differentiate V, and hence determine the optimum time to sell the wine.
(d) Explain intuitively why the wine should be kept longer if the interest rate is low.
(e) If the interest rate is 10%, should the Governing Body support the Bursar or the economist?

CHAPTER 7

Partial Differentiation

From the previous two chapters we know how to differentiate **functions of one variable**. But many functions in economics depend on several variables: output depends on both capital and labour, for example. In this chapter we show how to differentiate **functions of several variables** and apply this to find the **marginal products of labour and capital, marginal utilities, and elasticities of demand**. We use **differentials** to find the gradients of isoquants, and hence determine **marginal rates of substitution**.

1. Partial Derivatives

The derivative of a function of one variable, such as $y(x)$, tells us the gradient of the function: how y changes when x increases. If we have a function of more than one variable, such as:

$$z(x, y) = x^3 + 4xy + 5y^2$$

we can ask, for example, how z changes when x increases but y doesn't change. The answer to this question is found by thinking of z as a function of x, and differentiating, treating y as if it were a constant parameter:

$$\frac{\partial z}{\partial x} = 3x^2 + 4y$$

This process is called partial differentiation. We write $\frac{\partial z}{\partial x}$ rather than $\frac{dz}{dx}$, to emphasize that z is a function of another variable as well as x, which is being held constant.

$\dfrac{\partial z}{\partial x}$ is called the partial derivative of z with respect to x

$\frac{\partial z}{\partial x}$ is pronounced "partial dee z by dee x".

Similarly, if we hold x constant, we can find the partial derivative with respect to y:

$$\frac{\partial z}{\partial y} = 4x + 10y$$

Remember from Chapter 4 that you can think of $z(x, y)$ as a "surface" in 3 dimensions. Imagine that x and y represent co-ordinates on a map, and $z(x, y)$ represents the height of the land at the point (x, y). Then, $\frac{\partial z}{\partial x}$ tells you the gradient of the land as you walk in the direction of increasing x, keeping the y co-ordinate constant. If $\frac{\partial z}{\partial x} > 0$, you are walking uphill; if it is negative you are going down. (Try drawing a picture to illustrate this.)

EXERCISES 7.1: Find the partial derivatives with respect to x and y of the functions:

(1) $f(x,y) = 3x^2 - xy^4$

(2) $h(x,y) = (x+1)^2(y+2)$

(3) $g(x,y) = \dfrac{\ln x}{y}$

EXAMPLES 1.1: For the function $f(x,y) = x^3e^{-y}$:

(i) Show that f is increasing in x for all values of x and y.

$$\frac{\partial f}{\partial x} = 3x^2e^{-y}$$

Since $3x^2 > 0$ for all values of x, and $e^{-y} > 0$ for all values of y, this derivative is always positive: f is increasing in x.

(ii) For what values of x and y is the function increasing in y?

$$\frac{\partial f}{\partial y} = -x^3e^{-y}$$

When $x < 0$ this derivative is positive, so f is increasing in y. When $x > 0$, f is decreasing in y.

1.1. Second-order Partial Derivatives

For the function in the previous section:

$$z(x,y) = x^3 + 4xy + 5y^2$$

we found:

$$\frac{\partial z}{\partial x} = 3x^2 + 4y$$
$$\frac{\partial z}{\partial y} = 4x + 10y$$

These are the *first-order* partial derivatives. But we can differentiate again to find *second-order* partial derivatives. The second derivative with respect to x tells us how $\frac{\partial z}{\partial x}$ changes as x increases, still keeping y constant.

$$\frac{\partial^2 z}{\partial x^2} = 6x$$

Similarly:

$$\frac{\partial^2 z}{\partial y^2} = 10$$

We can also differentiate $\frac{\partial z}{\partial x}$ with respect to y, to find out how it changes when y increases. This is written as $\frac{\partial^2 z}{\partial y \partial x}$ and is called a *cross-partial* derivative:

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = 4$$

Similarly:

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = 4$$

Note that the two cross-partial derivatives are the same: it doesn't matter whether we differentiate with respect to x first and then with respect to y, or vice-versa. This happens with all "well-behaved" functions.

A function $z(x, y)$ of two variables has four second-order partial derivatives:

$$\frac{\partial^2 z}{\partial x^2}, \quad \frac{\partial^2 z}{\partial y^2}, \quad \frac{\partial^2 z}{\partial x \partial y} \quad \text{and} \quad \frac{\partial^2 z}{\partial y \partial x}$$

$$\text{but } \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}$$

1.2. Functions of More Than Two Variables

A function of n variables:

$$f(x_1, x_2, \ldots, x_n)$$

has n first-order partial derivatives:

$$\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}$$

and n^2 second-order partial derivatives:

$$\frac{\partial^2 f}{\partial x_1^2}, \frac{\partial^2 f}{\partial x_2 \partial x_1}, \frac{\partial^2 f}{\partial x_3 \partial x_1}, \ldots$$

Again we find that it doesn't matter which variable we differentiate with respect to first: the cross-partials are equal. For example:

$$\frac{\partial^2 f}{\partial x_3 \partial x_1} = \frac{\partial^2 f}{\partial x_1 \partial x_3}$$

1.3. Alternative Notation

There are several different ways of writing partial derivatives. For a function $f(x, y)$ the first-order partial derivatives can be written:

$$\frac{\partial f}{\partial x} \text{ or } f_x \quad \text{and} \quad \frac{\partial f}{\partial y} \text{ or } f_y$$

and the second-order partial derivatives are:

$$\frac{\partial^2 f}{\partial x^2} \text{ or } f_{xx} \quad \text{and} \quad \frac{\partial^2 f}{\partial y^2} \text{ or } f_{yy} \quad \text{and} \quad \frac{\partial^2 f}{\partial x \partial y} \text{ or } f_{xy}$$

For a function $f(x_1, x_2, x_3, \ldots)$ we sometimes write:

$$f_1 \text{ for } \frac{\partial f}{\partial x_1}, \quad f_2 \text{ for } \frac{\partial f}{\partial x_2}, \quad f_{11} \text{ for } \frac{\partial^2 f}{\partial x_1^2}, \quad f_{13} \text{ for } \frac{\partial^2 f}{\partial x_1 \partial x_3} \text{ etc.}$$

EXERCISES 7.2:

(1) Find all the first- and second-order partial derivatives of the function $f(x, y) = x^2 + 3yx$, verifying that the two cross-partial derivatives are the same.

(2) Find all the first- and second-order partial derivatives of $g(p, q, r) = q^2 e^{2p+1} + rq$.

(3) If $z(x, y) = \ln(2x + 3y)$, where $x > 0$ and $y > 0$, show that z is increasing in both x and y.

(4) For the function $f(x, y, z) = (x^2 + y^2 + z^2)^3$ find the partial derivatives f_x, f_{xx}, and f_{xz}.

(5) If $F(K, L) = AK^\alpha L^\beta$, show that $F_{KL} = F_{LK}$.

(6) For the utility function $u(x_1, x_2, x_3) = x_1 + 2x_2x_3$, find the partial derivatives u_{23} and u_{12}.

Further Reading and Exercises

- *Jacques* §5.1
- *Anthony & Biggs* §11.1 and §11.2

2. Economic Applications of Partial Derivatives, and Euler's Theorem

2.1. The Marginal Products of Labour and Capital

Suppose that the output produced by a firm depends on the amounts of labour and capital used. If the production function is

$$Y(K, L)$$

the partial derivative of Y with respect to L tells us the the marginal product of labour:

$$MPL = \frac{\partial Y}{\partial L}$$

The marginal product of labour is the amount of extra output the firm could produce if it used one extra unit of labour, but kept capital the same as before.
Similarly the marginal product of capital is:

$$MPK = \frac{\partial Y}{\partial K}$$

Examples 2.1: For a firm with production function $Y(K, L) = 5K^{\frac{1}{3}}L^{\frac{2}{3}}$:

(i) Find the marginal product of labour.

$$MPL = \frac{\partial Y}{\partial L} = \frac{10}{3}K^{\frac{1}{3}}L^{-\frac{1}{3}}$$

(ii) What is the MPL when $K = 64$ and $L = 125$?

$$MPL = \frac{10}{3}(64)^{\frac{1}{3}}(125)^{-\frac{1}{3}} = \frac{10}{3} \times 4 \times \frac{1}{5} = \frac{8}{3}$$

(iii) What happens to the marginal product of labour as the number of workers increases?

Differentiate MPL with respect to L: $\dfrac{\partial^2 Y}{\partial L^2} = -\dfrac{10}{9}K^{\frac{1}{3}}L^{-\frac{4}{3}} < 0$

So the MPL decreases as the labour input increases – the firm has diminishing returns to labour, if capital is held constant. This is true for *all* values of K and L.

(iv) What happens to the marginal product of labour as the amount of capital increases?

Differentiate MPL with respect to K: $\dfrac{\partial^2 Y}{\partial K \partial L} = \dfrac{10}{9}K^{-\frac{2}{3}}L^{-\frac{1}{3}} > 0$

So the MPL increases as the capital input increases – when there is more capital, an additional worker is more productive.

2.2. Elasticities of Demand

If a consumer has a choice between two goods, his demand x_1 for good 1 depends on its price, p_1, but also on the consumer's income, m, and the price of the other good. The demand function can by written:

$$x_1(p_1, p_2, m)$$

The *own-price elasticity of demand* is the responsiveness of demand to changes in the price p_1, defined just as in Chapter 6 except that we now have to use the partial derivative:

$$\epsilon_{11} = \frac{p_1}{x_1}\frac{\partial x}{\partial p_1}$$

But in addition we can measure the responsiveness of the demand for good 1 to changes in the price of good 2, or the consumer's income:

For the demand function $x_1(p_1, p_2, m)$

- The own-price elasticity is: $\epsilon_{11} = \dfrac{p_1}{x_1}\dfrac{\partial x_1}{\partial p_1}$
- The cross-price elasticity is: $\epsilon_{12} = \dfrac{p_2}{x_1}\dfrac{\partial x_1}{\partial p_2}$
- The income elasticity is: $\eta_1 = \dfrac{m}{x_1}\dfrac{\partial x_1}{\partial m}$

2.3. Euler's Theorem

Remember from Chapter 4 that a function of two variables is called *homogeneous* of degree n if $f(\lambda x, \lambda y) = \lambda^n f(x, y)$. If a production function is homogeneous of degree 1, for example, it has constant returns to scale. Euler's Theorem states that:

If the function $f(x, y)$ is homogeneous of degree n:

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf(x, y)$$

EXAMPLES 2.2: $f(x, y) = 2x^2y^2 + xy^3$

(i) Show that f is homogenous.

$$\begin{aligned} f(\lambda x, \lambda y) &= 2(\lambda x)^2(\lambda y)^2 + \lambda x(\lambda y)^3 \\ &= 2\lambda^4x^2y^2 + \lambda^4xy^3 \\ &= \lambda^4 f(x, y) \end{aligned}$$

(ii) Verify that it satisfies Euler's Theorem.

$$\frac{\partial f}{\partial x} = 4xy^2 + y^3 \quad \text{and} \quad \frac{\partial f}{\partial y} = 4x^2y + 3xy^2$$

$$\begin{aligned} \Rightarrow x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} &= 4x^2y^2 + xy^3 + 4x^2y^2 + 3xy^3 \\ &= 8x^2y^2 + 4xy^3 \\ &= 4f(x, y) \end{aligned}$$

EXERCISES 7.3: **Economic Applications of Partial Derivatives**

(1) For the production function $Q(K, L) = (K - 2)L^2$:
  (a) Find the marginal product of labour when $L = 4$ and $K = 5$.
  (b) Find the marginal product of capital when $L = 3$ and $K = 4$.
  (c) How does the marginal product of capital change as labour increases?

(2) Show that the production function $Y = (K^{\frac{1}{2}} + L^{\frac{1}{2}})^2$ has constant returns to scale. Verify that it satisfies Euler's Theorem.

(3) For the demand function $x_1(p_1, p_2, m) = \dfrac{p_2 m}{p_1^2}$
  (a) Find the own-price, cross-price, and income elasticities.
  (b) Show that the function is homogeneous of degree zero. Why is it economically reasonable to expect a demand function to have this property?

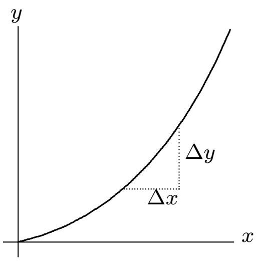
(4) Write down Euler's Theorem for a demand function $x_1(p_1, p_2, m)$ that is homogeneous of degree zero. What does this tell you about the own-price, cross-price, and income elasticities?

Further Reading and Exercises

- *Jacques* §5.2
- *Anthony and Biggs* §12.4

3. Differentials

3.1. Derivatives and Approximations: Functions of One Variable



If we have a function of one variable $y(x)$, and x increases by a small amount Δx, how much does y change? We know that when Δx is small:

$$\frac{dy}{dx} \approx \frac{\Delta y}{\Delta x}$$

Rearranging:

$$\Delta y \approx \frac{dy}{dx}\Delta x$$

Examples 3.1: If we know that the marginal propensity to consume is 0.85, and national income increases by £2bn, then from $\Delta C \approx \frac{dC}{dY}\Delta Y$, aggregate consumption will increase by $0.85 \times 2 = £1.7$bn.

3.2. Derivatives and Approximations: Functions of Several Variables

Similarly, if we have a function of two variables, $z(x, y)$, and x increases by a small amount Δx, but y doesn't change:

$$\Delta z \approx \frac{\partial z}{\partial x}\Delta x$$

But if both x and y change, we can calculate the change in z from:

$$\Delta z \approx \frac{\partial z}{\partial x}\Delta x + \frac{\partial z}{\partial y}\Delta y$$

Examples 3.2: If the marginal product of labour is 1.5, and the marginal product of capital is 1.8, how much will a firm's output change if it employs one more unit of capital, and one fewer worker?

$$\Delta Y \approx \frac{\partial Y}{\partial L}\Delta L + \frac{\partial Y}{\partial K}\Delta K$$

Putting: $\frac{\partial Y}{\partial L} = 1.5$, $\frac{\partial Y}{\partial K} = 1.8$, $\Delta K = 1$, and $\Delta L = -1$, we obtain: $\Delta Y = 0.3$.

3.3. Differentials

The approximation

$$\Delta z \approx \frac{\partial z}{\partial x}\Delta x + \frac{\partial z}{\partial y}\Delta y$$

becomes more accurate as the changes in the arguments Δx and Δy become smaller. In the limit, as they become *infinitesimally* small, we write dx and dy to represent infinitesimal changes in x and y. They are called *differentials*. The corresponding change in z, dz, is called the *total differential*.

The total differential of the function $z(x, y)$ is

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$$

EXAMPLES 3.3: Find the total differential of the following functions:

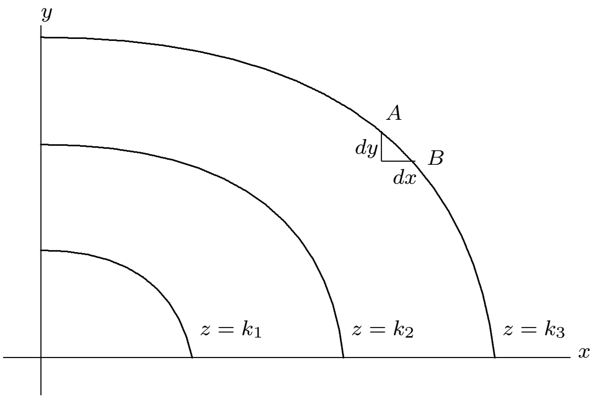
(i) $F(x,y) = x^4 + xy^3$

$$\frac{\partial F}{\partial x} = 4x^3 + y^3 \text{ and } \frac{\partial F}{\partial y} = 3xy^2$$
$$\Rightarrow dF = (4x^3 + y^3)dx + 3xy^2 dy$$

(ii) $u(x_1, x_2, x_3) = 2x_1 + 5x_2 + 9x_3$

$$du = 2dx_1 + 5dx_2 + 9dx_3$$

3.4. Using Differentials to Find the Gradient of an Isoquant



Remember from Chapter 4 that we can represent a function $z(x,y)$ by drawing the isoquants: the lines

$$z(x,y) = k$$

for different values of the constant k.

To find the gradient at the point A, consider a small change in x and y that takes you to a point B on the same isoquant.

Then the change in z is given by: $\quad dz = \dfrac{\partial z}{\partial x}dx + \dfrac{\partial z}{\partial y}dy$

But in moving from A to B, z does not change, so $dz = 0$: $\quad \dfrac{\partial z}{\partial x}dx + \dfrac{\partial z}{\partial y}dy = 0$

Rearranging this equation gives us $\frac{dy}{dx}$:

The gradient of the isoquant of the function $z(x,y)$ at any point (x,y) is given by:

$$\frac{dy}{dx} = -\frac{\frac{\partial z}{\partial x}}{\frac{\partial z}{\partial y}}$$

EXAMPLES 3.4: The diagram above shows the function $z = x^2 + 2y^2$.

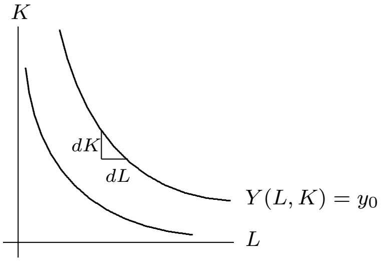
(i) Which isoquant passes through the point where $x = 2$ and $y = 1$?
At (2,1), $z = 6$, so the isoquant is $x^2 + 2y^2 = 6$.

(ii) Find the gradient of the isoquant at this point.

$$\begin{aligned} dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy &= 0 \\ \Rightarrow 2xdx + 4ydy &= 0 \\ \Rightarrow \frac{dy}{dx} &= -\frac{x}{2y} \end{aligned}$$

So at (2,1) the gradient is -1.

3.5. Economic Application: The Marginal Rate of Technical Substitution



For a production function $Y(L, K)$, the isoquants show the combinations of labour and capital that can be used to produce the same amount of output.

At any point (L, K), the slope of the isoquant is the *Marginal Rate of Technical Substitution* between labour and capital.

The $MRTS$ tells you how much less capital you need, to produce the same quantity of output, if you increase labour by a small amount dL.

Using the same method as in the previous section, we can say that along the isoquant output doesn't change:

$$\begin{aligned} dY = \frac{\partial Y}{\partial L}dL + \frac{\partial Y}{\partial K}dK &= 0 \\ \Rightarrow \frac{dK}{dL} &= -\frac{\frac{\partial Y}{\partial L}}{\frac{\partial Y}{\partial K}} \\ \Rightarrow MRTS &= -\frac{MPL}{MPK} \end{aligned}$$

The Marginal Rate of Technical Substitution between two inputs to production is the (negative of the) ratio of their marginal products.

EXERCISES 7.4: **Differentials and Isoquants**

(1) A firm is producing 1550 items per week, and its total weekly costs are £12000. If the marginal cost of producing an additional item is £8, estimate the firm's total costs if it increases weekly production to 1560 items.

(2) A firm has marginal product of labour 3.4, and marginal product of capital 2.0. Estimate the change in output if reduces both labour and capital by one unit.

(3) Find the total differential for the following functions:

   (a) $z(x, y) = 10x^3y^5$

   (b) $f(x_1, x_2, x_3, x_4) = x_1^2 + x_2x_3 - 5x_4^2 + 7$

(4) For the function $g(x, y) = x^4 + x^2y^2$, show that slope of the isoquant at the point (x, y) is $-\dfrac{2x^2 + y^2}{xy}$. What is slope of the indifference curve $g = 10$ at the point where $x = 1$?

(5) Find the Marginal Rate of Technical Substitution for the production function $Y = K(L^2 + L)$.

Further Reading and Exercises

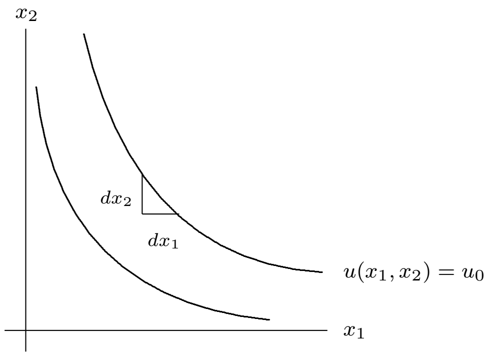
- *Jacques* §5.1 and §5.2.

4. Economic Application: Utility and Indifference Curves

If a consumer has well-behaved preferences over two goods, good 1 and good 2, his preferences can be represented by a utility function $u(x_1, x_2)$.

Then the marginal utilities of good 1 and good 2 are given by the partial derivatives:

$$MU_1 = \frac{\partial u}{\partial x_1} \text{ and } MU_2 = \frac{\partial u}{\partial x_2}$$



His indifference curves are the isoquants of u.

At any point (x_1, x_2), the slope of the isoquant is the *Marginal Rate of Substitution* between good 1 and good 2.

$$MRS = \frac{dx_2}{dx_1}$$

The MRS is the rate at which he is willing to substitute good 2 for good 1, remaining on the same indifference curve.

As before, we can find the marginal rate of substitution at any bundle (x_1, x_2) by taking the total differential of utility:

$$\begin{aligned} du = \frac{\partial u}{\partial x_1}dx_1 + \frac{\partial u}{\partial x_2}dx_2 &= 0 \\ \Rightarrow \frac{dx_2}{dx_1} &= -\frac{\frac{\partial u}{\partial x_1}}{\frac{\partial u}{\partial x_2}} \\ \Rightarrow MRS &= -\frac{MU_1}{MU_2} \end{aligned}$$

The Marginal Rate of Substitution between two goods is the (negative of the) ratio of their marginal utilities.

4.1. Perfect Substitutes

For the utility function: $u(x_1, x_2) = 3x_1 + x_2$
we find:

$$MU_1 = 3 \text{ and } MU_2 = 1 \text{ so } MRS = -3$$

The MRS is constant and negative, so the indifference curves are downward-sloping straight lines. The consumer is always willing to substitute 3 units of good 2 for a unit of good 1, whatever bundle he is consuming. The goods are perfect substitutes.

Any utility function of the form $u(x_1, x_2) = ax_1 + bx_2$, where a and b are positive parameters, represents perfect substitutes.

4.2. Cobb-Douglas Utility

For the utility function

$$u(x_1, x_2) = x_1^a x_2^b$$

where a and b are positive parameters, we find:

$$MU_1 = ax_1^{a-1}x_2^b \text{ and } MU_2 = bx_1^a x_2^{b-1}$$

$$\Rightarrow MRS = -\frac{ax_1^{a-1}x_2^b}{bx_1^a x_2^{b-1}} = -\frac{ax_2}{bx_1}$$

From this expression we can see:

- The MRS is negative so the indifference curves are downward-sloping.
- As you move along an indifference curve, increasing x_1 and decreasing x_2, the MRS becomes less negative, so the indifference curve becomes flatter. Hence the indifference curves are convex (as in the diagram on the previous page).

4.3. Transforming the Utility Function

If a consumer has a preference ordering represented by the utility function $u(x_1, x_2)$, and $f(u)$ is a monotonic increasing function (that is, $f'(u) > 0$ for all u) then $v(x_1, x_2) = f(u(x_1, x_2))$ is a utility function representing the same preference ordering, because whenever a bundle (x_1, x_2) is preferred to (y_1, y_2):

$$u(x_1, x_2) > u(y_1, y_2) \Leftrightarrow f(u(x_1, x_2)) > f(u(y_1, y_2))$$

The two utility functions represent the same preferences, provided that we are only interested in the consumer's *ordering* of bundles, and don't attach any significance to the utility numbers.

EXAMPLES 4.1: If a consumer has a Cobb-Douglas utility function:

$$u(x_1, x_2) = x_1^a x_2^b$$

$f(u) = \ln u$ is a monotonic increasing function so his preferences can also be represented by

$$v(x_1, x_2) = \ln\left(x_1^a x_2^b\right) = a \ln x_1 + b \ln x_2$$

Note that if we calculate the MRS for the function v we find:

$$MU_1 = \frac{a}{x_1} \text{ and } MU_2 = \frac{b}{x_2} \quad \text{so} \quad MRS = -\frac{ax_2}{bx_1}$$

Although their marginal utilities are different, v has the same MRS as u.

EXERCISES 7.5: Utility and Indifference curves

(1) Find the MRS for each of the following utility functions, and hence determine whether the indifference curves are convex, concave, or straight:
(a) $u = x_1^{\frac{1}{2}} + x_2^{\frac{1}{2}}$ (b) $u = x_1^2 + x_2^2$ (c) $u = \ln(x_1 + x_2)$

(2) Prove that for any monotonic increasing function $f(u)$, the utility function $v(x_1, x_2) = f(u(x_1, x_2))$ has the same MRS as the utility function u.
(*Hint*: Use the Chain Rule to differentiate v.)

Further Reading and Exercises

- *Jacques* §5.2.
- *Varian* "Intermediate Microeconomics" Chapter 4

5. The Chain Rule and Implicit Differentiation

5.1. The Chain Rule for Functions of Several Variables

If z is a function of two variables, x and y, and both x and y depend on another variable, t (time, for example), then z also depends on t. We have:

> If $z = z(x, y)$, and x and y are functions of t, then:
> $$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

Note the similarity of this rule to

- the Chain Rule for functions of one variable in Chapter 6, and
- the formula for the Total Differential in section 3.3

Examples 5.1: Suppose the aggregate output of a country is given by:

$$Y = AK^\alpha L^\beta$$

where K is the capital stock, and L is the labour force. If the capital stock and the labour force are each growing according to:

$$L(t) = L_0 e^{nt} \text{ and } K(t) = K_0 e^{mt}$$

(i) What are the proportional growth rates of capital and labour?
As in Chapter 6 we have:

$$\frac{dL}{dt} = nL_0 e^{nt} = nL(t) \text{ and similarly for } K(t)$$

So labour is growing at a constant proportional rate n, and capital is growing at constant proportional rate m.

(ii) What is the rate of growth of output?
Applying the Chain Rule:

$$\begin{aligned}
\frac{dY}{dt} &= \frac{\partial Y}{\partial K}\frac{dK}{dt} + \frac{\partial Y}{\partial L}\frac{dL}{dt} \\
\Rightarrow \frac{dY}{dt} &= \alpha AK^{\alpha-1}L^\beta \times mK + \beta AK^\alpha L^{\beta-1} \times nL \\
&= \alpha mAK^\alpha L^\beta + \beta nAK^\alpha L^\beta \\
&= (\alpha m + \beta n)Y
\end{aligned}$$

So output grows at a constant proportional rate $(\alpha m + \beta n)$.

5.2. Implicit Differentiation

The equation:

$$x^2 + 2xy + y^3 - 4 = 0$$

describes a relationship between x and y. If we plotted values of x and y satisfying this equation, they would form a curve in (x, y) space. We cannot easily rearrange the equation to find y as an *explicit* function of x, but we can think of y as an *implicit* function of x, and ask how y changes when x changes – or in other words, what is $\frac{dy}{dx}$? To answer this question, we can think of the left-hand side of the equation

$$x^2 + 2xy + y^3 - 4 = 0$$

as a function of two variables, x and y. And in general, if we have any equation:

$$f(x, y) = 0$$

we can find the gradient of the relationship between x and y in exactly the same way as we found the gradient of an isoquant in section 3.4, by totally differentiating:

$$\begin{aligned} df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy &= 0 \\ \Rightarrow \frac{\partial f}{\partial y}dy &= -\frac{\partial f}{\partial x}dx \\ \Rightarrow \frac{dy}{dx} = &= -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}} \end{aligned}$$

Finding the gradient of $y(x)$ when y is an implicit function of x is called *implicit differentiation*. The method explained here, using the total derivative, is the same as the one in *Jacques*. *Anthony and Biggs* use an alternative (but equivalent) method, applying the Chain Rule of the previous section.

Examples 5.2: *Implicit Differentiation*

(i) Find $\frac{dy}{dx}$ if $\quad x^2 + 5xy + y^3 - 4 = 0$

$$\begin{aligned} (2x + 5y)dx + (5x + 3y^2)dy &= 0 \\ (5x + 3y^2)dy &= -(2x + 5y)dx \\ \frac{dy}{dx} = &= -\frac{2x + 5y}{5x + 3y^2} \end{aligned}$$

(ii) Find $\frac{dq}{dp}$ if $\quad q^2 + e^{3p} - pe^{2q} = 0$

$$\begin{aligned} (2q - 2pe^{2q})dq + (3e^{3p} - e^{2q})dp &= 0 \\ (2q - 2pe^{2q})dq &= (e^{2q} - 3e^{3p})dp \\ \frac{dq}{dp} = &= \frac{e^{2q} - 3e^{3p}}{2(q - pe^{2q})} \end{aligned}$$

Exercises 7.6: **The Chain Rule and Implicit Differentiation**

(1) If $z(x, y) = 3x^2 + 2y^3$, $x = 2t + 1$, and $y = 4t - 2$, use the chain rule to find $\frac{dz}{dt}$ when $t = 2$.

(2) If the aggregate production function is $Y = K^{\frac{1}{3}}L^{\frac{2}{3}}$, what is the marginal product of labour? If the labour force grows at a constant proportional rate n, and the capital stock grows at constant proportial rate m, find the proportional growth rate of (a) output, Y and (b) the MPL.

(3) If $xy^2 + y - 2x = 0$, find $\dfrac{dy}{dx}$ when $y = 2$.

Further Reading and Exercises

- *Jacques* §5.1
- *Anthony and Biggs* §11.3, §12.1, §12.2 and §12.3.

6. Comparative Statics

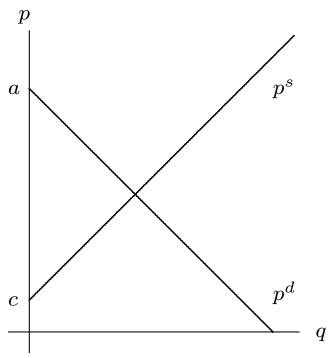
Examples 6.1:

(i) Suppose the inverse demand and supply functions for a good are:

$$p^d = a - q \quad \text{and} \quad p^s = bq + c \quad \text{where } b > 0 \text{ and } a > c > 0$$

How does the quantity sold change (a) if a increases (b) if b increases?

Putting $p^d = p^s$ and solving for the equilibrium quantity: $\quad q^* = \dfrac{a-c}{1+b}$



We can answer the question by differentiating with respect to the parameters:

$$\frac{\partial q^*}{\partial a} = \frac{1}{1+b} > 0$$

This tells us that if a increases – that is, there is an upward shift of the demand function – the equilibrium quantity will increase. (You can, of course, see this from the diagram.)

Similarly $\dfrac{\partial q^*}{\partial b} < 0$, so if the supply function gets steeper q^* decreases.

(ii) Suppose the inverse demand and supply functions for a good are:

$$p^d = a + f(q) \quad \text{and} \quad p^s = g(q) \quad \text{where } f'(q) < 0 \text{ and } g'(q) > 0$$

How does the quantity sold change if a increases?

Here we don't know much about the supply and demand functions, except that they slope up and down in the usual way. The equilibrium quantity q^* satisfies:

$$a + f(q^*) = g(q^*)$$

We cannot solve explicitly for q^*, but this equation defines it as an implicit function of a. Hence we can use implicit differentiation:

$$da + f'(q^*)dq^* = g'(q^*)dq^* \quad \Rightarrow \quad \frac{dq*}{da} = \frac{1}{g'(q^*) - f'(q^*)} > 0$$

Again, this tells us that q^* increases when the demand function shifts up.

The process of using derivatives to find out how an equilibrium depends on the parameters is called *comparative statics*.

Exercises 7.7: If the labour supply and demand functions are

$$l^s = aw \text{ and } l^d = \frac{k}{w} \quad \text{where } a \text{ and } k \text{ are positive parameters:}$$

(1) What happens to the equilibrium wage and employment if k increases?

(2) What happens to the equilibrium wage and employment if a *decreases*?

Further Reading and Exercises

- *Jacques* §5.3 gives some macroeconomic applications of Comparative Statics.

Solutions to Exercises in Chapter 7

EXERCISES 7.1:

(1) $\frac{\partial f}{\partial x} = 6x - y^4$ $\frac{\partial f}{\partial y} = -4xy^3$

(2) $\frac{\partial h}{\partial x} = 2(x+1)(y+2)$
$\frac{\partial h}{\partial y} = (x+1)^2$

(3) $\frac{\partial g}{\partial x} = \frac{1}{xy}$ $\frac{\partial g}{\partial y} = -\frac{\ln x}{y^2}$

EXERCISES 7.2:

(1) $f_x = 2x + 3y$, $f_y = 3x$,
$f_{xx} = 2$, $f_{yy} = 0$,
$f_{yx} = f_{xy} = 3$.

(2) $g_p = 2q^2 e^{2p+1}$,
$g_q = 2qe^{2p+1} + r$,
$g_r = q$,
$g_{pp} = 4q^2 e^{2p+1}$,
$g_{qq} = 2e^{2p+1}$,
$g_{rr} = 0$,
$g_{qp} = g_{pq} = 4qe^{2p+1}$,
$g_{rp} = g_{pr} = 0$,
$g_{rq} = g_{qr} = 1$.

(3) $\frac{\partial z}{\partial x} = \frac{2}{2x+3y} > 0$
$\frac{\partial z}{\partial y} = \frac{3}{2x+3y} > 0$

(4) $f_x = 6x(x^2 + y^2 + z^2)^2$
$f_{xx} = 6(x^2 + y^2 + z^2)^2 + 24x^2(x^2 + y^2 + z^2)$
$= 6(x^2 + y^2 + z^2)$
$\times (5x^2 + y^2 + z^2)$
$f_{xz} = 24xz(x^2 + y^2 + z^2)$

(5) $F_K = \alpha A K^{\alpha-1} L^{\beta}$
$F_L = \beta A K^{\alpha} L^{\beta-1}$
$F_{KL} = F_{LK}$
$= \alpha\beta A K^{\alpha-1} L^{\beta-1}$

(6) $u_{23} = 2$, $u_{12} = 0$

EXERCISES 7.3:

(1) (a) $\frac{\partial Q}{\partial L} = 2L(K-2)$
$= 24$
(b) $\frac{\partial Q}{\partial K} = L^2 = 9$
(c) It increases:
$\frac{\partial^2 Q}{\partial L \partial K} = 2L > 0$

(2) $Y(\lambda K, \lambda L)$
$= (\lambda^{\frac{1}{2}} K^{\frac{1}{2}} + \lambda^{\frac{1}{2}} L^{\frac{1}{2}})^2$
$= (\lambda^{\frac{1}{2}} (K^{\frac{1}{2}} + L^{\frac{1}{2}})^2$
$= \lambda (K^{\frac{1}{2}} + L^{\frac{1}{2}})^2$
$= \lambda Y(K, L)$
$F_K = K^{-\frac{1}{2}}(K^{\frac{1}{2}} + L^{\frac{1}{2}})$
$F_L = L^{-\frac{1}{2}}(K^{\frac{1}{2}} + L^{\frac{1}{2}})$
$\Rightarrow KF_K + LF_L$
$= (K^{\frac{1}{2}} + L^{\frac{1}{2}})^2$

(3) (a) $\epsilon_{11} = -2$, $\epsilon_{12} = 1$,
$\eta_1 = 1$
(b) $x_1(\lambda p_1, \lambda p_2, \lambda p_m)$
$= \frac{\lambda p_2 \lambda m}{\lambda^2 p_1^2}$
$= \frac{p_2 m}{p_1^2}$
If all prices double, but income doubles too, demands shouldn't change.

(4) $p_1 \frac{\partial x_1}{\partial p_1} + p_2 \frac{\partial x_1}{\partial p_2} + m \frac{\partial x_1}{\partial m} = 0$
Dividing by x_1 gives:
$\epsilon_{11} + \epsilon_{12} + \eta_1 = 0$

EXERCISES 7.4:

(1) $\Delta C = \frac{dC}{dq} \Delta q = 8 \times 10$
$= 80$
$\Rightarrow C = £12000 + £80$
$= £12080$

(2) $\Delta Y = 3.4 \times -1 + 2 \times -1$
$= -5.4$

(3) (a) $dz = 30x^2 y^5 dx$
$+ 50x^3 y^4 dy$
(b) $df = 2x_1 dx_1$
$+ x_3 dx_2 + x_2 dx_3$
$- 10x_4 dx_4$

(4) $(4x^3 + 2xy^2)dx$
$+ 2x^2 y dy = 0$
$\Rightarrow \frac{dy}{dx} = -\frac{4x^3 + 2xy^2}{2x^2 y}$
$= -\frac{2x^2 + y^2}{xy}$
When $x = 1$ and $g = 10$, $y = 3$.
Then $\frac{dy}{dx} = -\frac{11}{3}$

(5) $MRTS = -\frac{2KL+K}{L^2+L}$
$= -\frac{K(2L+1)}{L(L+1)}$

EXERCISES 7.5:

(1) (a) $MRS = -(\frac{x_2}{x_1})^{\frac{1}{2}}$
Convex
(b) $MRS = -(\frac{x_1}{x_2})$
Concave
(c) $MRS = -1$
Straight

(2) $\frac{\partial v}{\partial x_1} = f'(u) \frac{\partial u}{\partial x_1}$ and
$\frac{\partial v}{\partial x_2} = f'(u) \frac{\partial u}{\partial x_2}$
so the ratio of the MUs is the same for u and v.

EXERCISES 7.6:

(1) $\frac{dz}{dt} = 6x \times 2 + 6y^2 \times 4$
$= 6 \times 5 \times 2 + 6 \times 36 \times 4 = 924$

(2) $MPL = \frac{2}{3} K^{\frac{1}{3}} L^{-\frac{1}{3}}$
(a) $\frac{1}{3}m + \frac{2}{3}n$
(b) $\frac{1}{3}m - \frac{1}{3}n$

(3) When $y = 2$, $x = -1$.
$\frac{dy}{dx} = \frac{2-y^2}{2xy+1} = \frac{2}{3}$

EXERCISES 7.7:

(1) In equilibrium, $w = \sqrt{\frac{k}{a}}$ and $l = \sqrt{ak}$.
$\frac{\partial w}{\partial k} = \frac{1}{2\sqrt{ak}} > 0$ and
$\frac{\partial l}{\partial k} = \frac{1}{2}\sqrt{\frac{a}{k}} > 0$
so both w and l increase when k increases.

(2) $\frac{\partial w}{\partial a} = -\frac{1}{2}\sqrt{k} a^{-\frac{3}{2}} < 0$ and
$\frac{\partial l}{\partial a} = \frac{1}{2}\sqrt{\frac{k}{a}} > 0$
so w increases and l decreases when a decreases.

[1]This Version of Workbook Chapter 7: September 25, 2014

Worksheet 7: Partial Differentiation

Quick Questions

(1) Obtain the first-order partial derivatives of each of the following functions:

$$f(x,y) = 4x^2y + 3xy^3 + 6x; \qquad g(x,y) = e^{2x+3y}; \qquad h(x,y) = \frac{x+y}{x-y}.$$

(2) Show that the function

$$f(x,y) = \sqrt{xy} + \frac{x^2}{y}$$

is homogeneous of degree 1. Find both partial derivatives and hence verify that it satisfies Euler's Theorem.

(3) A firm has production function $Q(K,L) = KL^2$, and faces demand function $P^d(Q) = 120 - 1.5Q$.
  (a) Find the marginal products of labour and capital when $L = 3$ and $K = 4$.
  (b) Write down the firm's revenue, R, as a function of output, Q, and find its marginal revenue.
  (c) Hence, using the chain rule, find the marginal revenue product of labour when $L = 3$ and $K = 4$.

(4) A consumer has a quasi-linear utility function $u(x_1,x_2) = x_1^{\frac{1}{2}} + x_2$. Find the marginal utilities of both goods and the marginal rate of substitution. What does the MRS tell us about the shape of the indifference curves? Sketch the indifference curves.

(5) For a firm with Cobb-Douglas production function $Y(K,L) = aK^{\alpha}L^{\beta}$, show that the marginal rate of technical substitution depends on the capital-labour ratio. Show on a diagram what this means for the shape of the isoquants.

(6) The supply and demand functions in a market are

$$Q^s = kp^2 \quad \text{and} \quad Q^d = D(p)$$

where k is a positive constant and the demand function slopes down: $D'(p) < 0$.
  (a) Write down the equation satisfied by the equilibrium price.
  (b) Use implicit differentiation to find how the price changes if k increases.

Longer Questions

(1) A consumer's demand for good 1 depends on its own price, p_1, the price of good 2, p_2, and his income, y, according to the formula

$$x_1(p_1,p_2,y) = \frac{y}{p_1 + \sqrt{p_1p_2}}.$$

  (a) Show that $x_1(p_1,p_2,y)$ is homogeneous of degree zero. What is the economic interpretation of this property?
  (b) The own-price elasticity of demand is given by the formula

$$\epsilon_{11}(p_1,p_2,y) = \frac{p_1}{x_1}\frac{\partial x_1}{\partial p_1}.$$

  Show that $|\epsilon_{11}|$ is always less than 1.

(2) Tony and Gordon have utility functions:

$$\begin{aligned} \text{Tony} &: \quad U_T(x,y) = x^4y^2 \\ \text{Gordon} &: \quad U_G(x,y) = \ln x + \tfrac{1}{2}\ln y \end{aligned}$$

(a) Find their marginal utilities for both goods, and show that they both have the same marginal rate of substitution.
(b) What is the equation of Tony's indifference curve corresponding to a utility level of 16? Show that it is convex, and draw it.
(c) Does Gordon have the same indifference curves? Explain carefully how their indifference curves are related.
(d) Show that Gordon has "diminishing marginal utility", but Tony does not.
(e) Do Tony and Gordon have the same preferences?

(3) Robinson Crusoe spends his 9 hour working day either fishing or looking for coconuts. If he spends t hours fishing, and s hours looking for coconuts, he will catch $f(t) = t^{\frac{1}{2}}$ fish, and collect $c(s) = 2s^{\frac{1}{3}}$ coconuts.

(a) If Crusoe's utility function is $U(f,c) = \ln(f) + \ln(c)$, show that the marginal rate at which he will substitute coconuts for fish is given by the expression:

$$MRS = -\frac{c}{f}$$

(b) Show that production possibility frontier of the economy is:

$$f^2 + \left(\frac{c}{2}\right)^3 = 9$$

(c) Find an expression for the marginal rate of transfomation between fish and coconuts.
(d) Crusoe plans to spend 1 hour fishing, and the rest of his time collecting coconuts. If he adopts this plan:
   (i) How many of each can he consume?
   (ii) What is his MRS?
   (iii) What is his MRT?
   (iv) Would you recommend that he should spend more or less time fishing? Why? Draw a diagram to illustrate.
(e) What would be the optimal allocation of his time?

CHAPTER 8

Unconstrained Optimisation Problems with One or More Variables

In Chapter 6 we used differentiation to solve optimisation problems. In this chapter we look at more economic applications of this technique: **profit maximisation** in **perfect competition** and **monopoly**; and **strategic optimisation** problems that arise in **oligopoly** or when there are **externalities**. Then we show how to **maximise or minimise a function of more than one variable**, and again look at applications.

1. The Terminology of Optimisation

Suppose that an economic agent wants to choose some value y to maximise a function $\Pi(y)$. For example, y might be the quantity of output and $\Pi(y)$ the profit of a firm. From Chapters 5 and 6 we know how to solve problems like this, by differentiating.

The agent's optimisation problem is:

$$\max_y \Pi(y)$$

$\Pi(y)$ is called the agent's *objective function* - the thing he wants to optimise.

To solve the optimisation problem above, we can look for a value of y that satisfies two conditions:

The first-order condition: $\dfrac{d\Pi}{dy} = 0$

and the second-order condition: $\dfrac{d^2\Pi}{dy^2} < 0$

But remember that if we find a value of y satisfying these conditions it is not necessarily the optimal choice, because the point may not be the *global* maximum of the function. Also, some functions may not have a global maximum, and some may have a maximum point for which the second derivative is zero, rather than negative. So it is always important to think about the shape of the objective function.

An optimisation problem may involve minimising rather than maximising – for example choosing output to minimise average cost:

$$\min_q AC(q)$$

The first- and second-order conditions are, of course, $\frac{dAC}{dq} = 0$ and $\frac{d^2AC}{dq^2} > 0$, respectively.

2. Profit Maximisation

Suppose a firm has revenue function $R(y)$ and cost function $C(y)$, where y is the the quantity of output that it produces. Its profit function is:

$$\Pi(y) = R(y) - C(y)$$

The firm wants to choose its output to maximise profit. Its optimisation problem is:

$$\max_y \Pi(y)$$

The first-order condition for profit maximisation is:

$$\frac{d\Pi}{dy} = R'(y) - C'(y) = 0$$

which is the familiar condition that *marginal revenue* equals *marginal cost*.

EXAMPLES 2.1: A monopolist has inverse demand function $P(y) = 35 - 3y$ and cost function $C(y) = 50y + y^3 - 12y^2$.

(i) What is the profit function?

$$\begin{aligned} \Pi(y) &= yP(y) - C(y) \\ &= (35 - 3y)y - (50y + y^3 - 12y^2) \\ &= -y^3 + 9y^2 - 15y \end{aligned}$$

(ii) What is the optimal level of output?
The first-order condition is:

$$\frac{d\Pi}{dy} = -3y^2 + 18y - 15 = 0$$

There are two solutions: $y = 1$ or $y = 5$.
Check the second-order condition:

$$\frac{d^2\Pi}{dy^2} = -6y + 18$$

When $y = 1$, $\frac{d^2\Pi}{dy^2} = 12$ so this is a minimum point.
When $y = 5$, $\frac{d^2\Pi}{dy^2} = -12$ so this is the optimal level of output.

(iii) What is the market price, and how much profit does the firm make?
The firm chooses $y = 5$. From the market demand function the price is $P(5) = 20$.
The profit is $\Pi(5) = 25$.

2.1. Perfect Competition

A firm operating in a perfectly competitive market is a price-taker. If p is the market price, its revenue function is $R(y) = py$ and its profit function is:

$$\Pi(y) = py - C(y)$$

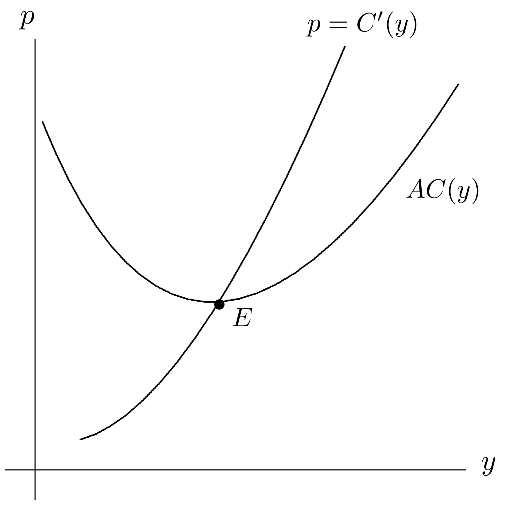
Again, the firm's optimisation problem is:

$$\max_y \Pi(y)$$

The first-order condition (FOC) is:

$$\begin{aligned} \frac{d\Pi}{dy} = p - C'(y) &= 0 \\ \Rightarrow p &= C'(y) \end{aligned}$$

A competitive firm alway chooses output so that price = marginal cost.



$p = C'(y)$ is the inverse supply function for a competitive firm.

Remember from Chapter 6 that the MC function passes through the minimum point of the average cost function.

In the long-run, firms enter a competitive market until (for the marginal firm) profit is zero:

$$\begin{aligned} py - C(y) &= 0 \\ \Rightarrow p &= \frac{C(y)}{y} = AC(y) \end{aligned}$$

Point E is the long-run equilibrium.

At point E: $p = MC$ (the firm is profit-maximising) and $p = AC$ (it is making zero-profit).

EXAMPLES 2.2: In a competitive market, all firms have cost-functions $C(y) = y^2 + y + 4$. The market demand function is $Q = 112 - 2p$. Initially there are 40 firms.

(i) What is the market supply function?
If the market price is p, each firm has profit function:

$$\Pi(y) = py - (y^2 + y + 4)$$

The firm chooses output to maximise profit. The FOC is:

$$p = 2y + 1$$

This is the firm's inverse supply function. (The SOC is satisfied: $\frac{d^2\Pi}{dy^2} = -2$.)
The firm's supply function is:

$$y = \frac{p-1}{2}$$

There are 40 firms, so total market supply is given by:

$$\begin{aligned} Q &= 40y \\ \Rightarrow Q &= 20(p-1) \end{aligned}$$

(ii) What is the market price and quantity in the short-run?
Equating market demand and supply:

$$112 - 2p = 20(p-1) \quad \Rightarrow \quad p = 6$$

Then from the demand function $Q = 112 - 2p = 100$.
The market price is 6, and the total quantity is 100.
Since there are 40 firms, each firm produces 2.5 units of output.

(iii) How much profit do firms make in the short-run?

$$\Pi(y) = py - (y^2 + y + 4)$$

When $p = 6$ and $y = 2.5$, profit is 2.25.

(iv) What happens in the long-run?
Since firms are making positive profits, more firms will enter. In the long-run, all firms will produce at minimum average cost:

$$\begin{aligned} AC(y) &= y + 1 + \frac{4}{y} \\ \frac{dAC}{dy} &= 1 - \frac{4}{y^2} = 0 \\ \Rightarrow y &= 2 \text{ and } AC = 5 \end{aligned}$$

So the long-run market price is $p = 5$, and every firm produces 2 units of output.
At this price, market demand is: $Q = 112 - 2p = 102$.
Hence there will be 51 firms in the market.

2.2. Comparing Perfect Competition and Monopoly

For all firms the FOC for profit maximisation is *marginal revenue* = *marginal cost*:

$$R'(y) = C'(y)$$

For a *competitive firm*:

$$\begin{aligned} R(y) &= py \\ \Rightarrow R'(y) &= p \end{aligned}$$

Hence, at the optimal choice of output, price equals marginal cost:

$$p = C'(y)$$

A *monopolist* faces a downward-sloping inverse demand function $p(y)$ where $p'(y) < 0$:

$$\begin{aligned} R(y) &= p(y)y \\ \Rightarrow R'(y) &= p(y) + p'(y)y \quad \text{(Chain Rule)} \\ &< p(y) \end{aligned}$$

So marginal revenue is less than price.
Hence, at the optimal choice of output, price is greater than marginal cost:

$$C'(y) = R'(y) < p(y)$$

Examples 2.3: Compare the price and marginal cost for the monopolist in Examples 2.1.

$$\begin{aligned} C(y) &= 50y + y^3 - 12y^2 \\ \Rightarrow C'(y) &= 50 + 3y^2 - 24y \end{aligned}$$

We found that the optimal choice of output was $y = 5$ and the market price was $P = 20$.
The marginal cost is $C'(5) = 5$.

Exercises 8.1: **Perfect Competition and Monopoly**

(1) A monopolist has fixed cost $F = 20$ and constant marginal cost $c = 4$, and faces demand function $Q = 30 - P/2$.

(a) What is the inverse demand function?

(b) What is the cost function?

(c) What is the firm's profit function?

(d) Find the firm's optimal choice of output.

(e) Find the market price and show that it is greater than marginal cost.

(2) What is the supply function of a perfectly competitive firm with cost function $C(y) = 3y^2 + 2y + 4$?

(3) What is the long-run equilibrium price in a competitive market where all firms have cost function $C(q) = 8q^2 + 128$?

(4) In a competitive market the demand function is $Q = 120/P$. There are 40 identical firms, each with cost function $C(q) = 2q^{\frac{3}{2}} + 1$. Find:

(a) The inverse supply function of each firm.

(b) The supply function of each firm.

(c) The market supply function.

(d) The market price and quantity.

Is the market in long-run equilibrium?

(5) A monopolist faces inverse demand function $p = 720 - \frac{1}{4}q^3 + \frac{19}{3}q^2 - 54q$, and has cost function $C(q) = 540q$. Show that the profit function has three stationary points: $q = 3$, $q = 6$, and $q = 10$. Classify these points. What level of output should the firm choose? Sketch the firm's demand function, and the marginal revenue and marginal cost functions.

Further Reading and Exercises

- *Jacques* §4.6 and §4.7
- *Anthony & Biggs* §§9.2, 9.3, 10.1 and 10.2
- *Varian* Chapters 19-24

3. Strategic Optimisation Problems

In some economic problems an agent's *payoff* (his profit or utility) depends on his own choices *and* the choices of other agents. So the agent has to make a strategic decision: his optimal choice depends on what he expects the other agents to do.

Suppose we have two agents. Agent 1's payoff is $\Pi_1(y_1, y_2)$ and agent 2's payoff is $\Pi_2(y_1, y_2)$. y_1 is a choice made by agent 1, and y_2 is a choice made by agent 2. Suppose agent 1 expects agent 2 to choose y_2^e. Then his optimisation problem is:

$$\max_{y_1} \Pi_1(y_1, y_2^e)$$

The first- and second-order conditions for this problem are:

$$\frac{\partial \Pi_1}{\partial y_1} = 0 \quad \text{and} \quad \frac{\partial^2 \Pi_1}{\partial y_1^2} < 0$$

Note that we use partial derivatives, because agent 1 treats y_2^e as given - as a constant that he cannot affect. Similarly agent 2's problem is:

$$\max_{y_2} \Pi_2(y_1^e, y_2)$$

We can solve both of these problems and look for a pair of optimal choices y_1 and y_2 where the agents' expectations are fulfilled: $y_1 = y_1^e$ and $y_2 = y_2^e$. This will be a *Nash equilibrium*, in which both agents make optimal choices given the choice of the other.

3.1. Oligopoly

When there are few firms in a market, the decisions of one firm affect the profits of the others. We can model firms as choosing quantities or prices.

EXAMPLES 3.1: *The Cournot Model: Simultaneous Quantity Setting*
There are two firms in a market where the demand function is $Q = 20 - \frac{1}{2}P$. Each firm has cost function $C(q) = 4q + 5$. The firms choose their quantities simultaneously.

(i) If firm 1 chooses quantity q_1, and firm 2 chooses q_2, what is the market price?

The inverse demand function is $P = 40 - 2Q$. The total quantity is $Q = q_1 + q_2$. Hence the market price is:

$$P = 40 - 2(q_1 + q_2)$$

(ii) What is firm 1's profit, as a function of q_1 and q_2?

$$\begin{aligned} \Pi_1(q_1, q_2) &= q_1(40 - 2(q_1 + q_2)) - (4q_1 + 5) \\ &= 36q_1 - 2q_1^2 - 2q_1q_2 - 5 \end{aligned}$$

(iii) If firm 1 expects firm 2 to choose q_2^e, what is its optimal choice of output?

Firm 1's optimisation problem is:

$$\max_{q_1} \Pi_1(q_1, q_2^e) \quad \text{where} \quad \Pi_1(q_1, q_2^e) = 36q_1 - 2q_1^2 - 2q_1q_2^e - 5$$

The first-order condition is:

$$\begin{aligned} \frac{\partial \Pi_1}{\partial q_1} &= 36 - 4q_1 - 2q_2^e = 0 \\ \Rightarrow q_1 &= 9 - \tfrac{1}{2}q_2^e \end{aligned}$$

This is the firm's reaction function. Note that $\frac{\partial^2 \Pi_1}{\partial q_1^2} = -4$, so the second-order condition is satisfied.

(iv) If firm 2 expects firm 1 to choose q_1^e, what is its optimal choice of output?
In just the same way: $q_2 = 9 - \frac{1}{2}q_1^e$

(v) How much does each firm produce in equilibrium?

In equilibrium, both firms' expectations are fullfilled: $q_1 = q_1^e$ and $q_2 = q_2^e$. Hence:

$$q_1 = 9 - \tfrac{1}{2}q_2 \quad \text{and} \quad q_2 = 9 - \tfrac{1}{2}q_1$$

Solving this pair of simultaneous equations we find:

$$q_1 = q_2 = 6$$

(vi) How much profit does each firm make?

From the profit function above: $\Pi_1(6, 6) = 67$. The profit for firm 2 is the same.

Examples 3.2: *The Stackelberg Model*
Suppose that, in the market of the previous example, with demand function $Q = 20 - \frac{1}{2}P$ and cost functions $C(q) = 4q + 5$, firm 1 chooses its quantity first, and firm 2 observes firm 1's choice before choosing its own quantity.

(i) What is firm 2's reaction function?

When firm 2 makes its choice, it already knows the value of q_1. So its optimisation problem is:

$$\max_{q_2} \Pi_2(q_1, q_2) \quad \text{where} \quad \Pi_2(q_1, q_2) = 36q_2 - 2q_2^2 - 2q_1q_2 - 5$$

As in the previous example its reaction function is:

$$q_2 = 9 - \tfrac{1}{2}q_1$$

(ii) What is firm 1's optimal choice?

As before, firm 1's profit is: $\Pi_1(q_1, q_2) = 36q_1 - 2q_1^2 - 2q_1q_2 - 5$
but this time it knows that firm 2 will choose $q_2 = 9 - \frac{1}{2}q_1$.
So we can write the problem as

$$\max_{q_1} 36q_1 - 2q_1^2 - 2q_1(9 - \tfrac{1}{2}q_1) - 5$$

Solving this problem we obtain $q_1 = 9$.

(iii) How much output does each firm produce?

We know that firm 1 will choose $q_1 = 9$. Therefore, from its reaction function, firm 2 chooses $q_2 = 4.5$.

3.2. Externalities

An externality occurs when the decision of one agent directly affects the profit or utility of another. It usually causes inefficiency, as the following example illustrates.

EXAMPLES 3.3: *A mutual positive consumption externality*
Two neighbours, A and B, each get pleasure $u(R) = 24R^{\frac{1}{2}}$ from the number of roses they can see. If A has r_A roses in her garden, and B has r_B roses in his, then the number of roses that A can see is $R_A = r_A + \frac{1}{2}r_B$. Similarly B can see $R_B = r_B + \frac{1}{2}r_A$. The cost of planting one rose is $c = 2$. Each chooses the number of roses to plant to maximise their own net utility, taking the other's choice as given. Hence, for example, A's optimisation problem is:

$$\max_{r_A} u(R_A) - cr_A \quad \text{where} \quad R_A = r_A + \tfrac{1}{2}r_B$$

(i) If A believes that B will choose r_B^e, how many roses should she plant?

We can write A's optimisation problem as:

$$\max_{r_A} 24(r_A + \tfrac{1}{2}r_B^e)^{\frac{1}{2}} - 2r_A$$

The first-order condition is:

$$12(r_A + \tfrac{1}{2}r_B^e)^{-\frac{1}{2}} = 2$$

Rearranging we obtain:

$$r_A = 36 - \tfrac{1}{2}r_B^e$$

This is A's reaction function to B's choice. The second-derivative of the objective function is: $-6(r_A + \frac{1}{2}r_B^e)^{-\frac{3}{2}} < 0$, so the second order-condition is satisfied.

(ii) If B believes that A will choose r_A^e, how many roses should he plant?
In just the same way,

$$r_B = 36 - \tfrac{1}{2}r_A^e$$

(iii) How many will each plant in equilibrium?

In equilibrium, the expectations of each neighbour are fulfilled: $r_A = r_A^e$ and $r_B = r_b^e$. We have a pair of simultaneous equations:

$$r_A = 36 - \tfrac{1}{2}r_B \quad \text{and} \quad r_B = 36 - \tfrac{1}{2}r_A$$

Solving gives: $r_A = r_B = 24$

(iv) Suppose the neighbours could make an agreement that each will plant the same number of roses, r. How many should they choose?

If r roses are planted by each, each of them will be able to see $\frac{3}{2}r$, so each will obtain net utility:

$$24\left(\tfrac{3}{2}r\right)^{\frac{1}{2}} - 2r$$

Choosing r to maximise this, the first-order condition is:

$$18\left(\tfrac{3}{2}r\right)^{-\frac{1}{2}} = 2$$

Solving, we find that each should plant $r = 54$ roses. This is an efficient outcome, giving both neighbours higher net utility than when they optimise separately.

Exercises 8.2: **A Strategic Optimisation Problem**
Two firms produce differentiated products. Firm 1 produces good 1, for which the demand function is:

$$q_1(p_1, p_2) = 10 - 2p_1 + p_2$$

where p_1 is the price of good 1 and p_2 is the price of good 2. Similarly firm 2's demand function is:

$$q_2(p_1, p_2) = 10 - 3p_2 + p_1$$

Both firms have constant unit costs $c = 3$. They each choose their own price, taking the other's price as given.

(1) Write down the profit function for each firm, as a function of the prices p_1 and p_2.

(2) If firm 1 expects firm 2 to choose p_2^e, what price will it choose?

(3) Similarly, find firm 2's reaction function to firm 1's price.

(4) What prices are chosen in equilibrium, and how much of each good is sold?

Further Reading and Exercises

- *Varian* (6th Edition) Chapters 27, 28 and 33

4. Finding Maxima and Minima of Functions of Two Variables

For a function of two variables $f(x, y)$ we can try to find out which pair of values, x and y, give the largest value for f. If we think of the function as a land surface in 3-dimensions, with $f(x, y)$ representing the height of the land at map-coordinates (x, y), then a maximum point is the top of a hill, and a minimum point is the bottom of a valley. As for functions of one variable, the gradient is zero at maximum and minimum points - whether you move in the x-direction or the y-direction:

> The **first order conditions** for a maximum or minimum point of $f(x, y)$ are:
> $$\frac{\partial f}{\partial x} = 0 \quad \text{and} \quad \frac{\partial f}{\partial y} = 0$$

The gradient may be zero at other points which are not maxima and minima - for example you can imagine the three-dimensional equivalent of a point of inflexion, where on the side of a hill the land momentarily flattens out but then the gradient increases again. There might also be a *saddle point* - like the point in the middle of a horse's back which if you move in the direction of the head or tail is a local minimum, but if you move sideways is a maximum point.

Having found a point that satisfies the first-order conditions (a stationary point) we can check the second-order conditions to see whether it is a maximum, a minimum, or something else. Writing the second-order partial derivatives as $f_{xx} = \frac{\partial^2 f}{\partial x^2}$, $f_{xy} = \frac{\partial^2 f}{\partial x \partial y}$ etc:

> **Second-order conditions:**
> A stationary point of the function $f(x, y)$ is:
> a maximum point if $f_{xx} < 0$, $f_{yy} < 0$ and $f_{xx}f_{yy} - f_{xy}^2 > 0$
> a minimum point if $f_{xx} > 0$, $f_{yy} > 0$ and $f_{xx}f_{yy} - f_{xy}^2 > 0$
> a saddle-point if $f_{xx}f_{yy} - f_{xy}^2 < 0$

Why are the second-order conditions so complicated? If you think about a maximum point you can see that the gradient must be decreasing in the x- and y-directions, so we must have $f_{xx} < 0$ and $f_{yy} < 0$. But this is not enough - it is possible to imagine a stationary point where if you walked North, South, East or West you would go down-hill, but if you walked North-East you would go uphill. The condition $f_{xx}f_{yy} - f_{xy}^2 > 0$ guarantees that the point is a maximum in *all* directions.

Note that, as for functions of one variable, we find local optima by solving the first- and second-order conditions. We need to think about the shape of the whole function to decide whether we have found the global optimum.

4.1. Functions of More Than Two Variables

Similarly, for a function of several variables $f(x_1, x_2, \ldots, x_n)$ the first-order conditions for maxima and minima are that all the first-order partial derivatives are zero: $f_1 = 0, f_2 = 0, \ldots, f_n = 0$. The second-order conditions are beyond the scope of this *Workbook*.

EXAMPLES 4.1: Find and classify the stationary points of the following functions:

(i) $f(x,y) = -x^2 + xy - 2y^2 - 3x + 12y + 50$
The first-order conditions are:

$$f_x = -2x + y - 3 = 0 \quad \text{and} \quad f_y = x - 4y + 12 = 0$$

Solving, there is one stationary point: $x = 0, y = 3$.
Find the second-order partial derivatives:

$$f_{xx} = -2 \quad f_{yy} = -4 \quad f_{xy} = 1$$

So at $(0,3)$ (and everywhere else, for this function):

$$f_{xx} < 0 \quad f_{yy} < 0 \quad f_{xx}f_{yy} - f_{xy}^2 = 8 - 1 > 0$$

Hence the function has a maximum point at $(0,3)$, and the maximum value is $f(0,3) = 68$.

(ii) $g(x,y) = y^3 - 12y + x^2e^y$

The first-order conditions are:

$$g_x = 2xe^y = 0 \quad \text{and} \quad g_y = 3y^2 - 12 + x^2e^y = 0$$

From the first of these, $x = 0$ (since e^y is positive for all values of y).
Putting $x = 0$ in the second equation, and solving, gives $y = \pm 2$.
So there are two stationary points, at $(0,2)$ and $(0,-2)$.

To classify them, find the second-order partial derivatives:

$$g_{xx} = 2e^y \quad g_{yy} = 6y + x^2e^y \quad g_{xy} = 2xe^y$$

At $(0,2)$, $g_{xx} > 0$, $g_{yy} > 0$ and $g_{xx}g_{yy} - g_{xy}^2 > 0$.
– This is a minimum point, and $g(0,2) = -16$.
At $(0,-2)$, $g_{xx}g_{yy} - g_{xy}^2 < 0$.
– This is a saddle point, and $g(0,-2) = 16$.

Thinking about this function, we can see that there is no global maximum or minimum, because when $x = 0$, $g(x,y) \to \infty$ as $y \to \infty$ and $g(x,y) \to -\infty$ as $y \to -\infty$.

EXERCISES 8.3: **Finding Maxima and Minima of Functions of Two Variables**

(1) Find and classify the stationary point of $f(x,y) = x^2 + 4y^2 + 11y - 7x - xy$

(2) Show that the function $g(x,y) = e^{xy} + 3x^2 + xy - 12x - 4y$ has a minimum point where $y = 0$.

(3) Find and classify the stationary points of $h(x,y) = 3x^2 - 12y - 2(x - 2y)^3$.
Hint: it is easier to solve this if you *don't* multiply out the brackets.

Further Reading and Exercises

- *Jacques* §5.4
- *Anthony & Biggs* Chapter 13

5. Optimising Functions of Two Variables: Economic Applications

5.1. Joint Products

Sometimes two different goods can be produced more cheaply together, by one firm, than by two separate firms – for example, if one good is a by-product of the production process of another.

EXAMPLES 5.1: Suppose a firm produces two goods with joint cost function

$$C(q_1, q_2) = 2q_1^2 + q_2^2 - q_1 q_2$$

The markets for both products are competitive, with prices p_1 and p_2.
Find the firm's supply functions for the two goods.

(i) The firm's maximisation problem is:

$$\max_{q_1,q_2} \Pi(q_1, q_2) \quad \text{where} \quad \Pi(q_1, q_2) = p_1 q_1 + p_2 q_2 - (2q_1^2 + q_2^2 - q_1 q_2)$$

(ii) The first-order conditions are:

$$\frac{\partial \Pi}{\partial q_1} = p_1 - 4q_1 + q_2 = 0 \quad \text{and} \quad \frac{\partial \Pi}{\partial q_2} = p_2 - 2q_2 + q_1 = 0$$

(iii) The second-order partial derivatives are:

$$\frac{\partial^2 \Pi}{\partial q_1^2} = -4 \quad \frac{\partial^2 \Pi}{\partial q_2^2} = -2 \quad \frac{\partial^2 \Pi}{\partial q_1 \partial q_2} = 1$$

So if we can find a stationary point, it must be a maximum.

(iv) Solving the first-order conditions for q_1 and q_2 gives:

$$q_1 = \frac{2p_1 + p_2}{7} \quad \text{and} \quad q_2 = \frac{4p_2 + p_1}{7}$$

These are the firm's optimal choices of quantity, as functions of the prices: they are the firm's supply functions for the two products. Note that:

$$\frac{\partial q_1}{\partial p_2} > 0 \quad \text{and} \quad \frac{\partial q_2}{\partial p_1} > 0$$

The goods are complements in production - when the price of one product goes up, more is produced of both.

5.2. Satiation

When modelling consumer preferences we usually assume that "more is always better" – the consumer wants as much as possible of all goods. But in some cases "enough is enough"– the consumer reaches a satiation point.

EXAMPLES 5.2: A restaurant serving pizzas and chocolate cakes offers "as much as you can eat for £10". Your utility function is $u(p, c) = 4\ln(p + c) - p - \frac{1}{3}c^2$. If you go there, how many pizzas and cakes will you consume?

(i) Your maximisation problem is:

$$\max_{p,c} u(p,c)$$

(Note that the price does not affect your choice – it is a sunk cost).

(ii) The first-order conditions are:

$$\begin{aligned}\frac{\partial u}{\partial p} &= \frac{4}{p+c} - 1 = 0\\ \frac{\partial u}{\partial c} &= \frac{4}{p+c} - \tfrac{2}{3}c = 0\end{aligned}$$

Solving: $c = 1\frac{1}{2}$ and $p = 2\frac{1}{2}$.
For the second-order conditions:

$$\frac{\partial^2 u}{\partial p^2} = -\frac{4}{(p+c)^2} < 0 \quad \text{and} \quad \frac{\partial^2 u}{\partial c^2} = -\frac{4}{(p+c)^2} - \tfrac{2}{3} < 0$$

$$\frac{\partial^2 u}{\partial c \partial p} = -\frac{4}{(p+c)^2} \quad \Rightarrow \quad \frac{\partial^2 u}{\partial p^2}\frac{\partial^2 u}{\partial c^2} - \left(\frac{\partial^2 u}{\partial c \partial p}\right)^2 = \tfrac{2}{3}\frac{4}{(p+c)^2} > 0$$

So this is a maximum; you will consume $2\frac{1}{2}$ pizzas and $1\frac{1}{2}$ cakes.

5.3. Price Discrimination

A monopolist who sells in two different markets, where there are no arbitrage opportunities (it is not possible for a third-party to buy in one market and then sell in the other), can usually raise profits by selling at different prices in the two markets. This is *third-degree* price discrimination.

EXAMPLES 5.3: A car-manufacturer has cost function $C(q) = 24q^{\frac{1}{2}}$, and can sell to its domestic and foreign customers at different prices. It faces demand functions $q_d = 36p_d^{-2}$ in the domestic market, and $q_f = 256p_f^{-4}$ in the foreign market.

(i) Find the price elasticity of demand in the two markets.

$$\begin{aligned}\frac{dq_d}{dp_d} &= -72p_d^{-3}\\ \Rightarrow \frac{p_d}{q_d}\frac{dq_d}{dp_d} &= \frac{p_d}{36p_d^{-2}} \times -72p_d^{-3} = -2\end{aligned}$$

So the price elasticity of demand in the domestic market is -2, and similarly the foreign elasticity is -4.

(ii) Write down the firm's profit as a function of the number of cars sold in each market.
The inverse demand functions for the two goods are: $p_d = 6q_d^{-\frac{1}{2}}$ and $p_f = 4q_f^{-\frac{1}{4}}$. Hence profit is:

$$\begin{aligned}\Pi(q_d, q_f) &= p_d q_d + p_f q_f - C(q_d + q_f)\\ &= 6q_d^{\frac{1}{2}} + 4q_f^{\frac{3}{4}} - 24(q_d + q_f)^{\frac{1}{2}}\end{aligned}$$

(iii) How many cars will be sold in each market?
The first-order conditions are:

$$3q_d^{-\frac{1}{2}} = 12(q_d + q_f)^{-\frac{1}{2}} \quad \text{and} \quad 3q_f^{-\frac{1}{4}} = 12(q_d + q_f)^{-\frac{1}{2}}$$

(Note that MR must be the same in both markets.)
From these two equations:

$$3q_d^{-\frac{1}{2}} = 3q_f^{-\frac{1}{4}} \quad \Rightarrow \quad q_f = q_d^2$$

Substituting for q_f in the first equation and solving:

$$q_d^{-\frac{1}{2}} = 4(q_d + q_d^2)^{-\frac{1}{2}} \quad \Rightarrow \quad q_d = 15, \quad q_f = 225$$

Checking the second-order conditions is messy, but it can be verified that this is a maximum.

(iv) Compare the domestic and foreign prices.
Substituting the optimal quantities into the inverse demand functions:

$$p_d = \frac{6}{\sqrt{15}} \quad \text{and} \quad p_f = \frac{4}{\sqrt{15}}$$

So the price is lower in the foreign market, where demand is more elastic.

Exercises 8.4: **Economic Applications**

(1) A firm producing two goods has cost function $C(q_1, q_2) = q_1^2 + 2q_2(q_2 - q_1)$. Markets for both goods are competitive and the prices are $p_1 = 5$ and $p_2 = 8$. Find the profit-maximising quantities.

(2) Show that a consumer with utility function $u(x, y) = 2ax - x^2 + 2by - y^2 + xy$ has a satiation point.

(3) A monopolist has cost function $C(q)$ and can sell its product in two different markets, where the inverse demand functions are $p_1(q_1)$ and $p_2(q_2)$. Assuming that the monopolist can price-discriminate, write down its maximisation problem. Hence show that the ratio of prices in the two markets satisfies:

$$\frac{p_1}{p_2} = \frac{1 - \frac{1}{|\epsilon_2|}}{1 - \frac{1}{|\epsilon_1|}}$$

where ϵ_1 and ϵ_2 are the price elasticities of demand.

Further Reading and Exercises

- *Jacques* §5.4
- *Anthony and Biggs* §13.1 and §13.2

Solutions to Exercises in Chapter 8

EXERCISES 8.1:

(1) (a) $P = 60 - 2Q$
(b) $C(Q) = 4Q + 20$
(c) $\Pi = 56Q - 2Q^2 - 20$
(d) $\Pi' = 56 - 4Q = 0$
$\Rightarrow Q = 14$ and $\Pi'' = -4$
(e) $P = 32$, MC=4

(2) $y = \frac{p-2}{6}$

(3) $AC = 8Q + \frac{128}{Q}$ so $\frac{dAC}{dQ} = 8 - \frac{128}{Q^2} = 0$
$\Rightarrow Q = 4, P = AC = 64$

(4) (a) $P = 3q^{\frac{1}{2}}$
(b) $q = \frac{P^2}{9}$
(c) $Q = 40\frac{P^2}{9}$
(d) $\frac{40}{9}P^2 = \frac{120}{P}$
$\Rightarrow P = 3,\ Q = 40,\ q = 1$
Find minimum AC: $AC = 2q^{\frac{1}{2}} + \frac{1}{q}$
$\frac{dAC}{dQ} = q^{-\frac{1}{2}} - \frac{1}{q^2} \quad \Rightarrow q = 1,\ AC = 3.$
It is in LR equilibrium.

(5) $\Pi = 720q - \frac{1}{4}q^4 + \frac{19}{3}q^3 - 54q^2 - 540q$
$\Pi' = 180 - q^3 + 19q^2 - 108q$
$q = 3 \Rightarrow \Pi' = 0$ so this is a stationary point. Similarly for $q = 6$ and $q = 10$.
$\Pi'' = -3q^2 + 38q - 108$
$q = 3 \Rightarrow \Pi'' = -21$; $q = 6 \Rightarrow \Pi'' = 12$; and $q = 10 \Rightarrow \Pi'' = -28$.
3 and 10 are maxima.
$q = 3 \Rightarrow \Pi = 204.75$
$q = 10 \Rightarrow \Pi = 233.33$
$q = 10$ is the optimal choice.
For sketch note that demand slopes down, but MR *increases* for some q.

EXERCISES 8.2:

(1) $\Pi_1 = p_1q_1 - 3q_1$
$= (p_1 - 3)(10 - 2p_1 + p_2)$
and similarly
$\Pi_2 = (p_2 - 3)(10 - 3p_2 + p_1)$

(2) $\frac{\partial \Pi_1}{\partial p_1} = 16 - 4p_1 + p_2^e \Rightarrow p_1 = \frac{16+p_2^e}{4}$

(3) $p_2 = \frac{19+p_1^e}{6}$

(4) $p_1 = 5,\ p_2 = 4,\ q_1 = 4,\ q_2 = 3$

EXERCISES 8.3:

(1) $f_x = 2x - 7 - y,\ f_y = 8y + 11 - x$
One stationary point: $x = 3, y = -1$.
$f_{xx} = 2,\ f_{yy} = 8, f_{xy} = -1, \Rightarrow$ minimum.

(2) $g_x = ye^{xy} + 6x + y - 12,\ g_y = xe^{xy} + x - 4$
When $y = 0$, $g_x = 6x - 12$ and $g_y = 2x - 4$.
So $(2, 0)$ is a stationary point.
$g_{xx} = y^2e^{xy} + 6,\ g_{yy} = x^2e^{xy}$
$g_{xy} = e^{xy} + xye^{xy} + 1$. At (2,0): $g_{xx} = 6$, $g_{yy} = 4,\ g_{xy} = 2 \Rightarrow$ minimum.

(3) $h_x = 6x - 6(x - 2y)^2 = 0$
$h_y = -12 + 12(x - 2y)^2 = 0$
From second equation $(x - 2y)^2 = 1$
So from first equation $x = 1$.
Then $(1 - 2y)^2 = 1 \Rightarrow y = 0$ or $y = 1$.
Stationary points are: (1,0) and (1,1).
$h_{xx} = 6 - 12(x - 2y),\ h_{yy} = -48(x - 2y)$, $h_{xy} = 24(x - 2y)$. At (1,0): $h_{xx} = -6$, $h_{yy} = -48,\ h_{xy} = 24$: saddlepoint.
At (1,1): $h_{xx} = 18,\ h_{yy} = 48,\ h_{xy} = -24$: minimum.

EXERCISES 8.4:

(1) $\Pi = 5q_1 + 8q_2 - (q_1^2 + 2q_2(q_2 - q_1))$
$\frac{\partial \Pi}{\partial q_1} = 5 - 2q_1 + 2q_2,\ \frac{\partial \Pi}{\partial q_2} = 8 - 4q_2 + 2q_1$
$\Rightarrow q_1 = 9,\ q_2 = 6.5.$ SOCs: $\frac{\partial^2 \Pi}{\partial q_1^2} = -2$,
$\frac{\partial^2 \Pi}{\partial q_2^2} = -4,\ \frac{\partial^2 \Pi}{\partial q_1 \partial q_2} = 2 \Rightarrow$ max.

(2) $\frac{\partial u}{\partial x} = 2a - 2x + y,\ \frac{\partial u}{\partial y} = 2b - 2y + x$
$\Rightarrow x = \frac{4a+2b}{3},\ y = \frac{2a+4b}{3}.$
$\frac{\partial^2 u}{\partial x^2} = -2,\ \frac{\partial^2 u}{\partial y^2} = -2\ \frac{\partial^2 u}{\partial x \partial y} = 1 \Rightarrow$ max.

(3) $\max_{q_1,q_2} p_1q_1 + p_2q_2 - C(q_1 + q_2)$
FOCs: $p_1 + p_1'q_1 = C'(q_1 + q_2)$
and $p_2 + p_2'q_2 = C'(q_1 + q_2)$
$\Rightarrow p_1 + p_1'q_1 = p_2 + p_2'q_2$
$\Rightarrow p_1\left(1 + \frac{q_1}{p_1}\frac{dp_1}{dq_1}\right) = p_2\left(1 + \frac{q_2}{p_2}\frac{dp_2}{dq_2}\right)$
$\Rightarrow$ Result.

[1]This Version of Workbook Chapter 8: September 25, 2006

Worksheet 8: Unconstrained Optimisation Problems with One or More Variables

(1) In a competitive industry, all firms have cost functions $C(q) = 32 + 2q^2$

(a) Write down the profit function for an individual firm when the market price is P, and hence show that the firm's supply curve is

$$q = \frac{P}{4}$$

(b) Find the firm's average cost function, and determine the price in long-run industry equilibrium.

The market demand function is $Q = 3000 - 75P$.

(c) In the short-run there are 200 firms. Find the industry supply function, and hence the equilibrium price, and the output and profits of each firm.

(d) How many firms will there be in long-run equilibrium?

Suppose that a trade association could limit the number of firms entering the industry.

(e) Find the equilibrium price and quantity, in terms of the number of firms, n.

(f) What upper limit would the trade association set if it wanted to maximise industry revenue?

(2) Find and classify the stationary points of the function: $f(x, y) = x^3 + y^3 - 3x - 3y$

(3) Two competing toothpastes are produced by a monopolist. Brand X costs 9 pence per tube to produce and sells at P_X, with demand (in hundreds/day) given by: $X = 2(P_Y - P_X) + 4$. Brand Y costs 12 pence per tube to produce and sells at P_Y pence, with demand given by: $Y = 0.25P_X - 2.5P_Y + 52$. What price should she charge for each brand if she wishes to maximize joint profits?

(4) A firm's production function is given by

$$Q = 2L^{\frac{1}{2}} + 3K^{\frac{1}{2}}$$

where Q, L and K denote the number of units of output, labour and capital. Labour costs are £2 per unit, capital costs are £1 per unit, and output sells at £8 per unit.

(a) What is the firm's profit function?

(b) Find the maximum profit and the values of L and K at which it is achieved.

(5) Suppose there are two firms, firm 1 which sells product X, and firm 2 which sells product Y. The markets for X and Y are related, and the inverse demand curves for X and Y are

$$\begin{aligned} p^X &= 15 - 2x - y, \text{ and} \\ p^Y &= 20 - x - 2y \end{aligned}$$

respectively. Firm 1 has total costs of $3x$ and firm 2 has total costs of $2y$.

(a) Are X and Y substitutes or complements?

(b) If the two firms are in Cournot competition (i.e. each maximizes its own profits assuming the output of the other is fixed), how much should each produce?

(6) Two workers work together as a team. If worker i puts in effort e_i, their total output is $y = A\ln(1 + e_1 + e_2)$. Their pay depends on their output: if they produce y, each receives $w = w_0 + ky$. A, w_0 and k are positive constants. Each of them chooses his own effort level to maximise his own utility, which is given by:

$$u_i = w - e_i$$

(a) Write worker 1's utility as a function of his own and worker 2's effort.
(b) If worker 1 expects worker 2 to exert effort $\hat{e_2}$, what is his optimal choice of effort?
(c) How much effort does each exert in a symmetric equilibrium?
(d) How could the two workers obtain higher utility?

(7) Two firms sell an identical good. For both firms the unit cost of supplying that good is c. Suppose that the quantity demanded of the good, q, is dependent on its price, p, according to the formula $q = 100 - 20p$.

(a) Assuming firm 2 is supplying q_2 units of the good to the market, how many units of the good should firm 1 supply to maximize its profits? Express your answer in terms of c and q_2.
(b) Find, in terms of c, the quantity that each firm will supply so that it is maximizing its profit given the quantity supplied by the other firm; in other words, find the Cournot equilibrium supplies of the two firms. What will be the market price of the good?
(c) Suppose that these two firms are in fact retailers who purchase their product from a manufacturer at the price c. This manufacturer faces the cost function

$$C(q) = 20 + q + \frac{1}{40}q^2,$$

where q is the output level. Assuming that the manufacturer knows that the retail market for his good has a Cournot equilibrium outcome, at what price should he supply the good in order to maximize his profits?

CHAPTER 9

Constrained Optimisation

Rational economic agents are assumed to make choices that maximise their utility or profit. But their **choices are usually constrained** – for example the consumer's choice of consumption bundle is constrained by his income. In this chapter we look at methods for solving optimisation problems with constraints: in particular the **method of Lagrange multipliers**. We apply them to **consumer choice**, **cost minimisation**, and other economic problems.

1. Consumer Choice

Suppose there are two goods available, and a consumer has preferences represented by the utility function $u(x_1, x_2)$ where x_1 and x_2 are the amounts of the goods consumed. The prices of the goods are p_1 and p_2, and the consumer has a fixed income m. He wants to choose his consumption bundle (x_1, x_2) to maximise his utility, subject to the constraint that the total cost of the bundle does not exceed his income. Provided that the utility function is strictly increasing in x_1 and x_2, we know that he will want to use all his income.

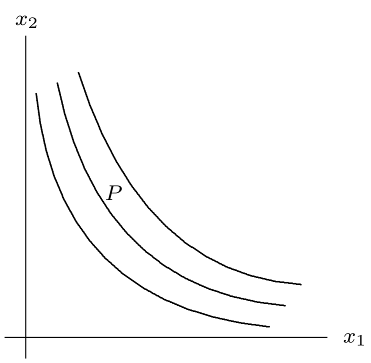
The consumer's optimisation problem is:

$$\max_{x_1,x_2} u(x_1, x_2) \quad \text{subject to} \quad p_1x_1 + p_2x_2 = m$$

- The objective function is $u(x_1, x_2)$
- The choice variables are x_1 and x_2
- The constraint is $p_1x_1 + p_2x_2 = m$

There are three methods for solving this type of problem.

1.1. Method 1: Draw a Diagram and Think About the Economics



Provided the utility function is *well-behaved* (increasing in x_1 and x_2 with strictly convex indifference curves), the highest utility is obtained at the point P where the budget constraint is tangent to an indifference curve.

The slope of the budget constraint is: $-\dfrac{p_1}{p_2}$

and the slope of the indifference curve is: $MRS = -\dfrac{MU_1}{MU_2}$

(See Chapters 2 and 7 for these results.)

Hence we can find the point P:

(1) It is on the budget constraint:

$$p_1x_1 + p_2x_2 = m$$

(2) where the slope of the budget constraint equals the slope of the indifference curve:

$$\frac{p_1}{p_2} = \frac{MU_1}{MU_2}$$

In general, this gives us two equations that we can solve to find x_1 and x_2.

Examples 1.1: A consumer has utility function $u(x_1, x_2) = x_1x_2$. The prices of the two goods are $p_1 = 3$ and $p_2 = 2$, and her income is $m = 24$. How much of each good will she consume?

(i) The consumer's problem is:

$$\max_{x_1,x_2} x_1x_2 \quad \text{subject to} \quad 3x_1 + 2x_2 = 24$$

The utility function $u(x_1, x_2) = x_1x_2$ is Cobb-Douglas (see Chapter 7). Hence it is well-behaved. The marginal utilities are:

$$MU_1 = \frac{\partial u}{\partial x_1} = x_2 \quad \text{and} \quad MU_2 = \frac{\partial u}{\partial x_2} = x_1$$

(ii) The optimal bundle:

(1) is on the budget constraint:

$$3x_1 + 2x_2 = 24$$

(2) where the budget constraint is tangent to an indifference curve:

$$\begin{aligned} \frac{p_1}{p_2} &= \frac{MU_1}{MU_2} \\ \Rightarrow \frac{3}{2} &= \frac{x_2}{x_1} \end{aligned}$$

(iii) From the tangency condition: $3x_1 = 2x_2$. Substituting this into the budget constraint:

$$\begin{aligned} 2x_2 + 2x_2 &= 24 \\ \Rightarrow x_2 &= 6 \\ \text{and hence } x_1 &= 4 \end{aligned}$$

The consumer's optimal choice is 4 units of good 1 and 6 units of good 2.

1.2. Method 2: Use the Constraint to Substitute for one of the Variables

If you did A-level maths, this may seem to be the obvious way to solve this type of problem. However it often gives messy equations and is rarely used in economic problems because it doesn't give much economic insight. Consider the example above:

$$\max_{x_1,x_2} x_1x_2 \quad \text{subject to} \quad 3x_1 + 2x_2 = 24$$

From the constraint, $x_2 = 12 - \frac{3x_1}{2}$. By substituting this into the objective function, we can write the problem as:

$$\max_{x_1} \left(12x_1 - \frac{3x_1^2}{2}\right)$$

So, we have transformed it into an *unconstrained* optimisation problem in one variable. The first-order condition is:

$$12 - 3x_1 = 0 \quad \Rightarrow \quad x_1 = 4$$

Substituting back into the equation for x_2 we find, as before, that $x_2 = 6$. It can easily be checked that the second-order condition is satisfied.

1.3. The Most General Method: The Method of Lagrange Multipliers

This method is important because it can be used for a wide range of constrained optimisation problems. For the consumer problem:

$$\max_{x_1,x_2} u(x_1, x_2) \quad \text{subject to} \quad p_1x_1 + p_2x_2 = m$$

(1) Write down the Lagrangian function:

$$\mathcal{L}(x_1, x_2, \lambda) = u(x_1, x_2) - \lambda(p_1x_1 + p_2x_2 - m)$$

λ is a new variable, which we introduce to help solve the problem.
It is called a *Lagrange Multiplier*.

(2) Write down the first-order conditions:

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial x_1} &= 0 \\ \frac{\partial \mathcal{L}}{\partial x_2} &= 0 \\ \frac{\partial \mathcal{L}}{\partial \lambda} &= 0 \end{aligned}$$

(3) Solve the first-order conditions to find x_1 and x_2.

Provided that the utility function is well-behaved (increasing in x_1 and x_2 with strictly convex indifference curves) then the values of x_1 and x_2 that you obtain by this procedure will be the optimum bundle.[1]

Examples 1.2: A consumer has utility function $u(x_1, x_2) = x_1x_2$. The prices of the two goods are $p_1 = 3$ and $p_2 = 2$, and her income is $m = 24$. Use the Lagrangian method to find how much of each good she will she consume.

(i) The consumer's problem is, as before:

$$\max_{x_1,x_2} x_1x_2 \quad \text{subject to} \quad 3x_1 + 2x_2 = 24$$

Again, since the utility function is Cobb-Douglas, it is well-behaved.

(ii) The Lagrangian function is:

$$\mathcal{L}(x_1, x_2, \lambda) = x_1x_2 - \lambda(3x_1 + 2x_2 - 24)$$

[1] In this *Workbook* we will simply use the Lagrangian method, without explaining why it works. For some explanation, see *Anthony & Biggs* section 21.2.

(iii) First-order conditions:

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial x_1} &= x_2 - 3\lambda & \Rightarrow && x_2 &= 3\lambda \\
\frac{\partial \mathcal{L}}{\partial x_2} &= x_1 - 2\lambda & \Rightarrow && x_1 &= 2\lambda \\
\frac{\partial \mathcal{L}}{\partial \lambda} &= -(3x_1 + 2x_2 - 24) & \Rightarrow && 3x_1 + 2x_2 &= 24
\end{aligned}$$

(iv) Eliminating λ from the first two equations gives:

$$3x_1 = 2x_2$$

Substituting for x_1 in the third equation:

$$2x_2 + 2x_2 = 24 \quad \Rightarrow \quad x_2 = 6 \quad \text{and hence} \quad x_1 = 4$$

Thus we find (again) that the optimal bundle is 4 units of good 1 and 6 of good 2.

1.4. Some Useful Tricks

1.4.1. *Solving the Lagrangian first-order conditions.* In the example above, the first two equations of the first-order conditions are:

$$\begin{aligned} x_2 &= 3\lambda \\ x_1 &= 2\lambda \end{aligned}$$

There are several (easy) ways to eliminate λ to obtain $3x_1 = 2x_2$. But one way, which is particularly useful in Lagrangian problems when the equations are more complicated, is to *divide* the two equations, so that λ cancels out:

$$\frac{x_2}{x_1} = \frac{3\lambda}{2\lambda} \quad \Rightarrow \quad \frac{x_2}{x_1} = \frac{3}{2}$$

Whenever you have two equations: $\begin{aligned} A &= B \\ C &= D \end{aligned}$

where A, B, C and D are non-zero algebraic expressions, you can write: $\dfrac{A}{C} = \dfrac{B}{D}$.

1.4.2. *Transforming the objective function.* Remember from Chapter 7 that if preferences are represented by a utility function $u(x_1, x_2)$, and f is an increasing function, then $f(u)$ represents the same preference ordering. Sometimes we can use this to reduce the algebra needed to solve a problem. Suppose we want to solve:

$$\max_{x_1,x_2} x_1^{\frac{3}{4}} x_2^{\frac{1}{4}} \quad \text{subject to} \quad 3x_1 + 2x_2 = 24$$

Preferences are represented by a Cobb-Douglas utility function $u = x_1^{\frac{3}{4}} x_2^{\frac{1}{4}}$, so they could equivalently be represented by:

$$\ln u = \tfrac{3}{4} \ln x_1 + \tfrac{1}{4} \ln x_2 \quad \text{or} \quad 4 \ln u = 3 \ln x_1 + \ln x_2$$

You can check that solving the problem:

$$\max_{x_1,x_2} 3 \ln x_1 + \ln x_2 \quad \text{subject to} \quad 3x_1 + 2x_2 = 24$$

gives exactly the same answer as the original problem, but that the equations are simpler.

1.5. Well-Behaved Utility Functions

We have seen that for the optimisation methods described above to work, it is important that the utility function is *well-behaved*: increasing, with strictly convex indifference curves. Two examples of well-behaved utility functions are:

- The Cobb-Douglas Utility Function
$$u(x_1, x_2) = x_1^a x_2^b$$
- The Logarithmic Utility Function
$$u(x_1, x_2) = a \ln x_1 + b \ln x_2$$

(We know from Chapter 7 that the Cobb-Douglas function is well-behaved. The logarithmic function is a monotonic increasing transformation of the Cobb-Douglas, so it represents the same preferences and is also well-behaved.) Two other useful forms of utility function are:

- The CES Utility Function[2]
$$u(x_1, x_2) = \left(a x_1^{-\rho} + b x_2^{-\rho}\right)^{-\frac{1}{\rho}}$$
where a and b are positive parameters, and ρ is a parameter greater than -1.
- The Quasi-Linear Utility Function
$$u(x_1, x_2) = v(x_1) + x_2$$
where $v(x_1)$ is any increasing concave function.

Both of these are well-behaved. To prove it, you can show that:

- the function is increasing, by showing that the marginal utilities are positive
- the indifference curves are convex as we did for the Cobb-Douglas case (Chapter 7 section 4.2) by showing that the MRS gets less negative as you move along an indifference curve.

Exercises 9.1: **Consumer Choice**

(1) Use Method 1 to find the optimum consumption bundle for a consumer with utility function $u(x, y) = x^2 y$ and income $m = 30$, when the prices of the goods are $p_x = 4$ and $p_y = 5$. Check that you get the same answers by the Lagrangian method.

(2) A consumer has a weekly income of 26 (after paying for essentials), which she spends on restaurant Meals and Books. Her utility function is $u(M, B) = 3M^{\frac{1}{2}} + B$, and the prices are $p_M = 6$ and $p_B = 4$. Use the Lagrangian method to find her optimal consumption bundle.

(3) Use the trick of transforming the objective function (section 1.4.2) to solve:
$$\max_{x_1, x_2} x_1^{\frac{3}{4}} x_2^{\frac{1}{4}} \quad \text{subject to} \quad 3x_1 + 2x_2 = 24$$

Further Reading and Exercises

- *Varian* "Intermediate Microeconomics", Chapter 5, covers the economic theory of Consumer Choice. The Appendix to Chapter 5 explains the same three methods for solving choice problems that we have used in this section.
- *Jacques* §5.5 and §5.6
- *Anthony & Biggs* §§21.2, 22.2, and 22.3

[2] "Constant Elasticity of Substitution" – this refers to another property of the function

2. Cost Minimisation

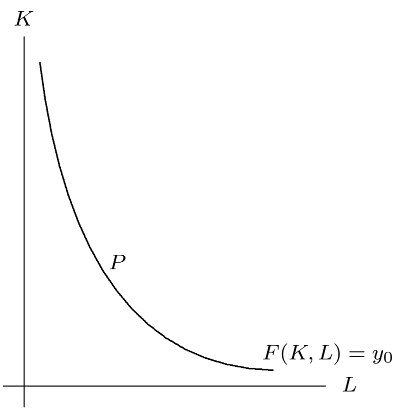
Consider a competitive firm with production function $F(K, L)$. Suppose that the wage rate is w, and the rental rate for capital is r. Suppose that the firm wants to produce a particular amount of output y_0 at minimum cost. How much labour and capital should it employ?

The firm's optimisation problem is:
$$\min_{K,L}(rK + wL) \quad \text{subject to} \quad F(K, L) = y_0$$

- The objective function is $rK + wL$
- The choice variables are K and L
- The constraint is $F(K, L) = y_0$

We can use the same three methods here as for the consumer choice problem, but we will ignore method 2 because it is generally less useful.

2.1. Method 1: Draw a Diagram and Think About the Economics



Draw the isoquant of the production function representing combinations of K and L that can be used to produce output y_0. (See Chapter 7.)

The slope of the isoquant is: $\quad MRTS = -\dfrac{MPL}{MPK}$

Draw the isocost lines where $rK + wL = constant$.

The slope of the isocost lines is: $\quad -\dfrac{w}{r}$

Provided the isoquant is convex, the lowest cost is achieved at the point P where an isocost line is tangent to the isoquant.

Hence we can find the point P:

(1) It is on the isoquant:
$$F(K, L) = y_0$$
(2) where the slope of the isocost lines equals the slope of the isoquant:
$$\frac{w}{r} = \frac{MPL}{MPK}$$

This gives us two equations that we can solve to find K and L.

EXAMPLES 2.1: If the production function is $F(K, L) = K^{\frac{1}{3}}L^{\frac{2}{3}}$, the wage rate is 5, and the rental rate of capital is 20, what is the minimum cost of producing 40 units of output?

(i) The problem is:
$$\min_{K,L}(20K + 5L) \quad \text{subject to} \quad K^{\frac{1}{3}}L^{\frac{2}{3}} = 40$$

The production function $F(K,L) = K^{\frac{1}{3}}L^{\frac{2}{3}}$ is Cobb-Douglas, so the isoquant is convex. The marginal products are:

$$MPK = \frac{\partial F}{\partial K} = \tfrac{1}{3}K^{-\frac{2}{3}}L^{\frac{2}{3}} \quad \text{and} \quad MPL = \frac{\partial F}{\partial L} = \tfrac{2}{3}K^{\frac{1}{3}}L^{-\frac{1}{3}}$$

(ii) The optimal choice of inputs:

(1) is on the isoquant:

$$K^{\frac{1}{3}}L^{\frac{2}{3}} = 40$$

(2) where an isocost line is tangent to it:

$$\begin{aligned} \frac{w}{r} &= \frac{MPL}{MPK} \\ \Rightarrow \frac{5}{20} &= \frac{\frac{2}{3}K^{\frac{1}{3}}L^{-\frac{1}{3}}}{\frac{1}{3}K^{-\frac{2}{3}}L^{\frac{2}{3}}} \\ \Rightarrow \frac{1}{8} &= \frac{K}{L} \end{aligned}$$

(iii) From the tangency condition: $L = 8K$. Substituting this into the isoquant:

$$\begin{aligned} K^{\frac{1}{3}}(8K)^{\frac{2}{3}} &= 40 \\ \Rightarrow K &= 10 \\ \text{and hence } L &= 80 \end{aligned}$$

(iv) With this optimal choice of K and L, the cost is: $\quad 20K + 5L = 20 \times 10 + 5 \times 80 = 600.$

2.2. The Method of Lagrange Multipliers

For the cost minimisation problem, the method is analagous to the one for consumer choice:

$$\min_{K,L}(rK + wL) \quad \text{subject to} \quad F(K,L) = y_0$$

(1) Write down the Lagrangian function:

$$\mathcal{L}(K,L,\lambda) = rK + wL - \lambda(F(K,L) - y_0)$$

(2) Write down the first-order conditions:

$$\frac{\partial \mathcal{L}}{\partial K} = 0, \quad \frac{\partial \mathcal{L}}{\partial L} = 0, \quad \frac{\partial \mathcal{L}}{\partial \lambda} = 0$$

(3) Solve the first-order conditions to find K and L.

Provided that the isoquants are convex, this procedure obtains the optimal values of K and L.

Examples 2.2: If the production function is $F(K,L) = K^{\frac{1}{3}}L^{\frac{2}{3}}$, the wage rate is 5, and the rental rate of capital is 20, use the Lagrangian method to find the minimum cost of producing 40 units of output.

(i) The problem is, as before:

$$\min_{K,L}(20K + 5L) \quad \text{subject to} \quad K^{\frac{1}{3}}L^{\frac{2}{3}} = 40$$

and since the production function is Cobb-Douglas, the isoquant is convex.

(ii) The Lagrangian function is:

$$\mathcal{L}(K,L,\lambda) = 20K + 5L - \lambda(K^{\frac{1}{3}}L^{\frac{2}{3}} - 40)$$

(iii) First-order conditions:

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial K} &= 20 - \tfrac{1}{3}\lambda K^{-\frac{2}{3}}L^{\frac{2}{3}} &\Rightarrow \quad 60 &= \lambda K^{-\frac{2}{3}}L^{\frac{2}{3}} \\
\frac{\partial \mathcal{L}}{\partial L} &= 5 - \tfrac{2}{3}\lambda K^{\frac{1}{3}}L^{-\frac{1}{3}} &\Rightarrow \quad 15 &= 2\lambda K^{\frac{1}{3}}L^{-\frac{1}{3}} \\
\frac{\partial \mathcal{L}}{\partial \lambda} &= -(K^{\frac{1}{3}}L^{\frac{2}{3}} - 40) &\Rightarrow \quad K^{\frac{1}{3}}L^{\frac{2}{3}} &= 40
\end{aligned}$$

(iv) Dividing the first two equations to eliminate λ:

$$\frac{60}{15} = \frac{K^{-\frac{2}{3}}L^{\frac{2}{3}}}{2K^{\frac{1}{3}}L^{-\frac{1}{3}}} \quad \text{which simplifies to} \quad L = 8K$$

Substituting for L in the third equation:

$$K^{\frac{1}{3}}(8K)^{\frac{2}{3}} = 40 \quad \Rightarrow \quad K = 10 \quad \text{and hence} \quad L = 80$$

(v) Thus, as before, the optimal choice is 10 units of capital and 80 of labour, which means that the cost is 600.

EXERCISES 9.2: **Cost Minimisation**

(1) A firm has production function $F(K, L) = 5K^{0.4}L$. The wage rate is $w = 10$ and the rental rate of capital is $r = 12$. Use Method 1 to determine how much labour and capital the firm should employ if it wants to produce 300 units of output. What is the total cost of doing so?
Check that you get the same answer by the Lagrangian method.

(2) A firm has production function $F(K, L) = 30(K^{-1} + L^{-1})^{-1}$. Use the Lagrangian method to find how much labour and capital it should employ to produce 70 units of output, if the wage rate is 8 and the rental rate of capital is 2.
Note: the production function is CES, so has convex isoquants.

Further Reading and Exercises

- *Jacques* §5.5 and §5.6
- *Anthony & Biggs* §§21.1, 21.2 and 21.3
- *Varian* "Intermediate Microeconomics", Chapter 20, covers the economics of Cost Minimisation.

3. The Method of Lagrange Multipliers

To try to solve any problem of the form:

$$\max_{x_1,x_2} F(x_1,x_2) \quad \text{subject to} \quad g(x_1,x_2)=c$$

or:

$$\min_{x_1,x_2} F(x_1,x_2) \quad \text{subject to} \quad g(x_1,x_2)=c$$

you can write down the Lagrangian:

$$\mathcal{L}(x_1,x_2,\lambda)=F(x_1,x_2)-\lambda(g(x_1,x_2)-c)$$

and look for a solution of the three first-order conditions:

$$\frac{\partial \mathcal{L}}{\partial x_1}=0, \quad \frac{\partial \mathcal{L}}{\partial x_2}=0, \quad \frac{\partial \mathcal{L}}{\partial \lambda}=0$$

Note that the condition $\frac{\partial \mathcal{L}}{\partial \lambda}=0$ always gives you the constraint.

3.1. Max or Min?

But if you find a solution to the Lagrangian first-order conditions, how do you know whether it is a maximum or a minimum? And what if there are several solutions? The answer is that in general you need to look at the second-order conditions, but unfortunately the SOCs for constrained optimisation problems are complicated to write down, so will not be covered in this *Workbook*.

However, when we apply the method to economic problems, we can often manage without second-order conditions. Instead, we think about the *shapes* of the objective function and the constraint, and find that either it is the type of problem in which the objective function can *only* have a maximum, or that it is a problem that can only have a minimum. In such cases, the method of Lagrange multipliers will give the required solution.

If you look back at the examples in the previous two sections, you can see that the Lagrangian method gives you just the same equations as you get when you "draw a diagram and think about the economics" – what the method does is to find tangency points.

But when you have a problem in which either the objective function or constraint doesn't have the standard economic properties of convex indifference curves or isoquants you cannot rely on the Lagrangian method, because a tangency point that it finds may not be the required maximum or minimum.

EXAMPLES 3.1: *Standard and non-standard problems*

(i) If the utility function is well-behaved (increasing, with convex indifference curves) the problem:

$$\max_{x_1,x_2} u(x_1,x_2) \quad \text{subject to} \quad p_1x_1+p_2x_2=m$$

can be solved by the Lagrangian method.

(ii) If the production function has convex isoquants the problem:

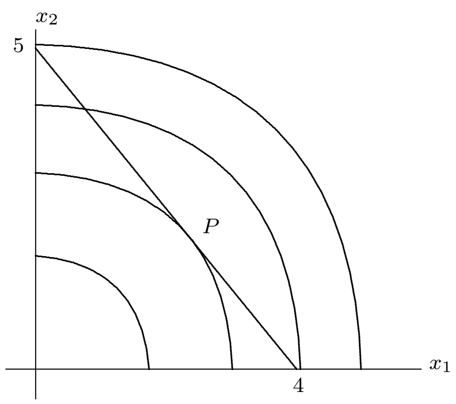
$$\min_{K,L}(rK+wL) \quad \text{subject to} \quad F(K,L)=y_0$$

can be solved by the Lagrangian method.

(iii) Consider a consumer choice problem with *concave* indifference curves:

$$\max_{x_1,x_2} u(x_1,x_2) \quad \text{subject to} \quad 5x_1 + 4x_2 = 20$$

where $u(x_1,x_2) = x_1^2 + x_2^2$.



The tangency point P is the point on the budget constraint where utility is *minimised.*

Assuming that the amounts x_1 and x_2 cannot be negative, the point on the budget constraint where utility is maximised is $x_1 = 0$, $x_2 = 5$.

If we write the Lagrangian:

$\mathcal{L}(x_1,x_2,\lambda) = x_1^2 + x_2^2 - \lambda(5x_1 + 4x_2 - 20)$

the first-order conditions are:

$$2x_1 = 5\lambda, \quad 2x_2 = 4\lambda, \quad 5x_1 + 4x_2 = 20$$

Solving, we obtain $x_1 = 2.44$, $x_2 = 1.95$. The Lagrangian method has found the tangency point P, which is where utility is minimised.

3.2. The Interpretation of the Lagrange Multiplier

In the examples of the Lagrangian method so far, we did not bother to calculate the value of λ satisfying the first-order conditions – we simply eliminated it and solved for the choice variables. However, this value does have a meaning.

> In an optimisation problem with objective function $F(x,y)$ and constraint $g(x,y) = c$, let F^* be the value of the objective function at the optimum. The value of the Lagrange multipler λ indicates how much F^* would increase if there were a small increase in c:
>
> $$\lambda = \frac{dF^*}{dc}$$

This means, for example, that for the cost minimisation problem:

$$\min_{K,L}(rK + wL) \quad \text{subject to} \quad F(K,L) = y$$

the Lagrange multiplier tells us how much the cost would increase if there were a small increase in the amount of output to be produced – that is, the marginal cost of output.

We will prove this result using differentials (see Chapter 7). The Lagrangian is:

$$\mathcal{L} = rK + wL - \lambda(F(K,L) - y)$$

and the first-order conditions are:

$$\begin{aligned} r &= \lambda F_K \\ w &= \lambda F_L \\ F(K,L) &= y \end{aligned}$$

These equations can be solved to find the optimum values of capital K^* and labour L^*, and the Lagrange multiplier λ^*. The cost is then $C^* = rK^* + wL^*$.

K^*, L^* and C^* all depend on the the level of output to be produced, y. Suppose there is small change in this amount, dy. This will lead to small changes in the optimal factor choices and the cost, dK^*, dL^*, and dC^*. We can work out how big the change in cost will be:

First, taking the differential of the constraint $F(K^*, L^*) = y$:

$$dy = F_K dK^* + F_L dL^*$$

then, taking the differential of the cost $C^* = rK^* + wL^*$ and using the first-order conditions:

$$\begin{aligned} dC^* &= rdK^* + wdL^* \\ &= \lambda^* F_K dK^* + \lambda^* F_L dL^* \\ &= \lambda^* dy \\ \Rightarrow \frac{dC^*}{dy} &= \lambda^* \end{aligned}$$

So the value of the Lagrange multiplier tells us the marginal cost of output.

Similarly for utility maximisation:

$$\max_{x_1,x_2} u(x_1, x_2) \quad \text{subject to} \quad p_1x_1 + p_2x_2 = m$$

the Lagrange multiplier gives the value of $\frac{du^*}{dm}$, which is the marginal utility of income.

This, and the general result in the box above, can be proved in the same way.

Examples 3.2: *The Lagrange Multiplier*

In Examples 2.2, for the production function $F(K, L) = K^{\frac{1}{3}}L^{\frac{2}{3}}$, with wage rate 5 and rental rate of capital 20, we found the minimum cost of producing 40 units of output. The Lagrangian and first-order conditions were:

$$\mathcal{L}(K, L, \lambda) = 20K + 5L - \lambda(K^{\frac{1}{3}}L^{\frac{2}{3}} - 40)$$

$$\begin{aligned} 60 &= \lambda K^{-\frac{2}{3}}L^{\frac{2}{3}} \\ 15 &= 2\lambda K^{\frac{1}{3}}L^{-\frac{1}{3}} \\ K^{\frac{1}{3}}L^{\frac{2}{3}} &= 40 \end{aligned}$$

Solving these we found the optimal choice $K = 10$, $L = 80$, with total cost 600.
Substituting these values of K and L back into the first first-order condition:

$$\begin{aligned} 60 &= \lambda 10^{-\frac{2}{3}}80^{\frac{2}{3}} \\ \Rightarrow \lambda &= 15 \end{aligned}$$

Hence the firm's marginal cost is 15. So we can say that producing 41 units of output would cost approximately 615. (In fact, in this example, marginal cost is constant: the value of λ does not depend on the number of units of output.)

3.3. Problems with More Variables and Constraints

The Lagrangian method can be generalised in an obvious way to solve problems in which there are more variables and several constraints. For example, to solve the problem:

$$\max_{x_1,x_2,x_3,x_4} F(x_1,x_2,x_3,x_4) \quad \text{subject to} \quad g_1(x_1,x_2,x_3,x_4) = c_1 \text{ and } g_2(x_1,x_2,x_3,x_4) = c_2$$

the Lagrangian would be:

$$\mathcal{L}(x_1,x_2,x_3,x_4,\lambda_1,\lambda_2) = F(x_1,x_2,x_3,x_4) - \lambda_1(g_1(x_1,x_2,x_3,x_4) - c_1) - \lambda_2(g_2(x_1,x_2,x_3,x_4) - c_2)$$

giving six first-order conditions to solve for the optimal choice.

Exercises 9.3: **The Method of Lagrange Multipliers**

(1) Find the Lagrange Multiplier and hence the marginal cost of the firm in Exercises 9.2, Question 2.

(2) Find the optimum consumption bundle for a consumer with utility $u = x_1x_2x_3$ and income 36, when the prices of the goods are $p_1 = 1$, $p_2 = 6$, $p_3 = 10$.

(3) A consumer with utility function $u(x_1, x_2)$ has income $m = 12$, and the prices of the goods are $p_1 = 3$ and $p_2 = 2$. For each of the following cases, decide whether the utility function is well-behaved, and determine the optimal choices:
(a) $u = x_1 + x_2$ (b) $u = 3x_1^{2/3} + x_2$ (c) $u = \min(x_1, x_2)$

(4) A firm has production function $F(K, L) = \frac{1}{4}\left(K^{1/2} + L^{1/2}\right)$. The wage rate is $w = 1$ and the rental rate of capital is $r = 3$.
   (a) How much capital and labour should the firm employ to produce y units of output?
   (b) Hence find the cost of producing y units of output (the firm's cost function).
   (c) Differentiate the cost function to find the marginal cost, and verify that it is equal to the value of the Lagrange multiplier.

(5) By following similar steps to those we used for the cost minimisation problem in section 3.2 prove that for the utility maximisation problem:

$$\max_{x_1,x_2} u(x_1,x_2) \quad \text{subject to} \quad p_1x_1 + p_2x_2 = m$$

the Lagrange multiplier is equal to the marginal utility of income.

Further Reading and Exercises

- *Anthony & Biggs* Chapter 22 gives a general formulation of the Lagrangian method, going beyond what we have covered here.

4. Some More Examples of Constrained Optimisation Problems in Economics

4.1. Production Possibilities

Robinson Crusoe spends his 8 hour working day either fishing or looking for coconuts. If he spends t hours fishing and s hours looking for coconuts he will catch $f(t) = t^{\frac{1}{2}}$ fish and find $c(s) = 3s^{\frac{1}{2}}$ coconuts. Crusoe's utility function is $u = \ln f + \ln c$.

The optimal pattern of production in this economy is the point on the production possibility frontier (ppf) where Crusoe's utility is maximised.

The time taken to catch f fish is $t = f^2$, and the time take to find c coconuts is $s = \left(\frac{c}{3}\right)^2$. Since he has 8 hours in total, the ppf is given by:

$$f^2 + \left(\frac{c}{3}\right)^2 = 8$$

To find the optimal point on the ppf we have to solve the problem:

$$\max_{c,f}(\ln c + \ln f) \quad \text{subject to} \quad f^2 + \left(\frac{c}{3}\right)^2 = 8$$

Note that the ppf is concave, and the utility function is well-behaved (it is the log of a Cobb-Douglas). Hence the optimum is a tangency point, which we could find either by the Lagrangian method, or using the condition that the marginal rate of substitution must be equal to the marginal rate of transformation – see the similar problem on Worksheet 7.

4.2. Consumption and Saving

A consumer lives for two periods (work and retirement). His income is 100 in the first period, and zero in the second. The interest rate is 5%. His lifetime utility is given by:

$$U(c_1, c_2) = c_1^{\frac{1}{2}} + 0.9c_2^{\frac{1}{2}}$$

If he consumes c_1 in the first period he will save $(100 - c_1)$. So when he is retired he can consume:

$$c_2 = 1.05(100 - c_1)$$

This is his lifetime budget constraint. Rearranging, it can be written:

$$c_1 + \frac{c_2}{1.05} = 100$$

Hence the consumer's optimisation problem is:

$$\max_{c_1,c_2}\left(c_1^{\frac{1}{2}} + 0.9c_2^{\frac{1}{2}}\right) \quad \text{subject to} \quad c_1 + \frac{c_2}{1.05} = 100$$

which can be solved using the Lagrangian:

$$\mathcal{L}(c_1, c_2, \lambda) = c_1^{\frac{1}{2}} + 0.9c_2^{\frac{1}{2}} - \lambda\left(c_1 + \frac{c_2}{1.05} - 100\right)$$

to determine the optimal consumption and saving plan for the consumer's lifetime.

4.3. Labour Supply

A consumer has utility

$$U(C, R) = 3 \ln C + \ln R$$

where C is her amount of consumption, and R is the number of hours of leisure (relaxation) she takes each day. The hourly wage rate is $w = 4$ (measured in units of consumption). She has a non-labour income $m = 8$ (consumption units). She needs 10 hours per day for eating and sleeping; in the remainder she can work or take leisure.

Suppose that we want to know her labour supply – how many hours she chooses to work. First we need to know the budget constraint. If she takes R units of leisure, she will work for $14 - R$ hours, and hence earn $4(14 - R)$. Then she will be able to consume:

$$C = 4(14 - R) + 8$$

units of consumption. This is the budget constraint. Rearranging, we can write it as:

$$C + 4R = 64$$

So we need to solve the problem:

$$\max_{C,R} (3 \ln C + \ln R) \quad \text{subject to} \quad C + 4R = 64$$

to find the optimal choice of leisure R, and hence the number of hours of work.

Exercises 9.4: **More Constrained Optimisation Problems**
Complete the solution of each of three problems discussed in this section.

5. Determining Demand Functions

5.1. Consumer Demand

In the consumer choice problems in section 1 we determined the optimal consumption bundle, given the utility function and *particular values* for the prices of the goods, and income. If we solve the same problem for *general values* p_1, p_2, and m, we can determine the consumer's demands for the goods as a function of prices and income:

$$x_1 = x_1(p_1, p_2, m) \quad \text{and} \quad x_2 = x_2(p_1, p_2, m)$$

EXAMPLES 5.1: A consumer with a fixed income m has utility function $u(x_1, x_2) = x_1^2 x_2^3$. If the prices of the goods are p_1 and p_2, find the consumer's demand functions for the two goods.

The optimisation problem is:

$$\max_{x_1, x_2} x_1^2 x_2^3 \quad \text{subject to} \quad p_1 x_1 + p_2 x_2 = m$$

The Lagrangian is:

$$\mathcal{L}(x_1, x_2, \lambda) = x_1^2 x_2^3 - \lambda (p_1 x_1 + p_2 x_2 - m)$$

Differentiating to obtain the first-order conditions:

$$\begin{aligned} 2x_1 x_2^3 &= \lambda p_1 \\ 3x_1^2 x_2^2 &= \lambda p_2 \\ p_1 x_1 + p_2 x_2 &= m \end{aligned}$$

Dividing the first two equations:

$$\frac{2x_2}{3x_1} = \frac{p_1}{p_2} \quad \Rightarrow \quad x_2 = \frac{3p_1 x_1}{2p_2}$$

Substituting into the budget constraint we find:

$$\begin{aligned} p_1 x_1 + \frac{3p_1 x_1}{2} &= m \\ \Rightarrow \quad x_1 &= \frac{2m}{5p_1} \\ \text{and hence} \quad x_2 &= \frac{3m}{5p_2} \end{aligned}$$

These are the consumer's demand functions. Demand for each good is an increasing function of income and a decreasing function of the price of the good.

5.2. Cobb-Douglas Utility

Note that in Examples 5.1, the consumer spends $\frac{2}{5}$ of his income on good 1 and $\frac{3}{5}$ on good 2:

$$p_1 x_1 = \frac{2}{5}m \quad \text{and} \quad p_2 x_2 = \frac{3}{5}m$$

This is an example of a general result for Cobb-Douglas Utility functions:

A consumer with Cobb-Douglas utility: $u(x_1, x_2) = x_1^a x_2^b$
spends a constant fraction of income on each good:
$p_1 x_1 = \frac{a}{a+b}m$ and $p_2 x_2 = \frac{b}{a+b}m$

You could prove this by re-doing example 5.1 using the more general utility function $u(x_1, x_2) = x_1^a x_2^b$.

This result means that with Cobb-Douglas utility a consumer's demand for one good does not depend on the price of the other good.

5.3. Factor Demands

Similarly, if we solve the firm's cost minimisation problem:

$$\min_{K,L}(rK + wL) \quad \text{subject to} \quad F(K, L) = y_0$$

for general values of the factor prices r and w, and the output y_0, we can find the firm's *Conditional Factor Demands* (see Varian Chapter 20) – that is, its demand for each factor as a function of output and factor prices:

$$K = K(y_0, r, w) \quad \text{and} \quad L = L(y_0, r, w)$$

From these we can obtain the firm's cost function:

$$C(y, r, w) = rK(y, r, w) + wL(y, r, w)$$

Exercises 9.5: **Determining Demand Functions**

(1) A consumer has an income of y, which she spends on Meals and Books. Her utility function is $u(M, B) = 3M^{\frac{1}{2}} + B$, and the prices are p_M and p_B. Use the Lagrangian method to find her demand functions for the two goods.

(2) Find the conditional factor demand functions for a firm with production function $F(K, L) = KL$. If the wage rate and the rental rate for capital are both equal to 4, what is the firm's cost function $C(y)$?

(3) Prove that a consumer with Cobb-Douglas utility spends a constant fraction of income on each good.

Further Reading and Exercises

- *Anthony & Biggs* §§21.3 and 22.4
- *Varian* "Intermediate Microeconomics", Chapters 5 and 6 (including Appendices) for consumer demand, and Chapter 20 (including Appendix) for conditional factor demands and cost functions.

Solutions to Exercises in Chapter 9

Exercises 9.1:

(1) $MU_x = 2xy$, $MU_y = x^2$
Tangency Condition: $\frac{2xy}{x^2} = \frac{4}{5} \Rightarrow y = \frac{2}{5}x$
Budget Constraint: $4x + 5y = 30$
$\Rightarrow 6x = 30 \Rightarrow x = 5$, $y = 2$.

(2) $\max_{M,B} \left(3M^{\frac{1}{2}} + B\right)$ s.t. $6M + 4B = 26$
$\mathcal{L} = \left(3M^{\frac{1}{2}} + B\right) - \lambda(6M + 4B - 26)$
$\Rightarrow \frac{3}{2}M^{-\frac{1}{2}} = 6\lambda$, $1 = 4\lambda$, $6M + 4B = 26$
Eliminate $\lambda \Rightarrow M = 1$
Then b.c. $\Rightarrow 6 + 4B = 26 \Rightarrow B = 5$.

(3) $\mathcal{L} = (3\ln x_1 + \ln x_2) - \lambda(3x_1 + 2x_2 - 24)$
$\Rightarrow \frac{3}{x_1} = 3\lambda$, $\frac{1}{x_2} = 2\lambda$, $3x_1 + 2x_2 = 24$
Solving $\Rightarrow x_2 = 3$, $x_1 = 6$.

Exercises 9.2:

(1) $\min_{K,L}(12K + 10L)$ s.t. $5K^{0.4}L = 300$
$MPL = 5K^{0.4}$, $MPK = 2K^{-0.6}L$
Tangency: $\dfrac{10}{12} = \dfrac{5K^{0.4}}{2K^{-0.6}L} \Rightarrow L = 3K$
Isoquant: $5K^{0.4}L = 300 \Rightarrow 15K^{1.4} = 300$
$\Rightarrow K = (20)^{\frac{1}{1.4}} = 8.50$, $L = 25.50$
Cost$=12K + 10L = 357$

(2) $\min_{K,L}(8L + 2K)$ s.t. $\dfrac{30}{L^{-1} + K^{-1}} = 70$
$\mathcal{L} = (8L + 2K) - \lambda\left(\frac{3}{L^{-1}+K^{-1}} - 7\right) \Rightarrow$
$8 = \frac{3\lambda}{L^2(L^{-1}+K^{-1})^2}$, $2 = \frac{3\lambda}{K^2(L^{-1}+K^{-1})^2}$
Eliminate $\lambda \Rightarrow K = 2L$. Substitute in:
$\frac{3}{L^{-1}+K^{-1}} = 7 \Rightarrow L = 3.5$, $K = 7$

Exercises 9.3:

(1) From 2^{nd} FOC in previous question:
$3\lambda = 2K^2(L^{-1} + K^{-1})^2$.
Substituting $L = 3.5$, $K = 7 \Rightarrow \lambda = 0.6$

(2) $\max_{x_1,x_2,x_3} x_1x_2x_3$ s.t. $x_1 + 6x_2 + 10x_3 = 36$
$\mathcal{L} = x_1x_2x_3 - \lambda(x_1 + 6x_2 + 10x_3 - 36)$
$\Rightarrow$ (i) $x_2x_3 = \lambda$, (ii) $x_1x_3 = 6\lambda$,
(iii) $x_1x_2 = 10\lambda$, (iv) $x_1+6x_2+10x_3 = 36$
Solving: (i) and (iii) $\Rightarrow x_1 = 10x_3$, and (ii) and (iii) $\Rightarrow x_2 = \frac{5}{3}x_3$.
Substituting in (iv): $30x_3 = 36$
$\Rightarrow x_3 = 1.2$, $x_1 = 12$ and $x_2 = 2$.

(3) (a) No. Perfect substitutes: u is linear, not strictly convex. Consumer buys good 2 only as it is cheaper: $x_1 = 0$, $x_2 = 6$.
(b) Yes. Quasi-linear; $x_1^{2/3}$ is concave.
$\mathcal{L} = (6x_1^{2/3} + x_2) - \lambda(3x_1 + 2x_2 - 12)$
$\Rightarrow 2x_1^{-1/3} = 3\lambda$, $1 = 2\lambda$,
and $3x_1 + 2x_2 = 12$
Eliminating $\lambda \Rightarrow 2x_1^{-1/3} = \frac{3}{2}$
$\Rightarrow x_1^{1/3} = \frac{4}{3} \quad \Rightarrow x_1 = 2.37$
and from budget constraint:
$7.11 + 2x_2 = 12 \quad \Rightarrow x_2 = 2.45$
(c) No. Perfect complements.
Optimum is on the budget constraint where $x_1 = x_2$:
$3x_1 + 2x_2 = 12$ and $x_1 = x_2$
$\Rightarrow x_1 = x_2 = 2.4$

(4) (a) $\mathcal{L} = 3K+L-\lambda\left(\frac{1}{4}\left(L^{1/2} + K^{1/2}\right) - y\right) \Rightarrow$
$24 = \lambda K^{-1/2}, 8 = \lambda L^{-1/2}$ and
$L^{1/2} + K^{1/2} = 4y$. Solving:
$K = y^2$, $L = 9y^2$.
(b) $C = 3K + L = 12y^2$
(c) $C' = 24y$.
From f.o.c. $\lambda = 24K^{1/2} = 24y$

(5) $\mathcal{L} = u(x_1, x_2) - \lambda(p_1x_1 + p_2x_2 - m)$
f.o.c.s: $u_1{=}\lambda p_1$, $u_2{=}\lambda p_2$, $p_1x_1 + p_2x_2{=}m$
Constraint: $dm = p_1dx_1 + p_2dx_2$
Utility: $du = u_1dx_1 + u_2dx_2 = \lambda p_1dx_1 + \lambda p_2dx_2 = \lambda dm \Rightarrow \frac{du}{dm} = \lambda$

Exercises 9.4:

(1) $\mathcal{L} = (\ln c + \ln f) - \lambda\left(f^2 + \left(\dfrac{c}{3}\right)^2 - 8\right)$
$\Rightarrow \dfrac{1}{f} = 2\lambda f$, $\dfrac{1}{c} = \dfrac{2\lambda c}{9}$, $f^2 + \left(\frac{c}{3}\right)^2 = 8$
Solving $\Rightarrow f = 2$, $c = 6$.

(2) $0.5c_1^{-\frac{1}{2}}{=}\lambda$, $0.45c_2^{-\frac{1}{2}}{=}\frac{\lambda}{1.05}$, $c_1 + \frac{c_2}{1.05} = 100$
Elim. $\lambda \Rightarrow c_2{=}.893c_1$ so $c_1 + \frac{.893}{1.05}c_1{=}100$
$\Rightarrow c_1 = 54.04$, $c_2 = 48.26$

(3) $\frac{3}{C} = \lambda$, $\frac{1}{R} = 4\lambda$, $C + 4R = 64$. Solving:
$C = 12R$, $\Rightarrow R = 4$, $C = 48$, 10 hrs work.

Exercises 9.5:

(1) $\mathcal{L} = (3M^{1/2}+B)-\lambda(p_MM+p_BB-y) \Rightarrow$
$\frac{3}{2}M^{-\frac{1}{2}} = \lambda p_M, 1 = \lambda p_B, p_MM{+}p_BB = y$
Solving for the demand functions:
$M = 2.25p_B^2/p_M^2$, $B = y/p_B - 2.25p_B/p_M$

(2) $\mathcal{L} = (rK + wL) - \lambda(KL - y) \Rightarrow w = \lambda K$, $r = \lambda L$ and $KL = y$. Solving: $L = \sqrt{\frac{ry}{w}}$, $K = \sqrt{\frac{wy}{r}}$, $C(y, 4, 4) = 8\sqrt{y}$

(3) $\mathcal{L} = x_1^a x_2^b - \lambda(p_1 x_1 + p_2 x_2 - m)$ $\Rightarrow ax_1^{a-1}x_2^b = \lambda p_1$, $bx_1^a x_2^{b-1} = \lambda p_2$ and $p_1 x_1 + p_2 x_2 = m$. Elim. $\lambda \Rightarrow \frac{ax_2}{bx_1} = \frac{p_1}{p_2}$. Subst. in b.c.: $p_1 x_1 + \frac{b}{a} p_1 x_1 = m$ $\Rightarrow p_1 x_1 = \frac{a}{a+b} m$, $p_2 x_2 = \frac{b}{a+b} m$

[3]This Version of Workbook Chapter 9: September 15, 2003

Worksheet 9: Constrained Optimisation Problems

Quick Questions

(1) A consumer has utility function $u(x_1, x_2) = 2\ln x_1 + 3\ln x_2$, and income $m = 50$. The prices of the two goods are $p_1 = p_2 = 1$. Use the MRS condition to determine his consumption of the two goods. How will consumption change if the price of good 1 doubles? Comment on this result.

(2) Repeat the first part of question 1 using the Lagrangian method and hence determine the marginal utility of income.

(3) Is the utility function $u(x_1, x_2) = x_2 + 3x_1^2$ well-behaved? Explain your answer.

(4) A firm has production function $F(K, L) = 8KL$. The wage rate is 2 and the rental rate of capital is 1. The firm wants to produce output y.
  (a) What is the firm's cost minimisation problem?
  (b) Use the Lagrangian method to calculate its demands for labour and capital, in terms of output, y.
  (c) Evaluate the Lagrange multiplier and hence determine the firm's marginal cost.
  (d) What is the firm's cost function $C(y)$?
  (e) Check that you obtain the same expression for marginal cost by differentiating the cost function.

Longer Questions

(1) A rich student, addicted to video games, has a utility function given by $U = S^{\frac{1}{2}}N^{\frac{1}{2}}$, where S is the number of Sega brand games he owns and N is the number of Nintendo brand games he possesses (he owns machines that will allow him to play games of either brand). Sega games cost £16 each and Nintendo games cost £36 each. The student has disposable income of £2,880 after he has paid his battels, and no other interests in life.
  (a) What is his utility level, assuming he is rational?
  (b) Sega, realizing that their games are underpriced compared to Nintendo, raise the price of their games to £36 as well. By how much must the student's father raise his son's allowance to maintain his utility at the original level?
  (c) Comment on your answer to (b).

(2) George is a graduate student and he divides his working week between working on his research project and teaching classes in mathematics for economists. He estimates that his utility function for earning £W by teaching classes and spending R hours on his research is:
$$u(W, R) = W^{\frac{3}{4}}R^{\frac{1}{4}}.$$
He is paid £16 per hour for teaching and works for a total of 40 hours each week. How should he divide his time between teaching and research in order to maximize his utility?

(3) Maggie likes to consume goods and to take leisure time each day. Her utility function is given by $U = \frac{CH}{C+H}$ where C is the quantity of goods consumed per day and H is

the number of hours spent at leisure each day. In order to finance her consumption bundle Maggie works $24 - H$ hours per day. The price of consumer goods is £1 and the wage rate is £9 per hour.

(a) By showing that it is a CES function, or otherwise, check that the utility function is well-behaved.
(b) Using the Lagrangian method, find how many hours Maggie will work per day.
(c) The government decides to impose an income tax at a rate of 50% on all income. How many hours will Maggie work now? What is her utility level? How much tax does she pay per day?
(d) An economist advises the government that instead of setting an income tax it would be better to charge Maggie a lump-sum tax equal to the payment she would make if she were subject to the 50% income tax. How many hours will Maggie work now, when the income tax is replaced by a lump-sum tax yielding an equal amount? Compare Maggie's utility under the lump-sum tax regime with that under the income tax regime.

(4) A consumer purchases two good in quantities x_1 and x_2; the prices of the goods are p_1 and p_2 respectively. The consumer has a total income I available to spend on the two goods. Suppose that the consumer's preferences are represented by the utility function

$$u(x_1, x_2) = x_1^{\frac{1}{3}} + x_2^{\frac{1}{3}}.$$

(a) Calculate the consumer's demands for the two goods.
(b) Find the own-price elasticity of demand for good 1, $\frac{p_1}{x_1}\frac{\partial x_1}{\partial p_1}$. Show that if $p_1 = p_2$, then this elasticity is $-\frac{5}{4}$.
(c) Find the cross-price elasticity $\frac{p_2}{x_1}\frac{\partial x_1}{\partial p_2}$ when $p_1 = p_2$.

(5) There are two individuals, A and B, in an economy. Each derives utility from his consumption, C, and the fraction of his time spent on leisure, l, according to the utility function:

$$U = \ln(C) + \ln(l)$$

However, A is made very unhappy if B's consumption falls below 1 unit, and he makes a transfer, G, to ensure that it does not. B has no concern for A. A faces a wage rate of 10 per period, and B a wage rate of 1 per period.

(a) For what fraction of the time does each work, and how large is the transfer G?
(b) Suppose A is able to insist that B does not reduce his labour supply when he receives the transfer. How large should it be then, and how long should A work?

CHAPTER 10

Integration

Integration can be thought of as **the opposite of differentiation** but is also a method for **finding the area under a graph**. It is an important mathematical technique, which will be familiar if you have done A-level maths. In this chapter we look at techniques for integrating standard functions, including **integration by substitution** and **by parts** and at economic applications including calculation of **consumer surplus**.

1. The Reverse of Differentiation

If we have a function $y(x)$, we know how to find its *derivative* $\frac{dy}{dx}$ by the process of *differentiation*. For example:

$$\begin{aligned} y(x) &= 3x^2 + 4x - 1 \\ \Rightarrow \quad \frac{dy}{dx} &= 6x + 4 \end{aligned}$$

Integration is the reverse process:

When you know the derivative of a function, $\frac{dy}{dx}$, the process of finding the original function, y, is called *integration*.

For example:

$$\begin{aligned} \frac{dy}{dx} &= 10x - 3 \\ \Rightarrow \quad y(x) &= ? \end{aligned}$$

If you think about how differentiation works, you can probably see that the answer could be:

$$y = 5x^2 - 3x$$

However, there are many other possibilities; it could be $y = 5x^2 - 3x + 1$, or $y = 5x^2 - 3x - 20$, or ... in fact it could be any function of the form:

$$y = 5x^2 - 3x + c$$

where c is a constant. We say that $(5x^2 - 3x + c)$ is the *integral* of $(10x - 3)$ and write this as:

$$\underbrace{\int}_{\text{The integral of}} (10x - 3) \underbrace{dx}_{\text{with respect to } x} = 5x^2 - 3x + c$$

c is referred to as an "arbitrary constant" or a "constant of integration". More generally:

Integration is the reverse of differentiation.
If $f'(x)$ is the derivative of a function $f(x)$, then the integral of $f'(x)$ is $f(x)$ (plus an arbitrary constant):

$$\int f'(x)dx = f(x) + c$$

1.1. Integrating Powers and Polynomials

In the example above you can see that since differentiating powers of x involves reducing the power by 1, integrating powers of x must involve increasing the power by 1. The rule is:

Integrating Powers of x:

$$\int x^n dx = \frac{1}{n+1}x^{n+1} + c \qquad (n \neq -1)$$

It is easy to check that this rule works by differentiating:

$$\frac{d}{dx}\left(\frac{1}{n+1}x^{n+1} + c\right) = x^n$$

You can also see from this that the rule doesn't work when $n = -1$. But it works for other negative powers, for zero, and for non-integer powers – see the examples below.

We can apply this rule to integrate polynomials. For example:

$$\begin{aligned}
\int (4x^2 + 6x - 3)dx &= \int (4x^2 + 6x - 3x^0)dx \\
&= 4 \times \frac{1}{3}x^3 + 6 \times \frac{1}{2}x^2 - 3x^1 + c \\
&= \frac{4}{3}x^3 + 3x^2 - 3x + c
\end{aligned}$$

It is easy to make mistakes when integrating. You should always check your answer by differentiating it to make sure that you obtain the original function.

EXAMPLES 1.1: *Integrating Powers and Polynomials*

(i) What is the integral of $x^4 - 2x + 5$?

$$\int (x^4 - 2x + 5)dx = \frac{1}{5}x^5 - x^2 + 5x + c$$

(ii) Integrate $2 - \frac{t^5}{5}$.

$$\int \left(2 - \frac{t^5}{5}\right) dt = 2t - \frac{t^6}{30} + c$$

(iii) If $\dfrac{dy}{dx} = (2-x)(4-3x)$, what is y?

$$\begin{aligned} y &= \int (2-x)(4-3x)dx \\ &= \int (8-10x+3x^2)dx = 8x-5x^2+x^3+c \end{aligned}$$

(iv) Integrate $1+\dfrac{10}{z^3}$.

$$\begin{aligned} \int \left(1+\frac{10}{z^3}\right)dz &= \int \left(1+10z^{-3}\right)dz \\ &= z+10\times\frac{1}{-2}z^{-2}+c \\ &= z-\frac{5}{z^2}+c \end{aligned}$$

(v) If $f'(x) = 3\sqrt{x}$ what is $f(x)$?

$$\begin{aligned} f(x) &= \int 3\sqrt{x}dx \\ &= \int 3x^{\frac{1}{2}}dx \\ &= 3\times\frac{2}{3}x^{\frac{3}{2}}+c \\ &= 2x^{\frac{3}{2}}+c \end{aligned}$$

(vi) Integrate the function $3ax^2+2tx$ with respect to x.

$$\int (3ax^2+2tx)dx = ax^3+tx^2+c$$

(In this example there are several variables or parameters. We say "with respect to x" to clarify which one is to be treated as the variable of integration. The others are then treated as constants.)

EXERCISES 10.1: **Integrating Powers and Polynomials**

(1) Find: (a) $\int 8x^3dx$ (b) $\int (2z-z^3+4)dz$ (c) $\int (1+3t^{-8})dt$
(d) $\int (a+bx)dx$ (e) $\int \left(\frac{q^2}{2}-\frac{18}{q^4}\right)dq$

(2) Integrate (a) $5x^{1.5}$ (b) $\sqrt{4z}$

(3) What is the integral of z^{3a-1} with respect to z?

(4) If $g'(p) = \alpha p^{\beta}$, what is $g(p)$?

1.2. Economic Application

Suppose we know that a firm's marginal cost of producing output y is $8y+3$, and also that the firm has a fixed cost of 10. Then we can integrate the marginal cost function to find the firm's total cost function:

$$\begin{aligned} C'(y) &= 8y+3 \\ \Rightarrow C(y) &= 4y^2+3y+c \end{aligned}$$

As usual, integration gives us an arbitrary constant, c. But in this case, we have another piece of information that tells us the value of c – the cost of producing zero output is 10:

$$\begin{aligned} C(0) &= 10 \quad \Rightarrow \quad c=10 \\ \Rightarrow C(y) &= 4y^2+3y+10 \end{aligned}$$

1.3. More Rules for Integration

Remember the rules for differentiating logarithmic and exponential functions (Chapter 6):

$$y=\ln x \quad \Rightarrow \quad \frac{dy}{dx}=\frac{1}{x}$$
$$y=e^{ax} \quad \Rightarrow \quad \frac{dy}{dx}=ae^{ax}$$

By reversing these we can obtain two more rules for integration:

$$\boxed{\begin{gathered} \int \frac{1}{x}\,dx=\ln x+c \\ \int e^{ax}dx=\frac{1}{a}\,e^{ax}+c \end{gathered}}$$

(The first of these rules tells us how to integrate x^n when $n{=}{-1}$, which we couldn't do before.)

EXAMPLES 1.2:

(i) $\displaystyle\int\left(6x+\frac{3}{x}\right)dx=\int\left(6x+3\times\frac{1}{x}\right)dx=3x^2+3\ln x+c$

(ii) $\displaystyle\int\left(4e^{2x}+15e^{-3x}\right)dx=4\times\frac{1}{2}e^{2x}+15\times\frac{1}{-3}e^{-3x}+c=2e^{2x}-5e^{-3x}+c$

Looking at the examples above you can also see that the following general rules hold. In fact they are obvious from what you know about differentiation, and you may have been using them without thinking about it.

$$\boxed{\begin{gathered} \int af(x)dx=a\int f(x)dx \\ \int (f(x)\pm g(x))dx=\int f(x)dx\pm\int g(x)dx \end{gathered}}$$

The rules for integrating $\frac{1}{x}$ and e^x can be generalised. From the Chain Rule for differentiation we can see that, if $f(x)$ is a function, then:

$$\frac{d}{dx}\left(\ln f(x)\right)=\frac{f'(x)}{f(x)} \quad \text{and} \quad \frac{d}{dx}\left(e^{f(x)}\right)=f'(x)e^{f(x)}$$

Reversing these gives us two further rules:

$$\int \frac{f'(x)}{f(x)}\,dx = \ln f(x) + c$$

$$\int f'(x)e^{f(x)}dx = e^{f(x)} + c$$

To apply these rules you have to notice that an integral can be written in one of these forms, for some function $f(x)$.

EXAMPLES 1.3:

(i) $\int xe^{3x^2}dx$ We can rewrite this integral:

$$\begin{aligned}\int xe^{3x^2}dx &= \frac{1}{6}\int 6xe^{3x^2}dx \quad \text{and apply the } 2^{nd} \text{ rule above with } f(x) = 3x^2\\ &= \frac{1}{6}e^{3x^2} + c\end{aligned}$$

(ii) $\int \frac{1}{y+4}\,dx$ Applying the 1^{st} rule directly with $f(y) = y + 4$:

$$\int \frac{1}{y+4}\,dy = \ln(y+4) + c$$

(iii) $\int \frac{z+3}{z^2+6z-5}\,dz$

$$\begin{aligned} &= \tfrac{1}{2}\int \frac{2z+6}{z^2+6z-5}\,dz \qquad (f(z) = z^2 + 6z - 5)\\ &= \tfrac{1}{2}\ln(z^2 + 6z - 5) + c\end{aligned}$$

EXERCISES 10.2: **Integrating Simple Functions**

(1) Find: (a) $\int 10e^{3x}dx$ (b) $\int \left(9y^2 - \frac{4}{y}\right) dy$ (c) $\int \left(\frac{1}{z} + \frac{1}{z^2}\right) dz$
(d) $\int t^2e^{-t^3}dt$ (e) $\int \frac{1}{2q-7}\,dq$

(2) If $f'(t) = 1 - e^{6t}$, what is $f(t)$?

(3) Integrate: $4x^2 - 2\sqrt{x} + 8x^{-3}$.

(4) If a firm has no fixed costs, and its marginal cost of producing output q is $(9q^{0.8} - 2)$, find the firm's total cost function $C(q)$.

(5) Find the integral with respect to x of $x^a + e^{ax} + x^{-a}$, assuming $-1 < a < 1$ and $a \neq 0$.

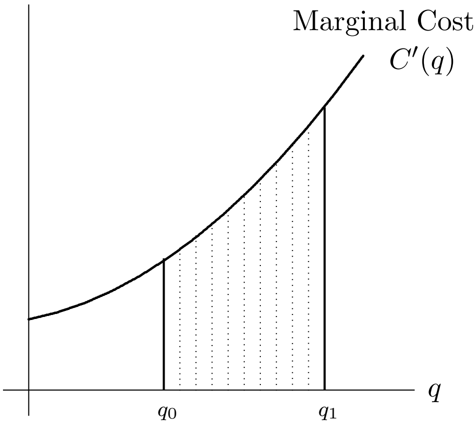
Further Reading and Exercises

- *Jacques* §6.1
- *Anthony & Biggs* §25.3

2. Integrals and Areas

In economics we often use areas on graphs to measure total costs and benefits: for example to evaluate the effects of imposing a tax. Areas on graphs can be calculated using integrals.

2.1. An Economic Example



The area under a firm's marginal cost curve between q_0 and q_1 represents the *total cost* of increasing output from q_0 to q_1: it adds up the marginal costs for each unit of output between q_0 and q_1.

So, the area represents: $C(q_1) - C(q_0)$.

To calculate this area we could:

- Integrate the marginal cost function $C'(q)$ to find the function $C(q)$
- Evaluate $C(q)$ at q_0 and q_1 to obtain $C(q_1) - C(q_0)$

Note that $C(q)$ will contain an arbitrary constant, but it will cancel out in $C(q_1) - C(q_0)$. (The constant represents the fixed costs – not needed to calculate the increase in costs.)

We write this calculation as:

$$\text{Area } = C(q_1) - C(q_0) = \int_{q_0}^{q_1} C'(q)dq$$

2.2. Definite Integration

$$\int_a^b f(x)dx$$

- represents the area under the graph of $f(x)$ between a and b.
- It is called a *definite integral.*
- a and b are called the *limits of integration.*
- To calculate it, we integrate, evaluate the answer at each of the limits, and subtract.

The type of integration that we did in the previous section is known as *indefinite* integration. For example, $\int(4x+1)dx = x^2 + x + c$ is an indefinite integral. The answer is a function of x

containing an arbitrary constant. In *definite* integration, in contrast, we evaluate the answer at the limits.

Examples 2.1: *Definite Integrals*

(i) $\int_{-1}^{4} (4x+1)dx$

This integral represents the area under the graph of the function $f(x) = 4x+1$ between $x = -1$ and $x = 4$.

$$\begin{aligned} \int_{-1}^{4} (4x+1)dx &= \left[2x^2 + x + c\right]_{-1}^{4} \\ &= \left(2 \times 4^2 + 4 + c\right) - \left(2 \times (-1)^2 - 1 + c\right) \\ &= (36 + c) - (1 + c) \\ &= 35 \end{aligned}$$

It is conventional to use square brackets here

From now on we will not bother to include the arbitrary constant in a definite integral since it always cancels out.

(ii) $\int_{9}^{25} 3\sqrt{y}\,dy$

$$\begin{aligned} \int_{9}^{25} 3\sqrt{y}\,dy &= \int_{9}^{25} 3y^{\frac{1}{2}}\,dy \\ &= \left[2y^{\frac{3}{2}}\right]_{9}^{25} \\ &= \left(2 \times 25^{\frac{3}{2}}\right) - \left(2 \times 9^{\frac{3}{2}}\right) \\ &= 250 - 54 \\ &= 196 \end{aligned}$$

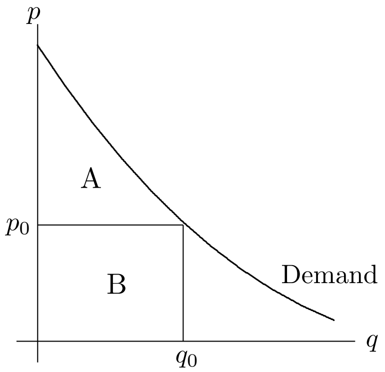
(iii) $\int_{1}^{a} \left(3 + \frac{2}{q}\right) dq$ where a is a parameter.

$$\begin{aligned} \int_{1}^{a} \left(3 + \frac{2}{q}\right) dq &= [3q + 2\ln q]_{1}^{a} \\ &= (3a + 2\ln a) - (3 \times 1 + 2\ln 1) \\ &= (3a + 2\ln a) - 3 \\ &= 3a - 3 + 2\ln a \end{aligned}$$

(iv) For a firm with marginal cost function $MC = 3q^2 + 10$, find the increase in costs if output is increased from 2 to 6 units.

$$\begin{aligned} C(6) - C(2) &= \int_{2}^{6} (3q^2 + 10)dq \\ &= \left[q^3 + 10q\right]_{2}^{6} \\ &= \left(6^3 + 60\right) - \left(2^3 + 20\right) \\ &= 276 - 28 \\ &= 248 \end{aligned}$$

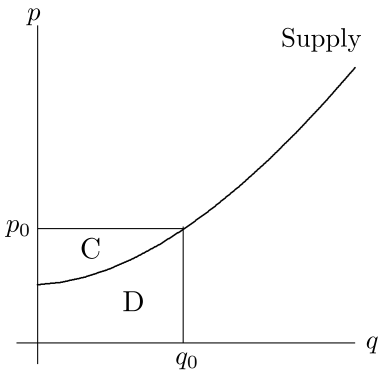
2.3. Economic Application: Consumer and Producer Surplus



The diagram shows a market demand function. When the market price is p_0, the quantity sold is q_0 and Area A represents net consumer surplus.

If the inverse demand function is $p = D(q)$, we can calculate Consumer Surplus by:

$$CS = \underbrace{\int_0^{q_0} D(q)dq}_{\text{Area(A+B)}} - \underbrace{p_0 q_0}_{\text{Area B}}$$



Similarly, Area C represents net producer surplus.

If the inverse supply function is $p = S(q)$, we can calculate Producer Surplus by:

$$PS = \underbrace{p_0 q_0}_{\text{Area(C+D)}} - \underbrace{\int_0^{q_0} S(q)dq}_{\text{Area D}}$$

Exercises 10.3: **Definite Integrals**

(1) Evaluate the definite integrals: (a) $\int_1^4 (2x^2+1)dx$ (b) $\int_{-1}^1 e^{3x}dx$

(2) For a market in which the inverse demand and supply functions are given by: $p^d(q) = 24 - q^2$ and $p^s(q) = q + 4$, find:
(a) the market price and quantity (b) consumer surplus (c) producer surplus.

(3) Evaluate the definite integrals:
$$\int_0^1 (x^2 - 3x + 2)dx, \quad \int_1^2 (x^2 - 3x + 2)dx, \quad \int_0^2 (x^2 - 3x + 2)dx.$$
Explain the answers you obtain by sketching the graph of $y = x^2 - 3x + 2$.

Further Reading and Exercises

- *Jacques* §6.2. In particular §6.2.3 looks at investment, another economic application of definite integrals.
- *Anthony & Biggs* §§25.1, 25.2 and 25.4

3. Techniques for Integrating More Complicated Functions

The rules in sections 1.1 and 1.3 only allow us to integrate quite simple functions. They don't tell you, for example, how to integrate xe^x, or $\sqrt{3x^2+1}$.

Whereas it is possible to use rules to differentiate any "sensible" function, the same is not true for integration. Sometimes you just have to guess what the answer might be, then check whether you have guessed right by differentiating. There are some functions that cannot be integrated algebraically: the only possibility to is to use a computer to work out a numerical approximation.[1]

But when you are faced with a function that cannot be integrated by the simple rules, there are a number of techniques that you can try – one of them might work!

3.1. Integration by Substitution

Consider the integral:

$$\int (3x+1)^5 dx$$

We know how to integrate powers of x, but not powers of $(3x+1)$. So we can try the following procedure:

- Define a new variable: $t = 3x+1$
- Differentiate: $\dfrac{dt}{dx} = 3 \quad \Rightarrow dt = 3dx \Rightarrow dx = \dfrac{1}{3}dt$
- Use these expressions to substitute for x and dx in the integral:

$$\int (3x+1)^5 dx = \int t^5 \frac{1}{3} dt$$

- Integrate with respect to t, then substitute back to obtain the answer as a function of x:

$$\begin{aligned} \int t^5 \frac{1}{3} dt &= \frac{1}{18} t^6 + c \\ &= \frac{1}{18}(3x+1)^6 + c \end{aligned}$$

You can check, by differentiating, that this is the integral of the original function. If you do this, you will see that integration by substitution is a way of reversing the Chain Rule (see Chapter 6).

Thinking about the Chain Rule, we can see that Integration by Substitution works for integrals that can be written in a particular form:

[1]The function $e^{-x^2/2}$, which is important in statistics, is an example.

When an integral can be written in the form:

$$\int f(t)\frac{dt}{dx}dx$$

where t is some function of x

then it can be integrated by substituting t for x:

$$\int f(t)\frac{dt}{dx}dx = \int f(t)dt$$

Sometimes you will be able to see immediately that an integral has the right form for using a substitution. Sometimes it is not obvious - but you can try a substitution and see if it works.

Note that the logarithmic and exponential rules that we found at the end of section 1.3 are a special case of the method of Integration by Substitution. The first example below could be done using those rules instead.

EXAMPLES 3.1: *Integration by substitution*

(i) $\int xe^{3x^2}dx$ For this integral we can use the substitution $t = 3x^2$:

$$t = 3x^2 \quad \Rightarrow dt = 6xdx$$

Substituting for x and dx:

$$\begin{aligned}\int xe^{3x^2}dx &= \int e^{3x^2}xdx\\ &= \int e^t\tfrac{1}{6}dt\\ &= \tfrac{1}{6}e^t + c = \tfrac{1}{6}e^{3x^2} + c\end{aligned}$$

(ii) $\int (3x^2+1)^5 dx$

This example looks similar to the original one we did, but unfortunately the method does not work. Suppose we try substituting $t = 3x^2 + 1$:

$$t = 3x^2 + 1 \quad \Rightarrow dt = 6xdx$$

To substitute for x and dx we also need to note that $x = \sqrt{\frac{t-1}{3}}$. Then:

$$\int (3x^2+1)^5dx = \int t^5\frac{1}{6\sqrt{\frac{t-1}{3}}}dt$$

Doing the substitution has made the integral more difficult, rather than less.

(iii) $\int \frac{4y}{\sqrt{y^2-3}}dy$ Here we can use a substitution $t = y^2 - 3$:

$$t = y^2 - 3 \quad \Rightarrow dt = 2ydy$$

Substituting for y and dy:

$$\begin{aligned}\int \frac{4y}{\sqrt{y^2-3}}dy &= \int 2\frac{1}{\sqrt{y^2-3}}2ydy\\ &= \int 2t^{-\frac{1}{2}}dt\\ &= 4t^{\frac{1}{2}} + c = 4\sqrt{y^2-3} + c\end{aligned}$$

EXERCISES 10.4: **Integration by Substitution**

(1) Evaluate the following integrals using the suggested substitution:

(a) $\displaystyle\int x^2(x^3-5)^4 dx \qquad t = x^3 - 5$

(b) $\displaystyle\int (2z+1)e^{z(z+1)} dz \qquad t = z(z+1)$

(c) $\displaystyle\int \frac{3}{\sqrt{2y+3}} dy \qquad t = 2y+3$

(d) $\displaystyle\int \frac{x}{x^2+a} dx \qquad t = x^2 + a$ where a is a parameter

(e) $\displaystyle\int \frac{1}{(2q-1)^2} dq \qquad t = 2q - 1$

(f) $\displaystyle\int \frac{p+1}{3p^2+6p-1} dp \qquad t = 3p^2 + 6p - 1$

(2) Evaluate the following integrals by substitution or otherwise:

(a) $\displaystyle\int (4x-7)^6 dx$ (b) $\displaystyle\int \frac{2q^3+1}{q^4+2q}\, dq$ (c) $\displaystyle\int xe^{-kx^2} dx$ (k is a parameter.)

3.2. Integration by Parts

Remember the Product Rule for differentiation:

$$\frac{d}{dx}\Big(u(x)v(x)\Big) = u(x)v'(x) + u'(x)v(x)$$

Integrating this we get:

$$u(x)v(x) = \int u(x)v'(x)dx + \int u'(x)v(x)dx$$

and rearranging gives us the following formula:

Integration by Parts:

$$\int u'(x)v(x)dx = u(x)v(x) - \int u(x)v'(x)dx$$

Hence if we start with an integral that can be written in the form $\int u'(x)v(x)dx$ we can evaluate it using this rule provided that we know how to evaluate $\int u(x)v'(x)dx$.

EXAMPLES 3.2: *Integration by Parts*

(i) $\displaystyle\int xe^{5x} dx$ Suppose we let $v(x) = x$ and $u'(x) = e^{5x}$.

$$\begin{aligned} u'(x) &= e^{5x} &\Rightarrow\quad u(x) &= \tfrac{1}{5}e^{5x} \\ v(x) &= x &\Rightarrow\quad v'(x) &= 1 \end{aligned}$$

Applying the formula:

$$\begin{aligned}\int u'(x)v(x)dx &= u(x)v(x) - \int u(x)v'(x)dx\\ \int xe^{5x}dx &= \tfrac{1}{5}xe^{5x} - \int \tfrac{1}{5}e^{5x}dx\\ &= \tfrac{1}{5}xe^{5x} - \tfrac{1}{25}e^{5x} + c\end{aligned}$$

(Check this result by differentiation.)

(ii) $\int (4y+1)(y+2)^3 dy$ Let $v(y) = 4y+1$ and $u'(y) = (y+2)^3$:

$$\begin{aligned}u'(y) &= (y+2)^3 &\Rightarrow\quad u(y) &= \tfrac{1}{4}(y+2)^4\\ v(y) &= 4y+1 &\Rightarrow\quad v'(y) &= 4\end{aligned}$$

(If this is not obvious to you it could be done by substitution)

Applying the formula:

$$\begin{aligned}\int u'(y)v(y)dy &= u(y)v(y) - \int u(y)v'(y)dy\\ \int (4y+1)(y+2)^3dy &= \tfrac{1}{4}(4y+1)(y+2)^4 - \int (y+2)^4dy\\ &= \tfrac{1}{4}(4y+1)(y+2)^4 - \tfrac{1}{5}(y+2)^5 + c\end{aligned}$$

(iii) $\int \ln x\, dx$

This is a standard function, but cannot be integrated by any of the rules we have so far. Since we don't have a product of two functions, integration by parts does not seem to be a promising technique. However, if we put:

$$\begin{aligned}u'(x) &= 1 &\Rightarrow\quad u(x) &= x\\ v(x) &= \ln x &\Rightarrow\quad v'(x) &= \tfrac{1}{x}\end{aligned}$$

and apply the formula, we obtain:

$$\begin{aligned}\int u'(x)v(x)dx &= u(x)v(x) - \int u(x)v'(x)dx\\ \int \ln x\, dx &= x\ln x - \int x\frac{1}{x}dx\\ &= x\ln x - \int 1.dx\\ &= x\ln x - x + c\end{aligned}$$

Again, check by differentiating.

Exercises 10.5: Use integration by parts to find:

(1) $\int (x+1)e^x dx$ (2) $\int 2y\ln y\, dy$

3.3. Integration by Substitution and by Parts: Definite Integrals

3.3.1. *Integration by Substitution.* If we want to find:

$$\int_0^1 \frac{2x}{x^2+8}dx$$

we could integrate by substituting $t = x^2+8$ to find the indefinite integral, and then evaluate it at the limits. However, a quicker method is to substitute for the limits as well:

$$\begin{aligned} t = x^2+8 &\Rightarrow dt = 2xdx \\ x = 0 &\Rightarrow t = 8 \\ x = 1 &\Rightarrow t = 9 \end{aligned}$$

Hence:

$$\begin{aligned} \int_0^1 \frac{2x}{x^2+8}dx &= \int_0^1 \frac{1}{x^2+8}2xdx \\ &= \int_8^9 \frac{1}{t}dt \\ &= [\ln t]_8^9 \\ &= \ln 9 - \ln 8 = \ln 1.125 \end{aligned}$$

3.3.2. *Integration by Parts.* Similarly, the method of integration by parts can be modified slightly to deal with definite integrals. The formula becomes:

$$\int_a^b u'(x)v(x)dx = \Big[u(x)v(x)\Big]_a^b - \int_a^b u(x)v'(x)dx$$

Consider, for example: $\displaystyle\int_0^1 (1-x)e^x\,dx$

We can integrate by parts using:
$$\begin{aligned} u'(x) &= e^x &\Rightarrow\quad u(x) &= e^x \\ v(x) &= 1-x &\Rightarrow\quad v'(x) &= -1 \end{aligned}$$

Applying the formula:

$$\begin{aligned} \int_0^1 (1-x)e^x\,dx &= \Big[(1-x)e^x\Big]_0^1 + \int_0^1 e^x dx \\ &= -1 + \big[e^x\big]_0^1 \\ &= e - 2 \end{aligned}$$

EXERCISES 10.6: **Definite Integrals**

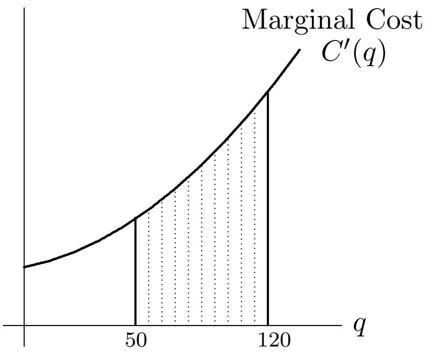
(1) Integrate by substitution: (a) $\displaystyle\int_0^2 (2x+1)^3 dx$ (b) $\displaystyle\int_{-1}^1 e^{1+3y}dy$

(2) Integrate by parts: $\displaystyle\int_0^1 4ze^{-2z}\,dz$

Further Reading and Exercises

- *Anthony & Biggs* §§26.1, 26.2 and 26.3

4. Integrals and Sums



Suppose a firm with marginal cost curve $C'(q)$ increases its output from 50 units to 120. The total increase in cost is given by the integral of the marginal cost curve:

$$C(120) - C(50) = \int_{50}^{120} C'(q)dq$$

We can think of this as the sum of all the marginal costs of the units of output between 50 and 120, so:

$$\int_{50}^{120} C'(q)dq \approx C'(51) + C'(52) + \cdots + C'(120)$$

So we can see that:

$$\int_{50}^{120} C'(q)dq \approx \sum_{q=51}^{100} C'(q)$$

These two expressions are only approximately equal because we have a *continuous* marginal cost curve, allowing for fractions of units of output.

In general, we can think of integrals as representing the equivalent of a *sum*, used when we are dealing with continuous functions.

4.1. Economic Application: The Present Value of an Income Flow

Remember from Chapter 3 that when interest is compounded continously at annual rate i, the present value of an amount A received in t years time is given by:

$$Ae^{-it}$$

Suppose you receive an annual income of y for T years, and that, just as interest is compounded continuously, your income is paid continuously. This means that in a short time period length Δt, you will receive $y\Delta t$. For example, in a day ($\Delta t = \frac{1}{365}$) you would get $\frac{1}{365}y$. In an infinitesimally small time period dt your income will be:

$$ydt$$

which has present value:

$$ye^{-it}dt$$

Then the present value of the whole of your income stream is given by the "sum" over the whole period:

$$V = \int_0^T ye^{-it}dt$$

We can calculate the present value in this way even if the income stream is not constant – that is, if $y = y(t)$.

Examples 4.1: *The Present Value of an Income Flow*

(i) An investment will yield a constant continuous income of £1000 per year for 8 years. What is its present value if the interest rate is 2%?

$$\begin{aligned} V &= \int_0^8 1000e^{-0.02t}dt \\ &= 1000\left[-\frac{1}{0.02}e^{-0.02t}\right]_0^8 \\ &= 50000(-e^{-0.16}+1) = £7393 \end{aligned}$$

(ii) A worker entering the labour market expects his annual earnings, y, to grow continuously according to the formula $y(t) = £12000e^{0.03t}$ where t is length of time that he has been working, measured in years. He expects to work for 40 years. If the interest rate is $i = 0.05$, what is the present value of his expected lifetime earnings?

$$\begin{aligned} V &= \int_0^{40} y(t)e^{-it}dt \\ &= \int_0^{40} 12000e^{0.03t}e^{-0.05t}dt \\ &= 12000\int_0^{40} e^{-0.02t}dt \\ &= 12000\left[-\frac{1}{0.02}e^{-0.02t}\right]_0^{40} \\ &= 600000\left[-e^{-0.02t}\right]_0^{40} \\ &= 600000\left(-e^{-0.8}+1\right) = £330403 \end{aligned}$$

Exercises 10.7: **The present value of an income flow**

(1) What is the present value of a constant stream of income of £200 per year for 5 years, paid continuously, if the interest rate is 5%?

(2) A worker earns a constant continuous wage of w per period. Find the present value, V, of his earnings if he works for T periods and the interest rate is r. What is the limiting value of V as $T \to \infty$?

Further Reading and Exercises

- *Jacques* §6.2.4.
- *Anthony & Biggs* §§25.1, 25.2 and 25.4

Solutions to Exercises in Chapter 10

EXERCISES 10.1:

(1) (a) $2x^4 + c$
(b) $z^2 - \frac{1}{4}z^4 + 4z + c$
(c) $t - \frac{3}{7}t^{-7} + c$
(d) $ax + \frac{1}{2}bx^2 + c$
(e) $\frac{q^3}{6} + \frac{6}{q^3} + c$

(2) (a) $2x^{2.5} + c$
(b) $\frac{4}{3}z^{3/2} + c$

(3) $\frac{1}{3a}z^{3a} + c$

(4) $\frac{\alpha}{\beta+1}p^{\beta+1} + c$

EXERCISES 10.2:

(1) (a) $\frac{10}{3}e^{3x} + c$
(b) $3y^3 - 4\ln y + c$
(c) $\ln z - \frac{1}{z} + c$
(d) $-\frac{1}{3}e^{-t^3} + c$
(e) $\frac{1}{2}\ln(2q - 7) + c$

(2) $f(t) = t - \frac{1}{6}e^{6t} + c$

(3) $\frac{4}{3}x^3 - \frac{4}{3}x^{3/2} - 4x^{-2} + c$

(4) $C(q) = 5q^{1.8} - 2q$

(5) $\frac{1}{a+1}x^{a+1} + \frac{1}{a}e^{ax} + \frac{1}{1-a}x^{1-a} + c$

EXERCISES 10.3:

(1) (a) $\left[\frac{2}{3}x^3 + x\right]_1^4 = 45$
(b) $\left[\frac{1}{3}e^{3x}\right]_{-1}^1 = \frac{1}{3}(e^3 - e^{-3}) = 6.68$

(2) (a) $q = 4$, $p = 8$
(b) $\left[24q - \frac{1}{3}q^3\right]_0^4 - 32 = 42\frac{2}{3}$
(c) $32 - \left[\frac{1}{2}q^2 + 4q\right]_0^4 = 8$

(3) $\left[\frac{1}{3}x^3 - \frac{3}{2}x^2 + 2x\right]_0^1 = \frac{5}{6}$
$\left[\frac{1}{3}x^3 - \frac{3}{2}x^2 + 2x\right]_1^2 = -\frac{1}{6}$
$\left[\frac{1}{3}x^3 - \frac{3}{2}x^2 + 2x\right]_0^2 = \frac{2}{3}$
Between $x = 1$ and $x = 2$ the graph is below the x-axis. So the area between the graph and the axis here is negative.

EXERCISES 10.4:

(1) (a) $dt = 3x^2dx$
$\int t^4\frac{1}{3}dt = \frac{1}{15}t^5 + c = \frac{1}{15}(x^3 - 5)^5 + c$
(b) $dt = (2z + 1)dz$
$\int e^tdt = e^t + c = e^{z(z+1)} + c$
(c) $dt = 2dy$
$\int \frac{3}{2t^{\frac{1}{2}}}dt = 3t^{\frac{1}{2}} + c = 3\sqrt{2y + 3} + c$
(d) $dt = 2xdx$
$\int \frac{1}{2t}dt = \frac{1}{2}\ln t + c = \frac{1}{2}\ln(x^2 + a) + c$
(e) $dt = 2dq$
$\int \frac{1}{2t^2}dt = -\frac{1}{2t} + c = -\frac{1}{4q-2} + c$
(f) $dt = (6p + 6)dp = 6(p + 1)dp$
$\int \frac{1}{t}\frac{1}{6}dt = \frac{1}{6}\ln t + c$
$= \frac{1}{6}\ln(3p^2 + 6p - 1) + c$

(2) (a) $t = 4x - 7$, $dt = 4dx$
$\frac{1}{28}(4x - 7)^7 + c$
(b) $t = q^4 + 2q$, $dt = (4q^3 + 2)dq$
$\frac{1}{2}\ln(q^4 + 2q) + c$
(c) $t = \pm kx^2$, $dt = \pm 2kxdx$
$-\frac{1}{2k}e^{-kx^2} + c$

EXERCISES 10.5:

(1) $u' = e^x$, $v = x + 1$
$xe^x + c$

(2) $u' = 2y$, $v = \ln y$
$y^2\ln y - \frac{1}{2}y^2 + c$

EXERCISES 10.6:

(1) (a) $t = 2x + 1 \Rightarrow \frac{1}{8}\left[t^4\right]_1^5 = 78$
(b) $t = 1+3y \Rightarrow \frac{1}{3}\left[e^t\right]_{-2}^4 = \frac{1}{3}\left(e^4 - e^{-2}\right)$

(2) $u' = e^{-2z}$, $v = 4z \Rightarrow$
$\left[-2ze^{-2z}\right]_0^1 + \left[-e^{-2z}\right]_0^1 = 1 - 3/e^2$

EXERCISES 10.7:

(1) $\int_0^5 200e^{-0.05t}dt = 200\left[-\frac{1}{0.5}e^{-0.05t}\right]_0^5$
$= 4000(1 - e^{-0.25}) = £884.80$

(2) $V = \int_0^T we^{-rt}dt$
$= w\left[-\frac{1}{r}e^{-rt}\right]_0^T = \frac{w}{r}\left(1 - e^{-rT}\right)$
$V \to \frac{w}{r}$ as $T \Rightarrow \infty$

[2]This Version of Workbook Chapter 10: September 24, 2010

Worksheet 10: Integration, and Further Optimisation Problems

Integration

(1) Evaluate the following integrals:

$$\text{(a)} \int (x-3)(x+1)dx \qquad \text{(b)} \int_0^1 e^{-5y}dy \qquad \text{(c)} \int \frac{2z+3}{z^2}dz$$

(2) Evaluate the following integrals, using a substitution if necessary:

$$\text{(a)} \int (3x+1)^9 dx \qquad \text{(b)} \int \frac{y-3}{y^2-6y+1}dy \qquad \text{(c)} \int_0^1 \frac{z^3}{\sqrt{z^2+1}}dz$$

(3) A competitive firm has inverse supply function $p = q^2 + 1$ and fixed costs $F = 20$. Find its total cost function.

(4) An investment will yield a continuous profit flow $\pi(t)$ per year for T years. Profit at time t is given by:

$$\pi(t) = a + bt$$

where a and b are constants. If the interest rate is r, find the present value of the investment. (Hint: you can use integration by parts.)

(5) The inverse demand and supply functions in a competitive market are given by:

$$p^d(q) = \frac{72}{1+q} \text{ and } p^s(q) = 2 + q$$

(a) Find the equilibrium price and quantity, and consumer surplus.
(b) The government imposes a tax $t = 5$ on each unit sold. Calculate the new equilibrium quantity, tax revenue, and the deadweight loss of the tax.

Further Problems

(1) An incumbent monopoly firm Alpha faces the following market demand curve:

$$Q = 96 - P,$$

where Q is the quantity sold per day, and P is the market price. Alpha can produce output at a constant marginal cost of £6, and has no fixed costs.

(a) What is the price Alpha is charging? How much profit is it making per day?
(b) Another firm, Beta, is tempted to enter the market given the high profits that the incumbent, Alpha, is making. Beta knows that Alpha has a cost advantage: if it enters, its marginal costs will be twice as high as Alpha's, though there will be no fixed costs of entry. If Beta does enter the market, is expects Alpha to act as a Stackelberg leader (i.e. Beta maximizes its profits taking Alpha's output as given; Alpha maximizes its profits taking into account that Beta will react in this way). Show that under these assumptions, Beta will find it profitable to enter, despite the cost disadvantage. How much profit would each firm earn?
(c) For a linear demand curve of the form $Q = a - bP$, show that consumers' surplus is given by the expression $CS = \frac{1}{2b}Q^2$. Evaluate the benefit to consumers of increased competition in the market once Beta has entered.

(2) A monopolist knows that to sell x units of output she must charge a price of $P(x)$, where $P'(x) < 0$ and $P''(x) < 0$ for all x. The monopolist's cost of producing x is $C(x) = A + ax^2$, where A and a are both positive numbers. Let x^* be the firm's profit-maximizing output. Write down the first- and second-order conditions that x^* must satisfy. Show that x^* is a decreasing function of a. How does x^* vary with A?

(3) An Oxford economic forecasting firm has the following cost function for producing reports:

$$C(y) = 4y^2 + 16$$

where y is the number of reports.

(a) What are its average and marginal cost functions?
(b) At what number of reports is its average cost minimized?
(c) Initially the market for economic forecasts in Oxford is extremely competitive and the going price for a report is £15. Should the firm continue to produce reports? Why or why not?
(d) Suppose the price rises to £20. How many reports will the firm supply? Illustrate diagrammatically and comment on your answer.
(e) Suddenly all competitor forecasting firms go out of business. The demand for reports is such that $p = 36 - 6y$, where p is the price of a report. How many reports will the Oxford firm produce? What profit will it earn?

(4) The telecommunications industry on planet Mercury has an inverse demand curve given by $P = 100 - Q$. The marginal cost of a unit of output is 40 and fixed costs are 900. Competition in this industry is as in the Cournot model. There is free entry into and exit from the market.

(a) How many firms survive in equilibrium?
(b) Due to technical advance, fixed costs fall to 400 from 900. What happens to the number of firms in the industry?
(c) What is the effect on price of the fall in fixed costs? Explain your answer.

(5) Students at St. Gordon's College spend all their time in the college bar drinking and talking on their mobile phones. Their utility functions are all the same and are given by $U = DM - O^2$, where D is the amount of drink they consume, M is the amount of time they talk on their mobile phones and O is the amount of time each other student spends on the phone, over which they have no control. Both drink and mobile phone usage cost £1 per unit, and each student's income is £100.

(a) How much time does each student spend on the phone and how much does each drink?
(b) What is the utility level of each student?
(c) The fellows at the college suggest that mobile phone usage should be taxed at 50 pence per unit, the proceeds from the tax being used to subsidize the cost of fellows' wine at high table. Are the students better off if they accept this proposal?
(d) The economics students at the college suggest that the optimal tax rate is twice as large. Are they right? What other changes might the students propose to improve their welfare?