The Nine Chapters on the History of Chinese Mathematics

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Abstract

This article explores Chinese mathematics from the first archeological evidence of numbers on oracle bones (14^{th} century BC) to the time Chinese mathematics became a part universal mathematics (halfway the 19^{th} century AD).

First a concise overview of Chinese history and in philosophy is given. The ethical oriented Confucianism was the dominant philosophy and consequently little attention was given to the natural world, hindering the development of natural sciences and mathematics. Due to historical and philosophical reasons, Chinese mathematics took quite a different path than its Western counterpart: Chinese mathematics was focused on algebra and practical applications instead of geometry and theoretical reasoning.

The *Nine Chapters on the Mathematical Art* (ca. 1^{st} century AD) is by far the most influential work: it would shape Chinese mathematics for centuries to come. Between the 3^{rd} and the 11^{th} century AD, Buddhist and Indian ideas got a firm grip on China, yet curiously Chinese mathematics is barely influenced. In the 'Chinese Renaissance' and the subsequent Mongol occupation between the 12^{th} and 14^{th} century Chinese mathematics will reach its zenith.

In the 15^{th} and 16^{th} centuries mathematical development waned and important achievements were forgotten. Only after the arrival of European missionary-scientists at the end of the 16^{th} and during the 17^{th} century mathematics made progress again. The Opium Wars of the 19^{th} century mark the end of the classical China and the indigenous Chinese mathematics would be assimilated by universal mathematics.

Two conclusions are reached: (i) war seems to be good for mathematical progress and (ii) from the 15^{th} century on Chinese mathematics passed away in particular due ot the lack of a general algebraic structure, which prevented structural insight.
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Introduction

China is one of the oldest civilizations, comparable only to Egypt and Babylonia, which were well versed in mathematics (for a discourse on Babylonian

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and Egyptian mathematics, see *Klassieke algebra door de tijden heen* by Van der Elsken, N. and Van der Pol, E.; for an extensive treatment of Egypt, see *Wiskunde voor het leven op papyrus beschreven* by Jaarsma, K. and Van Oost, K.). But unlike these nations, China developed upon her own until modern times.
Since it is impossible to Rule an empire as large as or larger than Rome without arithmetic and given the millennia that Chinese culture developed relatively unimpeded, her mathematics achievements are intriguing.

In this article the authors aim to share their discoveries about those achievements from the very start of mathematical development until its assimilation by Western mathematics.
To do so, we have divided the history of Chinese mathematics into six periods, treating them from chapter 3 to chapter 8:

- Chapter 3, Prelude (< ca. -200): As will be explained later, not much is known about the mathematics before the 2^{nd} century BC in China. This chapter will focus on the things we *do* know.
- Chapter 4, Foundations (ca. -200, ca. 600): Central theme of this chapter will be the most influential book in Chinese mathematical history. Also some wonderful achievements during the Chinese Dark Ages will be discussed. This book together with the mathematical developments during the Dark Ages would form the foundations of Chinese mathematics.
- Chapter 5, Buddhism (ca. 600 - ca. 1200): After the Dark Ages a new dynasty took control of China, who reformed the educational system. Meanwhile Buddhistic and Indian influences were at their best.
- Chapter 6, Zenith (ca. 1200 - ca. 1400): A new war was unleashed, after which China was conquered by the Mongols. In this period of war and bloodshed, ironically, Chinese mathematics flourished more than ever; its achievements would be the zenith of Chinese mathematics, unparalleled in Chinese history both before and after. When compared to the West, the Chinese where centuries ahead.
- Chapter 7, Decline (ca. 1400 - ca. 1850): China freed herself from her Mongolian invaders, but her mathematics would drop N points. She would, however, slowly recover with the aid of Europeans.
- Chapter 8, Westernization (ca. 1850 - ca. 1920): The end of Chinese mathematics. At gunpoint, Chinese mathematics was assimilated by its Western counterpart.

But before we immerse ourselves in the development of China's mathematics, we should keep in mind the following quote:

> *Our initial view of the science as a gradual accumulation of isolated discoveries, connected only by their common end is in the last analysis teleological. It can point A as a step on the way to B, but it cannot lead to an understanding of how B evolved out of the inner necessity, historical experience, and social consequences of A. ... A did not appear at a certain time because of some final cause ordained its time had come, but rather because it was coherent with other ideas, attitudes, and prejudices of its time.*

In other words:
Science has shaped human history. The advent of agriculture, the mastery of metallurgy, the development of a script, all this proto-scientific developments changed the course of events. Especially after the Scientific Revolution at the end of the European Renainssance, scientific advancements have played a crucial role in determining the balance of power; it is above all the high level of science and technology that gives the Western world her current dominant position in the world.

But the converse is true as well: the development of science – just as any other human enterprise – is inextricably linked with the social context in which those who develop the science live. The value a society attaches to science and the manner in which scientists are regarded is of profound influence on scientific development.

Therefore the first two chapters will discuss the political history (chapter 1) and the Chinese philosophy (chapter 2), since these may explain why Chinese mathematics developed the way she did.

Our summary and conclusions can be found in chapter 9.

1 The History of China

Contrary to Western history and philosophy, we expect that most of the readers will only have rudimentary knowledge of their Chinese counterparts. Therefore, our exploration of the history of Chinese mathematics will start with an overview of the history and philosophy of China itself.

1.1 Linear vs. Cyclic historiography: historical patterns

> *When history is viewed from the western side of Eurasia, the natural pattern seems to be for civilizations and empires to rise and wane. By 2000 BC Egypt had eclipsed Sumeria; by 1000 BC Egypt was on the decline but Babylonia was an impressive power. In time, however, it would be surpassed by the Persians, then the Greeks, then the Romans. (...) Underlying this view of history is an unspoken analogy between civilizations and human lives or perhaps the lives of competitive individual warrior-heroes. Civilizations have an early, creative, aggressive stage followed by the strong, mature age, but over time lose vigour and become less flexible until eventually they are defeated in battle or succumb of old age.*
> ...
> *When history is viewed from the eastern edge of Eurasia, a very different pattern emerges as natural and normal. There is no sense that younger civilizations supplant older ones, but that civilization progresses through a series of yin-yang-like reversals of direction from excessive disorder to excessive order and back again.*

(Ebrey, pg.33)

Few historians today accept the cyclic view of history, but nevertheless the history which will unfold itself in the coming paragraphs shows some striking patterns. A large part of Chinese history can indeed be described as a succession of dynasties. In general, a capable military leader unites a divided China under a strongly centralized civil government, with a scholar-official bureaucracy. Often, China itself is enlarged, because the momentum of the of internal unification victories leads the armies beyond the Chinese borders. "Their successors, however, would not all be supermen able to prevent power struggles at court, keep the cost of defense and local administration low, and preserve or enhance sources of revenue, all the while inspiring loyalty through their bearing and virtue. In this view of history, men of ability and integrity – both emperors and their counselors – could arrest decline or even temporarily reverse it, but inevitably the dynasty would weaken and eventually fall." (Ebrey, pg. 135.) Consequently, barbarian tribes who are eager to invade the rich China cannot be hold back any longer. These tribes conquer China and found own dynasty. But because (i) their lack of experience in governing a large state; (ii) their much smaller population size and (iii) their attempts to accommodate the Chinese, Chinese scholars are appointed to organize this new state. With the state based on Confucian values and the immersion of the foreign elite in the Chinese culture, the barbarian tribes are sinified within a few generations, effectively making the alien dynasty a Chinese one. This dynasty would be strong at first,

though slowly decline and eventually there would be civil war, till a capable military leader again unites China, etc.
So ironically, with the occupation of China the conquerers would not end Chinese civilization, but their own. As a result, Chinese cultural tradition, with its first traces in 8 000 BC, has survived almost ten millennia, although China has been occupied by foreign peoples many times. In the coming paragraphs we will outline the ten millennia of Chinese civilization, starting by the invention of the family and ending in the 19^{th} century, when the last dynasty falls.

1.2 In the beginning...

not some Divine Creator, but brilliant human beings where responsible for the steps that transformed the Chinese from primitive people to a civilization, or so the Chinese creation legends say (Ebrey, pg. 10). The Ox-farmer Fu Xi invented the family and the domestication of animals, Shen Nong, (whose name means literally Divine Farmer) invented agriculture and commerce and the Yellow Emperor expelled barbarian tribes from the Yellow River plane, giving his people a place to live. Furthermore, the Yellow Emperor ordered 7 of his subjects to respectively observe the sun, observe the moon, observe the stars, fix the musical scales, construct a sexagimal system[1], create arithmetic and to use the previous 6 to develop a calendar (Mikami,pg. 2).[2]

The view that the cosmos came into being on its own, without a Creator to set everything in motion, would become a fundamental difference between the Chinese and Western philosophical schools, having major repercussions on the development of Chinese and Western science, as we will see in chapter 2.

But despite a fundamental different cosmological view, it is safe to say that the Chinese are among the most ancient civilizations, comparable only to the Babylonians and Egyptians (Mikami, pg. 1).
Archeological excavations reveal that around 8000 BC a transition from a hunter-gatherer to an agricultural society occurred (Gernet, pg. 39), around two thousand years later the agricultural civilization was well entrenched. Stone, bone and (later) ceramic tools were used. The first bronze alloys are dated at 1700 – 1600 BC (Ebrey, pg. 25) and gradually the Chinese mastered the techniques to work with bronze to an unparalleled degree of sophistication (Gernet, pg. 40).

The first proof of a written language stems from 'oracle bones' used for religious purposes. Most inscriptions are inquiries from the royalty to spirits, as to inform when was a proper time to start a war, the worshipping of ancestors and divinities, agricultural campaigns, etcetera. The writing was rather complex: on the about 100 000 oracle bones found so far, with in total 5 000 different characters, of which 1 500 have been deciphered. On these bones also the first numbers are found, as will be described in further detail in chapter 3. (Gernet, pg. 47 and Schirokauer, pg. 9.)

[1]A numerical system with 60 numbers in its base.
[2]Actually, there are Chinese creation myths which stage a Creator who sets everything in motion. However, these myths are younger than the legends regarding the Yellow Emperor described here (Schirokauer, pg. 23).

The first interesting dynasty is the Zhou dynasty, established around 1050 BC. In the following centuries, Zhou Kings attained larger and larger territory, legitimizing their rule by religion (Ebrey, pg. 31). In absorbed states, relatives of the Zhou King or local chiefs were put on the throne as representatives of the King. The local representatives had significant freedom and their own armies; because their positions became hereditary, an influential aristocratic class was created. From 800 BC on, the Zhou Kings could no longer prevent wars between their local domains and incursions from neighboring people (Shirokauer, pg. 25).
Although the Zhou dynasty would continue to be for almost 500 more years, their role became ceremonial. On the political side, China proper was thrown into chaos by the continuous power-struggle between the different regions.

> *The ruthlessness of the competition among the regional powers, although uniformly lamented, nevertheless served to foster social, technological, and economical advances. ... Those engaged in advising rulers about state affairs began analyzing basic principles of human society and the natural order. Soon the most reflective officials were questioning established assumptions and values. Their ideas began to be written down, and the circulation of these treatises further stimulated intellectual debate.*

(Ebrey, pg. 38.)

1.3 The age of the philosophers

The era of declining power of the Zhou Kings[3] was a bloody one, the regional leaders all strived for hegemony over the Chinese world. In the struggle for survival, new techniques of warfare developed. The chivalrous, aristocratic type of warfare (mainly dependent on the aristocratic military class) vanished with the introduction of cavalry and (gigantic) infantry armies; at 300 BC states send out infantry armies of hundreds of thousands of soldiers. Since the victor was the leader who could raise the largest army, supply them with the best weapons and had the most extensive resources, regional rulers experimented with new types of government, military strategies and technologies. (Ebrey, pg. 41) These leaders, eager to conquer and afraid to be conquered, employed men of ideas on their courts to advice them on government affairs, in the process advancing the art of oratory, the development of ethics and the study of logic.

The foundations of the Chinese philosophy stem from this era which perhaps provides an explanation for the focus on ethical issues by Chinese philosophers. Four main schools developed: Confucianism, Taoism, Mohism and Legalism, chapter two will explore their different ideas and the implications for the development of science in more detail. For now, we just note that although the four schools differ significantly, when compared with the philosophical ideas of other civilizations it is the similarity that is striking (Ebrey, pg. 58).
"Among all Chinese thinkers of the period, order was viewed as inextricably connected to rulers, indeed to cosmically based universal kings. It is the universal king who embodies political order ... Law, by contrast, was not granted

[3]Historians often divide the era in two periods: the Spring and Autumn Period (722 BC – 403 BC) and the Warring States period (403 BC – 221 BC).

comparable power by any Chinese thinker. Whether from a Confucian, a Legalist or even a Taoist perspective, law was viewed as an expedient, not as something noble or inviolable, or something that exists above and beyond the ruler. They also shared an underlying assumption that the cosmos came into being on its own, without a creator of the sort so important in most Western thinking." (Ebrey, pg. 58 and 59.)

The power-hungry rulers didn't only look to each others territories for expansion, the peripheral regions also send expeditionary forces to the 'barbarious' neighboring tribes. This, combined with increased commercial trade with borderlands, prepared the way for the enormous military expansion that was to follow after the Zhou dynasty was finally disposed of in 256 BC. (Gernet, pg. 73.)

1.4 The first empires: the Qin and the Han dynasties

Based on Legalist ideals, the Qin state underwent several reorganizations: the aristocratic hierarchical system was replaced by a hierarchy determined by the state, settlers from other states were offered land and houses, serfdom was abolished and a severe penal code was put in place. (Ebrey, pg. 51.) The power of the Qin state rose steadily, till in a series of military victories from 230 – 221 BC Qin united China. The leader of Qin took the illustrious title First Emperor and did not only unify China, he also imposed uniformity on it. The Legalist-based administrative system of Qin was extended to all of China, measures of length and capacity were unified, standard characters were introduced, etc.

But although serfdom was abolished, peasants were still forced to work, not for someone from the nobility but for the Qin state. The First Emperor initiated numerous public work projects: a network of Imperial roads and irrigation cannels, connecting several parts of the now Great Wall, an Imperial Palace and several tombs. (Ebrey, pg. 61 and Gernet, pg. 106.)

Criticism was not tolerated and to prevent critics to spread their ideas in writing, all books except for those related to topics like agriculture, medicine or divination were burned in 213 BC, followed by the execution of 400 scholars who opposed the Emperor. (Ebrey, pg. 61). Although some books were hidden and survived, as a result of this book burning our knowledge of the Chinese civilization and science before 213 BC is limited.

However, China was not only unified under one government, it was also unified in its discontent with the unifying government. The continuous military operations, the labor intensive public works, extreme rigor of the Legalist penal system (see Chapter 2.4), the impossibility for scholars to express their opinion and the hatred of the replaced aristocracy made the regime very dependent on the strength of character of the person on the throne (Ebrey, pg. 63). When the First Emperor died in 210 BC, the unified China decayed into a several years of civil war. In 202 BC, the leader of the Han emerged victorious, crowned himself emperor and the Han dynasty was founded.

Instead of harsh Legalist laws, the Han dynasty used the Confucian notion of

hierarchy and ethics to control its civil servants. (Ebrey, pg. 79). To realise this, officials had to be trained in Confucian classics and should have a respected character. To train potential officials the Imperial Academy was established around 100 BC. This rapidly growing Academy had a few tens of students enrolled at its start, over 30 000 around the second century AD. Since the selection criteria for government-officials stressed Confucian learning and virtues, Confucianism would become the most influential philosophy in elite circles. That is not to say that other philosophies disappeared. First, Han Confucians adapted Confucianism to the needs of the state. (Ebrey, pg. 78) They developed cosmological ideas of the cosmos as being a self-generating and self-sustaining organism. Imperial rule was legitimized by ascribing the emperor with the unique ability to link heaven, earth and human beings. Natural disasters where seen as signs that the emperor didn't fulfill its role well.
Second, Taoism[4] remained of great influence, both at the court and with peasants (Gernet, pg. 160).
To 'accurately' make prophecies, care was taken to the observation of natural phenomena. This had a profound influence on the development of calendrical theories and, indirectly, on mathematics, used to predict the positions of planets and stars. As we will see in chapter three, the first Chinese work containing mathematics deals with astrometry and is entitled 'the circular paths of the heavens'.

Under the Han Emperors of the second century BC, the Han Empire expanded enormously. Plagues of nomadic tribes were nothing new, but at the fall of the Qin dynasty the nomadic tribes had organized. This led the Han Empire to send several military expedition forces of more than 100 000 soldiers to Mongolia, effectively ending the nomadic threat around 115 BC. (Gernet, pg. 120.) The sweet victories tasted for more, and the Han Empire adopted a policy of expansion. Huge armies conquered the east and centre of present-day China, parts of Korea and parts of Vietnam. After territory had been acquired, hundreds of thousands of Chinese were relocated to these areas to quicken the assimilation of the local population. Diplomacy was also used to increase the influence of the Empire: states who excepted a tributary status were given large amounts of silk, copper money, etc.

The second half of the second century AD was characterized by several heavy floods, which due to weakened imperial rule resulted in severe food crises. Several military organized religious cults revolted, while the generals sent to suppress the revolts used their army for their own purposes. In 189 the poet general Cao Cao took over the capital and made the emperor his puppet and later, the emperor was forced to abdicate. In this coup d'etat, the government library and archives were lost in flames. (Ebrey, pg. 84.) The Han dynasty had fallen, China entered its 'Middle-Ages'.

Nevertheless, the cultural influence of the Chinese remained. Neighboring tribes had all adopted Chinese script, so works written in Chinese were accessible for all. The cultural integration was magnified by the forced relocation of Chinese

[4]Taoism is a traditional Chinese philosophical and religious current which will be treated in chapter two.

and by giving Chinese brides to tribe leaders. (Ebrey, pg. 85 and Gernet, pg. 135.) The Han Empire had fallen, but China had been increased permanently...

1.5 The Age of Division: the Dark Ages

The Dark Ages are one of the most complex times in Chinese history. The era is characterized by a bloody, continuous struggle between short-lived states and in particular by a division between north and south. After the forced abdication of the last Han emperor, Cao Cao founded the Wei dynasty in the North of China, but in the south two other kingdoms emerged. Decades of war were the result, till in 263 one and in 280 respectively the other rival kingdoms were conquered, and China was briefly unified again! This dynasty is named the Jin, because during the wars with the rival kingdoms the Wei dynasty itself had been replaced by the son of a victorious general, who founded the Jin dynasty. Although the Jin had unified China, it never succeeded in the organization of autocratic imperial institutions. Consequently, the imperial power was constantly challenged by influential families. Further decrease of central power was stimulated by the Jin policy of parceling out large regions to imperial princes, which fed their hunger for power: during 291 and 305 a full-scale civil war raged around the capital, located in the North of China. (Ebrey, pg. 89.)

During the Han, Mongol tribes at the northern frontier were kept in check by military expeditions, wooing with generous gifts and their settling within the northern boundaries of China. During the civil war, however, the Chinese were too occupied with themselves to pay much attention to the Mongols, who rebelled. Tribe leader after tribe leader joined the civil war, making the North of China one big battleground: there were sixteen regional, overlapping kingdoms between 304 and 439.

Millions, including wealthy and influential families, fled the northern regions, hoping for a better existence in the south. From 317 till 589, five dynasties would subsequently rule the south of China. The emperors of the dynasties were capable of holding the government while alive, but were unable to successfully transfer their power to their heirs. This was partially caused by the rise of a hereditary aristocracy, which held the real political power at court. These aristocratic trained families descended from prominent families who had emigrated from the north and "provided a centre around which Chinese culture could adhere during a period when no state could serve that function" (Ebrey, pg. 91).

The exodus of millions caused severe problems in the north: much arable land remained uncultivated due to labor shortage. This problem was solved by one of the Mongol tribes by the forced relocation of populations to cultivate these lands. The resulting increase of states revenues made them powerful enough to defeat their northern opponents and in 439 the Northern Wei dynasty was founded, which would last till 534. But these Mongols, now in control of a vast empire, had no experience in organizing a large, agricultural state. To control such a state and to gain acceptability from the significantly larger indigenous Chinese population (ca. 1.5 million Mongols vs. more than 20 million Chinese), the task of setting up government institutions was given to the Chinese literati.

Furthermore, just as the dynasties before it, the Northern Wei was plagued by several incursions of (other) 'barbaric' tribes, the state revenue depended more and more on agriculture and the Mongol elite began to develop a taste for luxury; i.e.: the environment within the Northern Wei became like that of the old Chinese empires and the Mongols sinified within several generations. But the sinification of Mongol tribes should not be seen as the total victory of Chinese culture over other cultures. Certainly, Chinese culture was dominant, but during the Dark Ages and the subsequent Tang Empire, the Chinese world was remarkably open for new ideas and techniques on every mode of life: from the use of the saddle and ways to build bridges to the cultivation and weaving of cotton. (Gernet, pg. 197.) Yet by far the most substantial influence on Chinese culture was the introduction of Buddhism.

Buddhism settled in China in 65 AD, but was originally regarded as a version of Taoism, among others since the first monks used Taoist terms to convey religious and metaphysical concepts which had no real Chinese words.
Over the course of centuries, more and better translations of Buddhist text became available. In the south, the aristocracy displayed curiosity towards the religion; to gain acceptance in the north monks used tricks of magic to convince barbarian leaders that Buddhism was more powerful than the Shamanism they were accustomed to. Furthermore, in the north Buddhism was attractive to the Mongol leaders because of its universal claims: Buddhism could unify a mixed (Chinese and Mongol) ethnical population. But above all: in the medieval centuries of bloody warfare, and in its wake banditry, famine and social inequality, when life was cheap on every level, the confidence in Confucianism declined, provoking political and philosophical reflection. Many were appealed by Buddhism, which offered the prospect of salvation, addressed questions of life and dead with a directness unseen in Chinese traditions and had an ethical code that forbade the taking of life. (Ebrey, pg. 97)
At the beginning of the fifth century, the religion was firmly entrenched in the Chinese mind, a position it would hold till the beginning of the eleventh century. Buddhism enriched and transformed Chinese religious concepts; its increasing influence and number of followers were a driving force behind the emergence of Taoism from mystical shamanism to higher religion. (Ebrey, pg. 101)

In the wake of Buddhism secular aspects of Indian civilization followed, among others Indian science, who's influenced peaked between the beginning of the seventh till the middle of the eight century. (Gernet, pg. 231) Astronomy was significantly influenced and considering the interdependency between astronomy and mathematics, Indian influence in Chinese mathematics is to be expected.

The increasing sinification of the Mongol tribes who had conquered the northern part of China, increased the tension with their armies who protected the China – Mongol border, since these armies continued to live as nomads. Less and less interest was shown in the military, resulting in the rebellion of their own border armies in 523. Eleven years of civil war were the result, in 534 the Wei empire was divided between the old sinified Mongol elite and the nomadic military. Several more decades of war and bloodshed followed, till a general founded the Sui dynasty and reunified China (Ebrey, pg. 93 and Gernet, pg. 194). Due to court intrigue and the murder of several emperors, the dynasty changed name to the Tang dynasty in 626.

1.6 The Tang dynasty

Under the Tang dynasty, China would be at peace and its culture would prosper. The Mongol tribes who had controlled the northern parts of China sinified more and more, till their own language and culture disappeared completely and they were absorbed by Chinese civilization. (Ebrey, pg. 108) Apart from this, China's influence expanded in other ways as well. With the system of divisional militia, an army of volunteer soldier-farmers, the costs needed to uphold the military was kept relatively low and the Tang quickly gained enormous quantities of territory. During the Tang dynasty, China would reach its zenith of cultural influence. It would become a cosmopolitan civilization colored by influences of central Asia, India and Iran. The Chinese were receptive for foreign influence, which was stimulated through intensive trade. The presence of foreign merchants considerably increased the knowledge of the outside world; foreign goods, games, arts, hairstyles were all enthusiastically copied (Ebrey, pg. 118).

To strengthen the Tang empire, institutions were standardized and the power of the aristocracy constrained. The ministry of personnel was given the power to fill posts and officials were not allowed to serve in their home prefecture and more than one term in one prefecture. Another tool to curb the influence of the aristocracy was the focus on Confucian values as duty to the people and loyalty towards the ruler. To determine real Confucians, state schools were established and written civil service exams were introduced. (Ebrey, pg. 112) Somewhere between 628 and 656 mathematics was established as a subject on the Imperial Academy, but never for a long time. Quite often, mathematics as course on the Academy was cancelled and assigned to the Institute of Astronomy or the Institute of Records. The government issued standard textbooks for all courses, including mathematics. The prescribed textbooks for mathematics were the books in the *Ten Mathematical Manuals*, among other the *Zhoubi*, the *Nine Chapters of the Mathematical Art*, *Sea Island*, *Master Sun* and *Five Governments*; these books will be treated in more detail later. A significant part of the exam, however, consisted of recitation: candidates were given one or two sentences from one of the books and were asked to continue reciting from the given passage. Of course, this kind of examinations don't require great understanding or creativity and can't contribute to the development of science. Those who passed the mathematics exam would enter the lowest class in the official rank (LD, pg. 106), greater esteem was given to subjects like history and literature. On the most prestigious civil service exam poetic composition was tested.

Despite the system of state exams the gentry remained influential because they did well at the exams and could consequently be found everywhere in the civil service, especially at top positions. Nevertheless, it did shape the way they prepared for government service and it did shape their way of thinking. During the Tang, Confucian scholarship of many sorts flourished and Confucian ideology was seen as compatible with Buddhism. (Ebrey, pg. 120)

It was also during the Tang dynasty, with its outwardly cosmopolitan mood, that Buddhism became an integral part of Chinese life. Buddhist institutions organized schools for children, were a social meeting point for important occasions and became an economic force to be reckoned with; the large tracts of land which monasteries owned gave them vast financial resources. China even became the centre of Buddhism, which by then was the universal religion in Asia, and this was a major factor on which China's widespread influence was based. Nevertheless, the most popular Buddhist sects in China were the thoroughly sinified ones, some of which would also have tremendous influence on the development of Buddhism outside China. (Gernet, pg. 277 and Ebrey, pg. 121.) Being a prominent nation, China received embassies from many countries, among others from the Tazy, an Arabic nation founded in 582. Next to embassies, merchants of all directions arrived in China for trade. Records tell about Arab, Persian, Indian, Christian and many other merchants in the Chinese ports (Mikami, pg. 98).

The cosmopolitan character of the Tang empire didn't mean that they were at peace with all their neighbors. As mentioned above, the Tang expanded its borders enormously, but it had also to protect both its original territory and its conquered domains from among others Arabs, Turks and Tibetans. In the eighth century, the system of militia was replaced by the formation of large, professional armies, who's leaders had great autonomy.
Among others due to court intrigue some generals received a lot of favors from the court, further enhancing their power, influence and ambitions. One of these generals, An Lu-shan, rebelled in 755, marched to the capital and took it without a blow. The emperor fled and to restore control (i) he had to ask two neighboring 'allies' for assistance and (ii) rebel leaders were pardoned, often appointing them as military governors in the area where they had surrendered. This marks a turning point for the Tang dynasty. First, the 'allies' occupied large parts of Chinese territory and had to be paid of to prevent them from plundering the capital. (Gernet, pg. 260). Second, by appointing the ex-rebel leaders as military governors the central state effectively lost control of the regions under their command.

The decline of the Tang caused China to turn in itself, xenophobia and the thought that Chinese purity had been corrupted by foreign influences became a prevailing thoughts. "In the late Tang dynasty, however, a profound sense of cultural crisis pervaded intellectual life. The sense that some sort of action had to be taken motivated many of the best minds to rethink basic issues concerning the Chinese state and Chinese culture, in the process reinvigorating Confucian thinking."(Ebrey, pg. 131.) The excessive wealth of the Buddhist church provoked anti-Buddhist feelings, which was seen by some as 'foreign religion', although it had been on Chinese soil for almost 800 years. In 845, a proscription accused Buddhism of moral and economic enfeeblement and hundreds of thousands of monks were forced to return to normal life. Although Buddhism would overcome the blow, it would become "a church that had outlived itself and that seems to have lost its soul, for its bodies of learned monks had been dispersed and the traditions of its schools had been interrupted." (Gernet, pg. 295)

Around 900, the independent regional leaders named themselves king, several years later they founded their own dynasty and became emperor; civil war followed (Gernet, pg. 270 and Ebrey, pg. 129). The Tang dynasty had risen and fallen, a new circle in the cyclic interpretation of Chinese history was complete.

1.7 The Song and Yuan dynasties

If we instead of the cyclic interpretation of history use a progressive one, the process of reinvigorating Confucian thinking would herald the Chinese Renaissance. Confucianism would dominate once again, even more than it had ever done in Chinese history. Of all the dynasties, the Song dynasty can be said to approach the Confucian ideal of a strong centralized state with moral respectable leaders most closely.

The Song dynasty was founded in 960 by Taizu, a general who reunited most of China with the force of arms.
To prevent a new military leader to rise, he put his army under civil control after he had defeated his contenders for the Chinese throne. But having defeated all his contenders didn't mean he controlled all of China. Parts of China were taken by foreign powers, among others the Mongol tribe the Khitans; several attempts to retake this occupied territory were in vain, so peace agreements were reached in 1004 and 1044, which obliged the Song to make significant annual payments. I.e.: the Song bought off peace. Nevertheless, the security threat remained imminent and at its top, three-quarters of the state revenue was used by the military, by then the army would consist of 1 250 000 men. The constant threat had several important consequences. First, it stimulated the centralization of government. Second, it stimulated technical development and creativity. Siege machines, gunpowder and cannons were invented during the Song – Khitan conflict. (Ebrey, pg. 138)

The civil service examination system would become the cornerstone for the selection of government officials. Not ancestry, but capability became the selection criterion. Also members of the aristocratic class had to pass the exams to be given any serious post. To assure personal judgment of the candidates wouldn't influence the examiners, the papers were copied by clerks and identified only by number. Consequently, the number of participants grew quickly and would at its peak reach 400 000 participants per year. "Leading members of this new scholar-official elite were often men of remarkably intellectual breadth. ... By the end of the Song dynasty, the scholar-official had attained remarkable social, political, and cultural importance and marked China as different from other major societies of Eurasia. ... The identification of Chinese civilization with the literati ideal was strengthened, thanks in part to the rivalry between the Song and the military stronger nomadic and semi-nomadic peoples to the north." (Ebrey, pg. 149.) One of the most outstanding scholar-officials was Shen Kuo. He headed the Bureau of Astronomy, calculated the effects of currency policies, supervised military defense preparations and wrote on a wide range of topics, including history, military strategy and mathematics. We will meet him again in chapter five.

In 1115 the Jurchens, agricultural and herding people, came to power in the regions north of China and declared war to the Khitans, who still held a part of Chinese territory. Enthusiastically, the Song joint forced with the Jurchen and in 1123 the Khitans were defeated. But the Chinese weren't as luckily they might have thought, only two years later their former allies the Jurchen attacked the Song capital, which, although defended by 48 000 military, fell. The Song government fled south and for over two decades their was a struggle between the Jurchen, who had occupied the northern part of China, and the Song, still controlling of the southern part[5]. The situation stabilized slightly when a treaty was signed in 1142. After the loss of the north, China turned again in itself. Further revitalization of Confucianism was promoted and metaphysical ideas were added as counterweight to Buddhism. (Ebrey, pg. 151)

In the north, the Jurchen moved their capital to the former Song capital in 1161 and most Jurchen were forcibly relocated to China, so they could control the vast Chinese population. Lacking experience in controlling such a population, they turned to Chinese advisors and incorporated more and more Chinese institutions. Steadily, also Chinese customs, language and rituals were adopted. In 1191 the emperor even forbad to refer to Jurchen as 'people of the border area' to prevent them from being seen as outsiders. At the end of the Jurchen dynasty sinification would be largely accomplished: the Jurchen were assimilated in Chinese society.
(Ebrey, pg. 168.)

Yet, it were not the Chinese who heralded the end of the Jurchen dynasty. During the latter half of the twelfth century, the Mongol steppes were plagued by food shortage due to the drop of the mean annual temperature. In this setting, the brutal but military genial Genghis Khan united the Mongol tribes in 1206 and one of the most astounding military campaigns in world history would be initiated. First, nearby countries were subjugated, after that Central Asia was invaded, followed by the rest of Mongolia, Manchuria, Korea and numerous other regions. Genghis' army was seasoned and could move with incredible speed, his strategy was ruthless: countries were asked to surrender, if they did they were left intact, otherwise the population would be massacred and the city would be laid to ashes. One of his sons subdued the Jurchen in 1234 and around 1236 most of the Southern Song territory was under his control. After years of fierce battle, the Song dynasty finally surrendered in 1275 and the last Song loyalists were crushed in 1279. China was again under Mongol control. At the time the Southern Song was conquered, however, the Mongol empire created by Genghis Kahn was fallen apart in different Khanates. The Khanate which controlled China moved its capital to China and adopted a Chinese name: the Yuan dynasty.
(Ebrey, pg. 170- 173)

> *The Chinese might not have welcomed alien rule but at all social levels they found ways to adapt creatively to their new situations. The Khitan, Jurchen, and Mongol rulers all needed men capable of handling the paperwork that made centralized bureaucratic government possible ... and (Chinese scholars) saw that Mongol rule would be more palatable if Chinese scholars were to be the administrators. Moreover, they anticipated that the Mongols would gradually*

[5]The reign of the Song restricted to the southern part of China is often named the Southern Song.

become more sinified as the Jurchens had, and could view themselves as shielding Chinese society from the most brutal effects of Mongol rule.

(Ebrey, pg. 176)

Nevertheless, perhaps with the intention to learn from history, the Mongols purposely avoided social and political Chinese traditions and held to their own traditions, but the Chinese were not forced to abandon their traditions either. "The presence of an alien elite controlling the government did not diminish the prestige of the literati within Chinese society, and they continued to be accepted by ordinary Chinese as the natural leaders of local society, ... looking on themselves as trustees of the Confucian tradition." (Ebrey, pg. 179) Academies thrived as centers of learning beyond government control.

The continuous battles between the Song and the Jurchen in the latter half of the twelfth and the start of the thirteenth century gave a boost to technical science. It was followed by an era of oppression by the Yuan, who were also interested in technical science. Combine this with increased interest of the Chinese in their own culture and learning as reaction at the Mongol dominance, and perhaps we have an explanation for the fact that in this bloody and uncertain times of constant war, followed by alien subjugation, the Chinese mathematics, always oriented practically, would reach its zenith. The level Chinese mathematics achieved during the Song-Jurchen-Yuan times would be unparalleled elsewhere in Chinese history, both before and after. On some branches of mathematics, it would be around 800 years ahead of rest of the world, as we will see in chapter 6.

The enormous variety of cultures in the large Mongol Empire under Genghis Khan and his sons created interaction between these cultures. In the twelfth and thirteenth century, several Christian missionaries were sent to China, although they failed to convert a large number of believers. (Gernet, pg. 374) The Islam was more successful. In the Yuan epoch Moslem communities were established in the north of China and Arabs were treated with the same esteem as Mongols themselves. Next to the Chinese Astronomical Observatory, an Arabic Astronomical Observatory was set up in Peking, significantly influencing Chinese astronomy. Considering the interdependence between astronomy and mathematics, the last might be affected as well. (Gernet, pg. 382) Especially the development of proto-trigonometry is said to be based on Arab science (Mikami, pg. 101).

On the political level, similar to events we have seen before, mighty emperors were followed by weaker ones and palace intrigue and coup d'etats became the order of the day. Meanwhile, popular discontent grew due to the harsh treatment of the Chinese, more and more corrupted local authorities and the rise in prices. Food shortage and epidemics due to floods of the Yellow River lead to widespread rebellion in 1351. In 1367, one of the rebel leaders, Zhu Yuanzhang, marched to the capital. Not being able to win the war, the Yuan emperor fled to Mongolia and Zhu founded the Ming dynasty in 1368.
(Ebrey, pg. 173 and Gernet, pg. 373)

1.8 The age of decline: the Ming and Qing dynasties

For the first time in centuries, all of China was under a domestic Chinese dynasty again. But the Ming dynasty is mostly considered in negative light. The civil service examinations were ceased to be hold for about ten years, and afterwards they were notable for their narrowness, testing above all knowledge of just four books.
Despite the narrowness of the exams, however, officials were still well versed in Confucianism. And as good Confucianists do, they protested to the throne against bad officials and harmful policies. Good emperors should consider the critism of their officials and take action if the critisims was justified. Most Ming emperors, however, didn't appreciate officials who courageously spoke out and many were put to flogged or even executed. But this didn't stop the Confucianists from protesting and the Ming dynasty shows some mass demonstrations of officials. For example in 1519, when an emperor "announced plans to tour the southern provinces, officials submitted a flood of negative memorials and over a hundred officials staged a protest by kneeling in front of the palace. Outraged, the emperor ordered that as punishment they kneel there for five days, after which he also had them flogged. Eleven eventually died of the beatings. Only a few years later in 1524 hundreds of officials again gathered at the palace gate..." (Ebrey, pg. 123).

Although as the example above demonstrates Confucianism prevailed and only a few Ming intellectuals showed any interest in western ideas and knowledge, Europeans ideas began to filter in, especially through Christian missionaries. A missionary with large influence was Matteo Ricci, who entered China in 1583. By now, both the most modern Chinese calendar and the Islamic calendar had not been reformed for a long time and had become very inaccurate. Ricci adviced the Christian church to present missionaries as scholars rather than monks to win the trust of the people they were trying to convert. Since it was only with help of Europeans that the calendar was both improved and 'synchronized' again, European science was hold in high esteem within court circles; several Europeans even became a member of the Chinese Astronomical Observatory. (Mikami, pg. 112 – 114.)
Although there was a relatively lively interest in Western ideas and knowledge, there were many obstacles on the road to understanding. Christianity in particular formed a source of hostility: even open-minded Chinese found some ideas difficult to accept, like the dogma that the universe was created by a Supreme Being. Furthermore, many Chinese scholars could not accept the idea that they should give up their concubines, which was considered ruthless towards both the women and their children. (Ebrey, pg. 212)

So, although there was an influx of Western science during the Ming, it was during this dynasty that Chinese culture and science stagnated. Many important achievements during the Song and Yuan times were not further expanded and even raplidly declined, among others the mathematical accomplishments in the form of the 'Method of the four unknowns' and the method for solving arbitrarely degree polynomial equations. "How did this state of affiars arise? There are many reasons for it. Most importantly, ... we cannot find any factual, relevant problems (which these methods can solve) when we look for them in the light of the social and economic requirements of society at that time.” (LD, pg. 175). But wathever the causes might be, at the end of the Ming dynasty the dazzling mathematical achievements of Song and Yuan time were completely forgotten.

The decline of the Ming can be seen in the light of the classical cycle. When time progressed, the administration became less efficient and the expenditure of the Imperial family got out of hand. Combine this with the enormous costs of militairy campaigns and one has the most important causes for the fact that in the early seventeenth century, the dynasty was nearly bankrupt. Without financial resources, the Ming could not provide relief when food shortages occurred due to the drop in average temperatures, also at the beginning of the seventeenth century. Rebels emerged all over China, but the Imperial army didn’t succeed crushing them. After several years of famine, banditrary and civil war, one of the rebel leaders took over the capital, the last Ming emperor committed suicide, ending the Ming dynasty. None of the rebel leaders was able to conquer all of China and a violent time followed.
(Ebrey, pg. 214)

Meanwhile, north of the Great Wall the Manchus, nomadic horsemen just like the Mongols, had established an empire and when the Ming weakend, they occupied parts of Ming territory while steadily adopting Chinese institutions. Consequently, after the dead of the emperor several Chinese generals defected and with their help the Manchus squashed the rebels and established the Qing dynasty in 1644. As Ebrey puts it: “The three Manchus who ruled in the course of the eighteenth century ... proved excellent managers, and by many measures (though not a mathematical one) that century was the hight point of traditional Chinese civilization.” (pg. 222). The first of this three, Kang Xi (r. 1662 – 1722), deserves special attention. He patronized the Chinese literati, was familiar with Chinese literati culture and became fascinated by Western science. When the predictions of Jesuit monks proved to be more accurate than that of their Chinese counterparts, Jesuits were even appointed as head of the Chinese Astronomical Observatory. He accepted the presence of Christianity as long as it allowed its converts to perform ancestrial rites. This was exactly what the Vatican forbad and therefore most missionarries were expelled. To prevent foreigners from disrupting Chinese society, China initiated a closed door policy around the 1750s. European trade became restricted to Canton only. (Ebrey, pg. 235).

European nations, England in particular, were disgruntled and became even more unhappy when Chinese officials outlawed the use of Opium. In the 1730s Great Brittain had set up opium manufacture facilities for opium–sale with China as a way to even their balance of payments with China. It proved a profitable business, from 200 chests of opium in 1729 till 40 000 chests in 1840, but it devastated Chinese society. In 1839 China adopted tough anti-opium policies, much to the chagrin of British merchants, who effectively lobbied for war. In this war, the Opium War, Great Brittain sent two military expeditions to China and with their modern warships, the Chinese had no choice but to sue for peace. The peace agreements in the Treaty of Nanjing were very unfavorable for China. They had to pay an enormous amount of money; opium trade remained allowed; five treaty ports were opened at a fixed and low tariff; British subjects were answerable only to British law even when they were on Chinese territory (extraterritoriality); a 'most-favored-nation'-clause assured that every privilege China gave to another nation, would automatically also be extended to Brittain. Other Western nations followed swiftly. (Ebrey, pg. 239)

Besides the disagreements with foreigners, even more urgent problems plagued the Qing. Inside China enormous population growth and mismanagement, shortages of food and work, an excess of men (many parents didn't have the financial resources to bring up many children, boys were preferred over girls) and the humiliation by the repeated victories of foreign powers were feeding grounds for popular discontent and the number and intensity of uprisings grew steadily. One of the best known rebellions is the Taiping Rebellion, which started in 1850 and lasted till 1864. The rebel leader was an insane charismatic religious zealot who said he had seen visions of Jesus' younger brother and declared himself the king of the Heavenly Kingdom of Great Peace (Taiping). When his followers numbered in the tens of thousands, he took over several cities, brutally murdered all Manchus living there and issued utopian calls for a new society. It took years, but the Ming was able to defeat the Taiping, yet the price was high. Not only would the Rebellion cost the lives of twenty million, the evident weakness of the imperial power inflamed other discontent groups to revolt as well. (Ebrey, pg. 242.)

With both increasing internal unrest and the foreign pressure, the Qing dynasty stood on the brink of collapse. In government circles, discussion raged as how the problems could be best handled. Ideas regarding solutions varied enormously, reform-minded scholars suggested adoption of technology and science from the west, while more conservative literati held on to Confucianism. "One respected neo-Confucian scholar, the grand secretary Woren, objected to the establishment of an interpreters' college on the grounds that 'from ancient down to modern times' there had never been 'anyone who could use mathematics to raise a nation from a state of decline or to strengthen it in times of weakness'." (Ebrey, pg. 245) He couldn't have been more wrong. "Mathematics has a particularly important role in the history of science because the great Scientific Revolution, which brought all modern science, ecumenical and universal, into existence, depended essentially upon the mathematization of hypotheses about Nature, combined with relentless experimentation." (Sir Joseph Needham in the foreword of LD.) It was this modern science and the technological development in the industrial revolution that followed in its wake that were the keys to the rise of the West in the eighteenth and nineteenth century.

Not surprisingly, the reform-minded scholars won the argument. After further humiliation of China even the conservatives saw action was needed.
Western style technology and in its wake Western science where adopted at high pace and consequently Chinese science, including mathematics, would be absorbed and replaced by its then superior Western counterpart.

But why didn't this mathematization of hypotheses about Nature also develop in China? And, when the ideas where already worked out by Western scientists, why was there such a resistance to accept these ideas? The answer to these questions is inextricable linked with the development of mathematics itself, the topic of this article. As you will see, Chinese mathematics took a different road than its Western counterpart, and this road would *not* lead Chinese sciences to the mathematization highway. The next question we can ask ourselves is why Chinese mathematics took such a different road. To answers this questions, we have to know something more of the Chinese way of thinking, of Chinese philosophy. Therefore, now slightly familiar with the history of the Chinese civilization, we are ready to explore the Chinese mind in chapter two.

1.9 Epilogue

Though we feel that since we went through about 9900 years of Chinese civilization, from the first archeological evidence of civilization around 8000 BC till the final decades of the last dynasty at end of the nineteenth century AD, we can't leave the last 100 years of the Grand Narrative of China untold. Therefore we will end this chapter with a short description of the development of Chinese civilization from where we left in the previous paragraph till our own time.

After a Chinese delegation visited several Western capitals in the end of the nineteenth century, knowledge and acceptance of Western ideas, technologies and sciences increased gradually and attempts were taken to modernize Chinese economy, but largely in vain. This in contrary to Japan, which after the Meiji revolution (Japan's 'industrial revolution') in 1867 had accomplished much: its old feudal structure had been abolished and replaced by an constitutional monarchy which was lead by reform minded aristocrats, a universal school system had been set up and the army and navy were westernized. But besides Western technology and organization of state, also the Western imperialistic attitude was copied. The strength of its army was in the same order of magnitude as of Western nations, so Japan considered its time as Imperialistic nation had come. In the 1870s it invaded the Ryukuy Islands, in 1876 Korea and in 1894 China. Other western powers, seeing how easily China was defeated and all afraid their neighbour would get a bit more then they, began to pick parts from China as well. In 1912, due to a rebellion of several army officers, the Qing dynasty, the last dynasty of China, would meet its end. One and a half decade of civil war followed, till the Nationalist government was established in 1928, which led several modernization programs. But they couldn't tackle the large problems China had. Warlords continued to be a threat, a large proportion of the population lived in extreme poverty and government corruption grew.

Inspired by Marxist and Communist ideas, the Chinese Communist Party was established in 1921. At first, the Nationalists and Communists worked together to defend China against Japan, which seized more and more parts of China in the 1930s. But in essence, both parties were each others rivals, since they both had as ultimate goal the total domination over China.

In 1947, a full scale civil war ignited. The Nationalists seemed to have the better cards: its army had considerably more men and more advanced arms, but their two decades of government had made them that unpopular, that they lost the civil war within two years and fled to Taiwan.

The Chinese Communist Party (CCP) became the government of the Chinese mainland, with Mao Zedong as its leader. Massive modernization projects started: modern factories, railroads, schools, hospitals, etc. were set up and China transformed from an agricultural civilization to an industrial one. Also at the social level, major reforms took shape: the literati were not longer seen as the holder of Chinese culture, who acted according to elevated principles, but were labeled exploiters and part of the antiquated 'feudal' order. Together with landlords, richer merchants and enemies of the regime, millions were prosecuted. The exact number of casualties is unknown, but it ranges between the hundreds of thousands to tens of millions.

In order to accelerate the growth and industrialization of China, Mao proposed the *Great Leap Forward* in 1957. But instead of a leap forward it became a giant fall back. Scholars had been forced to work on the countryside or had been executed, so many projects were carried out without technical knowledge and many mistakes were made. The effect on agriculture was even more devastating: the large scale reform programs resulted in serious food crises, around 30 million died as a result. Since Mao was responsible for the tragedy; his grip on the CCP declined. As a result, he set in motion the *Cultural Revolution* in 1966, a program intended to purge China from 'capitalistic anti-revolutionary elements'. Youth and not educated workers were instructed to seize power from 'corrupt' party leaders and officials. Soon, the revolution went totally out of control and the military was needed to restore order; millions would have scars for the rest of their lives. Mao died in the end 1970s and soon after Deng Xiaoping rose to power. Under his leadership, economic development became a top priority and in the decades that followed, China's economy grew vastly. In the 1990s, capitalists could become a member of the CCP and a middle class emerged. But several problems threaten the 'harmonious' development of China: the social inequality has increased tremendously over the last decades, the country stands on the brink of an environmental disaster and corruption is deeply rooted.
(Ebrey, pg. 245 – 332)

Nevertheless, the future prospects of China might be sunny after all: the communiqué of the 6th Plenum of the 16th Central Committee of the Communist Party, hold in 2006, states:

> *The plenum puts forward the main objectives and tasks for building a harmonious socialist society by 2020, which are as follows: the socialist democratic and legal system is further improved; the fundamental principle of administering the country according to law is implemented in an all-around way; people's rights and interests enjoy concrete respect and guarantee...*

(CPC Embassy to UK, 2006)
I.e.: China is on its way to become a democracy.

> *The world would do well to wish this group (the current president and prime minister) success (with their plans) because there can be no harmonious world without an harmonious China.*

(Woo, 2007)

2 Chinese Philosophy

Science searches to explain the world we experience around us, but nothing can be explained without some basic assumptions on the interpretation of our senses and what is to be a correct form of logic. Therefore, in every culture essential steps towards the development of science are the construction of a cosmological view and a system of reasoning, hence, a philosophical system. As we saw in the first chapter, contrary to Western philosophers the Chinese didn't assume a creator, so important in Western thinking. "Instead of focusing on mechanisms that set things into motion, which are important where there is an assumption of a creator, these (Chinese) thinkers emphasized the organismic interconnections among all the constituent parts, stressing relationships and concurrences much more than causes." (Ebrey, pg. 59) Chinese philosophy then, is also much more ethical and social than metaphysical.

The direct result of this philosophical orientation is the lack of interest in phenomena and laws beyond human civilization. The concept of these Laws didn't fully develop, shaping Chinese mathematics in such a way that is was "Apparently not concerned with abstract geometry independent of concrete numbers, and consisting of theorems and propositions capable of proof, given only certain fundamental postulates at the outset. Numbers might be unknown, or they might not be any particular number, but numbers there had to be." (Needham II, pg. 21) (the passage will be quoted more completely later on).

This chapter will cover the four major philosophical currents in China. We start with *Confucianism*, which became the dominant way of thinking and played a decisive role on Chinese history. Confucianism is a mostly ethical philosophy, entailing deep concern for the well-begin of others. Convention governed actions and hierarchical differentiation, both within society at large as within the family, should according to its supporters results in a harmonious civilization. With the focus on hierarchical harmony, it approved of the feudal order during the Zhou reign. (Ebrey, pg. 43). *Taoism* on the contrary, which we will treat as second, strongly opposed the feudal order and defended private life and wanted rulers to leave people alone. Not human beings were placed at the centre of the cosmos, but the Tao[6] (Ebrey, pg. 47). *Mohists* can best be described as military pacifists acknowledging the importance of observation and logic (Needham II, pg. 165); they will be treated as third. The last major philosophical school is that of the *Legalists*. Legalists claimed that a strong government doesn't depend on the moral qualities of the leader, as Confucianism did, but by the establishment of law.
After we are familiar with the major philosophical currents paving the Chinese civilization and character, we will end the chapter with a discussion about the development (or lack) of absolute laws.

[6]The literal translation of Tao is Way, but it has to be emphasized that with Tao *the Way the Cosmos Works* or *the Order of Nature*, as Needham puts it, is meant and has nothing to do with Christian or Muslim mysticism. (Needham II, pg. 37).

2.1 Confucianism

According to traditional dates, Confucius lived between 551 - 479 BC, during the bloody era of the ceremonial Zhou Kings. Human life was cheap on all levels, and when placed in this context Confucius' ideas were revolutionary. His ideas are passed on to us mainly through the recordings of his disciples in the *Analects*.

The basic principle of Confucianism is the love for others, as is illustrated by the following quotation from the *Analects*:

> *Zhonggong asked about humanity. The Master said, When you go out, treat everyone as if you were welcoming a great guest. Employ people as though you were conducting a great sacrifice. Do not do unto others what you would not have them do unto you. Then neither in your country nor in your family will there be complaints against you.*

(Ebrey, pg. 46)
According to Confucius the aim of a government should be the welfare and happiness of the entire population. To achieve this goal, not rigid law but a paternalistic administration was needed, with the ruler as personification of ethical principles. Or, to quote the *Analects*:

> *The Master said: Lead the people by means of government policies and regulate them through punishments, and they will be evasive and have no sense of shame. Lead them by means of virtue and regulate them through rituals and they will have a sense of shame and moreover have standards.*

(Ebrey, pg. 46)
As a result, Confucianism agreed with the idea of a feudal-bureaucratic society (and later on with the large centralized empires).

Superstition was strongly opposed, but as is visible from the last quotation, ritual was supported; not for the sake of the spirits or ancestors, but for the sake of the living. Furthermore, an a priori distinction based on class or wealth was condemned; one's later occupation should solemnly depend upon character and knowledge, i.e. aspects attainable through education. Education should be equally accessible for all. These all were aspects of the philosophy positive for scientific development.

However, these positive aspects were nullified by the single-minded interest in human affairs as opposed to interest in nature. This humanification of intellectual thought made, on the large scale, the contribution of Confucianism to the development of science almost wholly negative (Needham II, pg. 28). The attitude might be best described with the following passage from the Confucianist Hsun Chhing, which is directed against the Taoists.

> *You glorify Nature and mediate on her;*
> *Why not domesticate her and regulate her?*
>
> *You obey Nature and sing her praises;*

Why not control her course and use it?

You look on the seasons with reverence and await them;
Why not respond to them by seasonal activities?

You depend on things and marvel at them?
Why not unfold your own abilities and transform them?

You meditate on what makes a thing a thing;
Why not so order things that you do not waste them?

You vainly seek into the causes of things;
Why not appropriate and enjoy what they produce?

Therefore I say - To neglect man and speculate about Nature
Is to misunderstand the facts of the Universe.

(Needham II, pg. 28)
Note the negative attitude regarding wondering about the cause of the Cosmos in the second to last verse and in the focus on humans in the last. So this humanitarian philosophy, there was no place for science as long as it was not directly related to the human cause.

2.2 Taoism

Where Confucianism agreed with feudal society, Taoism had different political views. The ideal Taoist social structure is described as follows in a Taoist classical:

> *Take a small country with a small population. The sage could bring it about that though there were contrivances which saved labour ten or a hundred times over, the people would not use them. He could make the people ready to die twice over for their country rather than emigrate. There might still be boats and chariots but no one would ride in them. There might still be weapons of war but no one would drill with them. He could bring it about that the people should go back (from writing) to knotted cords, be contented with their food, pleased with their clothes, satisfied with their homes, and happy in their work and customs. The country over the border might be so near that one could hear the cocks cowing and the dogs barking in it, but the people would grow old and die without ever once troubling to go there.*

(Needham II, pg. 99)

Their ideal society was a cooperative, undifferentiated natural' condition of life, as was envisioned to have been the case before the time of social differentiation, before the institution of property. Humans would live together in small self-sufficing villages; there would be spontaneous collaboration instead of directive force.

Another point where Confucianism and Taoism were orthogonal to each other was the focus on Humans of the first and the focus on Nature of the second. The Taoist interest in nature is illustrated by the following quotation from a major Taoist philosopher:

> *How (ceaselessly) heaven revolves! How (constantly) earth abides at rest! Do the sun and the moon contend about their respective places? Is there someone presiding over and directing these things? Who binds and connects them together? Who causes and maintains them, without trouble or exertion? Or is there perhaps some secret mechanism, in consequence of which they cannot but be as they are? Is it that they move and turn without being able to stop of themselves? Then how does a cloud become rain, and the rain again form clouds? What diffuses them so abundantly? Is there someone with nothing to do who urges them on to all these things for his enjoyment? Winds rise in the north, one blows to the west, another to the east, while some rise upwards, uncertain of their direction. What is it sucking and blowing like this? Is there someone with nothing to do who thus shakes world I venture to ask about the causes?*

(Needham II, pg. 28)

The interest in Nature might be explained by the centre of the Taoist philosophy, the Tao. The Tao was some mystical energy, which was in both organic and inorganic matter. Lao Tzu (one of the greatest Taoist writers) described the Tao as follows:

> *The Tao is dark and elusive, difficult to describe. ... Light came from darkness, order from the formless. The Tao produces vital energy, and this gives birth to organic forms; ... Fathomless, it is like the sea. Awe-inspiring, beginning again in cycles ever new. Sustaining all things, it is never exhausted. ... What gives life to all creation and is itself inexhaustible - that is the Tao.*

(Needham II, pg. 39)

Another essential property of Taoism might be inferred from this passage: mysticism. Taoism contained shamanism and magic, the greatest Taoist writers wrote their ideas as poets.

“So wedded to the empiricism were they, so impressed by the boundless multiplicity of nature, so lacking in Aristotelian classificatory boldness, that they wholly dissociated themselves from the efforts ... to elaborate a logic suitable for science.” (Needham II, pg. 162) I.e.: Taoists were interested in nature, but didn’t consider it possible to rationally understand it.
In time and under Buddhist influence (see chapter 1.5), Taoism transformed from the agnostic naturalism with anti-feudal political orientation to a full-fledged theology with anti-foreigner secret societies. Although the seeds for the development of natural sciences based on natural laws were there, in the Confucian dominated political climate Taoism didn’t develop in this direction, but grew towards an extreme mysticism.

2.3 Mohism

Mohism was founded by Master Mo, who lived around 400 BC Mohists were politically active, one of their most important ethical doctrines was universal love and they preached pacifism, but only till a certain end. If a small state was threatened by a large state, they rushed to the help of the small state. With regard to political organization of the state, they did support some of the Taoist ideas regarding the self-sufficing villages, but they didn't really oppose feudalism, because it could be saved with their doctrine of universal love. (Needham II, pg. 165 and 168.)

To be able to rush to the aid of small states the Mohists trained themselves in military techniques and were famous for their fortification techniques. Needham suggests that this probably lead them to study the basic methods of science. In fact, their records regarding mechanics and optics are among the first records regarding Chinese science. But not only they knew much about physics, when examining their *Canons* it becomes evident that they had created the basis for an entire scientific system. The following passages from the Mohist classic Mo Tzu gives some prime examples (N.B.: the examples are from different chapters of the classic). C stands for Canon and E for Exposition, the text between ( ) is commentary of Needham:

> *Attributes*
> *C An attribute may be (added on to or) taken away from (something) without involving increase of reduction. The reason is given under 'origin' (or cause).*
> *E Both are the same one thing and no change has occurred. (This refers to subjective judgments as of a 'beautiful' flower which remains the same flower whether it is called beautiful or not.)*
>
> *Hardness and whiteness*
> *C Hardness and whiteness are not mutually exclusive.*
> *E Within a stone, the (qualities of) hardness and whiteness are diffused throughout its substance; thus we can say that the stone has these two qualities. But when they are then different places they do not pervade one another; not thus pervading they are then mutually exclusive.*
>
> *Classification*
> *C Applying the principles of classification is difficult. The reason is given under 'broad and narrow'.*
> *E For instance, animals of four feet' form a broader group than that of 'oxen and horses', while the group of 'things' is broader still. Everything may be classified in broader and narrower groups.*
>
> *C Things can be separated into different groups. The reason is given under 'responding respectively'. And different things can be combined into a single group. The reason is give under 'common point'.*
> *E The 'common point' is, for example, that the ox should be called ox and the horse horse. If the ox and horse are considered separately*

> *they make two things, but if the ox-horse group is considered, they make one thing. It is like counting fingers, each hand has five, but one can take one hand as one (thing).*
>
> *Causation*
> *C A cause is that with the obtaining of which something becomes (comes into existence).*
> *E Causes: A minor cause is one with which something may not necessarily be so, but without which it will never be so .For example, a point in a line. A major cause is one with which something will of necessity be so (and without which it will never be so). As in the case of the act of seeing which results in sight. (Clearly with discussions such as this we are in the very engine-room of scientific thinking. The minor cause here is a necessary condition we should say, rather than a cause.)*

(Needham II, pg. 172 - 176)

Many other statements are written down, regarding science an sich, reasoning, argumentation, mechanics biology and geometry (see chapter three); unfortunately this article lacks the space to present even a representative sample. The Mohists, not as the Taoists mistrusting human reason, tried to develop a method of scientific reasoning. They treated classification, causality, the role of perception, different ways of scientific reasoning, et cetera. (Some related philosophical groups even played with the concepts of special relativity.) What was missing, however, was some general theory of natural phenomena. (Needham II, pg. 182)

From the beginning, Mohism seemed to be better organized than both Confucianism and Taoism, however they completely vanished during the Qin unification. Why they disappeared remains unexplained. Needham suggests that the Confucius - Taoist polarization of social life forced the Mohists to side with one of them, thus effectively making them disappear. Furthermore, the ethical orientation of the intellectual debate didn't leave much room for discussions about logic. As an influential Confucian noted:

> *There is no reason why problems of 'hardness and whiteness', 'likeness and unlikeness', 'thickness or no thickness' should not be investigated, but the superior man does not discuss them; he stop at the limit of profitable discourse.*

(Needham II, pg. 202)

2.4 Legalism

On the extreme right of the political spectrum we have the legalists. Just as Confucianism, Legalism focuses on humans only, but the two firmly split when it comes to ethics. The first philosophy is based on it, the second considers it irrelevant. According to Legalists, the paternalistic Confucian administration was inadequate for an authoritarian government. They introduced absolute Laws to which everyone in the state was bound. Transgression of the Law was without exception intentionally cruelly punished. As Han Fei, a legalist philosopher, states:

Severe penalties are what the people fear, heavy punishments are what the people hate. Accordingly the sage promulgates what they fear in order to forbid the practice of wickedness, and establish what they hate in order to prevent villainous acts. Thus the state is safe and no outrage can occur. From this I know well that benevolence, righteousness, love, and favour are not worth adopting, while severe punishment and heavy penalties can maintain the state in order.

(Needham II, pg. 206)

The punishments were extremely cruel and didn't align with ethical ideas, even common human sense, but legalists were by choice completely amoral. Legalist virtue was not goodness or benevolence, but obedience to the Law of the state; there was no room for private opinions, all that counted was the opinion of the ruler who made the Law. However, here we also see that the Law was not really absolute, since the Law still depended on the rulers; there was no law that might set limits to what the rulers would do. (Ebrey, pg. 53).
Next to obedience to the Law, Legalists envisioned a society primarily based on warfare and agriculture. All other occupations were considered useless.

As mentioned in chapter one, the state of Qin adapted Legalist policies and with its expansion, Legalist law was spread through much of China. After the revolt against the successors of the First Emperor, however, the public had accumulated such an aversion against Legalist law, that it was abolished for once and forever. The Legalists had failed, their ideas were abandoned. From the Han Empire on, Chinese law was reduced to the absolute minimum and became based on what was considered to be easily demonstrable ethical principles.

Needham suggests that this aversion from codification of law, and thus the relative dominance which was given to 'natural ethical law' might be one of the factors which prevented a fruitful intellectual climate for the development of a system of scientific thought (Needham II, pg. 204).

2.5 Laws of Nature and Abstraction

When comparing the major Chinese philosophical schools, the difference in attitude towards nature and logic is striking. The Legalists developed ideas about (semi-)absolute laws, but just as the Confucians they didn't have any interest in nature at all. The Taoists were interested in nature, but thought it could not be understood by logic and reason. The Mohists used logic and were interested in nature, but they disappeared completely. As Needham puts it: "One is tempted to think that perhaps the greatest tragedy in the history of Chinese science was that Taoist naturalist insight could not be combined with Mohist logic." (Needham II, pg. 182)

But why didn't the Chinese develop the concept of laws of nature? (Or, why the Western world did?) An important factor in Western thinking might be the fundamental notion of a Supreme Being. Everything in nature was regulated by the Laws of this Supreme Being, and since this Being was rational, so were its Laws. Therefore, the Laws of Nature could be revealed by using ratio (hence,

the very word 'revealed' is symptomatic for Western thinking).
In Chinese philosophy there was no notion of a Supreme lawgiver imposing laws on non-human nature.
"It was not that there was no order in Nature for the Chinese, but rather that it was not an order ordained by a rational personal being and hence there was no conviction that rational personal beings would be able to spell out in their lesser earthly languages the divine code of laws which he had decreed aforetime." (Needham II, pg. 581) The order in nature was attained by spontaneous harmony between all beings, each following its own internal necessities. The cosmos was to complex and subtle to be written in laws understandable for humans.

As to the emphasis on a Supreme Being in Western civilization and on spontaneity in Chinese civilization, it is suggested that the difference might arise depending on whether the civilization is based on cattle breeding and fishing or agriculture. The shepherd and the cowherd actively guide their animals. On fishing ships the life of all depends on obedience towards the experienced few. Analogously, the Supreme Being is often envisioned as a 'Shepherd' guiding his herd in the right direction.
In agriculture, however, the crops grow spontaneously; the less one interferes, the better. Similarly, order in nature is spontaneous and not necessarily understandable for humans. (Needham II, pg. 577).
In our opinion, however, this theory is interesting but requires elaboration and lacks proof.
"The development of the concept of precisely formulated abstract laws capable, because of the rationality of an Author of Nature, of being deciphered and restated, did not therefore occur." (Needham II, pg. 582) As a result, Chinese science (and Chinese mathematics) would always remain concrete and applied; high forms of abstractions didn't occur. Abstraction was also not considered necessary,

> *... because 'man has a wisdom of analogy' that is to say, after understanding a particular line of argument one can infer various kinds of similar reasoning, or in other words, by asking one question one can reach ten thousand things. When one can draw inferences about other cases from one instance and one is able to generalize, then one can say that one really knows how to calculate. ... To be able to deduce and then to generalize, that is the mark of an intelligent man.*

As is written in the second part of the Zhoubi. (Li, Y. and Du. S., 1987, pg. 28; hereafter LD)

Consequently, the path Chinese mathematics took was quite different than that of Western mathematics, as will become evident in the chapters to come.

3 Prelude

The first mathematical work is the so called *Nine Chapters on the Mathematical Art*, which will be dealt with in the next chapter. This chapter, however, will focus on the period in advance. About the Zhou reign we can learn that mathematics was considered one of the six gentlemanly arts. As the book *Zhou rites* records:

> *The ... official is to teach the ways to the children of the country, to teach them the six gentlemanly arts, the first of which is called the five rites, the second is the six modes of music, the third the five methods of archery, the fourth the five ways of horsemanship, the fifth the six ways of calligraphy and the sixth the nine calculations of mathematics.*

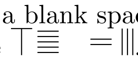
(LD, pg. 22)

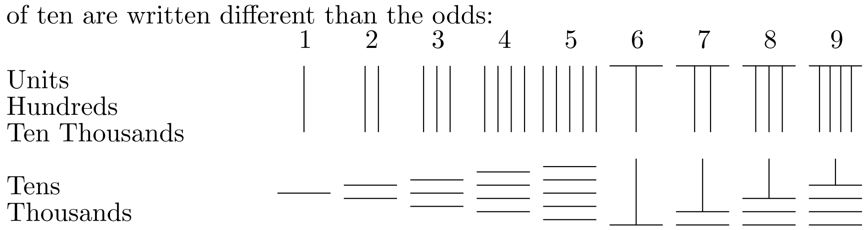
From this we can conclude that mathematics was already being practised a long time before the *Nine Chapters*, but unfortunately not much is known about the mathematics of that time, since the edict of 213 BC robbed history of her sources. Luckily we can learn a lot about numerical notations from old coins and oracle-bone inscriptions. But even better: the *Zhoubi*, mainly concerned with China's most important science, the study of the calendar, shows some mathematical skills of the ancient Chinese; they had already discovered the *Pythagoras theorem*. Lastly, the *Mohist Canon* learns us that some of the ancients were even studying geometry, while the following chapters will show that the Chinese were mainly dealing with algebra.
So this chapter will discuss these works and also the less scientific approach of numbers, but first it will deal with the numerical system.

3.1 Numerical System

When one would practice the mathematical art in ancient times, he would find the Hebrews just assigning a value to their letters, so that just summing the values of the letters of a word would give a number. The Romans did the same thing, but they also used some kind of place-value by saying $4 = IV \neq VI = 6$, which is known as the subtraction device. Also the Babylonian astronomers had a different system than we use today: it is thanks to their sexagesimal notation that hours are divided in 60 minutes and minutes likewise in seconds.

Knowing how ancient western civilizations wrote numbers, one should be struck with fear if he is preparing himself to learn the numerical notation of a culture which alphabet outnumbers ours like their population does. According to the 3000 year old oracle bones, however, the Chinese used some sort of decimal notation from the very beginning.
For example, 656 on the oracle bones was written like six hundreds, five tens, six. Where the symbol for 6 and the symbol for 100 where combined to a single symbol and likewise for the 5 and 10.
However if one thinks about this notation and recalls that many words in Chinese are written with a single symbol, one will recognize that this notation should rather be compared to our alphabetical notation than the numerical,

like we would write *six hundred and fifty-six.* This may explain why it was easier for the Chinese to work with a decimal system, since their writing of numbers was already pretty short in words, while western mathematicians preferred a shorter notation. Think of the Romans for example: **C** is much shorter than *centus.*
But mathematicians are known for their minimalistic nature: they will always try to do the greatest amount of work with the smallest amount of effort. So even the Chinese written form was considered too long, which resulted in an even shorter and easier notation: the *rod-numerals.*
These worked the same way as our system does today: they used some kind of frequency-method (see below) for the numbers *one* to *nine* and placed them behind each other, leaving a blank space where we would write 0. For example 65023 would be written like . It should be noted that the even powers of ten are written different than the odds:



This may have two reasons. First if both *ten* and *one* where written with a vertical line, one could confuse eleven with two, since both of them would be written with two vertical lines. So this would be more practical. The second reason may be found in the Chinese view of *yin* and *yang*, which will be discussed in paragraph 3.4.

Counting rods were used on a so called *countingboard*, which can be compared to using a scrappaper. For example if we calculate 256×623 on a scrappaper, it will look like:

			2	5	6
			6	2	3
			7	6	8
		5	1	2	0
1	5	3	6	0	0
1	5	8	4	8	8

So first calculate 3×256, 20×256 and 600×256, then take the sum.
The countingboard, was divided in rows and columns, by placing numbers beneath each other with counting rods, small sticks, one could perform similar operations on the countingboard.

About the origin of these rod-numerals less is known, although there is no doubt they are derived from the counting rods. A famous rebus in the *Zuo Zhuan* (542 BC) which may refer to the use of rods, is a passage concerning the determination of the age of an old man, in which the analysis of a certain character results in a two and three sixes after which may be concluded that the man is 2666 decades of days. This, of course, is no proof of the existence of the rod-numerals, but it shows the Chinese did have an understanding of the concept of place-value, which may mean it was known by then. Furthermore, if we consider the character *suan*, meaning mathematics, as a graph for an old arranged counting rod, then rod-numerals may even descent from the time of the oracle bones.

But whatever be the case, it is considered a fact that the different notation for even and odd powers of ten was standardized during the Qin and Han. This concludes the treatment of the advanced Chinese numerical system.

3.2 *Zhoubi*

Though the *Nine Chapters* is considered the oldest of the mathematical works, the first surviving work dealing with mathematics is considered to be the *Zhoubi Suanjing*. The reason we don't take an English translation for the title like we do with the *Nine Chapters* is because 'Zhou' can refer to the dynasty, but also to 'the circular paths of the heavens'.
Anyway the *Zhoubi* consists of two books, of which the second is treating only calendrical theories; therefore we will not pay any further attention to that part. The first part, however, shows how easily Chinese could work with fractions and, very important to note: they knew the Pythagoras theorem.
About the writer and the age is discussion, Needham says there is a tendency to look at it as a Han book, while Mikami suggests it is the result of different writers from different times. But, since the persons of interest really did exist, if one accepts the dialogs of these persons in the book really were of them, then questions about the author do not matter, since our interest is the state of mathematics at a certain time.
The first book starts with a dialog of Shang Gao and Zhou Gong, who ruled China as a regent during the first years of the Zhou dynasty, because his nephew was a little too young. Part of the dialog will be given, as it is quoted in Needham III (pg. 22 - 23) (completely with clarifying footnotes):

> *(1) Of old, Zhou Gong addressed Shang Gao, saying: "I have heard that the Grand Prefect is versed in the art of numbering. May I venture to enquire how Fu-Xi anciently established the degrees of the celestial sphere? There are no steps by which one may ascend the heavens, and the earth is not measurable with a foot-rule. I should like to ask you what was the origin of these numbers?"*

(2) Shang Gao replied: "The art of numbering proceeds from the circle and the square. The circle is derived from the square[7] *and the square from the rectangle.*

(3) The rectangle originates from (the fact that) $9 \times 9 = 81$ *(i.e. the multiplication table of numbers as such).*[8]

(4) Thus let us cut a rectangle (diagonally), and make the width 3 wide, and the length 4 long. The diagonal between the (two) corners will then be 5 long. Now after drawing a square on this diagonal, circumscribe it by half-rectangles like that which has been left outside, so as to form a (square) plate. Thus the (four) outer half-rectangles of width 3, length 4, and diagonal 5, together make two rectangles (of area 24); then (when this is subtracted from the square plate of area 49) the remainder is of area 25. This (process) is called 'pilling up the rectangles'.

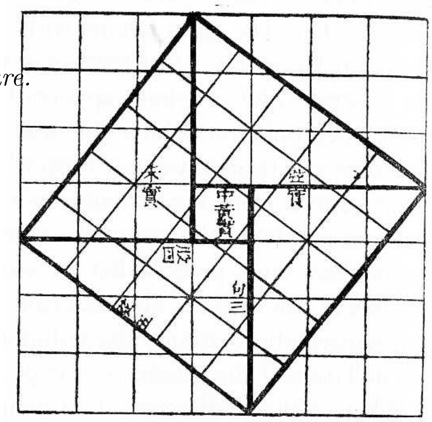


Figure 1: Proof of the Pythagoras Theorem (Needham III, pg. 22)

(5) The methods used by Yu the Great in governing the world were derived from these numbers."

(6) Zhou Gong exclaimed: "Great indeed is the art of numbering...."

It is also 'great' that the Chinese seem to be already familiar with the Pythagoras theorem, or the Gougu Rule in Chinese, six centuries before Pythagoras lived. So Mikami (pg. 7) can't help to suggest that Pythagoras found his theorem not by puzzling, but by travelling.
On the same page Mikami also says that the Chinese and Babylonians are thought to be related and that the latter also seem to have been in possession of that relation, that is: at least for the special case 3, 4 and 5.
Since the proof is delivered with the special case, one may wonder whether the relation was only known for this special case or also in general.
Needham explains (III, pg. 23-34):

> *No further commentary is needed on this classical passage, except by way of emphasis on what seems a deeply significant point, namely, the statement in sentence (3) that geometry arises from mensuration. As has already been indicated, this seems to show the Chinese arithmetical-algebraic mind at work from the earliest time, ap-*

[7] Presumably the writer was thinking of the diameter of a circle being equal to the diagonal of its inscribed square; perhaps also of exhaustion methods for getting π.

[8] Chao Chun-Chhing explains that it is necessary to know the properties of numbers before one can work with geometrical figures. Note how radically different this is from the Euclidean method, in which actual numerical values are irrelevant provided the basic axioms and postulates are accepted. Here an arithmetical square is significantly given.

parently not concerned with abstract geometry independent of concrete numbers, and consisting of theorems and propositions capable of proof, given only certain fundamental postulates at the outset. Numbers might be unknown, or they might not be any particular number, but numbers there had to be. In the Chinese approach, geometrical figures acted as a means of transmutation whereby numerical relations were generalized into algebraic form.

The next part of the *Zhoubi* is not of much interest to us, but in the passage which follows it there is given a case of dividing by a fraction, done by Lu Shi, an important minister from the Qin in the third century BC.
He wanted to divide 119000 by $182\frac{5}{8}$ and does that by first multiplying both numbers by 8 and then just two integers are left for division. What makes it so remarkable is that until the 17th century all western cultures tended to avoid fractions.
To make it even worse: the same Lu states that one years is divided in $365\frac{1}{4}$ days and a circumference of a circle in the same rational number of degrees.
So also at this point, ancient Chinese mathematics did not differ very much from the modern version.

3.3 *Mohist Canon*

When it comes to geometry, the Greeks are by far number one. The Chinese can't change that, for they never developed a theoretical geometry, but rather searched for facts concerning real life.
However, the *Mohist Canon*, tells us that also in China there existed a group dealing with this subject the theoretical way, or at least they seemed to. The book is speaking about subjects from physical science and lines, spaces and figures. Alas, most of it has been lost, but here follow statements concerning geometry which did escape the devouring jaws of time - again using Needham's classic, where C stands for Canon and E for Exposition:

C The definition of point is as follows: the line is separated into parts, and that part which has no remaining part (i.e. cannot be divided into smaller parts) and (thus) forms the extreme end (of a line) is a point.

E A point may stand at the end (of a line) or at its beginning like a head-presentation in childbirth. (As to its indivisibility) there is nothing similar to it.

C That which is not (able to be separated into) halves cannot be cut further and cannot be separated. The reason is given under point.

E If you cut a length continually in half, you go on forward until you reach the position that the middle (of a fragment) is not big enough to be separated anymore into halves; and then it is a point. Cutting away the front part (of a line) and cutting away the back part, there will (eventually remain an indivisible) point in the middle. Or if you keep on cutting into half, you will come to a stage in which there is an 'almost nothing', and since nothing cannot be halved, this can no more be cut.

C (Two things having the) same length, means that two straight lines finish at the same place.
E It is like a straight door-bolt which can be placed flush with the edge of the door.
E In both methods there is a fixed point so that the comparison may be carried out.

C Level means (being supported by props) of the same height.
E Like two persons carrying (a beam on their shoulders), who should be of the same height like brothers.

C Space includes all the different places.
E East, west, south and north, are all enclosed in space.

C Outside a bounded space no line can be included (because the edge of an area is a line, and beyond that line is outside the area).
E A plane area cannot include every line since it has a limit. But there is no line which could not be included if the area were unbounded.

C Having (two-dimensional) space means that something is contained in it.
E Like the 'door-ear', the space being part between the door and the jamb.

C A plane space does not reach its sides.
E The plane space is what is inserted between lines. The line in front of the plane but behind the point, but yet is not 'between' them.

C Rectangular shapes have their four sides all straight, and their four angles all right angles.
E Rectangular means using the carpenter's square so that the four lines all just meet (each other).
E Things which have no thickness (exemplify this principle).

C A circle can rest on any point of its circumference.
E (The Explanation is missing)

C A circle is a figure such that all lines drawn through the centre (and reaching the circumference) have the same length.
E A circle is that line described by a carpenter's compass which ends at the same point as which it started.

C (As for the circle, there is a) centre (from which the distance to any point on the circumference is) of the same length.
E The centre is (like a) heart, from which (a point moving to any part of the circumference) travels in all cases the same distance.

C (Every volume) has a thickness dimension which gives a size (to the body).

E Without a thickness dimension, there is no bodily size.

(Needham III, pg. 91 - 94)

Needham also refers to comparative parts of Euclid, which we have omitted, since our goal was just to show that also the Chinese had made a theoretical approach on geometry (for an extensive treatment of geometry, please see *In den beginne was er meetkunde* of Scherphof, P. and Tervaert, G.). How deep they went we cannot know with certainty for as already stated and as can be noted by he many brackets in the quote: the *Mohist Canon* did not come down to us in its best shape.
But it does not matter, for as Needham says: "... their deductive geometry remained the mystery of a particular school and had little or no influence on the main current of Chinese mathematics."(III, pg. 94)
So despite their work, the Mohists could not convince the Chinese mathematics to trade her practical approach for a philosophical one.

3.4 Practical Use of Numbers

The Chinese made extensive use of yin-yang-like oppositions: man versus woman, heaven versus earth and likewise odd versus even. Although very interesting, this article lacks the space to deal with this extensively. But if one keeps in mind that knowing the age of a woman and her conception-time the Chinese predicted whether it would be a boy or girl by distinguishing odd from even numbers, then one will remember how important a good calendar was to the ancient Chinese.
Perfect example of this yin-yang-view is the magical square:
"One of the components of the corpus of legends was the story that to help him in governing the empire the engineer-emperor Yu the Great was presented with two charts or diagrams by miraculous animals which emerged from the waters which alone he had been able to control."(Needham III, pg. 56)
The gifts were the Ho Thu from a dragon-horse from the Yellow River and the Luo Shu from a turtle from the River Lo. Mikami, however, says about the dragon-horse (pg. 3): "According to tradition the former (the Ho Thu) came down from the time of the Emperor Fu-hi in the 29^{th} century BC, when a dragon-horse had discovered himself at a river and his footsteps had left the figures of the *pa-kua*, ..." On the turtle they do agree. But whatever may be the story, fact is that two magical figures were born in China and that the last one, the Lo Shu, would also enchant the world as the so called magical square.

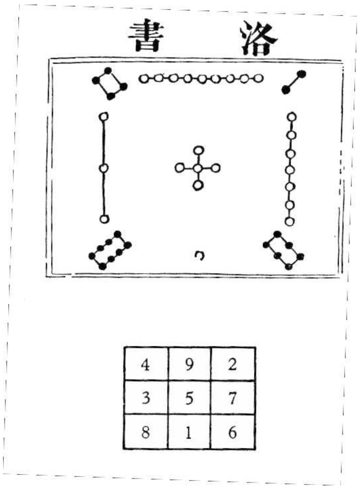


Figure 2: The Lo Shu siagram. (Needham III, pg. 57)

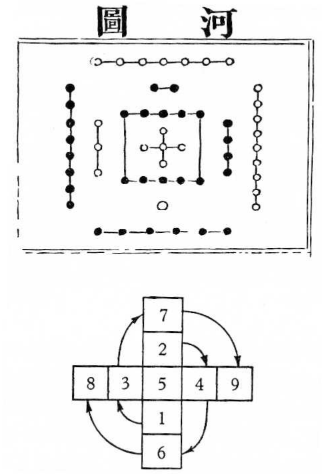


Figure 3: The Ho Thu diagram. (Needham III, pg. 57)

In the Ho Thu one will end up with 20 by following the arrows, while in the Luo Shu every straight line (horizontal, vertical or diagonal) sums to 15. In both figures the numbers *1* to *9* can be found, where the yang or odd numbers are represented by the white dots and the yin or even by the black ones. Note that in the Ho Thu 5 is surrounded by two times five black dots: they don not represent two times the number five, but stand for ten.
This concludes the Chinese view of numbers and prepares the reader for what follows next: the foundations of Chinese mathematics.

4 Foundations

The emperor was considered to have been given a 'mandate from the Heavens' and was seen as the 'Sun of the Heavens'. With a calendar, a link was created between the Emperor and the Heavens, consequently calendrical treatises and astrometry[9] were of fundamental importance in Chinese religion and were supported by the Emperor. As Needham notes:

> *While, among the Greeks, the astronomer was a private person, a philosopher ..., as often as not on uncertain terms with the priests of his city; in China, on the contrary, he was intimately connected with the sovereign pontificate of the Son of Heaven, part of an official government service, and ritually accommodated within the very walls of the imperial palace."*

So astronomy was given considerably attention and since mathematics was a necessity in making predictions about future positions of celestial objects, in the wake of astronomy mathematical knowledge grew as well. Furthermore, mathematics was used for practical applications like tax and land division.

The Foundations of Chinese mathematics are laid down in the extraordinary work *The Nine Chapters on the Mathematical Art*[10]. It is the first Chinese treatise specifically on mathematics, and in influence on the development of mathematics it is comparable only the Euclid's Elements (Needham III, pg. 25; Yan L. and Shiran, D., 1987, pg. 33; Kangshen S., LD, pg. vii)[11]. As the last work often is considered to be the basis of the Western branch of mathematics, so the Nine Chapters is considered the cornerstone of its Chinese counterpart. Where the Euclidian tradition is based on theorems followed by proof (again we refer to the article *In den Beginne was er Meetkunde* of Scherphof, P. and Tervaert, G.C. elsewhere in this bundle for a detailed treatise of the Euclidian geometry), the Nine Chapters provides arithmetical Rules focused on practical applications, compiled in question and answer form. Though it should be noted that the arithmetical rules were written in words, not in algebraic notation we are accustomed to today.

In ancient China, scholars often commented on classical work. Therefore, several comments have been given on the *Nine Chapters*. Undoubtedly the most important comments are those of the brilliant mathematician Liu Hui. He provided justification of the Rules and explained the methods the *Nine Chapters* used to reach an answer.

One of the other major contributions of Liu Hui is the *Sea Island Mathematical Manual*. Originally intended as an extension of the *Nine Chapters*, it became a work on its own. Named after the first problem, the manual treats the 'methods of double differences', used for surveying.

[9] The branch of astronomy that deals with the measurement of the position and motion of celestial bodies.

[10] Hereafter: the *Nine Chapters* or 9C.

[11] For brevity, hereafter (Kangshen S., Crossley J.N. and Lun, A.W.C, 1999) will be written as (KCL).

Two other noteworthy mathematicians of the Age of Division are Zu Chongzhi and Zu Geng, father and son. They were responsible for the development of the Da Ming calendar, the best available calendar of that time, calculated π to seven decimal places and pointed out errors in the *Nine Chapters* and Liu Hui's comments.

The *Nine Chapters* together with these comments would become the foundation of Chinese mathematics, its importance unparalleled in the history of Chinese Mathematics: for over a millennium to come, it would be used as mathematical textbook. To quote Liu Hui:

> *I read the Nine Chapters as a boy, and studied it in full detail when I was older. (I) observed the division between the dual natures of Yin and Yang (the positive and negative aspects) which sum up the fundamentals of mathematics. Thorough investigation shows the truth therein, which allows me to collect my ideas and take the liberty of commenting on it. ... Curious gentlemen, read the text thoroughly.*

(KCL, pg. 53.)

4.1 The Classics

Before immersing ourselves in the classics and Liu comments, we discuss the works in general. What is their general content, how is the content ordered, when are they written?
After questions like these are answered, we are free to focus on the most groundbreaking aspects of these books in the areas of arithmetic, algebra and geometry.

4.1.1 *The Nine Chapters on the Mathematical Art* (with comments)

As was mentioned in chapter three, the Nine Arithmetic Arts were one of the six types of the gentlemanly arts of the Zhou dynasty, and Liu Hiu mentions the *Nine Chapters* are based on it. But as to the exact time of its writing, even Liu, who lived in the third century AD, was unsure. In his preface to the work he mentions:

> *It is said that the Duke of Zhou formulated the nine rites and that consequently there was the calculation of the nines. The popularizing of the calculation of the nines led to the Nine Chapters. The tyrant Qin Emperor burned all the books and the classics were either lost or destroyed. It was not until later that Marquis Zhang Chang (? – 152 BC) and the Han and the Minister of Agriculture (Geng Souchang (first century BC)) became famous for their ability to calculate. Because the old text had a lot of lacunae or was incomplete and names were either missing or had been interpolated later, Marquis Zhang and other re-edited the book and determined what was authentic or not and re-wrote part of it.*

(LD, pg. 34).

Considering the tradition of commenting on classics, it is hard to determine the era from which certain parts of the text come, but it is rather certain the ‘original text’ (so excluding Liu Hui’s and later comments) was formed by several people, living in different times. The classic can therefore be seen as the collective effort of centuries of mathematicians, but the final ‘original text’ has at last been written in the first century AD. (LD, pg. 35) The oldest (partly) surviving edition is a woodblock print of the Song Dynasty, shortly after 1213 AD.

The *Nine Chapters* is written in question-answer (-explanation) form. The nine chapter of the work contains questions, 246 in total, closely related to practical life. Here a brief overview of the content per chapter.
Chapter I, *Field Measurements*, discusses the calculation of areas of cultivated land and computation with fractions.
Chapter II, *Millet and rice*, is about problems of proportion and exchanging different types of cereals.
Chapter III, *Proportional Distribution*, extends the proportion problems discussed in chapter II. In solving some questions, sequences are introduced, although not explicitly.
Chapter IV, *short width*, exhibits calculations to find a side or diameter if the area or volume is known. Methods to determine square and cube roots are discussed.
Chapter V, *Construction consultations*, focuses on finding the volumes of different solids and on the number of labourers needed to complete a certain task.
Chapter VI, *Fair levies*, further develops fraction and proportion problems discussed in chapters II and III, as well as inverse problems.
Chapter VII, *Excess and deficit*, is concerned with the solution of linear equations by use of the method of double false position.
Chapter VIII, *Rectangular arrays*, deals with systems of linear equations! All techniques correspond with modern techniques. Both determinate and indeterminate systems are treated and negative numbers are used.
Chapter IX, *Right-Angled Triangles*, introduces the Gougu’s theorem (‘Pythagoras’ theorem) and problems related to right-angled triangles. Also, general methods for solving quadratic equations are discussed.

From a pedagogical point of view, the chapters in 9C are well structured (KCL, pg. 135 and others). The questions begin simple and easy, but become more complex and difficult as the chapter progresses. Some problems are solved in more then one way, to emphasize several aspects of the solution. If necessary, diagrams are used to make relations visible. Particularly Liu Hui’s comments are instructive; as the authors of (KCL, pg. 308) note: “‘Comprehend by analogy and link old knowledge to new ideas’ is another important pedagogical method for active learning and knowledge consolidation, and Liu was clearly fully aware of the value of such links.”.

4.1.2 *Sea Island Arithmetic Manual*

The Sea Island Arithmetic Manual was originally a part of Liu Hui’s commentary on 9C, but during the Tang dynasty this part of Liu’s comments was separated and named after the first problem, surveying a sea island.
The classic deals with mathematical surveying: mensuration of distances, lengths, etc. from far away objects.

All problems can be solved with the method of double differences; we will explain the method using the name-giving problem in paragraph 4.4.

Surveying was important for both agriculture (to determine which area belonged to which farmer), merchants and the military. Monumental projects like the Great Wall and the Grand Canal couldn't have been constructed without accurate survey methods.

Although Liu Hui was one of the most brilliant mathematicians in human history, we know little of him. The only record about his life that has survived is a sentence in the *Book of Music and the Calendar* (written around 600 AD): "In the fourth year of the Jingyuan Reign of Prince Chenliu of the Wei (Dynasty) (263 AD), Liu Hui commented on the Nine Chapters." (KCL, pg. 3). Based on his comments on the *Nine Chapters*, he seemed to be a gifted, though modest man as well as an excellent teacher.

Figure 4: The ingenious mathematician Liu hui (O'Connor and Robertson, 2003; hereafter OR, 2003)

Unfortunately, Liu's derivation of the double differences method and the original diagrams of the text have been lost. Today's version is a compilation of the *Complete Library of the Four Branches of Literature* by Dai Zhen (1724 – 1777 AD), who copied it himself from another source. (LD, pg. 76.) This version contains 9 problems: measuring the distance to and height of an island, a hill, a city; the width of a river; the depth of a pool; etc., increasing in difficulty. For the easier problems only two observations are required; the more difficult ones require four observations. The style of writing is analogous to the *Nine Chapters* and others: problem, solution, method.

In what follows we briefly discuss the achievements in arithmetic, algebra and geometry. `This typewriting will be used for the original text of a classic`, this for comments of Liu, *this for comments of other Chinese mathematicians* and the current typewriting for our own comments or comments from present-day mathematicians and historians.

4.2 Arithmetic Achievements

The arithmetic knowledge of the ancient Chinese is impressive. The *Nine Chapters* with associated comments alone lay the foundation for Chinese arithmetic, so this paragraph deals only with this classic. The books gives an elaborate discourse on fractions: reduction, addition, subtraction, multiplication and division are treated. Furthermore, a Rule is given to solve first order linear equations. As illustration we will quote the first and the last aspects of calculation with fractions and an application of computations with fractions. Let 9C speak for itself:

```
(Problem 5 of chapter 1)
Now given a fraction 12/18.  Tell:  reducing it (to its lowest terms),
what is obtained?
Answer:  2/3.
(Problem 6 of chapter 1)
Given another fraction 49/91.  Tell:  reducing it, what is obtained?
Answer:  7/13
```

The Rule for Reduction of Fractions[a]

```
If (the denominator and numerator) can be halved, halve them.  If
not, lay down the denominator and numerator, subtract the smaller
number from the greater.  Repeat the process to obtain the Greatest
Common Divider (GCD). Reduce them by the GCD.
```
[b]

a. Liu: Reduce a fraction means: a quantity of things cannot always be an integer, but sometimes must be represented by a fraction. A fraction may be tedious to handle. Taking 2/4, 4/8 is tedious, while 1/2 is simple. They are written differently, but equal in value. In practice the divisor and dividend, (while) mutually related, are often mismatched, therefore in arithmetic we have to learn the rule for fractions.
b. Liu: To reduce a fraction by the GCD means to divide. Subtract the smaller number from the greater repeatedly, because the remainders are nothing but the overlaps of the GCD, therefore divide by the GCD.
(KCL, pg. 64)

The method for finding the GCD described here is named the Euclidian Algorithm in the West. For those not familiar with the algorithm, lets use problem 6 as an example. The numbers 49 and 91 can't be halved, so we continue to find the GCD(91,49). 91 >49, so we subtract 49 from 91 to get (42,49). Since 42 <49, we now subtract 42 from 49 to get (42,7). Subtracting 7 five times from 42 results in (7,7). I.e.: the GCD is 7. Dividing denominator and enumerator by 7 gives the answer. This process also works for any finite number of integers at the same time, the lesser numbers might be subtracted from the higher ones in any order.
The Euclidian Algorithm was used inter alia for the compilation of calendars. Zu Chongzhi used it for leap-year determination in the Da Ming calendar.

Several problems later, the Nine Chapters discusses division of fractions:

```
(Problem 17 of chapter 1)
Now given 7 persons share 8 (1/3) coins.  Tell:  how much does each
person get?
Answer:  Each gets 1 (4/21) coins.
(Problem 18 of chapter 1)

   Again given 3(1/3) persons share 6(1/3) and (3/4) coins Tell:  how
much does each person get?
Answer:  Each gets 2(1/8) coins.
```

The Rule for the Divisions of Fractions$^{\alpha}$

Take the number of persons as divisor, the number of coins as dividend. Divide. If either the dividend or the divisor is a (mixed) fraction, convert it to an improper fraction.[a] If both of them are (mixed) fractions, convert them to improper fractions with a common denominator.[b]

α. *All the rules following that for the Addition of Fractions involve the Homogenizing and Uniformizing Rule. Here ask directly how much each person gets. Divide what is shared by the number of persons, hence (this is) called jingfen. (In the Nine Chapters, this Rule was named the Jingfen Rule (literally: to divide directly).)*

a. Liu: Each numerator is multiplied by the denominator of the other fractions to homogenize the numerators; multiply the denominators to uniformize the denominators. ... The multiplication of both the numerator and denominator of the dividend and the divisor by the denominator of the divisor and dividend respectively is called sanfen.

b. Liu: Multiply the numerator and denominator of the dividend by the denominator of the divisor, and multiply those of the divisor by the denominator of the dividend. That is to say, if both the dividend and the divisor are fractions, we first convert them (as mixed fractions) to improper (fractions), and then multiply both the dividend and the divisor by the denominator of the other.

(KCL, pg. 81)

So, assume we have to divide $a + \frac{b}{c}$ by $d + \frac{e}{f}$. First the mixed fractions are converted to improper fractions:

$$a + \frac{b}{c} = \frac{ac + b}{c} \text{ and } d + \frac{e}{f} = \frac{df + e}{f}$$

Next, multiply the enumerator and denominator of both fractions with the enumerator of the other fraction, so the denominators of both fractions are equal, and then multiply both fractions with their common denominator:

$$\frac{(\frac{ac+b}{c})}{(\frac{df+e}{f})} = \frac{(\frac{f(ac+b)}{cf})}{(\frac{c(df+e)}{cf})} = \frac{f(ac + b)}{c(df + e)}$$

Working with fractions is applied throughout 9C. In chapter II (Millet and Rice), exchange rates between different kind of grains are discussed. Chapter III (Proportional Distribution) deals with problems how to divide hunting loots among officials of different class, who are entitled to different rates; how much silk one can get given a price per square meter and an amount of money; interest rates for lends; how to distribute investments in public works equally over several regions. Chapter V (Construction consultation) discusses problems related to the number of required workers to construct a dyke or canal of a certain volume. Chapter VI (Fair levies) treats proportionally problems related with the fair distribution of taxes and corveé labor, see for example the following problem:

(Problem 2 of chapter 6)
Now frontier guard-duty is distributed among five counties. County

A is near the frontier, with 1200 adults; County B is 1 day away from the frontier, with 1550 adults; County C, 2 days away, with 1280 adults; County D, 3 days away, with 990 adults; County E, 5 days away, with 1750 adults. The total service is 1200 soldiers a month. Assume the guard-duty is distributed in accordance with the number of adults and the distance from the frontier.
Tell: how many adults does each county dispatch?
Answer: County A, 229 adults; county B, 286 adults County C, 228 adults; County D, 171 adults; and County E, 286.
Method: Divide the number of adults of each county by the sum of days on guard-duty and in travel for each to get each rate of distribution, i.e. A, 4; B, 5; C, 4; D, 3; and E, 5.[a] Take their sum as divisor. Multiply the number of adults of each county by each rate as dividend. Divide.[b] If the quotients contain fractions, round them.[c]

a: Liu: Here we also consider the number of days as the basis for fair distribution and imitate the number of adults dispatched as a task of service in transportation. County A is no distance, with only 30 days (1 month) in service as rate. For the rate of proportional distribution, County B should send one more adult for every 30 adults that the service is based on one adult a day, and that brings us the rate of fair distribution. I.e.: travel days are counted as extra days in service. So the rates for distribution in the Problem are then:
$(A, B, C, D, E) = (\frac{1200}{30}, \frac{1550}{30+1}, \frac{1280}{30+2}, \frac{990}{30+3}, \frac{1750}{30+5}) = (228\frac{4}{7}, 285\frac{5}{7}, 228\frac{4}{7}, 171\frac{3}{7}, 285\frac{5}{7})$
or equivalently 4, 5, 4, 3, 5 respectively. (KCL, pg. 313)

b: Liu: Multiplying the number of adults from each county by the sum of days on guard-duty and in travel, and dividing it by the adults of each county gives the rate of adults, 5(5/7) days of individual service.
I.e.: The number of adults from County A who have to serve is 1200 × 4 / (4 + 5 + 4 + 3 + 5) = 1200 × (4/21) = 228(4/7). Thus, County A serves at the frontier for 1200 × 30 × (4/21) days, so each adult serves 30 × (4/21) = 5(5/7) days. Analogous for the other Counties.

c: Liu: Round the quotients with fractional parts. Here the fractional part of D is the smallest, so combine it with E, not with B, for D is near E. Combine the surplus (of E) with B. There is a comparatively smaller fractional part in C. Also combine it with B. Combine the surplus (of B) with A, which is now an integer. The fractional parts of A and C are equal, and the two are equidistant from B. Do not combine A with B because one has to combine something lesser with something greater.
N.B.: the rule for rounding is rather complex to assure that a total of 1,200 is reached (what would not be the case when one simply rounds the numbers). Considering the limit space we will not treat it here. See below for the reference. (KCL, pg. 315)

Fair distribution of burdens was considered important by the Confucian educated officials. Around 100 BC the Emperor appointed a Fair Taxes official, who had to determine taxes and corveé labor per province, depending on the number of inhabitants and distance to the Capital. (There was an inverse relation between distance to the capital and taxes.) (LD, pg. 40.)

Chapter VII, *Excess and deficit*, explains how to solve linear problems like

$$ax + b = c$$

where x has to be determined. a, b, c are not known, what is given are two pairs $\{\alpha_i, \gamma_i\}$ such that

$$\begin{aligned} a\alpha_1 + b &= \gamma_1 \\ a\alpha_2 + b &= \gamma_2 \end{aligned}$$

Subtracting the first from the second equation gives

$$a = \frac{\gamma_1 - \gamma_2}{\alpha_1 - \alpha_2} \text{ andthus } c - b = \frac{\alpha_2\gamma_1 - \alpha_1\gamma_2}{\alpha_1 - \alpha_2}$$

In modern mathematics, this method is known as the method of double false position. 9C treats all variations: $\gamma_1\gamma_2 < 0$ (so one excess and one deficit), $\gamma_1\gamma_2 > 0$ (two excesses or two deficits), $\gamma_1 > 0, \gamma_2 = 0$ or vice versa (one excess and one fit) and $\gamma_1 < 0, \gamma_2 = 0$ and, of course, vice versa (one deficit and one fit). (KCL, pg. 349)
Afterwards, applications are given: the place of encounter of two horses traveling at different speed; the number of different items obtained given a certain amount of money; the capital and interest of a businessman withdrawing money.

The ancient Chinese clearly were talented in arithmetic, but their algebraic achievements are astonishing. They could solve systems with Gaussian elimination; both determinate and indeterminate systems of equations were dealt with.[12]

4.3 Algebraic Achievements

Again, the *Nine Chapters* with associated comments are the fundaments of Chinese algebra, so in this paragraph only the last mentioned work is considered.
The procedure of Gaussian elimination is named the *method of Rectangular Arrays* in Chinese and is introduced in chapter VIII of 9C, together with the concept of negative numbers (negative numbers are essential in the Gaussian elimination procedure). Beside systems of linear equations, methods for extracting square and cubic roots are discussed. Lets turn our attention to 9C. Note that what we would write as row, the Chinese wrote as a column, the first equation right, the last left. For the convenience of the present day reader, Arabic numbers are used.

```
(Problem 1 of chapter 8)
Now given 3 bundles of top grade paddy, 2 bundles of medium grade paddy,
(and) 1 bundle of low grade paddy.  Yield:  39 dou[13] of grain.  2 bundles
of top grade paddy, 3 bundles of medium grade paddy, (and) 1 bundle
of low grade paddy, yield 34 dou.  1 Bundle of top grade paddy, 2 bundles
```

[12]It is an even more splendid achievement when compared with the development of mathematics in other regions. The Greek, the Indians and the Arabs discussed systems of linear equations as well. Yet the world besides the Chinese had to wait about 2 000 more years before Gauss published his solution in 1826. (KCL, pg. 388).

[13]A measure for volume. One dou is between 1.98 and 2.10 Litres (KCL, pg. 9).

of medium grade paddy, (and) 3 bundles of low grade paddy, yield 26 dou. Tell: how much paddy does one bundle of each grade yield? Answer: Top grade paddy yields 9(1/4) dou (per bundle); medium grade paddy 4(1/4) dou; (and) low grade paddy 2(3/4) dou.

The Array (Fangcheng)[a] Rule

(Let Problem 1 serve as example,) lay down in the right column 3 bundles of top grade paddy, 2 bundles of medium grade paddy, (and) 1 bundle of low grade paddy. Yield: 39 dou of grain. Similarly for the middle and the left column.
Or in modern algebraic notation, we have:

$$\begin{array}{rcrcrcl} 3x & + & 2y & + & z & = & 39 \\ 2x & + & 3y & + & z & = & 34 \\ x & + & 2y & + & 3z & = & 26 \end{array}$$

What we write as

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 2 \\ 3 & 1 & 1 \\ 26 & 34 & 39 \end{pmatrix}$$

Use (the number of bundles of) top grade paddy in the right column to multiply the middle column then merge.

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 2 \\ 3 & 1 & 1 \\ 26 & 34 & 39 \end{pmatrix} \to \texttt{middle} \times 3 - \texttt{right} \times 2 \to \begin{pmatrix} 1 & 0 & 3 \\ 2 & 5 & 2 \\ 3 & 1 & 1 \\ 26 & 24 & 39 \end{pmatrix}$$

Again multiply the next (and) follow by pivoting.[b]

$$\begin{pmatrix} 1 & 0 & 3 \\ 2 & 5 & 2 \\ 3 & 1 & 1 \\ 26 & 24 & 39 \end{pmatrix} \to \texttt{left} \times 3 - \texttt{right} \to \begin{pmatrix} 1 & 0 & 3 \\ 4 & 5 & 2 \\ 8 & 1 & 1 \\ 39 & 24 & 39 \end{pmatrix}$$

Then use the remainder of the medium grade paddy in the middle column to multiply the left column and pivot.[c]

$$\begin{pmatrix} 1 & 0 & 3 \\ 4 & 5 & 2 \\ 8 & 1 & 1 \\ 39 & 24 & 39 \end{pmatrix} \to \texttt{left} \times 5 - \texttt{middle} \times 4 \to \begin{pmatrix} 0 & 0 & 3 \\ 0 & 5 & 2 \\ 36 & 1 & 1 \\ 99 & 24 & 39 \end{pmatrix}$$

The remainder of the low grade paddy in the left column is the divisor (of the left column), the entry below is the dividend. The quotient is the yield of the low grade paddy.
So we have 99 dou of the low grade paddy.

To solve for the medium grade paddy, use the divisor (of the left column)

to multiply the shi[14] in the middle column then subtract the value of the low grade paddy.
The number of dou of medium grade paddy is $(24 \times 36 - 99)/5 = 153$

To solve for the top grade paddy also take the divisor to multiply the shi of the right column then subtract the values of the low grade paddy and the medium grade paddy.
The number of top grade paddy is $(39 \times 36 - 2 \times 153 - 99)/3 = 333$

Divide by the number of bundles of top grade paddy. This is the number of bundles of tip grade paddy. The constants are divided by the divisors. Each gives the dou of yield (in one bundle).
Thus, we have (99/36) dou of low grade paddy; (153/36) dou of medium grade paddy; (333/36) dou of top grade paddy.

a. Liu: The character cheng means comparing quantities. Given several different kinds of item, display (the number for each as a number in an array with their sums (at the bottom). Consider (the entries in) each column as rates, 2 items corresponds to a quantity twice, 3 items corresponds to a quantity 3 times, so the number of items is equal to the corresponding (number). They are laid out in columns (from right to left), (and) therefore called a rectangular array (fangcheng). Entries in each column are distinct from one another and (these entries) are based on practical examples.
b. Liu: Again eliminate the first entry in the left column.
c. Liu: Again, use the two adjacent columns to eliminate the medium grade paddy.
d. Here Liu suggests making a Reduced Row Echelon Form.

N.B.: For the sake of brevity, some of Liu's other comments are omitted.

Chapter 8 of the *Nine Chapters* gives several other problems regarding systems of linear equations, up to 5×5 systems. Problem 13 of the chapter even considers an indeterminate problem, with 5 equations and 6 unknowns. 9C gave only one particular solution, yet Liu Hui notes correct: "Discuss this (problem) in terms of rates." (KCL, p. 401).

When solving systems of linear equations with the Rectangular Array Rule one requires negative numbers; chapter 8 then also introduces negative numbers in the Sign Rule:

The Sign Rule

Like signs subtract. Opposite signs add. Positive without extra, make negative; negative without extra makes positive. Opposite signs subtract; same signs add; positive without extra, make positive; negative without extra, make negative.
(KCL, pg. 404)

[14] Shi is grain or fruit. The same Chinese word also means: constant or absolute term of an equation; dividend; area; volume.

To be able to distinguish positive and negative numbers with counting rods, red and black rods were used respectively.

The method for finding a square root and solving second order equations is similar and is stated by the Rule for Extracting the Square Root, used in chapter IV, *short width*, for the determination of lengths of a square field which surface area is known. To illustrate the procedure, we will use problem 12 of this chapter, which requires us to find the square root S of 55 225.

1. To estimate the first digit x_1 of the root S, we have to solve $x_1^2 \leq 5 \leq (x_1+1)^2$. In our example, we have $x_1 = 2$.
2. Subtract $100^2 \cdot x_1^2 = 100^2 \cdot 2^2$ from the total, so our remaining term is $55225 - 40000 = 15225$. This corresponds to removing the central square in figure.
3. Replace x_1 by $(x_1 + \tilde{x}_1)$, then we get $100^2(x_1 + \tilde{x}_1)^2 = 100^2(2 + \tilde{x}_1)^2 = 55225$, which can be rewritten to $10000\tilde{x}_1^2 + 40000\tilde{x}_1 = 15225$. Here, the $40000\tilde{x}_1$ corresponds to the grayed rectangles in our figure.
4. To estimate the second digit $\tilde{x}_1$ of the root, substitute $\tilde{x}_1$ by $10x_2$ and solve $4000x_2 \leq 15225 \leq 4000(x_2 + 1)$, giving $x_2 = 3$.

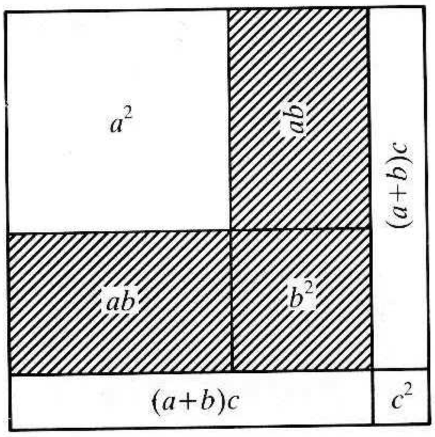


Figure 5: Diagram for the extraction of roots. (LD, pg. 53)

5. Subtract $100x_2^2 + 4000x_2 = 900 + 12000 = 12900$ from the remaining of the square: $15225 - 12900 = 2325$.
6. Replace $x_1 + 0.1x_2$ by $(x_1 + 0.1x_2 + 0.1\tilde{x}_2)$, then we get $100^2(x_1 + 0.1x_2 + 0.1\tilde{x}_2)^2 = 100^2(2.3 + 0.1\tilde{x}_2)^2 = 55225$, which can be rewritten to $100\tilde{x}_2^2 + 4600\tilde{x}_2 = 2325$.
7. To estimate the third digit $\tilde{x}_2$ of the root, substitute $\tilde{x}_2$ by $10x_3$ and solve $460x^3 \leq 2325 \leq 460(x_3 + 1)$, giving $x_3 = 5$.
8. Subtract $x_3^2 + 460x_3 = 2325$ from the remaining square: $2325 - 2325 = 0$.
9. I.e.: $\sqrt{55225} = x_1x_2x_3 = 235$. In case there would still be a remainder, the previous steps can be repeated to obtain an accuracy as high as required.

Since in the algorithm above a part of the square is rewritten as a second order equation, second order equations can be solved with a similar procedure.
Cube roots are solved analogously.
(KCL, pg. 204 – 209 and LD, pg. 53.)

The strength of Chinese mathematics was in arithmetic and algebra, but geometry was not neglected. Although Chinese geometry didn't have the axiomatic foundations the Greek geometry had, the Chinese determined the surface and volume of many geometrical shapes.

4.4 Geometric Achievements

The geometry arose from land mensuration, necessary when fertile land had to be divided, and astronomical observations (KCL, pg. 84). When time progressed, geometry became also used for estimating the volume of earthworks, storehouses, et cetera.

In chapter I of the *Nine Chapters*, named *Field Measurements*, the area for rectangular, triangular, trapezoid, circular, bow-like, segment-of-sphere and annular fields is given. As Liu Hui notes correctly is his comments, the formulas for the bow-like field is wrong and the formula for the segment-of-sphere is only a very crude approximation. The other areas are correct, but usage of the wrong value for π causes some small deviations. The approximation of π is dealt with in the next paragraph. (KCL, pg. 84 – 92 and pg. 117 – 133.)

Chapter IV of the same work, *Construction Consultations*, gives the volumes for city walls and dykes (frustums of prisms), square forts (rectangles), circular forts (cylinders), frustums of pyramids, pyramids, frustums for circular cones, circular cones, prisms, tetrahedrons and several other prism-like volumes. All formulae are correct, if one would use the right value for π.

The volume of a sphere, however, is incorrect: 9C gives $V = \frac{16}{9}R^3$. Liu Hui notes that the formula is wrong, as is the attempt of another mathematician: "The error is very serious. In an attempt to make his (the other mathematicians) statement consistent and harmonize his philosophy of yin and yang, and the doctrine of odd and even, he neglected the precision of the data. The statement runs counter the facts. It is quite fallacious."(KCL, pg. 230.) Honestly, Liu notes that he doesn't know the solution; in his comments on 9C he notes "let us leave the problem to whomever can tell the truth" (KCL, pg. 229).

But, although he couldn't tell the volume of the sphere, it is one of his only shortcomings. When considering the next topic of geometry, Pythagoras' theorem or the Gougu Rule, his contribution to one of the most important application of the Rule, surveying, is enormous.

The Gougu Rule was already used in the *Zhoubi*, but being an important aspect of mathematics, it is also stated in the *Nine Chapters* and various practical applications are given. The *Sea Island Arithmetic Manual*, written by Liu and originally a part of the *Nine Chapters*, discusses surveying in great detail. To illustrate the surveying method, we cite the first problem of the *Sea Island Manual*, including comments.

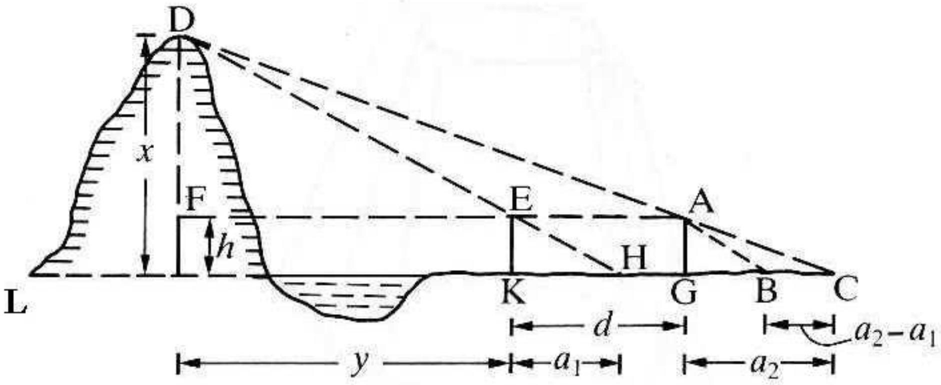


Figure 6: Diagram for problem 1 of Sea Island (LD, pg. 76)

(Problem 1)
Now survey a sea island. Erect two poles of the same height, 3 zhang, so that the front and rear poles are 1000 bu apart. They are aligned with the summit of the island. Move backwards 123 bu from the front pole, sighting at ground level, and find that the summit of the island coincides with the tip of the pole. Move backwards 127 bu from the rear pole, sighting at ground level, and find that the summit of the island also coincides with the tip of the pole. Tell: what are the height of the island and its distance from the (front) pole?
Answer: The height of the island is 4 li 55 bu; it is 102 li 150 bu[15] from the front pole.

Method: Multiply the distance between the poles by the height of a pole as dividend. Take the difference in distance from the points of observation (xiangduo), as divisor and divide. Adding the quotient of the height of a pole, get the height of the island.$^{\alpha}$ To find the distance of the island from the front pole, multiply the distance between the two poles (biaojian) by the distance moved backwards as divisor. Divide to obtain the distance between the island (summit) and the (front) pole in li.$^{\beta b}$
(KCL, pg. 539)

I.e., using the numbers of figure 6, the method gives the following formulas:

$$x = \frac{d \cdot h}{a_2 - a_1} + h \quad y = \frac{d}{a_2 - a_1} \cdot a_1$$

[15]Cun, fen, li, hao, miao and hu are units of length, each of the following units being one-tenth the previous one. In *Sea Island* also the length bu is used, where 1 li = 300 bu. 1 cun ≈ 23.0 – 24.255 cm, its exact length varied slightly throughout history. (KCL, pg. 8.)

As mentioned in paragraph 1, the proof is long lost. But for the convenience of the modern reader we give the proof here. (LD, pg. 77.)
The points H and C are determined by the positions where, as viewed from the ground, the top of the rods and the top of the hill coincide, i.e.: the eye, the top of a rod and the top of the hill are all on the line HD and CD respectively. With the points H and C known, the distances a_1, a_2 can be found. d is also known.

From A, construct AB // DE. Then $\triangle ABC$ is similar to $\triangle DEA$. Consequently,

$$\frac{AE}{CB} = \frac{d}{a_2 - a_1} = \frac{AD}{CA}$$

Note that $\triangle AGC$ is similar to $\triangle DFA$, so we get

$$\frac{AD}{AC} = \frac{DF}{AG} = \frac{DF}{h}$$

Combining the previous two formulae gives

$$\frac{d}{a_2 - a_1} = \frac{DF}{h}$$

Thus,

$$x = DF + h = \frac{d}{a_2 - a_1} \cdot h + h$$

And since $\triangle EKH$ is similar to $\triangle DLH$, we have

$$\frac{x}{y + a_1} = \frac{d \cdot h/(a_2 - a_1) - h}{h} = \frac{y + a_1}{a_1}$$

So

$$y = a_1(\frac{d}{a_2 - a_1} + 1) - a_1 = \frac{d}{a_2 - a_1} \cdot a_1$$

QED

α. *Here the summit of the island refers to the top of a hill. Poles are the tips of vertically standing rods. The line of sight passes through the tip of the pole and the summit of the island. The distance 123 bu from the pole is the length of shadow of the front pole. The line of sight also passes through the tip of the rear pole and the surveyor. The distance is 127 bu from the pole. The difference between the two distances from the poles is called the xiangduo, which is considered as divisor. The distance between the poles, 1000 bu, is called the biaojian. Multiply (it by) the height of a pole for the dividend. Divide and add the height of the pole to obtain the height of the island, 1255 bu. There are 300 bu in a li, so one gets 4 li 55 bu, the height of the island.*

β. *In the Method, it is better to say: multiply the distance between the poles by the distance from the front pole, giving 123000 (square) bu. Take the difference from the points of observation (xiangduo), 4 bu, as divisor. Divide to obtain 30750 bu. Divide again by 300 bu in a li to have 102 li 150 bu, the distance between the island and the (front) pole.*

4.5 The calculation of π

The *Nine Chapters* uses consequently $\pi = 3$, but for the area of a square this gives a value 12 r^2: "... which equals the area of the inscribed dodecagon[16]. It is not precise to take it as the area of the circle." (KCL, pg. 92) as Liu mentions in his comments on The Rule for Circular Fields. To compute π, Liu devises the *method of circle division*. The idea of the method is as follows:

1. Start with an inscribed regular polygon, which of course has a smaller area than the circle.

2. When one knows the side of an inscribed regular $2n$-gon, one can find the area of an inscribed regular $4n$-gon with the formula $A_{4n} = R\frac{2na_{2n}}{2}$.

The formula $A_{4n} = R\frac{2na_{2n}}{2} = 2n \cdot \frac{Ra_{2n}}{2}$ can easily be understood by considering figure The volume of the diamond OACB is given by $\frac{1}{2}Ra_{2n}$, there are $2n$ of such diamonds in the $4n$-gon.

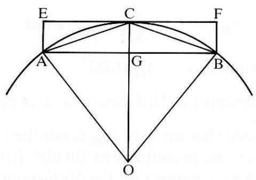


Figure 7: Diamond *method of circle division* (LD, pg. 76)

3. By repeated application of the Gougu Rule, the length of the side of an $4n$-gon can be found from the side of the $2n$-gon. Or in Liu's own words:

   The Rule for Calculating the dodecagon from hexagon: Let the diameter of a circle be 2 chi; and 1 chi the radius (one side of the hexagon) be the hypotenuse, and half the side, 5 cun, be the gou (base) of a right-angled triangle,

[16] A twelve-sided polygon. Here, it is used for the regular twelve-sides polygon.

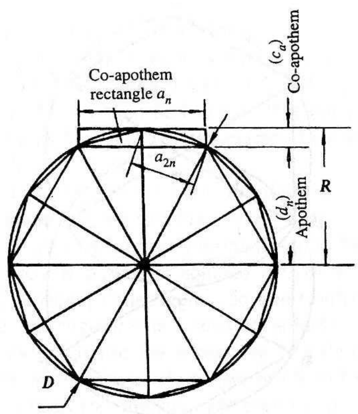


Figure 8: Diagram for the *method of circle division* (KCL, pg. 95)

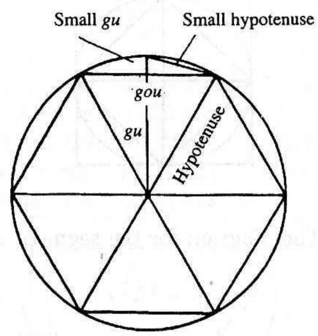


Figure 9: Circle with inscribed n-sided polygon (KCL, pg. 94)

now find the gu (altitude). From the square of the hypotenuse, subtract the square on the gou, or 25 (square) cun, and the remainder is 75 (square) cun. In present-day algebraic and decimal notation: $d_6^2 = R^2 - (\frac{1}{2}a_n)^2 = 1 - 0.25 = 0.75$

Extract the square root up to miao, or hu and one digit down once more to the tiny decimal, which is taken as numerator with 10 as its denominator, i.e. (2/5) hu. Now the root is the gu, 8 cun 6 fen 6 li 2 miao and 5(2/5) hu. In present-day notation: $d_6 = \sqrt{0.75} \approx 0.8860254 = 0.886025\frac{4}{10} = 0.886025\frac{2}{5}$

which is 1 cun 3 fen 3 li 9 hao 7 miao 4(3/5) hu less than the radius, the latter number is called the small base; half a side of a hexagon is called a small altitude.
$c_a = R - d_6 = 1 - 0.886025\frac{2}{5} = 0.133974\frac{3}{5}$

From these, one gets the (small) hypotenuse. The square on it would be 267 949 193 445 (square) hu, where the residue has been neglected
$a_{2n}^2(\frac{1}{2}a_n)^2 + c_n^2 = 0.267949193445$

Extracting the square root, get the side of the inscribed dodecagon. $a_{2n} = \sqrt{0.267949193445} = 0.517638$

The rule for calculating the 24-gon from the dodecagon: as above.
(KCL, pg. 89.)

4. The area of the circle is bounded by $A_{2n} < A < A_{2n} + (A_{2n} - A_n)$ and $\lim_{n\to\infty} A_n = A$, so the more-sided the polygon becomes, the better one approximates the surface of the circle, from which π follows.
(LD, pg. 67.)

Liu calculated subsequently the hexagon, the dodecagon, the 24-gon, the 48-gon, the 96-gon and the 192-gon, finding $\pi = 3.14\frac{64}{625}$. He was aware that this value was only an approximation, not the final answer.

A new height in Chinese mathematics was reached when Zu Chongzhi increased the accuracy in the determination of π significantly. He found that $3.1415926 < \pi < 3.1415927$, which is accurate to 6 decimal places. Most of the work of Zu Chongzhi is lost, so it is unknown how he found this value. If Liu's method of circle division was used, a $6 \cdot 10^{12}$-gon should have been calculated. With only counting rods as aid this would have been a rather tedious calculation, with a significant change on making errors.

Figure 10: *Zu Chongzhi (OR)*

Zu Chongzhi (429 – 501 AD) and Zu Geng (450 – 520)*, father and son, came from a talented family of astronomers; skills were handed down from father to son. In accordance with family tradition, Chongzhi entered into the imperial service and was appointed as officer in a city in Kiangsu, later he was transferred to Nanking, then as military officer. During this time he worked on astronomy and on a new Calendar, the Da Ming calendar (calendar of Great Brightness), which he proposed to the Emperor in 462 AC. He was opposed, however, by Dai Faxing, a minister of the Emperor. The debate between Zu and Dai can be seen as a battle between science and anti-science, between progress and conservatism. To Dai's opinion, that what was passed down from history was the product of Sages and thus should remain unchanged. Astronomy and the making of calendars can't be changed by ordinary humans: he accused Zu of "blaspheming against heaven and working against the Classics." (LD, pg. 81).

But Zu stood ground: "... not from spirits or from ghosts, but from careful observations and accurate mathematical calculations. ... people must be willing to hear and look at proofs in order to understand truth and facts." (O'Connor and Robertson, 2003, hereafter OR.)

Due to the opposition of Dai, it was only after Chongzhi's dead that Geng finally had the Da Ming calendar accepted. Zu Geng's own life was characterized by learning. According to legends, when thinking about problems when he walked, he bumped into other walkers.

* Years of birth and dead according to (OR), which is in slight disagreement with (LD), who give (429 – 500 AD) for Zu Chongzhi (no dates for Zu Geng).

Therefore, it cannot be excluded that Zu determined his approximation of π with another method, it be a method which he had devised himself, it be a method borrowed from someone else, perhaps even from outside China. Influence from the outside is even not that strange a thought. The era in which Liu Hui and both Zu-s lived, the Age of Division, was a tumultuous one. Boundaries changed on regular basis, nomadic clans which had been used as auxiliary armies during the Han took over large parts of the old Empire, inequality between different classes increased. As mentioned in chapter 1, Buddhism had been introduced in China at 65 AD, but only from the dark and chaotic times of the Chinese Middle Ages its influence increased significantly. Questions about death and suffering were discussed in a way unseen in Chinese history; there was a possibility of salvation; its code of conduct forbad the taking of life. In the wake of Buddhism, Indian culture and science followed. In the fifth and the sixth centuries, translated Indian works were read ten times more than the native classics. Indian things became a hype, both in literature, art and philosophy. (Mikami, pg. 57.)

At the rise of the new Empire in 589, Buddhism was firmly entrenched in China. In this cultural environment, Chinese science and mathematics wouldn't remain uninfluenced.

5 Rise of Buddhism

Although it is believed that during the 3rd century BC Buddhism started to influence China, it wasn't actively preached before 65 AD. In this year Ming Ti of the Han dynasty dreamt of a golden man, which was explained to be a vision of Buddha, so he sent messengers to India.
Nevertheless Buddhism does not seem to have influenced Chinese mathematics much the first centuries. Xu Yo devotes a part of his *Memoir on some Traditions of Mathematical Art*, written about 190 AD, to very large, impractical numbers. This is often seen as a Buddhistic influence, since Buddhists are talking about kalpa's: a world will last one kalpa - which was a very long time, namely 4 320 000 years - after that it is destroyed and a new world will be formed.
But as said in the concluding parts of the previous chapter: during the Dark Ages Buddhist, and therefore Indian, influence grew. So one may expect that enriching Chinese culture by Indian, also renown for their mathematical skills, influences resulted in a golden age for mathematics. Alas: mathematics was not given much credit by the Chinese, so the Indian knowledge would be used for Chinas more important science: the one concerning calendars.
Actually, it was not even stagnation: one can even speak a decline, since the work of Zu Chongzhi got lost and the very accurate value for π was traded for a less accurate one. Yet an important mathematical work occurred: the *Ten Mathematical Manuals* came into existence, saving ten books from getting lost. However, only one book was from the Tang-period itself, while the level was not very high in all cases. So not many achievements were made in mathematics.
Nevertheless an interesting mathematical achievement was reached during the Tang: Wang Xiaotong made a study on cubic equations and was able to replace the numerical solution by an algebraic one: a millennium before Cardano (see *Wie hebben de 2e, 3e en 4e orde vergelijkingen opgelost?* by Doetjes, P. and Bosman, J. for a treatment of Cardano and the algebraic solutions of 2nd, 3rd and 4th order polynomials in Europe). Another interesting achievement is the one of the astronomer Liu Zhuo with his *method of second differences interpolation.* After him the last person before the golden age will come to stage: Shen Kuo, who discussed, among many other subjects, plane geometry.
However, another important subject should also be discussed in this chapter: the zero.

5.1 The Ten Books of Mathematical Manuals

As was mentioned in paragraph 1.6, between 628 and 656 mathematics was established at the Imperial Academy. In order to teach the students, textbooks were needed. During the Tang the so-called *Ten Books of Mathematical Manuals* would become the standard mathematical textbook, which thanks to its passing down saved books like the *Nine chapters* from getting lost. In the book of LD, one can find which books made it to this ten (pg. 228-229): the *Nine Chapters*, the *Zhoubi* and the *Sea Island classic* already have been discussed. *Master Sun*, *Xiahou Yang* and *Zhang Qiujian* will be discussed this chapter together with the three books, which are credited to Zhen Luan: the *Five Governments*, the *Five Arithmetic classics* and *Memoir on some traditions of Mathematical art*, after which it is time for Wang Xiaotong and his cubic equations in the *Continuation of Ancient Mathematics.*

5.1.1 Master Sun's Mathematical Manual

What makes master Sun's manual especially interesting for this chapter is the following problem:
"There is a Buddhistic work consisting of 29 stanzas, each of which contains 63 ideographs. It is required to find how many ideographs will be contained in all?"(Mikami, pg. 26)
So it nicely illustrates the integration of Buddhism in Chinese society and science. But also mathematically this book is interesting: it repeats many subjects from the *Nine Sections*, but as Mikami says (pg. 29): "The explanation of the extraction of the square root found in the second book of Sunzi, although little differing from that in the *Nine Sections*, is exceedingly clear in meaning, unlike the obscure language in the latter. The technical terms used here were the same that became current both in China and in Japan in later years, if not in the same meanings as used by Sunzi."
However, Sun's manual is best known for: "..., the Sunzi gives the earliest example of a worked-out problem in indeterminate analysis." (Needham III, pg. 33) Mikami fails note this fact, but unlike his colleague, he delivers to us the whole problem (pg. 31-32):

> *A woman was washing dishes in a river, when an official whose business was overseeing the waters, demanded of her: 'Why are there so many dishes here?' - 'Because a feasting was entertained in the house', the woman replied. Thereupon the official inquired of the number of the guests. 'I don't know', the woman said, 'how many guests had been there; but every two used a dish for rice between them; every three a dish for broth; every four a dish for meat; and there were sixty-five dishes in all.*

To find the answer, Sun gives the following rule:

> *Rule. Arrange the 65 dishes, and multiply by 12, when we get 780. Divide by 13, and thus we obtain the answer.*

This can be verified easily: let N be the number of guests, then
$\frac{N}{2} + \frac{N}{3} + \frac{N}{4} = 65 \Rightarrow 6N + 4N + 3N = 12 \times 65$.
After this example, he goes immediately on with the next:

> *There are certain things whose number is unknown. Repeatedly divided by 3, the remainder is 2; by 5 the remainder is 3; and by 7 the remainder is 2. What will be the number?*

Solving this is the same as solving N in the following indeterminate system:

$$\begin{cases} N = r_1 \operatorname{mod} m_1 \\ N = r_2 \operatorname{mod} m_2 \\ N = r_3 \operatorname{mod} m_3 \end{cases}$$

Master Sun solves it thus:

> *The remainder divided by 3 is 2 and so take 140. The remainder divided by 5 is 3, and so take 63. The remainder divided by 7 is 2, and so take 30. Adding these together we get 233. Therefrom subtract 210, and we obtain the answer.*

After frowning his eyebrows, the reader may want to know the idea behind this. So luckily Sun is willing to explain:

> *In general take 70, when the remainder of the repeated division by 3 is 1; take 21 when the remainder of repeated of the repeated division by 5 is 1; and take 15, when the remainder of the repeated division by 7 is 1. When the sum of these numbers is above 106, subtract 105, before we get the answer.*

So returning to the indeterminate system, the idea is to find for every divisor m_i a number that is a multiple of both other divisors and when taken modulo by m_i gives 1. For example 70 is a multiple of 5 and 7 and gives 1 when taken modulo by 3. Then we multiply this number by the remainder of m_i, that is r_i, and call that S_i. Since the S_i of the other numbers are devisable by m_i, the sum $\sum_{j=1}^{3} S_j$ has the same remainder for m_i as S_i. The last step uses that we can add and subtract the product $\prod_{j=1}^{3} m_j = 3 \times 5 \times 7 = 105$ as much as we want, without affecting the remainders.

5.1.2 Xiahou Yang's mathematical manual

The book of Xiahou Yang is referred to by the next book (Zhang Qiujiang's) and therefore dated before it, meaning it is dated in the fourth century. However, some problems in the book must have been added later, which is noted by LD (pg. 97): "For example, in the book there are problems on the calculation of the two taxes on rice and the two taxes on household wealth, but the two-tax system was only made law at the beginning of the reign of Emperor Dai Zong of Tang (reigned 763-779 AD). (This 'two-tax'-system consisted of two leavies made in summer and autumn. It replaced a system of complex taxes and surcharges.)"
LD suggest that this may be because copyists in the meantime had added problems of their own periods. This theory is supported by Needham (III, pg. 34): "However, the book seems to have been wholly re-written by Han Yen (+780 to +804) in the Tang, and the original may have been better."
Especially, note the last part of the sentence: from the mathematical perspective this book is not very interesting. The book contains calculations of percentages and roots, the ordinary logistic operators and mistakes.
After this sentence, Needham continues and reports something, which is interesting in the light of this chapter: "Van Hée, who devoted a special article to the *Zhang Qiujiang Suanjing*, suspected an Indian origin for the names of certain fractions used therein, such as *chung pan* ('dead on half') for $\frac{1}{2}$, *shao pan* ('lesser half') for $\frac{1}{3}$, *thai pan* ('greater half') for $\frac{2}{3}$, and *jo pan* ('weak half') for $\frac{1}{4}$ ..." But Needham also adds that the arguments of Van Hée are not very convincing, since similar expressions were already common in astronomy, while the names for the thirds were even used in the *Nine Chapters*.

5.1.3 Zhang Qiujiang's mathematical manual

In the preface Zhang (500 AD) says: "In learning arithmetic, we are not troubled with the difficulties in multiplication and division, but we are troubled with the hardships of considering fractions." (Mikami, pg. 39), thus preparing the

reader for a sea full of fractions.
He also treats other subjects, already discussed in the classics, but is still original in the sense that he introduces new problems, which are connected with the weaving of textiles. What makes it so special is that he solved them in a general way, which can be understood by comparing the next problem with the problem of master Sun:
"There is a woman, who weaves 5 feet on the first day, and her weaving diminishes day after day, until she weaves 1 foot on the last day. Supposing she has woven 30 days, it is required to know the total amount of her weaving."
For this problem Zhang gives the following rule: "Add together the amounts woven on the first and last days, and take the half of the sum; then multiply thereto the number of days of weaving, when we get the answer."
This gives us the sum of an arithmetical progression by the product of half the sum of the first and last terms and the number of terms.
Another problem is:
"There is a weaver woman, whose weaving increases day after day. She weaves 5 feet on the first day and 9 $p'i$ and 30 feet in a month. It is required to find the amount of the daily increase of her weaving."
Here the author appends the rule for his solution:
"Arrange the number of feet woven, and divide it by the number of days in a month; double the quotient obtained, and subtract there from twice the number of feet woven on the first day; the result will be taken for the dividend. And divide it by the number of days in a month less one day. Then the quotient is the answer."

5.1.4 Zhen Luan

Zhen Luan Lived at the end of the 6^{th} century and is especially known for attacking the Taoist religion after his conversion to Buddhism. But he used his position also for the more mathematical aspects of live: he wrote many commentaries on ancient works. This way the *Mathematic Manual of the Five Governments* and the *Memoir on some Traditions of Mathematical Art* were preserved thanks to him and made it to the *Ten Manuals*. His other book, which made it to the *Ten Manuals*, is from his own hand and is called: *Arithmetic in the Five Classics.*

The *Five Governments* has probably come into existence during the 4^{th} century and was a manual for government officials. The book only uses the basic operations, but still manages to make mistakes, which makes Needham remark sarcastically, after discussing the works of the 3^{rd} to 6^{th} century (III, pg. 36): "It is interesting in connection with the governmental associations of the *Five Governments* that it was the worst of all."
From Mikami we can get an example (pg. 38):

> *"There is a quadrangular field, of which the eastern side is 35 paces, the western side is 45 paces, the southern side is 25 paces and the northern side is 15 paces. It is required to find the area of the field."*
> *This problem is solved by the following rule:*
> *"Adding the eastern and western side we get 80 paces, and halving it, we get 40 paces. Again add the southern and northern side to get*

40 paces, and halving it we have 20 paces. Multiply the two together, giving 800 (square) paces, ... which is the answer required."

So the *Five Governments* gives the following formula: area $= \frac{E+W}{2}\frac{S+N}{2}$. This, however, is generally not true as the following examples show:

1. Points: $(0,0),(25,0),(25,35),(0,45)$, area is $25 \times \frac{45+35}{2}$.
2. Points: $(0,0),(25,0),(15,45),(0,45)$, area is $45 \times \frac{15+15}{2}$.

The author probably thought to be smart by combining these two for cases when there are no parallel lines assumed.

The *Memoir on some Traditions* originally was not part of the *Ten Manuals*, but replaced the *Method of Interpolation* of Zu Ghongzi during the Song when that book was lost. The book originally belonged to Xu Yo, writing it around 190 AD. LD say (pg. 96): "The *Memoir* uses many Buddhist, Taoist, and mystical phrases in its explanation. This means the book contains a lot of mysticism and occultism. The contents are not easy to understand."
One of the things he discusses is the meaning of symbols for very large numbers, which seems not to have been been consistent during the previous era's. Another interesting subject is the magical square:

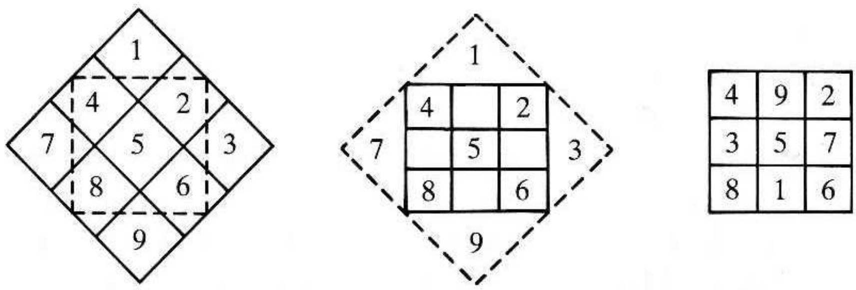


Figure 11: Forming a 'magic square' (LD, pg. 97)

The last book credited to Zhen Luan in the *Ten Manuals*, the *Arithmetic in the Five Classics*, is discussing statements about mathematics in the *Ancient History*, *Classic of Poetry*, *Book of Changes*, *Zhou Rites* and *Record of Rites* and also statements of Han-mathematicians. (LD, pg. 96) It is rather about writing mathematics, than mathematics itself.

5.2 Algebra: Cubic Equations

The last book belonging to the *Ten Manuals* is the one written by Wang Xiaotong: the *Continuation of Ancient Mathematics*, written about 625 AD and therefore the youngest member of the manuals. The book consists of 20 problems, among one very interesting one: in the *Nine Chapters* one can already see how a cube-root is transformed into a third-order equation and solved numerically, Wang, however, is the first one considering explicitly given cubic equations. Solving it, however, is done numerically, using the method of the *Nine Chapters*.

There is a right triangle, the product of whose two sides is $706\frac{1}{50}$*, and whose hypotenuse is greater than the first side by* $30\frac{9}{60}$*. It is required to know the lengths of the three sides."*
Wang gives the answer of this problem to be $14\frac{7}{20}$*,* $49\frac{1}{5}$ *and* $51\frac{1}{4}$*. He then proceeds to give the rule:*
"The product (P) being squared and being divided by twice the surplus (S), make the result the shih *or the constant class. Halve the surplus, and make it the* lien-fa *or the second degree class. And carry out the operation of evolution according to the extraction of a cube root. The result gives the first side. Adding the surplus to it, one gets the hypotenuse. Divide the product with the first side and the quotient is the second side."*

(Mikami, pg. 54)

5.3 Geometry: Shen Kuo

The only mathematician of interest between the *Ten Manuals* and the mathematical Zenith is Shen Kuo, who treated, among many other things, plane geometry.
When one uses different books to find information about Shen Kuo, he will encounter a problem. In Needham III, page 38-39, one can find the following information about him:
(Note: unlike in other quotes, names have not been transliterated by the authors of the article)

Between these men and the great figures of the Sung and Yuan algebraic period (+13th and +14th century) there stand few mathematicians of importance. The most interesting is Shen Kua, whose many sided genius has already been mentioned in the historical Section. His book, the Mêng Chhi Pi Than *(Dream Pool Essays) of +1086, is not a formal mathematical treatise, for it contains notes on almost every science known in his time, but there is much of algebraic and geometric interest to be found in it. In particular, Shen Kua, whose duties as a high official had placed him in charge of considerable engineering and survey works, made progress in plane geometry.*

When one searches Mikami for this Shen Kuo, he won't find him. However page 61 gives:

After I-hsing's (Yi Xing) time during the long reign of the dynasties of T'ang (Tang) and Sung, mathematics was undoubtedly cultivated and made some progress; but we have few particulars of these ages. Ch'ên Huo and Shieh Ch'a-wei are the only two men we can mention as belonging to this period, of whom we have some knowledge.
...
Chên Huo was a native of Ch'ien-t'ang and he was once the president of the Astronomical Board; he then held the ministerial rank. He died

in 1075 at the age of 64. As astronomer he was able.
He wrote the Mong-hsia Pi-t'an *or the "Talks recorded at Mong-hsia", in which he recorded two arithmetical contributions of his own.*

Finally, when one read the book of LD, he will encounter the next text:

> *One mathematician worth mentioning from the Northern Song period was Shen Kuo (1031-1095 AD). Shen Kuo's range of mathematics was very wide. He was well versed in mathematics and the sciences of the heavens. In his famous work* Dream Pool Essays (Meng qi bi tan) *there were several passages concerning mathematical problems. Also among the State Astronomers at that time, besides Shen Kuo, ...*

Taken into account that Needham says that in history Shen Kuo appears under several disguises (Needham III, pg. 38), the authors of this article share the opinion that all three books are discussing the same man. One could argue that the 'Ch'ên Huo' dying in 1075 AD never could be the same man as the author of the *Dream Pool Essays* written in 1086. But 'Ch'ên Huo' died when he was 64, which is the same age at which the 'Shen Kuo' of LD passed. Also all three writers talk about their man as being a statesman and astronomer. Furthermore the fact that no more interesting mathematicians lived during that time, leads us to the given conclusion.

This is historically interesting, but mathematically we are interested in his calculation of the length of a circular arc (Mikami, pg. 62):

> *Arrange the diameter of the circular field and halve it; then the result is to be taken as the hypotenuse. Let the difference, that arises when the radius is diminished by the divided part (or the sagitta), be the first side. Subtract the square of this side from that of the hypotenuse; extract the square-root of the remainder, and take the result as the second side. Twice this side is the chord of the segmental field. Square the divided part (or the sagitta) and lower its digits by one position, and double the result. Again, divide the result by the diameter and add the chord, which give the arc of the divided circular field.*

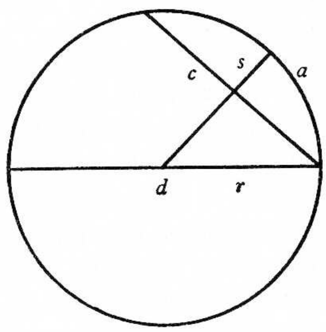


Figure 12: $a = c + 2s^2/d$. (Needham III, pg. 39)

This means: (arc) = (chord) + $2 \times \frac{(\text{sagitta})^2}{(\text{diameter})}$. Which is by the way not entirely true but is still a good approximation.

5.4 Astronomy: Interpolation

It is in Astronomy that the Indian names come to stage. We will continue on this foreign influence in the last section. For now focus will be on Liu Zhuo,

who was the first to use interpolation of the second degree.
To the Chinese it was very important to know the exact location of every star at every time, since they assumed to be able to predict the future by knowing those locations. Problem, however, is that at daylight, the sun is shining far too bright to make star spotting possible. Therefore the Chinese invented an interpolation method, which works quite simple:
Consider the following function: $f : (\text{time}) \to (\text{position}) : t \to 4t + 8$
To determine the position at time $t = 2.5$ one can simply do: $f(2.5) = f(2) + \frac{f(3)-f(2)}{2} = 18$
However this only works if the position is linear dependent on time. Quadratic cases will not give a desirable result:
$f(t) = t^2 \Rightarrow f(2.5) = 6.25 \neq f(2) + \frac{f(3)-f(2)}{2} = 4 + 2.5 = 6.5$
Jia Kui (92 AD) was the first to discover that the sun does not always move with the same speed. This is caused, as we know nowadays, because the earth follows an elliptic orbit around the sun, instead of a circular one. So a better interpolation algorithm was needed.
Suppose we observe the sun at regular time intervals $i, 2i, 3i$, with the results $f(i), f(2i), f(3i)$. Let $d_1 = f(2i) - f(i), d_2 = f(3i) - f(2i)$, etc. Then, for the 'Imperial Standard Calendar' Liu used the following formula:

$$f(i+n) = f(i) + \frac{n(d_1+d_2)}{2} + n(d_1 - d_2) - \frac{n^2}{2}(d_1 - d_2)$$

This formula equals the first two terms of Newton's interpolation formula:

$$f(i+n) = nd_1 + \frac{n(n-1)}{2}(d_2 - d_1)$$

and is therefore called the 'interpolation of second degree'. In the next chapter, we will see that a higher order generalization is found.
A century later the monk Yi Xing used an interpolation method for unequal time intervals and constructed a calendar which would beat all other until that time:

> *After Yi Xing was dead and his calendar system had been ordained for practice, two jealous men made an accusation that the calendar was nothing but a copy of an Indian calendar, and even worse than its original. Hereupon were made observational verifications of various calendars in the Imperial Observatory, but the calendar of Yi Xing was by far number one.*
> *During the Tang dynasty many calendars were made, becoming more and more accurate, "but they did not go as yet very exactly with the actual progress of the heavens. On the contrary the system of Yi Xing answered very minutely, and it was imitated by the calendarmakers for a long time in the succeeding centuries.*

(inspired by Mikami, pg. 60)

5.5 Education

From LD, page 104-106, we can learn about education during this area:
With the rise of the Tang dynasty during the sixth century the country was

unified and the need for uniform rules and regulations rose, which would also affect the educational system. This gave birth to the Imperial Academy, the same as a modern day State University, which was continued throughout the Tang dynasty.
Mathematics, however, which is considered so important today that innocent teenagers need to be traumatised by it, was looked upon very different during the period of Tang: sometimes it was taught, sometimes abolished, sometimes it was considered full, sometimes it was just a part of another science. When it was restored in 658 AD, it was immediately suggested: "Since mathematics ... leads only to trivial matters and everyone specializes in their own way, it distorts the facts and it is therefore decreed that it shall be abolished. " (LD, pg. 105) This was not just propaganda of an anti-mathematical linguïst. For we can learn from the administration of Yuan He that the total number of math students on three different universities was twelve, which is even less than the number of math students in the Netherlands.
About the *Ten Manuals* the New and Old Histories of the Tang:

> *At the beginning of Tang ... in his report to the Emperor, Wang Sibian said: A lot of the arguments in the* Five Governments *and other books of the* Mathematical Manuals *are contradictory. ... and others were ordered to write commentaries .... After the completion of these book Emperor Gaozong of Tang decreed their use throughout the State Institutes.*

(LD, pg. 105)
Thus the *Ten Manuals* became the official textbook, working as follows:

> *30 students are divided into two streams: 15 persons learn the* Nine Chapters, Sea Island, Master Sun, Five Governments, Five Classics, Zhang Qiujian, Xiahou Yang *and* Zhoubi *and 15 others learn the more complicated* Method of Interpolation *and* Continuation. *The allocation of time was* Master Sun *and* Five Governments: *one year;* Nine Chapters *and* Sea Island*: three years;* Zhang Qiujian *and* Xiahou Yang*: one year each;* Zhoubi *and* Five Classics*: one year - a total of seven years. People in the other stream had four years to learn* Method of Interpolation*, three years to learn the* Continuation*; also took seven years to graduation.*

(LD, pg. 106)
Completing such a course would give one rank 8: meaning he was allowed to teach children of very low officials and commoners. To the ones who could not follow the course, there was another exam (giving rank 9): the recitation. "What is called 'recitation' consisted of one or two sentences being picked at random from the text and the student being asked to continue from the given passage." (LD, pg. 106)
Such a method certainly does not sound very attractive to the real mathematical brain: "This sort of examination cannot advance the development of science much." (LD, pg. 106)
Yet many people became willing to improve their position, which resulted in 30,000 candidates during the time of Shen Kuo (of course not only for mathematics) and even tenfold a century later.

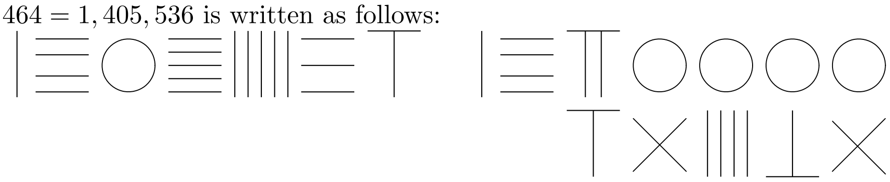
The ones searching for a little more originality could always become a hermit, as is seen in the life of Yi Xing:

> *(Before he was introduced to the emperor) Yi Xing had studied under Phu Chi at Sung shan. there was an entertainment of monks and stramanas from a hundred li around. Among them was a very learned one named Lu Hung. This hermit was asked to write an essay commemorating the meeting, and did so, using very difficult words, and saying that he would take as his pupil anyone who could read and understand it. Yi Xing came forward smiling, looked at it, and laid it down. Lu Hung despised his offhand manner, but when Yi Xing repeated it without one mistake, he was quite overcome and said to Phu Chi, 'This student is not a person you can teach, you had better let him travel'*

(Needham III, pg. 38)
And so he did.

5.6 Contact

The first circular symbol for zero in China is found in the book of Qin Jiushao (1247), who will be discussed in the next chapter. His subtraction $1,470,000 - 64,464 = 1,405,536$ is written as follows:



However there is a reason to believe that the zero was already in use during the Tang. For one of the Indian scholars, Qutan Xita, wrote around 729 AD (Needham III, pg. 12):

> *When one or other of the 9 numbers (is to be used to express a multiple of) 10 (lit. reaches 10), then it is entered in a column in front of (the unit digit). Whenever there is an empty space in a column (i.e. a zero), a dot is always placed (to signify it).*

So Qutan wanted to write a dot instead of an empty space, giving it the same function as the zero-symbol.
This may indicate that the zero in China originates from India.

But about the Indian influence we can conclude with a quote from Needham when talking about another imported astronomer (III, pg. 37): "who was there shortly after +650, occupied with the improvements of the calendar, as were most of his later Indian successors."
So since mathematics was not considered as full and since most Indian immigrants were rather astronomers than real mathematicians, Indians did not really get the chance to significantly influence Chinese mathematics. And unfortunatly, till their culture was overrun, the Chinese themselves didn't make much progress either...

In 1127 AC, although defended by 48000 military, the capital of the Song fell. The one-and-a-half century which layed ahead would be one of political tensions between the Southern Song and the Jin dynasty and of failing military expeditions on both sides trying to (re)gain territory (Ebrey, pg. 150). About hundred years later, however, both were conquered by the Mongols. But in the midst of all the political turmoil, Chinese mathematics would reach its Zenith. On some areas, the Chinese would be almost a millennium ahead of Europe.

6 Zenith

The apogee of Chinese mathematics can be placed in the 13^{th} and the beginning of the 14^{th} century, and would be shaped by four great mathematicians, two on both sides of the Song-Jin demarcation line. In the South, this were Qin Jiushao and Yang Hui, in the North Li Zhi and Zhu Shijie. Of those four, Zhu Shijie "of the Song and Yuan period, together with Qin Jiushao and Li Zhi (also named Li Ye) can be said to form a tripod. Qin contributed the positive and negative and the extraction of roots, Li contributed the celestial element and al those contributions stretch back into the past and will survive to thousands of future generations; Zhu includes everything, he has improved everything to such an extent that it is only understood by the gods and has surpassed the two schools of either Qin and Li" (LD, pg. 117), according to *Sequal to the Biographies* of *Mathematicians and Astronomers*, printed during the Qing Dynasty. Contemporary mathematicians agree, Zhu is one of the greatest mathematicians in Chinese history. As we will see (and might have expected) calendrical and astronomical requirements played a crucial role in defining the braches were Chinese mathematics would shine.

Qin Jiushao (1202(?) − 1261) was a genius, learned in astronomy, architecture, mathematics, poetry, ect. In his introduction to his *Mathematical Treatise in Nine Sections* he writes "In my youth I was living in the captical, so that I was enabled to study in the Board of Astronomy; subsequently I was instructed in mathematics by a recluse scholar." (Libbrecht, pg. 26). He held several minor government posts and military functions and wrote the *Nine Sections* in a "period when soldiers were everywhere and horses were running wild." (LD, pg. 113.) To escape the military problems, he moved to the southeast and would get several influental functions. But despite his cleverness he had a bad character. He used his government positions only to advance his own career and to enrich himself, (tried to) poison those who opposed or might betray him and exploited the population. Consequently, he was impeached for corruption and exploitation of the people, but some time later a friend gave him again a government position. Only after this friend was disgraced, Qin was transfered as a punitive measure to an impopular district. (Libbrecht, pg. 29.)

Qin had a mystical belief in numbers, in his preface of the *Nine Sections* he notes: "As for changes in the affairs of men, there was nothing that could not be accounted for. ... If you look at essential aspects of their meanings, the numbers and the Tao do not derive from two (different) bases."(Libbracht, pg. 57).

Li Zhi (1192 – 1279), who changed his name to Li Ye because he discovered one of the emperors of the Tang dynasty was named Li Zhi, had a completely different character. In Jurchen-occupied territory he took the civil service examinations in 1230 and was after graduation appointed as government clerk. Though this and his following post didn't last long, because the Jurchen were conquered by the Mongols. The next fifteen years he lived in poverty in a mountain region, but it was in this period that he published his exemplary *Sea Mirror of Circle Measurements*. He moved to Mount Fenglong where he became famous. The Mongol emperor at that time gave several audiences to him and Li was given several important posts, but he always resigned soon afterwards. (OR, 2003.) The last years of his live he lived again Mount Fenglong, where he lectured crouds of people. Li's view of mathematics is intriguing. In his preface to the *Sea Mirror of Circle Measurements* he writes: "Numbers are uncountable. I once tried to exhaust them by force but numbers do not obey orders, I could not get to the end and my strength was exhausted. ... Again, in this world there is something that exists beyond existence. Beyond existence are the natural numbers. It is not only the unnatural numbers, it is also the reason for nature. ... One can work with natural reasoning in order to understand the natural numbers, but then the mystery of the universe and of all other objects cannot be matched" (LD, pg. 114 – 115). LD interpret this as that Li points out "that 'numbers' reflects objective existence. In a lot of interesting phenomena their is the 'existence of something beyond'; this is the 'natural numbers' and this is exactly the reflection of 'natural law." (pg. 115). I.e.: Li seems to have had intuitive ideas similar to Natural Laws.

About Yang Hui (ca. 1238 – 1298) we know very little. He lived in Qiantang and he was a skilled writer of practical oriented mathematical books. His work is extremely important, because it collected problems and methods of computation of several mathematical works that are lost themselves, among others the method to compute arbitary degree polynomials. (LD, pg. 115.) He might have been a minor official or a mathematics teacher (OR, 2003).

Unfortunately, about the most brilliant of all, Zhu Shijie (ca. 1260 – 1320), we know almost nothing. All we do know is from the preface of another mathemician to Zhu's *Precious Mirror of the Four Elements*. This preface tells us that Zhu was a gifted and famous mathematics teacher, who travelled across China and, according to the preface, "the number of those who came to be taught by him increased each day." (OR, 2003).

In this section, we will explore the pearls of Chinese mathematics: the *method of the celestial element* and its generalization '*the method of four unknowns*' (a method to solve polynomial equations of arbitrary degree), indeterminate analysis (hence, the Chinese remainder theorem) and higher order equal difference series. Finally, we will consider the development (of lack thereof) in geometry and trigonometry. Before we embark on this exploration, however, we will shortly describe the landmarks of Chinese mathematical literature between the twelfth and fourteenth century.

6.1 Important literature

There is plenty of surviving literature from the twelfth, thirteenth and fourteenth centuries, but unfortunately only small amounts are translated in English. We will mainly focus on the most important works of the four great mathematicians. What is their content and how are they written? When this questions are answered, we will explore the apogee of Chinese mathematics.

6.1.1 Mathematical Treatise in Nine Sections

The first work of the four great mathematicians, *Mathematical Treatise in Nine Sections*, was written by Qin and is published in 1247 AD. It is divided in nine sections and each section has nine problems, but the division is unrelated to the *Nine Chapters*. The content of the *Nine Chapters* and the *Nine Sections*, however, does correspond partially: a significant part of the *Nine Sections* is a reorganization of problems derived from the *Nine Chapters*, applied to different and usually more difficult problems. Some problems though are far beyond the level of the *Nine Chapters* (Libbricht, pg. 64.). The text is written in the usual question-answer-method style, followed by diagrams of the required numerical operations on the counting board.
Here a brief overview of the content per chapter:
Chapter I, *Indeterminate analysis*, describes the solution of simultaneous linear congruences.
Chapter II, *Heavenly phenomena*, deals with the computation of the calendar and the measurement of rainfall, snowfall, etc.
Chapter III, *Boundaries of fields* (thus surveying), is concerned with the measurement of the area of fields.
Chapter IV, *Telemetry* (or measuring at a distance), is about right-angled triangles and double difference problems.
Chapter V, *Corvee labor*: taxation problems.
Chapter VI, *Money and grain*: transportation and storage of food.
Chapter VII, *Fortifications and buildings*: architectural constructions.
Chapter VIII, *Military affairs*: arrangements of tents end the supply of the army.
Chapter IX, *Commercial affairs*: calculations regarding interest and trade.
(LD, pg. 112)

It is interesting to note that the numbers used in the exercises are realistic and represent historically valid information. This further strengthens the image that Chinese mathematics was primary a tool, focussed on solving practical problems.

6.2 Sea Mirror of Circle Measurements

Around the same time, but at the opposite side of what had been the Song-Jin border, Li Zhi wrote the *Sea Mirror of Circle Measurements*, which was published in 1248 AD. The *Sea Mirror* is the first work giving a systematic description of the celestial element method. The central theme of all problems is to find the diameters of the circles inscribed in or cutting the sides of right-handed triangles from the lengths of the sides.

6.2.1 Introduction to mathematical studies

As the title of this work implies, it does not contain difficult or original ideas, it is a basic textbook. It was written by Zhu Shijie and was published in 1299. The work has 3 volumes, 20 chapters and 259 problems, starting with simple methods for multiplication and subsequently working with fractions, ratios, proportions, areas of various geometrical figures, volumes of various geometrical figures, pairs of linear equations, series, the method of double false position, linear simultaneous equations where the coefficients of the unknowns and the constant terms are arranged in the form of a matrix and finally root extraction of numerical equations up to the fifth degree and the method of the celestial element. (Lam Lay-Yong, 1979, pg. 4.) From a pedagogical point of view the structure and content are good: the problems go from easy to difficult; almost all aspects of mathematics known at the time were treated. (LD, pg. 116).

6.2.2 Precious Mirror of the Four Elements

Four years after the publication of *Introduction to mathematics studies*, Zhu published his notable book *Precious Mirror of the Four Elements.* Two main subjects are treated: solutions to systems of quadratic and higher degree equations and finding the sum of series with a finite number of terms. The book contains 36 problems on systems in two unknowns, 13 problems with three unknowns and 7 problems of four unknowns; in total it has three chapters and 288 problems. (LD, pg. 116)

The number of mathematical books which survived till today is greater than in the previous periods, the list above highlights only the most distinguished ones, and of those not even all. Still, a signficant proportion of mathematical works have been lost, we now only know of their existance through bibliographies of the official histories, by references in other writings and by parts that have been copied in other works. In this last class, one collosal achievement deserves special attention. (Needham III. pg. 18)

6.2.3 The Great Encyclopaedia of the Yung-Lo Reign-period

Issued by order of a Ming Emperor in 1403, the immense *Great Encyclopaedia of the Yung-Lo Reign-period* was compiled in the years 1403 – 1408. The Encyclopedia contained 11 095 volumes, divided in 22 211 parts, of which part 16 329 till 16 365 were devoted to mathematics. By now, only 370 volumes remain, among which parts 16 343 and 16 344 of the mathematics section (Libbrecht, pg. 38.).

Now we have a general idea of the principal mathematical works of the period, we will explore the accomplishments on various mathematical fields at the apex of Chinese mathematics.

6.3 Systems of polynomial equations – the techniques of the *celestial element* and the *four unknowns*

As was mentioned in chapter 4 the classic *the Nine Chapters* already describes how to solve square and cube equations. During the Song time this method was

generalized to polynomials of arbitrary degree, as we will see below.
But such methods are only of practical use if one can extract a polynomial equation from the given conditions of a problem, and this is exactly what one of the most fascinating and majestic techniques, the *technique of the celestial element*, did. A celestial element is an unknown, an variable. Where we would say: 'let x be such and such', the Chinese said 'establish the celestial element as such and such' (LD, pg. 135).
The *technique of the celestial element* was, among others, treated in Li Zhi's work *Sea Mirror of Circle Measurements*, from which we will quote an example below. But in this book, Li quoted two other books of as the source of his technique. Another mathematician, who commented on this work, names several more earlier treatises of the method. Based on these references, LD believes that the *technique of the celestial element* originates in "*the beginning of the* 13^{th} *century AD or even a little earlier*" (LD, pg. 139), Needham suggests even the 12^{th} century (Needham III, pg. 41).

To illustrate how this technique was used, we consider problem 2 of the *Sea Mirror of Circle Measurements*:

`(Assume their is a circular fort of unknown diameter and circumference), person A walks out of the south gate 135 steps and person B walks out of the east gate 16 steps and then they see one another. (What is the diameter?)` (LD, pg. 136.)

As is mentioned above, the Chinese worked with counting bords and words to describe operations. But for the sake of clearity for the modern reader, we will use instead present day algebraic descriptions. Li Zhi gave five solutions to the problem, we will give the second, as explained by LD, pg. 136 – 138.

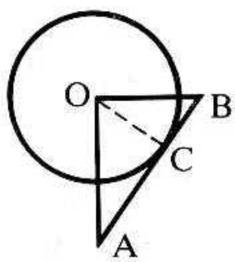


Figure 13: Diagram for problem 2 of the *Sea Mirror of Circle Measurements* (LD, pg. 136)

Let x be the radius of the circular fort then side $OA = x + 135$, side $OB = x + 16$.
Finding the diameter of a circular fort. $\mathrm{OA} \times \mathrm{OB} = (x + 135)(x + 16) = x^2 + 151x + 2160$,
Divide by x getting the hypotenuse[17] $= x + 151 + 2160x^{-1}$
(Note that the equality $\mathrm{OA} \times \mathrm{OB} = \mathrm{AB} \times \mathrm{OC}$ is used. This is valid since both sides of the equality give the surface area of the triangle.)

Multiply this by itself, getting

$(hyp)^2 = x^2 + 302x + 27121 + 652320x^{-1} + 4665600x^{-2}$

[17] The long side of a triangle

Again

$$OB^2 = (x+16)^2 = x^2 + 32x + 256$$
$$OA^2 = (x+135)^2 = x^2 + 270x + 18225$$

$$OB^2 + OA^2 = 2x^2 + 302x + 18481 = (\text{hyp})^2$$

Equate to equation 6.3 and simplify getting $-x^2 + 8640 + 652320x^{-1} + 4665600x^{-2} = 0$, rationalizing the equations we get

$$-x^4 + 8640x^2 + 652320x + 4665600 = 0. \tag{1}$$

Solving, we get $x = 120$ (steps) as the radius of the circular fort.

Equation 1 is a polynomial which, as mentioned above and explained below, could be solved by the Chinese.

The *technique of the celestial element* was a magnificant result, though in the thirteenth century it would even develop further: Zhu Shijie generalized the technique to the *'technique of the four unknowns'*, which could solve systems of polynomial equations up to four variables! Next to the celestial unknown, we would get the earth, human and thing unknown.

Though the evolution from one to four variables was not a sudden jump but rather a gradual development. In a comment on Zhu's work, we can read "Li Dezai of Pingyang region, having published *The Complete Collection for Heroes on Two Principles*, they then had the earth unknown. The critic Liu Runfu ... published *Heaven and Earth in a Bag*, which included two problems involving the human unknown. My friend, Master Zhu Hangqing (Zhu Shijie) of Yanshan district explained mathematics for many years. Having explored the mysteries of the three unknowns and sought out the hidden detials of the *Nine Chapters*, he set up the four unknowns according to heaven, earth, man, and matter". (LD, pg. 141)

The *'technique of four unknowns'* is essentially the same as the modern method of elimination and substitution. One of the equations is (mentally) rewritten and expressed in the other variables. In (one of the) other equations, this variable is substituted by its expression in other variables. This proces continues till we are left with two equations in two variables, of the following form:

$$P_1(z)x + P_0(z) = 0$$
$$Q_1(z)x + Q_0(z) = 0$$

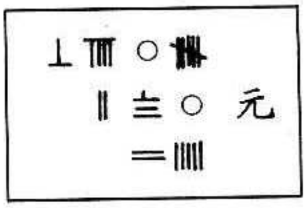
where $P_1(z), P_0(z), Q_1(z)$ and $Q_0(z)$ are polynomials of z. Next, by multiplying the upper equation with $Q_1(z)$, the lower equation with $P_1(z)$ and subtracting them we get:

$$P_0(z)Q_1(z) - P_1(z)Q_0(z) = 0 \tag{2}$$

which is a polynomial in z and as mentioned earlier, can be solved.

If it is not possible to write the polynomial directly in the form of equation 6.3 because P and Q depend on more than one variable, some more elaborate computations were used to derive the answer, but in the end the expression which was solved was always of the form of equation 6.3. (LD, pg. 148)

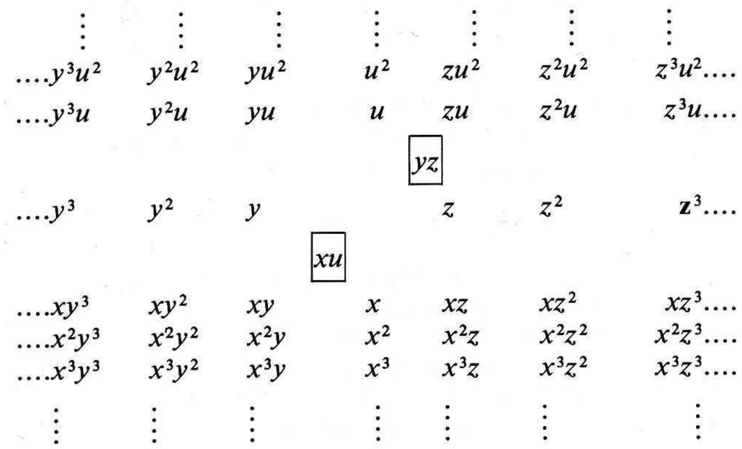
But one may ask oneselves, why didn't they go beyond 4 unknowns? The answer lies hidden in the way the Chinese calculated. Not with paper and pencil, but with counting bords (as was mentioned in chapter 3). On a counting bord, a single-variable polynomial would be denoted as a vector: the constant term first and the x^{th}-order term x places below. For example, $25x^2 + 280x - 6905 = 0$ can be recorded as[18](LD, pg. 140):



where the character 元 marks the first order term. The way polynomials of several variables are recorded is "to have the source of all unknowns (the constant term) in the centre and to put the celestial (unknown) element at the bottom, the earth unknown on the left, the human unknown on the right and the material unknown at the top" (in a preface to the *Precious Mirror*, LD pg. 141). For systems of equations, each equation had its own counting bord. I.e., let x, y, z, u be the four variables, then the single equation $x + y + z + u = 0$ is recorded as (LD, pg. 142):

n^{th}-orders of x are put n places below the constant term, n^{th}-orders of y are put n places left from the constant term, etc. The coefficients of adjacent crossterms, such as xy^3, are recoreded at the corresponding intersection lines. The coefficients of opposite crossterms, like x^2u^3 are placed in the holes of the grid. What happens with crossterms of more than two variables, for example xyz^2, is not mentioned in the literature we consulted. So the general lay-out of a counting bord is as follows (using Western algebraic notation, LD, pg. 142)

[18]The stripe through the 'five' of 6905 means that we have -6905 instead of +6905.



When you take a look at the diagram it is evident why the Chinese stopped at four variables: there just wasn't enough space at the counting bord.

Nevertheless, the *technique of the celestial element* and the *'technique of the four unknowns'* are majestic achievements and proof of the lofty level of Chinese mathematics during the Song and Yuan dynasties.
With these techniques a systems of multivariable polynomials can be rewritten to several one-variable polynomials, which will be solved in the next paragraph.

6.4 n^{th} degree root extraction and n^{th} degree polynomials

As early as in the *Nine Chapters*, square and cube roots are solved. The method was described in paragraph 4.3, and was used as well for solving second and third order polynomials. Great advancement was made in the Song and Yuan period: the technique was generalized for roots and polynomials of arbitrary degree, by means of iterated multiplication. The earliest book still around which mentions it, is the *Reclassicification of the Mathematical Method in the 'Nine Chapters'* of Yang Hui, although the method itself is credited to the astronomer Jia Xian, who lived in the middle of the 11^{th} century AD. In the West, the generalized method is known as Horner's method, which he published in 1819, almost 800 years afer the Chinese (LD, pg. 111).

As illustration of the procedure, we will use problem 3.1 of Qin's *Mathematical Studies in Nine Sections*, were we are asked to solve the polynomial

$$f(x) = -x^4 + 763200 \cdot x^2 - 40642560000 = 0 \qquad (3)$$

With counting rods, the equation was arranged as follows:

- 40 642 560 000	absolute term
0	1st degree
763 200	2nd degree
0	3rd degree
-1	4th degree

First, estimate the number of digits in the root (without decimals, of course) and the first digit of the root. In our example, the root has three digits. The first digit of the root can be found by solving $x_1^4 \leq 40642560000/10^4 \leq (x_1+1)^4$, so $x_1 = 8$.

The, equation 3 can be rewritten as

$$f(x) = (x-800)(-x^3-800\cdot x^2+123200\cdot x+98560000)+38205440000 = 0 \quad (4)$$

To perform the same equations on the counting bord, multiply the 4th degree (-1) number with 800 and add the result algebraically to the 3rd degree class. Subsequently, multiply the result by 800 and add to the 2nd degree, multiply this result by 800 and add to the 1st degree and multiply this result by 800 and add by the absolute term. This gives:

	- 40 642 560 000	
(98 560 000 · 800)	+ 78 848 000 000	absolute term
	= 38 205 440 000	
	0	
(123 200 · 800)	+ 98 560 000	1st degree
	= 98 560 000	
	763 200	
(-800 · 800)	- 640 000	2nd degree
	= 123 200	
	0	3rd degree
	- 800	
	= - 800	
	-1	4th degree

Next, define $x = y + 800$ and rewrite f to

$$f(x) = y\left[y\left(-x^2 - 1600\cdot x - 1156800\right) - 826880000\right] + 38205440000 = 0 \quad (5)$$

To perform this equations on the counting bord, subsequently multiply the 4th degree number with 800 and add the product to the 3rd degree, multiply the result by 800 and add the product to the 2nd degree, multiply this result by 800 and add the product to the 1st degree, giving:

	38 205 440 000	absolute term
	123 200	
(-1 1 56 800 · 800)	-925 440 000	1^{st} degree
	= 826 880 000	
	123 200	
(-1 600 · 800)	- 1 280 000	2^{nd} degree
	= -1 156 800	
	-800	
	- 800	3^{rd} degree
	= - 1 600	
	-1	4^{th} degree

Repeat this procedure to get

$$f(x) = y\,[y\,(y\,(-x - 2400) - 3076800) - 826880000] + 38205440000 = 0 \quad (6)$$

On the counting bord, subsequently multiply the 4^{th} degree number with 800 and add the product to the 3^{rd} degree, multiply the result by 800 and add the product to the 2^{nd} degree.

	38 205 440 000	absolute term
	826 880 000	1^{st} degree
	-1 156 800	
(-2 400 · 800)	- 1 920 000	2^{nd} degree
	= -3 076 800	
	-1 600	
	- 800	3^{rd} degree
	= -2 400	
	-1	4^{th} degree

Finally, rewrite f to

$$y\,[y\,(y\,(y(-1) - 3200) - 3076800) - 826880000] + 38205440000 = 0 \quad (7)$$

What corresponds on the counting bord with the multiplication of the 4^{th} degree number by 800 and adding this product to the 3^{rd} degree.

38 205 440 000	absolute term
826 880 000	1^{st} degree
-3 076 800	2^{nd} degree
-2 4000	
- 800	3^{rd} degree
= -3 200	
-1	4^{th} degree

The resulting equation can be written as

$$f(x) = -y^4 - 3200 \cdot y^3 - 3076800 \cdot y^2 - 826880000 \cdot y + 38205440000 = 0 \quad (8)$$

This equation is again a polynomial and we can repeat the procedure above to find the second digit of the root. Let $x_2 = y$, then solve $x_2^4 \leq 38205440000/100^4 \leq (x_2+1)^4$, then $x_2 = 4$.

Omitting the counting bord operations for briefness, we rewrite equation 8 to

$$f(x) = (y-40)(-y^3 - 3240 \cdot y^2 - 3206400 \cdot y - 955136000) = 0 \quad (9)$$

Define $z = y - 40$, then using the same procedure as above, we find $z = 0$. In the case under discussion, we now have the exact answer: $x = 840$. If this would not be the case successive repetitions of the procedure will give more and more digits.
(Libbrecht, pg. 181 - 189; Mikami, pg. 75)

The same method was used to evaluate square roots of arbitrary degree. Take for example a problem of Yang Hui's *A Detailed Analysis of the Methods of Computation*, which has survived till the present because (parts of) it were contained in part 16 344 of the Great Encyclopaedia. He asks to compute the fourth root of 1 336 336. In modern algebraic notation, we have $x = \sqrt[4]{1336336}$, which is equivalent with $x^4 = 1336336$ and this polynomial can be solved with Horner's method (LD, pg. 126).

When solving higher degree polynomials or roots, one needs the Binomial Theorem, which in modern algebraic notation is written as $(x+a)^n = x^n + C_{1,n}x^{n-1}a + C_{2,n}x^{n-2}a^2 + ... + C_{n,n}a^n$, with for a certain n, the coefficients $C_{i,n}$ are a horizontal line of numbers in "Pascal's" Triangle. Yes, **"Pascal's"** Triangle, because Pascal actually wasn't the first: Jia Xiang had devised the triange as early as the 11^{th} century. Unfortunately, his work is lost. But in a book of Yang Hui there is a diagram entitled *the source of the method of extracting roots*, this diagram has been copied to part 16 344 of the *Great Encyclopedia.* (Needham III, pg. 133; LD, pg. 121.)

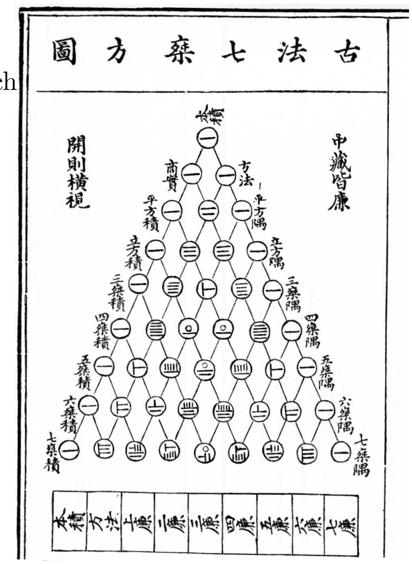


Figure 14: The Chinese 'Pascal triangle'. (Needham III, pg. 135)

6.5 Indeterminate analysis - the Chinese remainder theorem

In the previous chapter we discussed the manual of Master Sun. In it an interesting problem of indeterminate analysis occures. That is: find an N such that $N \bmod 3 = 2, N \bmod 5 = 3, N \bmod 7 = 2$. Solving this was done by finding for each divider a multiple of all the other dividers, which would yield 1 when taken modulo by that number. For example for 3 we find 70, which is a multiple of both 5 and 7, while $70 \bmod 3 = 1$. Then mulyiply this number by the remainder of the corresponding divider, in our example: $2 \times 70 = 140$, finally sum all these numbers and while the result is greater than the product of the dividers, which is $3 \times 5 \times 7 = 105$, subtract 105, until you get the answer. The reader is directed to sub-paragraph 5.1.1 or to his own creativity to check this is true.

Our interest, however, is in an important step: all steps are just straight forward, except the first one. For one may wonder how to find a number, which is modulo one of the dividers equal to 1, while modulo any other divider results in 0.

In the problem of Sun it is just 'trial and error', because numbers are not too large. But for making the calendar and determining the year, the monk Yi Xing solved the following equation: $1110343y = 44820 \bmod (60 \times 3040) = 49107 \pmod{89773}$, where y is the number of years past since the Grand Cycle (remember the Buddhistic *kalpa's*). (Libbrecht, pg. 281) In this case you rather not do 'trial and error'. So one may wonder whether there is an easy way to find a number, giving 1 when taken modulo a certain divider, while being the multiple of all other dividers.

In the *Nine Section* an easy way is revealed to find that number. To understand how it works, we first need some basic knowledge about the *greatest common divisor*.

To find a gcd, one can use a very easy to use method: $\gcd(a, b) = \gcd(b, a \bmod b)$. Thus we can construct an array $a, b, a \bmod b, b \bmod (a \bmod b), \ldots$ or in numbers: $27, 21, 6, 3$ meaning $\gcd(27, 21) = \gcd(21, 6) = \gcd(6, 3) = 3$.

Now comes the important step: keep writing the tail in terms of the two heads: $27, 21, 27 - 21, 21 - 3(27 - 21) = 4 \times 21 - 3 \times 27,$.

More general: $\forall a, b \exists m, n : ma + nb = \gcd(a, b)$, where a, b, m, n are all integers. Returning to the indeterminate problem and assume that the gcd of the dividers is equal to 1 (if not, then simplify dividing every divisor by the gcd), then we have to find an m_i such that $m_i \frac{D}{d_i} + n_i d_i = 1$, where the divisors d_i are known, just like their product D, while m_i, n_i are to be found.

Qin goes on like this:

1	G
0	m_i

(LD, pg. 163)

Here $G = \frac{M}{m_i} \bmod m_i$. The meaning of the numbers in the left column is to keep track of the number of G's in the right column. Important to note: the upper left keeps track of the *positive* amount of G's in the upper right, so would become negative, if the amount of G's in the upper right would become negative, while the lower left keeps track of the *negative* amount in the lower right. Then the work begins: divide the lower right by the upper rigth, multiply the integer value (meaning th k such that $m_i = kG + G \bmod m_i$) with the upper left and add it to the lower left. This gives:

$$\boxed{\begin{array}{cc} 1 & G \\ k_1 & m_i - k_1 G \end{array}}$$

Now do the same for the upper part: divide the upper right by the lower rigth, multiply the integer value with the lower left and add it to the upper left. Giving:

$$\boxed{\begin{array}{cc} 1 + k_1 k_2 & (k_2 k_1 + 1)G - k_2 m_i \\ k_1 & m_i - k_1 G \end{array}}$$

Now repeat the whole proces until we can make the upper right equal to 1. Then the upper left gives the result.

For example: the divisor is 39 527 and the product of the other divisors is 18 446. By what number should the product be multiplied to yield 1 when taken modulo by the divisor?

$$\boxed{\begin{array}{ll} 1 & 18446 \\ 0 & 39527 \end{array}} \Rightarrow \boxed{\begin{array}{lc} 1 & 18446 \\ 2 & 39527 - 2 \times 18446 = 2635 \end{array}}$$

$$\Rightarrow \boxed{\begin{array}{cc} 1 + 7 \times 2 = 15 & 18446 - 7 \times (39527 - 2 \times 18446) = 15 \times 18446 - 7 \times 39527 = 1 \\ 2 & 2635 \end{array}}$$

One can see that this works much nicer than ‘trial and error’.

As already touched on lightly, the Chinese remainder theorem has important astronomical applications. The sun, moon, planets and stars all have different orbital periods. Given the orbital period and the configuration of some celestial bodies at a certain time, the Chinese remainder theorem can be used to calculate at which part of their joint cycle you are. For example, suppose we have the fictitious planets α, β and γ with orbital periods $p_\alpha = 3, p_\beta = 4, p_\gamma = 5$ and when you make an observation, you find that they are at $\frac{3}{3}$, $\frac{2}{4}$ and $\frac{1}{5}$ of their orbits respectively. I.e., let x be the joint period, then we have

$$\begin{aligned} x &\equiv 0 \bmod 3 \\ x &\equiv 1 \bmod 4 \\ x &\equiv 2 \bmod 5 \end{aligned}$$

Application of the Chinese remainder theorem gives that the joint period = 60 days and that the planets are at the 6^{th} day of their mutual cycle.
But this theorem wasn’t the only astronomy-induced result. Inquiry into higher order equal difference series can also be attributed to the study of the stars.

6.6 Higher order equal difference series

A n^{th}-order equal difference series is a series where the differences between the subsequent terms don’t have to be the same, but the differences of the differences (of the differences, etc. n times) are. Take for example the series n^2, its first terms are

1		4		9		16		25	values
	3		5		7		9		1^{st} order difference Δ
		2		2		2			2^{nd} order difference Δ^2

has equal second differences, so is a second order equal difference series. If we look at the series $n^3 + n^2$ we have for the first terms:

2		12		36		80		150	values
	10		24		44		70		1^{st} order difference Δ
		14		20		26			2^{nd} order difference Δ^2
			6		6				3^{rd} order difference Δ^3

has equal third differences, so is a third order equal difference series. What you might have noted here is true in general: an n^{th} degree polynomial has an n^{th} order equal difference series.
Equal difference series can be used for interpolation, what was used to predict the positions of the sun, moon, and the five known planets for calendrical purposes. In fact, we already came across the use of a second order equal difference for interpolation by Yi Xing in paragraph 5.4. Considering the astronomical applications, research in these series was important; it was during the Song-Yuan era that significant improvements were made.
Though to understand these improvements, we first have to know something about the research into series.

The *Zhoubi* and the *Nine Chapters* already contain problems regarding series, but no general method is employed, so the mathematical value is limited (Libbrecht, pg. 172). The first real research into series was done by Shen Kuo, in his work *Dream Pool Essays* he mentions the technique of small increments, i.e. the problem of making stacks (for example stacks of wine vessels). Suppose one has a stack of wine vessels of n layers with on the top layer $a \times b$ vessels and on the bottom layer $c \times d$ vessels, how many objects are there in the pile? One gets the series (using modern algebraic notation)

$$S = ab + (a+1)(b+1) + (a+2)(b+2) + ... + cd \qquad (10)$$

As answer, Shen Kuo gives the following formula: $\frac{n}{6}((2b+d)a + (2d+b)c) + \frac{n}{6}(c-a)$. Some time later Yang Hui would discuss this kind of problems in slightly more detail. (LD, pg. 149 – 150)
Real progress is made by Zhu Shijie. In the *Precious mirror of the four elements* his considered the following type of series: A 'pile of reeds':

$$1 + 2 + 3 + ... + n = \sum_{i=1}^{n} i = \frac{1}{2!} n\,(n+1) \qquad (11)$$

A 'Triangular pile':

$$1 + 3 + 6 + 10 + ... + \frac{1}{2} n\,(n+1) = \sum_{i=1}^{n} \frac{1}{2} i(i+1)$$

$$= \frac{1}{3!} n\,(n+1)\,(n+2) \tag{12}$$

A 'pile of scattered stars:

$$1 + 4 + 10 + 20 + ... + \frac{1}{2} n\,(n+1)\,(n+2) = \sum_{i=1}^{n} \frac{1}{2} i(i+1)(i+2)$$

$$= \frac{1}{4!} n\,(n+1)\,(n+2)\,(n+3)$$

Etc., so in general we get:

$$\sum_{i=1}^{n} \frac{1}{p!} i(i+1)(i+12)...(i+p-1) = \frac{1}{(p+1)!} n(n+1)n+2...(n+p) \tag{13}$$

These sequences could be used for the interpolation of the position of celestial bodies. Suppose you know the angular position of the sun f at $f(i), f(i+1), f(i+2), ...$ and you want to know $f(i+n)$, with $(i+n)$ in between two known points. Then, one can use the first-, second-, third-, ... order differences to interpolate $f(i+n)$. If the differences become very small at the p-$^{\text{th}}$ order (or you assume that the angular motion of the sun can be described with a p$^{\text{th}}$ degree polynomial, the Chinese used a second and later a third order polynomial[19]) one can interpolate $f(i+n)$ in the following way

$$f(i+n) = f(i) + n\Delta + \frac{1}{2!} n(n-1)\Delta^2 + \frac{1}{3!} n(n-1)(n-2)\Delta^3 + \frac{1}{4!} n(n-1)(n-2)(n-3)\Delta^4 +$$

$$... + \frac{1}{p!} n(n-1)(n-2)...(n-(p+1))\Delta^p$$

with $\Delta = f(i+1) - f(i)$ = first order difference, $\Delta^2 = [f(i+2) - f(i+1)] - [f(i+1) - f(i)] = f(i+2) - 2f(i+1) + f(i)$ = second order difference, etc. Zhu gave this formula for a fourth order equal difference series and noted that the terms of this formula were just equal to the pile series. Therefore, it is quite plausible that he could have generalized the formula to arbitrary order equal difference series. In Europe, an approximation formula like this would be used for the first time about 400 years later. (LD, pg. 156 – 161.)

6.7 Geometric and trigonometric achievements

One of the only fields of mathematics where no significant progress was made was geometry. In for example the *Nine Sections*, the same erroneous formulas for bow-like and segment-of-sphere fields are used (Lam Lay-Yong, 1979, pg. 17). The determination of π by Zu Chonzhi in the fifth century AD, up to six decimal places, seems to be forgotten; in the Nine Sections, π is given as either 3, (22/7) or $\sqrt{10}$ (Libbrecht, pg. 97).

[19]Because the motion of the earth around the sun is chaotic, a third degree polynomial will of course not suffice for longer periods of time. Although the Chinese knew that their calendars had to be updated once and a while, they had no idea of chaotic systems. For an excellent treatment on this modern discipline we refer to *In de vlindervlucht door de chaos* by Hu, R.M. and Niezink, N. elsewhere in this bundle.

As was mentioned previously, the Chinese didn't develop trigonometry. Under the Mongol regime and the increasing influence of the Arabic astronomers, however, the Chinese became anxious to improve their calendrical and astronomical calculations (Needham III, pg. 109). For this trigonometry formed a powerfull tool. Therefore we find that during the Yuan dynasty, under influence of Arab science, some proto-trigonometry developed. In particular, the following three trigonometric formulas were found:

$$\begin{aligned} \sin\alpha &= \sin c \sin\alpha \\ \cos b &= \frac{\cos c}{\sqrt{\sin^2 c \cos^2 \alpha + \cos^2 c}} \\ \sin b &= \frac{\sin b \cos\alpha}{\sqrt{\sin^2 c \cos^2 \alpha + \cos^2 c}} \end{aligned}$$

although one should note that the sine and the cosine weren't known. Instead, the lengths of the corresponding sides were used (i.e. sin = opposite side / hypothenuse, etc.).

These formulas are in particular important to find the degrees of latitude and longitude (arc CE and BD in figure 15 respectively) when you know the position of the sun in terms of its angular position in the ecliptic (arc BD).
(LD, pg. 169 – 171).

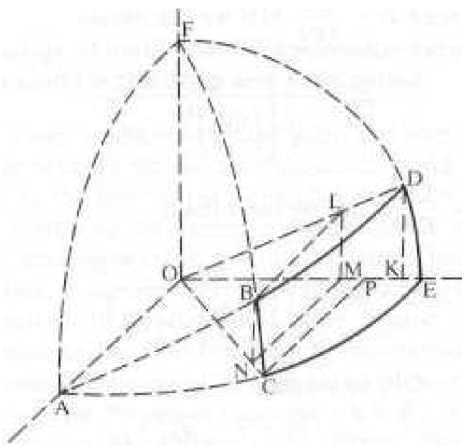


Figure 15: Diagram of relation between latitude, longitude and angular position of sun (LD, pg. 169)

Except geometry, the mathematical accomplishments described above clearly show the advanced level of Chinese mathematics in the 13^{th} and the beginning of the 14^{th} century. Unfortunately, however, Chinese mathematics would not maintain its grandiose growth, or would even be able to hold to its level. The fall of the Yuan marked its stagnation, soon to be followed by decline.

7 Decline

Like many never want to hear anything about mathematics again as soon as their examination results allow them to drop their scholarly burdens, likewise the liberation of the Mongolian yoke may have been very good to the Chinese, but not to their mathematical skills. It became even so terrible that when two centuries later the calendar had become severely inaccurate, China's most important science was unable to take action without foreign help.
Luckily, at the same time the Society of Jesus came into existence: founded in 1540 AD this Catholic army had the mission to spread Christianity all over the world. And although it is wrong to argue that this was the first time of western contact - think for example of Marco Polo - it was not until now that time that the Western civilization was getting a firm influence on China's. Especially Mattheo Ricci is to thank for it: he helped translating some important western works.
Mathematical interest increased among the Chinese and at the beginning of the 18^{th} century the *Collected Basic Principles of Mathematics* came into existence thanks to Emperor Kang Xi. However during that time Christianization had begun and the Catholic Church began to worry about the tradition of honouring ancestors and eventually in 1704 AD she forbad this practice. Kang Xi did not like that very much, so the Papal embassy was ordered to leave Chinese territory for Macao. The Catholic Church on her turn did not like this decision, so her missionaries were to help a pro-Catholic prince to succeed to the throne. But now the pro-Buddhist successor of Kang Xi was not pleased, so missionaries were forbidden to travel freely through the country, while only a few were even allowed to continue their work in the State Observatory. The closed door policy had begun and would last until the Opium Wars of 1840.

7.1 Algebra and Arithmetic

The first important arithmetic work is also the most important pre-Jesuit Ming work: the *Systematic Treatise on Arithmetic* from Chen Dawei, discussing the use of the abacus. Another milestone is the importation of the logarithm in 1653 by the missionary Nicolas Smogulęcki, being assisted by Xue Fengzuo. But it had little influence and was not used before more detailed discussion appeared in the *Collected Basic Principles of Mathematics* of 1723, which is discussed later. Therefore no further attention will be given to logarithms, this will be delayed to the next chapter. Here we'll start with the study of equations, to which Jen Luan devoted much of time.

7.1.1 Abacus

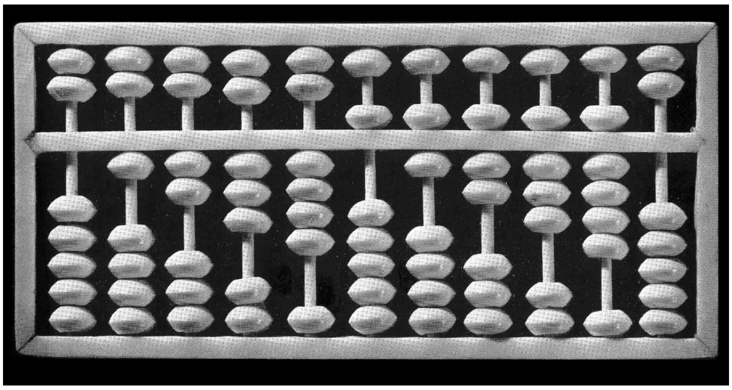


Figure 16: A Chinese abacus. (Needham III, pg. 74)

In 1592 AD Cheng Dawei completed his *Systematic Treatise on Arithmetic* in which extensive use is made of the abacus to solve problems. What makes it so special is told by Mikami (p. 110): "..., for it is the oldest work now extant that contains a diagram of the abacus ..."

Now it has to be said that historians suspect that the abacus was already used during the dark ages, since the *Memoir*, discussed two chapters ago, speaks about 'ball-arithmetic'. However, this is the first solid evidence of the use of the abacus.

The content is explained in the preface (LD, pg. 186): "I started to learn mathematics when I was young. About the age of 20 I was trading and travelling around the Wu and Chu districts and sought out expert teachers. Then I retired and studied very hard near the source of the river Shuai for more than 20 years in order to collect together the fundamentals until at last I felt I had achieved something."

He then tells about his hard work to create the perfect text book: 595 problems divided over 17 chapters, giving detailed instructions on how to use the abacus.

7.1.2 Equations

The coming part is based on LD, page 240-244:

The most interesting algebraic advancement occurred during the closed-door policy. So far the Chinese were of the opinion that as soon one solution was found, the problem was solved. However, through research Wang Lai discovered that an equation could have multiple solutions, so he started to study quadratic and cubic equations, hoping to find a patron. At the beginning of the 19^{th} century he listed 100 types of the following form:

'$bx + c = ax^2$ has only one positive root' or '$bx - c = x^2$ has more than one positive root'.

This inspired Li Rui to do some research upon equations, after which he was able to bring Wang Lai's 100 types down to 3, while even implementing polynomials of degree n:

1. If the coefficients of the various terms of the polynomial change sign only once, the equation has only one positive root.

2. If a_0 and a_n have different signs and the signs of $a_1, ..., a_{n-1}$ are alternately positive or negative and there is a positive root, then the equation can be refactorized to $(x - \alpha)\sum_{k=0}^{n-1} a'_k x^k$. If all a'_k have the same sign, the original equation has only one positive root. Otherwise there is more than one positive root.

3. If all coefficients have the same sign, then the equation can have more than one positive root.

Wang Lai was extremely impressed by Li Rui's deductions of his rules. Yet he considered the second rule a bit impractical, so again he studied on polynomials and discovered something like a determinant-rule. For example, if $x^2+px+q = 0$, then there is a positive root if and only if $q \leq (p/2)^2$.
Again this inspired Li Rui and he came up with the following rules:

1. When the coefficients in an equation change sign once it is possible to have one positive root;

2. Two changes of sign, two positive roots;

3. Three changes of sign, three or one positive root;

4. Four changes of sign, four or two positive roots;

He was also the first Chinese to consider the situation of extracting roots from negative numbers. He said: "Not possible to extract roots is called 'no number'. If in a 'no number' case there are two (roots), there is no 'no number' by itself." In other words: he discovered that complex roots never occur by themselves. For one can test that if an equation yields 0 for a given complex number, then it should also yield 0 for its conjugate (where the sign of the imaginary part is changed).

7.2 Geometry

As has been mentioned before, mathematics received a heavy blow when the Mongols were expelled. However, after a century and a half it slowly recovered. The first mathematician worth mentioning is Tang Shunzi. He lived at the first half of the 16$^{\text{th}}$ century and after a quite long argumentation concluded that $-(\text{arc})^2 + (\text{area})x^2 + (\text{diameter})x^3 - 1.25x^4 = 0$
Another geometrist lived during his time: Ku Ying-Hsiang. He found among others the following formula: $\frac{\text{chord}}{2} = \sqrt{(\frac{\text{diameter}}{2})^2 - (\frac{\text{diameter}}{2} - \text{sagitta})^2}$.
But no real interesting geometry occurred until Ricci translated part of Euclid. So further discussion will take place in the paragraph 'Western Influence'.

7.3 Astronomy

The main reasons of the European success were as LD state it (p. 191): "The most important requirements of the Ming government at that time were the emendation of the calendar and the manufacture and use of cannon." Coincidently both were known in Europe.
But the missionaries encountered a problem: nationalism. At that time there

were two different calendars, namely the Da Tong and the Islamic (which was native by that time). Both had become significantly inaccurate during the past centuries, so a new calendar was needed. This made Ricci decide to ask the Church to send missionaries specialized in the calendrical sciences. But nationalism caused part of the Chinese wise men to refuse to use the western methods. But in 1629 AD a solar eclipse occurred and while the native calendars were a bit too inaccurate, Xu Guangqi, a friend of Ricci, used European methods and did guess right. So, of course, the Emperor had to punish his calendrical officers for their inability. However, thanks to Xu a less painful (physically) solution was found: reform the calendar using European methods. This work was completed in 1634 and was after a practical test in 1643 accepted as the national calendar.
Johann Adam Schaal von Bell played an important part in the construction of the calendar, but he got into problems when the Manchu's grapped power, starting the Qing dynasty. When his calculations about a coming eclipse yielded true in 1645, however, he was restored in his position as president of the Astronomical board and the calendar he helped to construct was restored likewise. But still his problems were not over yet, when Yang Guangxian said: "It is better to have no decent calendar than have Westerners in China." (LD, pg. 205) With this the Emperor agreed, so this nationalist replaced Schaal as the president of the board and the Da Tong calendar was restored.
This time the Belgian Ferdinand Verbiest saved the day: in 1669 he challenged the Chinese astronomers to predict the length of shadows on a certain day. And while his prediction was accurate enough, Yang Guangxian proved that he was serious when saying to prefer a bad calendar above western methods: his prediction was not close at all. So Schaal was again restored, together with his calendar.

On the methods used for this calendar, fewer words will be spent, since they were European, not Chinese. It will suffice to say that one can find tables in them of goniometric functions and also that logarithms were imported.

7.4 Project History of Mathematics

During the Qing period two important works came into existence. The first one, the *Basic Principles*, combined the European mathematics and the Chinese mathematics, known at that time. The other work, lead by Ruan Yuan, contains biographies of ancient Chinese mathematicians and, of course, astronomers. Thus the Chinese began to investigate their own history of science and discovered that they had something to be proud of.
This pride is nicely shown by the following story of Mikami (p. 120):

> *Mei Juecheng was presented with an algebraically treatise by the Emperor Kang Xi, ... Upon that occasion the Emperor said to him: "The Europeans call this work 'a ar-ji-pa-la' (or algebra), which means method that has come from the East". Mei ... became convinced of the intimate resemblance it bears to the old celestial element method, ... He then adds: "It is unknowable to us why the celestial element method employed by the calendar-makers of the Yuen Dynasty should have been forgotten in the succeeding ages, but very*

luckily for the cause of science the lost knowledge has been restored from the people of far-off lands, because they ever strive after enlightenment. Besides, they refer to their science as of 'eastern origin', for they do not forget whence it has come. (Ruan's "Biographical Notices", Book 39)

Obviously the proud Chinese did not know that Algebra simply is derived from the Arabian word for fractions, which is 'Al-Jabr'. Nevertheless, the Age of Decline was nearing its end and the ancient texts were rediscovered.

7.4.1 The *Collected Basic Principles of Mathematics*

Although the European missionaries helped reviving mathematics, the mathematical progress blew over again after the doors were closed, if not thanks to one man: emperor Kang Xi. He was the second emperor of the Qing Dynasty and played an important role in restoring the economy and stabilizing of power of the Qing. (LD, pg. 216-217)
But what makes him most special is his crusade for science: "He was of an enlightened spirit and extraordinarily patronized learning" (Mikami, pg. 116-117) while "It is quite a rarity in Chinese history for an emperor in a feudal system to learn mathematics and other sorts of natural science." (LD, pg. 217)
The emperor was eager to learn and many scientists, both Chinese and European, were honoured by an invitation to court in order to teach him - some even stayed a few days in a row. Eventually one of these Chinese scientists, Chen Houyao, proposed the emperor "to produce a definitive edition of the mathematical texts in order to benefit the country", after which dozens of scientists were invited to produce such a work. The same Mei of the theory of 'a ar-ji-pa-la' was put in charge of this progress and after ten years the *Ocean of Calendrical and Acoustical Calculations*, consisting of three parts, was completed. But it the prints were not finished before the dead of the emperor, two years later.
Part 2 of this book was the *Collected Basic Principles of Mathematics*, which was divided into two parts: 5 chapters 'to establish the objectives and to understand the system', in other words a warming-up, and the second part consisting of 38 chapters, deals with the various subjects of mathematics, like geometry and algebra. (LD, pg. 218-219)

7.4.2 Ruan Yuan

The other important work during the Qing Dynasty is the one of Ruan Yuan: the *Biographies of Mathematicians and Astronomers.* The book came into existence at 1799 AD and consists of 46 chapters, dealing 243 Chinese mathematicians and calendar makers and in the appendix there is also room for 37 European ones.
Ruan was born in 1764 and became head of the astronomical board in 1789. Mikami tells about him that "Ruan Yuan was very fond of learning, and made various publications; he encouraged all sorts of learning." (pg. 124) This eagerness is shown by his construction of the *Biographies.* It needs to be said that he did not work alone: one of his companions was Li Rui.
(see LD, pg. 232-233, and Mikami, pg. 123-124)

7.5 Western Influence

The western influence on astronomy has already been discussed. This section will focus on two key-figures: Mattheo Ricci and Mei Wending.

7.5.1 Mattheo Ricci

Mattheo Ricci came to China in 1582 was the first Jesuit to enter China. Mikami tells about him (p. 113): "He was most enthusiastic in the mission and his teaching of the various sciences was all affected with the view of attaining this great goal." It was thanks to this enthusiasm that Ricci started the translation of the first six books of Euclid and the *Epitome of Practical Arithmetic* of Clavius. The translation of the first was completed in 1607 AD with the help of Xu Guangqi - the one of the calendar - and Wu Song, acting as scribes. Why Ricci only did the first six books is explained in the preface:

> *The Grand Scholar (that is, Xu Gangqi) is very enthusiastic, he wanted to complete the translation but I said: No, please first distribute it. Help those interested to study it and if it proves useful, then we shall translate the rest. The Grand Scholar said: Very well. If this book is useful it should be used. It does not have to be completed by us. So we shall stop the translation and publish it...*

(LD, pg. 194)
Exactly that happened where Ricci was afraid of: the book did not become a great success (at that time), probably because it was too theoretical to the practical attitude of the Chinese. This is nicely illustrated by Xu in a later edition: "... we do not know when and by whom this important task will be continued." (LD, pg. 194)
It took two and a half centuries before Wylie and Shanlan completed the job.

The other translation of Ricci, the one of the *Epitome of Practical Arithmetic*, was called the *Treatise on European Arithmetic (Tong Wen suan zhi)*, which was completed in 1631 AD, almost 25 years after the *Elements*, with the help of Li Zhizao. The reason why this work was translated instead of the other books of Euclid may find its cause in the word *Practical*. Another reason may be found in the preface, where Li describes the three parts, which are the introduction, general survey and the supplement, he says about the second part: "Brief exposition of examples which are easy to read, including work from the Nine Chapters which is not outside the scope of the original (Clavius's) text." In other words, the book would not seem too weird to the Chinese, yet it would help them to understand European mathematics a little better.

7.5.2 Mei Wending

Mei Wending is the grandfather of the Mei Juecheng of the previous section. He was not involved with mathematics or calendars before he was 27, but after that he spent the rest of his life with these sciences and wrote more than 80 works, which were eventually published by his grandson. What makes him so special is told by LD (pg. 215-216):

> *Mei Wending's works bear on almost all aspects of the western mathematics which had come in at the time and moreover, he was not just collecting, he also did the initial task of assimilating it and some further research. ..., he had already understood and blended these (the translations of Ricci and the books about the Schaal-calendar) together: he was not just copying.*
> *..., he himself could augment it and present something extra.*

But what may be most special about him is found in the concluding parts of LD (pg. 216):

> *Mei Wending used a simple and easily understood presentation to describe what most people would regard as difficult mathematical problems, this was his most valuable and rare quality and these are the points we should learn from.*

8 Westernization

In 1840 AD the Opium Wars began and the closed door policy was blown apart. The Chinese emperor has to accept the Europeans entering China, and where his predecessors previously were able to control their influence - even by sending them away - he had to suffer the western science weeping China's science away. China's light in these dark days can be found in the person of Li Shanlan, who played an important role in the modernization of Chinese science.
Li Shanlan was born in 1811 and when he was 10 years old he started to study the *Nine Chapters* and when he was 15, he read the *Elements*, that is the first six books of Euclid (see previous chapter). When he met Alexander Wylie in 1852 AD, he had already done much work by himself, so he know where he was talking about, while he was translating, just like Mei Wending was able to comprehend things.
There is, however, one difference between Li and Mei:

> *The works translated by Li Shanlan contained very new material, and contained much that was too difficult for the ordinary reader at that time. The circulation and influence of these translations was confined to a small number of lovers of mathematics. Consequently later on people criticized his translations, saying his aim was to 'show off his specialist learning'. Although this sort of comment was not fair, it is not entirely without foundation.*

(LD, pg. 258)
In other words: Li may be an able translator, he was missing the clarifying skills of M'ei.
Yet, when we speak of 'modernization' we do not intend to underestimate the Chinese mathematics under the closed door policy: rather we mean to say the adoption of a western style by Chinese mathematics. For especially Li Shanlan is known for his achievements before the Opium Wars ... maybe that is what made him the right person for this process of modernization. And although much of the Chinese achievements during the Qing were already known in Europe, see for example the research on equations in the previous chapter, one should not forget that tings like the 'method of Horner' were already discovered during the Zenith.

8.1 Discoveries of Li Shanlan

After the introduction, one may be curious to Li's research. This section will discuss three of them:

- Prime Numbers: In 1872 Li published the *Four Techniques of Determining Primes*. In it he discovers that if $a^k - 1$ is divisible by p and p is prime, then $p - 1$ is divisible by k. Thus, unaware of it, he was just proving the Little Theorem of Fermat: If p is prime, then for any $a : 1 < a < p \Rightarrow a^{p-1} \bmod p = 1$.

- Cones: Using Taylor, one can express a function in a polynomial form as precisely as he wants. Li constructed a method which worked about the same way: using cones, he was able to express a function like the logarithm in a polynomial form. In order to do so, he used cones, which had the following property: at height h the cross-sectional body is always equal to ah^n with $n \in N$. This results in a body with a volume of $\int_0^h ax^n dx = \frac{ah^{n+1}}{n+1}$. Combining multiple cones, one gets a volume of $\int_0^h (\sum_1^n a_k x^k) dx$.

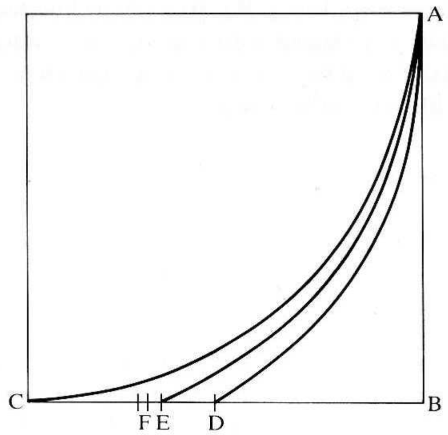


Figure 17: Method of Cones. (LD, pg. 253)

  To determine π one can use a 2D-version of this method. First note that the area of a circle is πr^2 and then take $r = 1$. Now looking at the figure and taking $CB = BA = 1$, then $\pi = 4(1 - CBA)$, where AC follows the curve. The last step to take is to look at CBA as a combination of cones, in fact: DBA, EDA, ... Thus one can get as close to π as he wants.

- Logarithms: In 1653 AD Smogulęcki, a missionary, taught Xue Fengzuo about logarithms. Xue was very enthusiastic, since they could be used to replace multiplication and division by addition and subtraction, therefore decreasing the change of errors. However, just like the *Elements* did not have an immediate impact, likewise the logarithms were used rarely at the start. This changed when the *Basic Principles* came into existence, but the method this book used was quite difficult, since it used successive roots. (LD, pg. 221)
  It is here that the conic method of Li proved to be very useful:

$$\log_e x = \sum_{k=1}^{\infty} \frac{1}{k} (\frac{x-1}{x})^k$$

8.2 Reformation of the Educational System

Now we have shown the most important and outstanding parts of the history of Chinese mathematics, we are left with the final transgression to modern day. As said, this transgression starts with the Opium Wars: by imperial edict, Europeans were forbidden to enter China, but now the Europeans were tired of this conservative view and the Qing government was forced to allow Europeans enter China again.
Mathematics gratefully accepted this change and brought the two right persons together. The first was our Li, who had to flee to Shanghai in 1852 because of Taiping rebellion. The other was Alexander Wylie who happened to be at the same place were missionaries had established St. John's College in 1845.
It may be clear by now that Li was very interested in mathematics and since

Wylie did not dislike it either, they translated some very important European, mathematical works, hoping to share their love for this science with the rest of China. The translation was, of course, the rest of *Euclid's Elements*, once started by Ricci and Xu Guangqi, which they did from 1852 to 1856. Another very important translation was the one of the *Elements of Algebra* by A. De Morgan, which was completed in 1859. It is in this translation that it becomes clear why Li Shanlan was exactly the right person for leading Chinese mathematics to the modern age. Instead of just copying the Western texts to Chinese, he kept in mind how his ancestors had previously dealt with the same topics, so wherever it was possible, he tried to use a more Chinese form of writing: "Although the form of the book was still conservative, nevertheless the content was essentially the same as that in ... the various countries of the world ..." (LD, pg. 265)
This is illustrated by the translation of a book concerning differentials, namely the *Eighteen Chapters of Elements of Analytical Geometry and of Differential and Integral Calculus* by E. Loomis. He does not just use the Latin forms, but some Chinese characters. For example: $\int 3x^2dx$ is written as: 禾三天^{二}彳天

But as said: Li may have had a gift of perfectly understanding things, he did not have the gift to make it comprehensive to the less brilliant minds. So there were troubles with his translations. Therefore we should not forget to mention Hua Hengfang, his buddy was John Fryer, who also translated several works and introduced Probability-theorem into China with the translation of the *Probabilities* by Galloway. His translations were easier to understand, so some people preferred his textbooks.

Meanwhile a pro-westernization movement had established the Foreign Languages Institute in Beijing in 1862, where Chinese could learn English, French and Russian. In 1866 the leader of this movement, prince Yi Xin, also founded a mathematical department, because at the moment China was negotiating about buying a steamship and he came to realize that not knowing the basics of science would make it difficult for the Chinese to operate such things. Finding a chief instructor was not a challenge: Li was already getting paid for private lessons. So it happened that in the fourth year students were introduced to mathematics and also learned *algebra.* Next year *Euclid* and *trigonometry* and finally there were *calculus* and *navigation.* (LD, pg. 211-212)
In 1898 a Reform Movement made plans to change the whole educational system to something we're more familiar with today: primary school - secondary school - university. Mathematics would not become something extra, but take an important position in every phase. In 1905 the civil service examinations - the recital examinations - were abolished and in 1912 the modern school system came into existence in 1912. Thus after a long struggle of acknowledgement, mathematics finally received here place between the primary sciences.

9 Summary and Conclusions

China belongs together with its Babylon and Egypt to the oldest civilizations in the world. Yet there is an important difference with the other two: Babylon and Egypt were overrun and assimilated by barbaric tribes, while China seems to have been able to keep her identity, at least till recent times. Tribal invaders have conquered China more than once, but in general the conquerors were assimilated in Chinese civilization within several generations. Consequently, Chinese culture could develop relatively uninterrupted for several millennia. Therefore it is no surprise that much can be said about Chinese mathematical achievements and that in some cases Chinese mathematics was way beyond other cultures, including its Western counterpart.

The first archaeological traces of Chinese culture are from 8 000 BC and the first scripts and numbers date back to 1 050 BC. According to Chinese legends, not some Divine Creator, but brilliant human beings where responsible for the steps that transformed the Chinese from primitive people to a civilization. This view would be of fundamental importance to the development of Chinese mathematics. Because no Rational Creator was assumed, the world didn't have to be understandable with ratio. Instead of focusing on mechanics that set things into motion, as Western philosophers did, Chinese philosophy would be centered around ethical and moral questions. The major Chinese philosophical currents were the Confucians, the Taoists, the Mohists and the Legalists.

- *Confucianism* would become the most important and influential philosophy. Its principles would determine the actions of about one-fifth of the world population for over 2 millennia. Confucianism has beautiful ethics, but due to its single-mindedness on human affairs, nature is neglected.

- *Taoists* were interested in nature, but to them nature could not be understood with intellect.

- *Mohism* seemed to be on the right track for the development of natural sciences, but for unknown reasons this philosophical school completely vanished during the Qin unification.

- *Legalism* did focus on absolute laws to regulate country, or actually semi-absolute, since they didnt apply to the emperor. But their laws were intentionally cruel and completely amoral. When the first empire fell, this philosophical school fell with it.

These three aspects: (i) nature doesn't have to be understandable by ratio, (ii) the focus on ethics and (iii) the aversion from absolute law, had as direct result that the concept of Absolute Natural Laws didn't occur to the Chinese philosophers. This would have profound consequences for the development of mathematics: it would focus on algebra and practical applications instead of geometry and theoretical reasoning (which characterize Western mathematics).

Alas, not much is known about the time before the book-burning in 213 BC, but luckily we can get an interesting picture thanks to remaining fragments.

- In the first place numerous oracle bones have been found, showing the Chinese way of writing numbers: it seems they used a decimal system from the very beginning. This is quite extraordinary, since the western cultures preferred alternative systems for a long time. An explanation may be that the Chinese symbolic script did not make it necessary to use shorter forms of writing numbers.
- Another important item is the Zhoubi. This is an astronomical book, but in it one can find the Pythagoras theorem - while the persons discussing the theorem lived about seven centuries *before* Pythagoras.
- The last important fragment really *is* fragmentary, namely the *Mohist Canon*. It is very special because it treats geometrics in a very theoretical way, while the Chinese were more practical in nature. The influence of the book, however, was rather small, so it would not impact the rest of China's mathematical future.

The first real mathematical treatise that has survived down to our time is the classic *The Nine Chapters on the Mathematical Art* which, together with the comments of the brilliant mathematician Liu Hui can be seen as the fundament of Chinese mathematics. The current version of the text in the *Nine Chapters* was formed at last in the first century AD, Liu's comments date from 263 AD. The most lofty accomplishments of the *Nine Chapters* are the rules for fractions: both addition, substraction, multiplication, division, reduction and averaging of fractions was treated; Gaussian elimination; numerical methods to solve second and third degree roots and polynomials; its discourse on surveying.
But Liu's time was one of the violent, bloody and uncertain times in Chinese history: the Chinese Dark Ages. And during this dark time, many sought light by the emerging religion of Buddhism. This religion discussed questions about death and suffering in a way unseen in Chinese history; its code of conduct forbad the taking of life. In the wake of Buddhism; there was a possibility of salvation. In the wake of Buddhism, Indian culture and science followed.

So during this time Chinese Buddhism was on the rise and at the beginning of the fifth century, the religion was firmly entrenched in the Chinese mind, a position it would hold till the beginning of the eleventh century. China and India now shared their knowledge, so one may expect some rather impressive achievements of these two advanced cultures. However, the Indian knowledge was not used for mathematics, but rather for astronomy. So China's mathematical achievements during the Medieval period and the subsequent Tang are rather disappointing. Most interesting is the construction of the *Ten Mathematical Manuals*. Although this is more from the historical point than a mathematical point of of view, since the level of some of these 'manuals' was rather low: combining mistakes with easy mathematics. The state of mathematics is also indicated by civil service exams: one was not expected to be able to solve problems, but rather to recite a text.

The Tang dynasty fell and after the usual stage of civil war, the Song emerged victorious and founded their own dynasty, which can be seen as the Chinese Renaissance. The civil service examinations became the dominant way of selecting

government officials. Consequently, more and more students would study the classics, so their view on the world would be shaped by the Confucian tradition. In 1125, the nomadic Jurchen would occupy the northern part of Chinese territory, pushing the Song back to the South. The century that followed would be characterized by mutual attempts to regain each others territory. In the process, the Jin sinified. This battle would never be fought to the end, because in 1234 and 1236 respectively the Jurchen and the Song were conquered by the Mongols, who founded the Yuan dynasty. Yet, in the era of continuous battles between the Song and the Jurchen, followed by a century of suppression by the Yuan, Chinese mathematics would reach its zenith, a level unequaled by ever before or after. In certain branches of mathematics, they would be about 800 years ahead of the West.
In this culmination point of mathematical knowledge, the Chinese had devised among others:

- The technique of the celestial element, extended by the technique of four unknowns. With this technique, systems of polynomial equations could be solved.
- A method to numerically solve polynomials and roots of arbitrary degree. This technique would be rediscovered in the West about 800 years later and named Horner's method.
- The Chinese remainder theorem.
- Ideas on how to compute certain finite series.

Ironically as it may seem, when the Chinese were freed from the Mongol yoke, their mathematical skill dropped enormously. So it isn't before the end of the 16^{th} century, when European missionaries enter China, that mathematics made progress again. At that time Chinese science was in a rather bad state, while the Chinese calendar needed to be reformed and their weapons could use an upgrade. Consequently, in general Europeans were welcomed with enthusiasm. During the 17^{th} century the Mongols return, which was good for mathematics. Emperor Kang Xi was fond of science and ordered the construction of a mathematical encyclopaedia. It may be thanks to him that when the relations with the west worsened, mathematics continued to improve. For when the people forbad the Chinese Catholics to partake any longer with ancestor-worshipping, a political powerplay was unleashed, eventually resulting in the banning of all Europeans from China. But even now still progress was made and like some European inventions had already been discovered by the Chinese during their mathematical Zenith, likewise the Chinese would invent things - independent of Europe - which were already known in the West, like the Rule of Signs.

In 1840 the Opium Wars broke out and China was forced to open her door once again for westerners. So again Europe got the change to influence China and so she did. Unlike the previous time, the emperor no longer had the power to control this influence, which resulted in a reformation of the educational system and the abolishment of the civil service examinations. Key figure during these troubled times is doubtless Li Shanlan. He already had done a lot of research before the Opium Wars and was therefore perfectly able to understand the European mathematics, so that he could translate some important works.

The translations of Western mathematical treatises would mark the end of the indigenous Chinese mathematics. Within decades, she would be assimilated in the universal mathematics we know today.

When considering the summary and conclusions above, two things are striking. The first is more of social nature: war seems to be good for mathematics. In almost all periods with major mathematical breakthroughs, violence, cruelty and bloodshed were of the order of the day.

- The philosophical foundations of the Chinese civilization were formulated during the five hundred years of civil war marking the fall of the Zhou kings.
- The creation of the definite version of the *Nine Chapters* took place at the end of the Han dynasty. Liu Hui as well as several other excellent mathematicians, like Zu Chongzhi and Zu Geng, lived during the Chinese Dark Ages.
- During the relatively peaceful Tang dynasty, no significant breakthroughs were made.
- This changed during the Song and Yuan dynasties, when Chinese mathematics would reach is Zenith. But qua peace-level, this era doesn't score high either: the Song and Jin dynasties was dead-locked in a struggle regarding their territory, only to be conquered and subjugated by a third party.
- When the Mongol yoke was thrown off, mathematical development was thrown away with it. The centuries of peace which were to follow would also be the centuries of mathematical decline.
- Chinese mathematics began to make progress again after the mathematics of the war-like Europeans entered China.

The second question that might come to your mind is why, after its grandiose accomplishments, Chinese mathematics would not only be surpassed by its Western counterpart, its progress would not only come to a halt, but it would even decline and crumble?

Mathematics is often compared with a building. Axioms give support to basic theorems, which in their turn support other theorems, etc. In Western, by now universal mathematics the problem is in the fundaments. Although the building seems relatively concrete we are not so sure about its philosophical foundation (for an extensive treatise regarding the philosophical foundation of mathematics, we refer to the article *What is this mathematics you speak of?* by De Vos, S. and Weitenberg, E. which can be found elsewhere in this bundle).

For over 2 000 years the dominant Chinese philosophy has been Confucianism. Due to its focus on Human affairs as opposed of Nature, the concept of

Natural Laws didn't occur; and mankind was seen as the center of the cosmos. Therefore science and mathematics served practical human needs, mathematical research for its own sake found little feeding ground and consequently "mathematics developed only very slowly; and if we take into account this practical orientation, we should not be surprised by what seems to be a real stagnation. Although Chinese mathematics was built up as a constellation of relations, the relations were unique, and there was no general algebraic structure. The main reason for this lack was the absence of general analysis, which could have lent structural insight. There are many beautiful bricks in Chinese mathematics, but there is no building." (Libbrecht, pg. 14.)

Without a building in which they could glitter, these beautiful bricks were left untouched and with the passing of time they would be covered with dust, became invisible and finally ... were forgotten.

Timeline

Period	Important Events, Persons and Books
-8000 *First Traces Civilization* -1050 to -770 *Zhou* -770 to -256 *Ceremonial Zhou* -221 to -210 Qin	*ca. -1000:* Gougu Theorem. *-551 - -479:* Confucius. Gradual accumulation of the *Zhoubi.* *-213:* Book burning.
-202 to 189 *Han* 189 to 626 *Dark Ages*	*ca. 100: Nine Chapters.* *ca. 250:* Liu Hui writes his commentaries on the *Nine Chapters* in the *Sea Island.* *ca. 500:* Zu Chongzhi and his son Zu Geng find a very accurate value for π.
626 to 907 *Tang* 960 to 1276 *Song*	*ca. 650: Ten Mathematical Manuals.* *ca. 650:* Yi Xing. *1031 - 1095:* Shen Kuo.
1126 to 1234 *Song vs. Jing* 1215 to 1367 *Yuan*	*1202(?) - 1261:* Qin Jiushao. *1192 - 1279:* Li Zhi. *1238 - 1298:* Yang Hui. *1247:* Qin completes his *Mathematical Treatise in Nine Sections.* *1248:* Li completes his *Sea Mirror of Circle Measurements.* *ca. 1260 - 1320:* Zhu Shijie. *1299:* Zhu completes his *Introdution to Mathematical Studies.* *1303:* Zhu completes his *Precious Mirror of the Four Elements.*
1367 to 1644 *Ming* 1644 to 1911 *Qing* 1579 to 1704 *Jesuit Influence*	*1582:* Entry of Ricci. *1662 to 1722:* Kang Xi. *1704:* Decline of Western relations. *1723: Collected Basic Principles.*
1840 to 1842 Opium Wars *1911:* End of Qin	*1811 - 1882:* Li Shanlan. *1852:* Li meets Alexander Wylie in Shanghai. *1905:* Civil service exams abolished. *1912:* Modernization of the educational system.
1927 to ... Republic of China *1949 to ...* People's Republic of China	*1949:* Nationalist governments flees to Taiwan.

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