THE NUCLEAR SHELL MODEL

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"In physics, three points always make a straight line" (I Talmi)

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1 One-Particle Excitations

1.1 Introduction

For a set of quantum numbers a and a complete coordinate system $\mathbf{r}$, the function $\varphi_a(\mathbf{r})$ is a solution to the Schrödinger one-body equation if

$$(T + U(r))\varphi_a(\mathbf{r}) = \varepsilon_a \varphi_a(\mathbf{r}).$$

For A independent nucleons, the Hamiltonian, its eigenfunctions and energy can be written as

$$H_0 = \sum_{i=1}^{A} (T_i + U(r_i))$$

$$\Psi_{a_1,\ldots,a_A}(\mathbf{r}_1, \ldots, \mathbf{r}_A) = \prod_{i=1}^{A} \varphi_{a_i}(\mathbf{r}_i)$$

$$E_0 = \sum_{i=1}^{A} \varepsilon_{a_i}.$$

If the nucleons are identical, the wave function needs to be antisymmetrised by means of a Slater determinant,

$$\Psi_{a_1,\ldots,a_A}(\mathbf{r}_1, \ldots, \mathbf{r}_A) = \frac{1}{\sqrt{A!}} \begin{vmatrix} \varphi_{a_1}(\mathbf{r}_1) & \ldots & \varphi_{a_1}(\mathbf{r}_A) \\ & \ddots & \\ \varphi_{a_A}(\mathbf{r}_1) & \ldots & \varphi_{a_A}(\mathbf{r}_A) \end{vmatrix}.$$

For two particles this becomes

$$\Psi_{a_1,a_2}(\mathbf{r}_1, \mathbf{r}_2) = \frac{1}{\sqrt{2}} (\varphi_{a_1}(\mathbf{r}_1)\varphi_{a_2}(\mathbf{r}_2) - \varphi_{a_1}(\mathbf{r}_2)\varphi_{a_2}(\mathbf{r}_1)).$$

The average field $U(r)$ is an approximation for the actual two-body interaction $V_{i,j}$ as defined by the residual Hamiltonian H_{res},

$$\begin{aligned} H &= \sum_{i=1}^{A} T_i + \frac{1}{2} \sum_{i,j=1}^{A} V_{i,j} \\ &= \sum_{i=1}^{A} (T_i + U(r_i)) + \frac{1}{2} \sum_{i,j=1}^{A} V_{i,j} - \sum_{i=1}^{A} U(r_i) \\ &= H_0 + H_{res}. \end{aligned}$$

The derivation of a good average field from a given two-body interaction is carried out by the Hartree-Fock algorithm. For this we need wave functions that are fair approximations to the actual wave function. We shall obtain them from the easier case of independent motion in a harmonic oscillator potential.

1.2 The Harmonic Oscillator

Using commutation relations one derives

$$\begin{aligned}
\mathbf{l}^2 &= (\mathbf{r}\times\mathbf{p})\cdot(\mathbf{r}\times\mathbf{p}) \\
&= r^2p^2 - \mathbf{r}(\mathbf{r}\cdot\mathbf{p})\cdot\mathbf{p} + 2i\hbar\mathbf{r}\cdot\mathbf{p} \\
&= r^2p^2 + \hbar^2\frac{\partial}{\partial r}(r^2\frac{\partial}{\partial r}).
\end{aligned}$$

The kinetic energy then becomes

$$\begin{aligned}
T &= \frac{\mathbf{p}^2}{2m} \\
&= \frac{\mathbf{l}^2}{2mr^2} - \frac{\hbar^2}{2mr^2}\frac{\partial}{\partial r}(r^2\frac{\partial}{\partial r}).
\end{aligned}$$

Setting $\mathbf{l}^2 = \hbar^2\lambda$ and choosing an eigenfunction $\varphi(\mathbf{r})$, the Schrödinger equation of a central potential is now given by

$$\left(-\frac{\hbar^2}{2mr^2}\frac{\partial}{\partial r}(r^2\frac{\partial}{\partial r}) + \frac{\hbar^2\lambda}{2mr^2} + U(r)\right)\varphi(\mathbf{r}) = E\varphi(\mathbf{r}).$$

We aim at a solution of the form

$$\varphi(\mathbf{r}) = \frac{u(r)}{r}Y(\theta,\varphi)$$

such that

$$\frac{r^3}{u(r)Y(\theta,\varphi)}\left(-\frac{\hbar^2}{2mr^2}\frac{\partial}{\partial r}(r^2\frac{\partial}{\partial r}) + \frac{\hbar^2\lambda}{2mr^2} + U(r) - E\right)\frac{u(r)}{r}Y(\theta,\varphi) = 0$$

$$-\frac{r^2}{u(r)}\frac{\hbar^2}{2m}\frac{\partial^2 u(r)}{\partial r^2} + \frac{1}{Y(\theta,\varphi)}\frac{\hbar^2\lambda}{2m}Y(\theta,\varphi) + (U(r)-E)r^2 = 0$$

and after separation of variables,

$$-\frac{\hbar^2}{2m}\frac{d^2u(r)}{dr^2} + \left(\frac{\ell(\ell+1)\hbar^2}{2mr^2} + U(r)\right)u(r) = Eu(r)$$

with boundary and normalisation conditions

$$u(\infty) = 0$$

$$u(0) = 0$$

$$\int_0^\infty u^2(r)dr = 1.$$

For a harmonic oscillator

$$U(r) = \frac{1}{2}m\omega^2r^2$$

this becomes a Laguerre equation

$$\frac{d^2u(r)}{dr^2} + \left(\frac{2m}{\hbar^2}E - \frac{m^2\omega^2r^2}{\hbar^2} - \frac{\ell(\ell+1)}{r^2}\right)u(r) = 0$$

the solutions of which are

$$u_{kl}(r) = N_{kl} r^{l+1} e^{-\nu r^2/2} L_k^{l+1/2}(\nu r^2)$$

where $\nu = m\omega/\hbar$ and $L_k^{l+1/2}(\nu r^2)$ are the associated Laguerre polynomials. The energy eigenvalues are found from the identification

$$E = \hbar\omega(2k + \ell + \frac{3}{2})$$

and therefore

$$\begin{array}{rcl} N = & 2k + \ell & = 0, 1, \ldots \\ \ell = & N - 2k & = N, N-2, \ldots, 1 \text{ or } 0 \\ k = & (N - \ell)/2 & = 0, 1, \ldots, (N-1)/2 \text{ or } N/2. \end{array}$$

Instead of the radial quantum number k, one often uses the number of nodes n in the interval $[0, \infty[$,

$$n = k + 1 = (N - \ell + 2)/2.$$

The energy degeneracy of the sets of (k, l) values for the same N gives rise to magic numbers at 2, 8, 20, 40, 70. The degeneracy, however, is lifted by introducing a spin-orbit term. Including spin space the unperturbed single-particle wave function is expressed by

$$|nljm\rangle = \frac{u_{nl}(r)}{r}[\mathbf{Y}_l(\theta, \varphi) \otimes \boldsymbol{\chi}^{1/2}(\boldsymbol{\sigma})]_m^{(j)}$$

and its unperturbed energy, independent of spin orientation, is given by

$$\langle nljm|h_0|nljm\rangle = \varepsilon_{nlj}^0.$$

The spin-orbit interaction is a perturbation in the Hamiltonian

$$\begin{aligned} h &= h_0 + \zeta(r)\, \mathbf{l} \cdot \mathbf{s} \\ &= h_0 + \zeta(r)\frac{1}{2}(\mathbf{j}^2 - \mathbf{l}^2 - \mathbf{s}^2) \end{aligned}$$

and the perturbed energy is found from

$$\begin{aligned} \varepsilon_{nlj} &= \varepsilon_{nlj}^0 + \langle nljm|\zeta(r)\frac{1}{2}(\mathbf{j}^2 - \mathbf{l}^2 - \mathbf{s}^2)|nljm\rangle \\ &= \varepsilon_{nlj}^0 + \frac{1}{2}\left(j(j+1) - \ell(\ell+1) - \frac{3}{4}\right)\int u_{nl}^2(r)\zeta(r)dr \\ &= \begin{cases} \varepsilon_{nlj}^0 + \dfrac{\ell}{2}\displaystyle\int u_{nl}^2(r)\zeta(r)dr \\ \varepsilon_{nlj}^0 - \dfrac{\ell+1}{2}\displaystyle\int u_{nl}^2(r)\zeta(r)dr. \end{cases} \end{aligned}$$

A common expression for the interaction is

$$\zeta(r) = -V_{ls} r_0^2\, \frac{1}{r}\frac{\partial U(r)}{\partial r}$$

with the Woods-Saxon potential

$$U(r) = \frac{-V_0}{1 + e^{(r-R)/a}}$$

in which a is called the diffusiveness (typically 0.5 fm) and $R = r_0 A^{1/3}$ is the nuclear radius ($r_0 \approx 1.2$ fm and A is the mass number). An additional term proportional to $\mathbf{l}^2$ is then added to recover the magic numbers 2, 8, 20, 28, 50, 82, 126 with correct spacings.

1.3 The Hartree-Fock Method

We want to derive the average field $U(r)$ from a microscopic starting point. The nuclear density is composed from the occupied single-particle states as

$$\varrho(\mathbf{r}) = \sum_{a \in F} \varphi_a^*(\mathbf{r}) \varphi_a(\mathbf{r}).$$

The potential at a point $\mathbf{r}$ generated by the two-body interaction $V(\mathbf{r}, \mathbf{r}')$ is found from

$$U_H(\mathbf{r}) = \sum_{a \in F} \int \varphi_a^*(\mathbf{r}') V(\mathbf{r}, \mathbf{r}') \varphi_a(\mathbf{r}') d\mathbf{r}'.$$

The precise Schrödinger equation, however, includes an exchange term,

$$-\frac{\hbar^2}{2m} \nabla^2 \varphi_i(\mathbf{r}) + \sum_{a \in F} \int \varphi_a^*(\mathbf{r}') V(\mathbf{r}, \mathbf{r}') (\varphi_a(\mathbf{r}') \varphi_i(\mathbf{r}) - \varphi_a(\mathbf{r}) \varphi_i(\mathbf{r}')) d\mathbf{r}' = \varepsilon_i \varphi_i(\mathbf{r}).$$

Introducing the notation

$$U_F(\mathbf{r}, \mathbf{r}') = \sum_{a \in F} \varphi_a^*(\mathbf{r}') V(\mathbf{r}, \mathbf{r}') \varphi_a(\mathbf{r})$$

we obtain the Hartree-Fock equations

$$-\frac{\hbar^2}{2m} \nabla^2 \varphi_i(\mathbf{r}) + U_H(\mathbf{r}) \varphi_i(\mathbf{r}) - \int U_F(\mathbf{r}, \mathbf{r}') \varphi_i(\mathbf{r}') d\mathbf{r}' = \varepsilon_i \varphi_i(\mathbf{r})$$

where the parts containing $U_H(\mathbf{r})$ and $U_F(\mathbf{r}, \mathbf{r}')$ are called the Hartree and Fock terms, or direct and exchange terms, respectively. For a chosen interaction $V(\mathbf{r}, \mathbf{r}')$ and a set of wave functions $\varphi_a(\mathbf{r})$ one then calculates iteratively the new eigenfunctions till the energy of the system is minimised,

$$\delta \, \langle \prod_{i=1}^{A} \varphi_i(\mathbf{r}_i) | H | \prod_{i=1}^{A} \varphi_i(\mathbf{r}_i) \rangle = 0.$$

This minimisation is only possible if strong short-range correlations and density-dependent interactions are excluded.

2 Two Identical Nucleons

2.1 Two-Particle Wave Functions

The two-particle wave function

$$\psi(j_1(1)j_2(2);JM) = \sum_{m_1,m_2} \langle j_1m_1, j_2m_2|JM\rangle \varphi_{j_1m_1}(1)\varphi_{j_2m_2}(2)$$

is an eigenfunction of the unperturbed Hamiltonian

$$H = H_0 + \sum_{i=1}^{2} h_0(i)$$

where H_0 describes a closed core and $h_0(i)$ the valence nucleon i. The energy shift induced by a residual interaction V_{12} is found from

$$\Delta E(j_1j_2;J) = \langle j_1j_2;JM|V_{12}|j_1j_2;JM\rangle.$$

However, since we consider identical particles, we should not forget to antisymmetrise. If $j_1 \neq j_2$,

$$\begin{aligned}
&\psi_{AS}(j_1j_2;JM)\\
&\qquad = N \sum_{m_1,m_2} \langle j_1m_1, j_2m_2|JM\rangle(\varphi_{j_1m_1}(1)\varphi_{j_2m_2}(2) - \varphi_{j_2m_2}(1)\varphi_{j_1m_1}(2))\\
&\qquad = \frac{1}{\sqrt{2}}\left(\psi(j_1j_2;JM) - (-)^{j_1+j_2-J}\psi(j_2j_1;JM)\right).
\end{aligned}$$

If $j_1 = j_2$,

$$\begin{aligned}
&\psi_{AS}(j^2;JM)\\
&\qquad = N' \sum_{m_1,m_2} \langle jm_1, jm_2|JM\rangle(\varphi_{jm_1}(1)\varphi_{jm_2}(2) - \varphi_{jm_2}(1)\varphi_{jm_1}(2))\\
&\qquad = N' \sum_{m_1,m_2} (\langle jm_1, jm_2|JM\rangle - \langle jm_2, jm_1|JM\rangle)\,\varphi_{jm_1}(1)\varphi_{jm_2}(2)\\
&\qquad = N'(1-(-)^{2j-J}) \sum_{m_1,m_2} \langle jm_1, jm_2|JM\rangle\varphi_{jm_1}(1)\varphi_{jm_2}(2)\\
&\qquad = \frac{1}{2}(1-(-)^{2j-J})\psi(j^2;JM).
\end{aligned}$$

For the latter case it follows that

$$J = 0,\ 2, \ldots,\ 2j-1.$$

The two-body matrix elements become

$$\Delta E(j_1j_2;J) = \langle j_1j_2;JM|V_{12}|j_1j_2;JM\rangle - (-)^{j_1+j_2-J}\langle j_1j_2;JM|V_{12}|j_2j_1;JM\rangle$$

$$\Delta E(j^2;J) = \langle j^2;JM|V_{12}|j^2;JM\rangle$$

or taking both formulae together, we can symbolically write

$$\Delta E(J) = \frac{1}{1+\delta_{j_1,j_2}}\Delta E(j_1j_2;J).$$

2.2 Two-Particle Residual Interaction

An effective interaction can be constructed empirically when the single-particle energies and the two-body matrix elements are taken as free parameters and fitted to experimental data. The Brussaard-Glaudemans and Cohen-Kurath interactions are well-known choices for the p shell, Wildenthal-Brown for the sd and Brown-Richter for the fp shells.

Realistic forces contain an interaction with an analytical structure, the parameters of which are fitted to experimental data to reproduce the proper strengths. The Hamada-Johnston potential is an example

$$V = V_C(r) + V_T(r)S_{12} + V_{LS}(r)\mathbf{l}\cdot\mathbf{S} + V_{LL}(r)L_{12}$$

$$S_{12} = \frac{3}{r^2}(\boldsymbol{\sigma}_1\cdot\mathbf{r})(\boldsymbol{\sigma}_2\cdot\mathbf{r}) - \boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2$$

$$L_{12} = (\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)\mathbf{l}^2 - \frac{1}{2}\left((\boldsymbol{\sigma}_1\cdot\mathbf{l})(\boldsymbol{\sigma}_2\cdot\mathbf{l}) + (\boldsymbol{\sigma}_2\cdot\mathbf{l})(\boldsymbol{\sigma}_1\cdot\mathbf{l})\right).$$

The local force comprises a central interaction V_C and a non-central tensor force $V_T S_{12}$. The tensor force only acts on the $S = 1$ state and is the formal equivalent of a classical dipole-dipole field. For a classical magnetic dipole field

$$\mathbf{B}(\mathbf{r}) = \frac{3\mathbf{n}(\mathbf{n}\cdot\boldsymbol{\mu}) - \boldsymbol{\mu}}{r^3}$$

the dipole-dipole interaction is indeed written as

$$-\boldsymbol{\mu}_1\cdot\mathbf{B}_2 = \frac{1}{r^3}\left(\boldsymbol{\mu}_1\cdot\boldsymbol{\mu}_2 - \frac{3(\mathbf{r}\cdot\boldsymbol{\mu}_1)(\mathbf{r}\cdot\boldsymbol{\mu}_2)}{r^2}\right).$$

The non-local forces are the spin-orbit interaction $V_{LS}\mathbf{l}\cdot\mathbf{S}$, again only for $S = 1$, and a quadratic spin-orbit contribution $V_{LL}L_{12}$. The radial functions are multiparameter functions that are essentially built from the Yukawa function

$$V(r) = \frac{e^{-\mu r}}{\mu r}$$

with $1/\mu = \hbar/m_\pi c$ the Compton wavelength of the pion. At large distances they must converge to the one-pion exchange potential

$$V(r) = -\frac{e^{-\mu r}}{\mu r}\left(1 + \frac{3}{\mu r} + \frac{3}{(\mu r)^2}\right).$$

However, the strong repulsive core of the potential renders Hartree-Fock and perturbation theory very difficult. This can be circumvented by introducing the Brückner G-matrix. If

$$H_0\varphi = E\varphi$$

then the Schrödinger equation can be rewritten

$$(H_0 - E)\psi = -V\psi$$

$$\psi = \varphi - \frac{1}{H_0 - E}V\psi.$$

If we define

$$G\varphi = V\psi$$

we obtain the Lippman-Schwinger equation

$$G\varphi = V\varphi - V\frac{1}{H_0 - E}G\varphi.$$

In the nuclear medium is it also known as the Bethe-Goldstone equation. In operator form it becomes

$$\begin{aligned} G &= V + V\frac{1}{E - H_0}G \\ &= V + V\frac{1}{E - H_0}V + \ldots \end{aligned}$$

where the higher-order terms express the hard core that can thus be removed by cutting off the series. An often used realistic force, where the nuclear potential is replaced by the G-matrix, is the Kuo-Brown interaction. Other field-theoretical forces include the Argonne, Bonn, Nijmegen, Paris, and Urbana potentials.

Schematic interactions aim at gaining basic insight in the nuclear force. They are simple mathematical expressions, such as the Yukawa potential, the Gaussian potential, the Square Well, or the Surface Delta Interaction. They stress the short range of the nuclear force and allow for analytic results. To mend their shortcomings, one may add the Bartlett exchange operator

$$P_\sigma = \frac{1}{2}(1 + \boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2).$$

Indeed

$$\mathbf{S} = \mathbf{s}_1 + \mathbf{s}_2$$

$$\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2 = 2(\mathbf{S}^2 - \mathbf{s}_1^2 - \mathbf{s}_2^2)$$

$$\langle \boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2 \rangle = 2S(S+1) - 3$$

such that for a parallel state the spin coordinates are symmetric and for an antiparallel state they are antisymmetric

$$\begin{aligned} P_\sigma \psi_{S=1} &= \psi_{S=1} \\ P_\sigma \psi_{S=0} &= -\psi_{S=0}. \end{aligned}$$

This allows to define the projection operators

$$P_{S,T} = \frac{1}{2}(1 \mp P_\sigma)$$

that select the spin singlet and triplet states, respectively. The Heisenberg operator is the equivalent of the Bartlett term in isospin space.

2.3 Two-Body Matrix Elements

If it is central, we expand the interaction in Legendre polynomials

$$V(|\mathbf{r}_1 - \mathbf{r}_2|) = \sum_{k=0}^{\infty} v_k(r_1, r_2) P_k(\cos\theta_{12})$$

with

$$P_k(\cos\theta_{12}) = \frac{4\pi}{2k+1} \sum_{m=-k}^{k} Y_k^{m*}(\Omega_1) Y_k^m(\Omega_2).$$

We separate the direct and exchange, radial and angular variables in the two-body matrix element as

$$\Delta E(j_1 j_2; J) = \sum_k f_k F^k - (-)^{j_1+j_2-J} \sum_k g_k G^k$$

such that, for the direct term,

$$f_k = \frac{4\pi}{2k+1} \langle j_1 j_2; JM | \mathbf{Y}_k(\Omega_1) \cdot \mathbf{Y}_k(\Omega_2) | j_1 j_2; JM \rangle$$

$$F^k = \int |u_{n_1 l_1}(r_1) u_{n_2 l_2}(r_2)|^2 v_k(r_1, r_2) dr_1 dr_2.$$

The Wigner-Eckart theorem yields

$$\begin{aligned} f_k &= \frac{4\pi}{2k+1} (-)^{J-M} \begin{pmatrix} J & 0 & J \\ -M & 0 & M \end{pmatrix} \langle j_1 j_2; J \| \mathbf{Y}_k(\Omega_1) \cdot \mathbf{Y}_k(\Omega_2) \| j_1 j_2; J \rangle \\ &= \frac{4\pi}{2k+1} \frac{1}{\hat{J}} \langle j_1 j_2; J \| \mathbf{Y}_k(\Omega_1) \cdot \mathbf{Y}_k(\Omega_2) \| j_1 j_2; J \rangle. \end{aligned}$$

The scalar product in this expression is a spherical tensor product of rank zero. Its reduced matrix element gives

$$f_k = \frac{4\pi}{2k+1} (-)^{j_1+j_2+J} \begin{Bmatrix} j_1 & j_2 & J \\ j_2 & j_1 & k \end{Bmatrix} \langle j_1 \| \mathbf{Y}_k \| j_1 \rangle \langle j_2 \| \mathbf{Y}_k \| j_2 \rangle.$$

For the exchange term

$$g_k = \frac{4\pi}{2k+1} (-)^{2j_2+J} \begin{Bmatrix} j_1 & j_2 & J \\ j_1 & j_2 & k \end{Bmatrix} \langle j_1 \| \mathbf{Y}_k \| j_2 \rangle \langle j_2 \| \mathbf{Y}_k \| j_1 \rangle$$

$$G^k = \int u_{n_1 l_1}(r_1) u_{n_2 l_2}(r_2) u_{n_1 l_1}(r_2) u_{n_2 l_2}(r_1) v_k(r_1, r_2) dr_1 dr_2.$$

F^k and G^k we call Slater integrals. We now choose a delta interaction, the multipole expansion of which is

$$\begin{aligned} \delta(\mathbf{r}_1 - \mathbf{r}_2) &= \sum_{k,m} \frac{\delta(r_1 - r_2)}{r_1 r_2} Y_k^{m*}(\hat{\mathbf{r}}_1) Y_k^m(\hat{\mathbf{r}}_2) \\ &= \sum_k \frac{\delta(r_1 - r_2)}{r_1 r_2} \frac{2k+1}{4\pi} P_k(\cos\theta_{12}). \end{aligned}$$

The Slater integrals become

$$\begin{aligned}
F^k &= \frac{2k+1}{4\pi}\int \frac{1}{r^2}|u_{n_1l_1}(r)u_{n_2l_2}(r)|^2 dr \\
&= (2k+1)F^0 \\
G^k &= (2k+1)F^0.
\end{aligned}$$

With the algebra that can be found in the appendix one can then show

$$\Delta E(j_1 j_2; J) = \frac{F^0}{2}(2j_1+1)(2j_2+1)(1+(-)^{l_1+l_2+J})\begin{pmatrix} j_1 & j_2 & J \\ \frac{1}{2} & -\frac{1}{2} & 0 \end{pmatrix}^2$$

$$\Delta E(j^2; J) = \frac{F^0}{2}(2j+1)^2\begin{pmatrix} j & j & J \\ \frac{1}{2} & -\frac{1}{2} & 0 \end{pmatrix}^2$$

so the residual interaction lifts the degeneracy of the J multiplet.

2.4 The Moshinsky Transformation

For a central residual interaction, we want to separate the relative from the centre-of-mass motion. We write

$$\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2, \quad \mathbf{R} = \frac{1}{2}(\mathbf{r}_1 + \mathbf{r}_2)$$

$$\mathbf{p} = \frac{1}{2}(\mathbf{p}_1 - \mathbf{p}_2), \quad \mathbf{P} = \mathbf{p}_1 + \mathbf{p}_2.$$

The harmonic oscillator potential and the kinetic energy become

$$\frac{m}{2}\omega^2(r_1^2 + r_2^2) = \frac{m}{4}\omega^2 r^2 + m\omega^2 R^2$$

$$\frac{1}{2m}(p_1^2 + p_2^2) = \frac{1}{m}p^2 + \frac{1}{4m}P^2$$

such that for the Hamiltonian

$$H = \frac{1}{m}p^2 + \frac{m}{4}\omega^2 r^2 + \frac{1}{4m}P^2 + m\omega^2 R^2.$$

We build the wave function from a relative and a center-of-mass part,

$$\psi = |nlm\rangle|N\Lambda M_\Lambda\rangle$$

related to the independent-particle description by the Moshinsky transformation,

$$|n_1l_1, n_2l_2; LM\rangle = \sum_{n,l,N,\Lambda}\langle nl, N\Lambda; L|n_1l_1, n_2l_2; L\rangle|nl, N\Lambda; LM\rangle.$$

It holds

$$\mathbf{l}_1 + \mathbf{l}_2 = \mathbf{l} + \mathbf{\Lambda}$$

$$2n_1 + \ell_1 + 2n_2 + \ell_2 = 2n + \ell + 2N + \Lambda.$$

2.5 Configuration Mixing

Knowing how to calculate matrix elements, we can set up the secular equation to obtain the eigenvalues. For a basis $|\psi_k^{(0)}\rangle$ with respect to an unperturbed Hamiltonian H_0, the wave function of an excited two-particle state is written

$$|\Psi_p\rangle = \sum_{k=1}^{n} a_{kp}|\psi_k^{(0)}\rangle.$$

We obtain the Schrödinger equation

$$(H_0 + H_{res})\sum_{k=1}^{n} a_{kp}|\psi_k^{(0)}\rangle = E_p \sum_{k=1}^{n} a_{kp}|\psi_k^{(0)}\rangle$$

$$\sum_{k=1}^{n}\langle\psi_l^{(0)}|H_0 + H_{res}|\psi_k^{(0)}\rangle a_{kp} = E_p a_{lp}.$$

We get $l = 1, \ldots, n$ equations for every p, in matrix language

$$\begin{cases} [E_p] \equiv E_p[I] \\ H_{lk} \equiv E_k^{(0)}\delta_{lk} + \langle\psi_l^{(0)}|H_{res}|\psi_k^{(0)}\rangle \end{cases}$$

$$[H][A_p] = [E_p][A_p].$$

The secular equation is

$$\begin{vmatrix} H_{11} - E_p & H_{12} & \ldots & H_{1n} \\ H_{21} & H_{22} - E_p & \ldots & H_{2n} \\ & & \ddots & \\ H_{n1} & H_{n2} & \ldots & H_{nn} - E_p \end{vmatrix} = 0.$$

From the orthonormalisation condition for $|\Psi_p\rangle$ we can diagonalise the Hamiltonian

$$\sum_{l=1}^{n} a_{lp}a_{lp'}^{*} = \delta_{pp'}$$

because the matrix equation above now becomes

$$\sum_{k,l=1}^{n} a_{lp'}^{*} H_{lk} a_{kp} = E_p \delta_{pp'}$$

$$[A_p^{\dagger}][H][A_p] = [E_p]$$

with $[A_p^{\dagger}]$ the conjugate transpose matrix. Common diagonalisation algorithms include the ones by Jacobi, Householder, or Lanczos. To alleviate the discussion we now set $n = 2$ and find the secular equation

$$\lambda^2 - \lambda(H_{11} + H_{22}) - H_{12}^2 + H_{11}H_{22} = 0$$

since $H_{12} = H_{21}$. Thus

$$\lambda = \frac{H_{11} + H_{22}}{2} \pm \frac{1}{2}\sqrt{(H_{11} - H_{22})^2 + 4H_{12}^2}$$

$$\Delta\lambda = \sqrt{(H_{11} - H_{22})^2 + 4H_{12}^2}.$$

If $|H_{11} - H_{22}| \gg |H_{12}|$ then

$$\begin{aligned}
\lambda &\approx \frac{H_{11} + H_{22}}{2} \pm \frac{H_{11} - H_{22}}{2} \left(1 + \frac{2H_{12}^2}{(H_{11} - H_{22})^2}\right) \\
&= \begin{cases} H_{11} + \dfrac{H_{12}^2}{H_{11} - H_{22}} \\ H_{22} + \dfrac{H_{12}^2}{H_{22} - H_{11}}. \end{cases}
\end{aligned}$$

Having determined the energies we calculate the wave functions

$$(H_{11} - \lambda_1)a_{11} + H_{12}a_{21} = 0$$

$$\frac{H_{12}^2}{(H_{11} - \lambda_1)^2} a_{21}^2 + a_{21}^2 = 1$$

$$a_{21} = \frac{1}{\sqrt{1 + \frac{H_{12}^2}{(H_{11} - \lambda_1)^2}}}$$

et cetera. So if $|H_{11} - H_{22}| \gg |H_{12}|$ then

$$\lambda_1 = H_{11} \text{ and } a_{21} = 1$$

implying a level swapping

$$|\Psi_1\rangle = |\psi_2^{(0)}\rangle \text{ and } |\Psi_2\rangle = |\psi_1^{(0)}\rangle.$$

3 Non-Identical Systems

3.1 Isospin Formalism

We define isospin from the Pauli isospin matrices

$$\tau_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ \tau_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},\ \tau_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\boldsymbol{\tau} = (\tau_x, \tau_y, \tau_z)$$

$$\mathbf{t} = \frac{\boldsymbol{\tau}}{2}$$

From spin algebra we may copy the following relations

$$[t_x, t_y] = it_z \text{ and cyclic permutations}$$

$$[\mathbf{t}^2, t_i] = 0,\ i = x, y, z$$

$$t_\pm = t_x \pm it_y$$

$$t_- t_+ = \mathbf{t}^2 - t_z(t_z + 1).$$

We write the proton and neutron wave functions

$$\varphi_n(\mathbf{r}) = \varphi(\mathbf{r}) \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\varphi_p(\mathbf{r}) = \varphi(\mathbf{r}) \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

It then follows

$$t_z \varphi_n(\mathbf{r}) = \frac{1}{2}\varphi_n(\mathbf{r}),\ t_z \varphi_p(\mathbf{r}) = -\frac{1}{2}\varphi_p(\mathbf{r})$$

$$t_- \varphi_n(\mathbf{r}) = \varphi_p(\mathbf{r}),\ t_+ \varphi_p(\mathbf{r}) = \varphi_n(\mathbf{r})$$

$$t_- \varphi_p(\mathbf{r}) = 0,\ t_+ \varphi_n(\mathbf{r}) = 0.$$

We introduce the charge operator

$$\frac{Q}{e} = \frac{1}{2}(1 - \tau_z)$$

$$\frac{Q}{e}\varphi_p(\mathbf{r}) = \varphi_p(\mathbf{r}),\ \frac{Q}{e}\varphi_n(\mathbf{r}) = 0.$$

For a many-nucleon system we define

$$\mathbf{T} = \sum_{i=1}^{A} \mathbf{t}_i$$

$$T_z = \sum_{i=1}^{A} (t_z)_i$$

so we obtain

$$T_z = \frac{1}{2}(N - Z)$$

and for a given nucleus

$$T = |T_z|, |T_z| + 1, \ldots, \frac{A}{2}.$$

Within an isospin multiplet it holds

$$T_\pm |T, T_z\rangle = \sqrt{T(T+1) - T_z(T_z \pm 1)}\; |T, T_z \pm 1\rangle.$$

3.2 Charge Independence

Conservation of charge implies

$$[H, T_z] = 0$$

while for charge independence

$$[H, T_\pm] = 0.$$

The isoscalar Hamiltonian becomes

$$H = -\sum_{i=1}^{N} \frac{\hbar^2}{2m_n}(\nabla_n^2)_i - \sum_{i=1}^{Z} \frac{\hbar^2}{2m_p}(\nabla_p^2)_i + \frac{1}{2}\sum_{i,j=1}^{A} V_{i,j} + \sum_{i<j}^{Z} \frac{e^2}{|\mathbf{r}_i - \mathbf{r}_j|}.$$

However, introducing a mass difference

$$m = \frac{m_n + m_p}{2}$$
$$\Delta m = m_n - m = m - m_p$$
$$\frac{1}{m_n} \approx \frac{1}{m}(1 - \frac{\Delta m}{m})$$
$$\frac{1}{m_p} \approx \frac{1}{m}(1 + \frac{\Delta m}{m})$$

and making use of

$$\frac{1}{4}(1 - (\tau_z)_i - (\tau_z)_j + (\tau_z)_i(\tau_z)_j) = \begin{cases} 1 \text{ for } i, j \text{ protons} \\ 0 \text{ otherwise} \end{cases}$$

we should rewrite

$$\begin{aligned} H \approx &-\sum_{i=1}^{A} \frac{\hbar^2}{2m}\nabla_i^2 + \sum_{i=1}^{A} \frac{\hbar^2 \Delta m}{m^2}(t_z)_i \nabla_i^2 + \frac{1}{2}\sum_{i,j=1}^{A} V_{i,j} \\ &+ \sum_{i<j}^{A} \frac{e^2}{4|\mathbf{r}_i - \mathbf{r}_j|} - \sum_{i<j}^{A} \frac{e^2}{2|\mathbf{r}_i - \mathbf{r}_j|}((t_z)_i + (t_z)_j) + \sum_{i<j}^{A} \frac{e^2}{|\mathbf{r}_i - \mathbf{r}_j|}(t_z)_i(t_z)_j. \end{aligned}$$

The Hamiltonian is now seen to contain isovector and isotensor terms and is no longer charge independent. Therefore

$$H = H_0^{(2)} + H_0^{(1)} + H_0^{(0)}$$

and we apply the Wigner-Eckart theorem

$$\begin{aligned}\langle T,T_z|H|T,T_z\rangle &= (-)^{T-T_z}\begin{pmatrix} T & 0 & T \\ -T_z & 0 & T_z\end{pmatrix}\langle T\|\mathbf{H}^{(0)}\|T\rangle \\ &+(-)^{T-T_z}\begin{pmatrix} T & 1 & T \\ -T_z & 0 & T_z\end{pmatrix}\langle T\|\mathbf{H}^{(1)}\|T\rangle \\ &+(-)^{T-T_z}\begin{pmatrix} T & 2 & T \\ -T_z & 0 & T_z\end{pmatrix}\langle T\|\mathbf{H}^{(2)}\|T\rangle\end{aligned}$$

such that we find the isospin dependence of the energy

$$\langle T,T_z|H|T,T_z\rangle = a(T) + b(T)T_z + c(T)T_z^2.$$

This is called the isobaric multiplet mass equation (IMME).

3.3 Isospin Wave Functions

The isospin spinors $\zeta^{1/2}_{t_z}$ allow us to construct the two-nucleon isospin eigenvectors

$$\zeta(\frac{1}{2}\frac{1}{2};TT_z) = \sum_{t_z,t'_z}\langle\frac{1}{2}t_z,\frac{1}{2}t'_z|TT_z\rangle\zeta^{1/2}_{t_z}(1)\zeta^{1/2}_{t'_z}(2)$$

in particular the symmetric triplet

$$\zeta(\frac{1}{2}\frac{1}{2};1,1) = \zeta^{1/2}_{+1/2}(1)\zeta^{1/2}_{+1/2}(2)$$

$$\zeta(\frac{1}{2}\frac{1}{2};1,-1) = \zeta^{1/2}_{-1/2}(1)\zeta^{1/2}_{-1/2}(2)$$

$$\zeta(\frac{1}{2}\frac{1}{2};1,0) = \frac{1}{\sqrt{2}}(\zeta^{1/2}_{+1/2}(1)\zeta^{1/2}_{-1/2}(2) + \zeta^{1/2}_{-1/2}(1)\zeta^{1/2}_{+1/2}(2))$$

and the antisymmetric singlet

$$\zeta(\frac{1}{2}\frac{1}{2};0,0) = \frac{1}{\sqrt{2}}(\zeta^{1/2}_{+1/2}(1)\zeta^{1/2}_{-1/2}(2) - \zeta^{1/2}_{-1/2}(1)\zeta^{1/2}_{+1/2}(2)).$$

For the proton-neutron case, the spatial-spin wave function can be written

$$\psi^{\pm}_{pn}(j_aj_b;JM) = N\sum_{m_a,m_b}\langle j_am_a,j_bm_b|JM\rangle\left(\varphi_a(1)\varphi_b(2)\pm\varphi_a(2)\varphi_b(1)\right).$$

We then impose antisymmetry on the full wave function

$$\begin{aligned}&\psi^{+}_{pn}(j_aj_b;JM)\\ &\quad= N\sum_{m_a,m_b}\langle j_am_a,j_bm_b|JM\rangle\left(\varphi_a(1)\varphi_b(2)+\varphi_a(2)\varphi_b(1)\right)\zeta(\frac{1}{2}\frac{1}{2};0,0)\end{aligned}$$

$$\begin{aligned}&\psi^{-}_{pn}(j_aj_b;JM)\\ &\quad= N\sum_{m_a,m_b}\langle j_am_a,j_bm_b|JM\rangle\left(\varphi_a(1)\varphi_b(2)-\varphi_a(2)\varphi_b(1)\right)\zeta(\frac{1}{2}\frac{1}{2};1,0).\end{aligned}$$

For two particles in a single j-shell we find

$$\psi(j^2; JM, TT_z) = N'(1-(-)^{2j-J+1-T}) \times \sum_{m,m'} \langle jm, jm'|JM\rangle \sum_{t_z,t'_z} \langle \frac{1}{2}t_z, \frac{1}{2}t'_z|TT_z\rangle \varphi_{jm}(1)\varphi_{jm'}(2)\zeta^{1/2}_{t_z}(1)\zeta^{1/2}_{t'_z}(2)$$

from which it follows

$$J+T = \text{ odd.}$$

3.4 Two-Body Matrix Elements

For an interaction

$$V = V_0 \delta(\mathbf{r}_1 - \mathbf{r}_2)$$

with Slater integral

$$A_0 = \frac{V_0}{4\pi} \int \frac{1}{r^2} |u_{n_1 l_1}(r) u_{n_2 l_2}(r)|^2 dr$$

one can prove that

$$\langle j_1 j_2; JM, TT_z|V|j_1 j_2; JM, TT_z\rangle = \frac{A_0}{2}(2j_1+1)(2j_2+1) \times \left((1-(-)^{\ell_1+\ell_2+J+T}) \begin{pmatrix} j_1 & j_2 & J \\ \frac{1}{2} & -\frac{1}{2} & 0 \end{pmatrix}^2 + (1+(-)^T) \begin{pmatrix} j_1 & j_2 & J \\ \frac{1}{2} & \frac{1}{2} & -1 \end{pmatrix}^2 \right)$$

and if $j_1 = j_2$

$$\langle j^2; JM, TT_z|V|j^2; JM, TT_z\rangle = \frac{A_0}{4}(2j+1)^2 \times \left((1-(-)^{J+T}) \begin{pmatrix} j & j & J \\ \frac{1}{2} & -\frac{1}{2} & 0 \end{pmatrix}^2 + (1+(-)^T) \begin{pmatrix} j & j & J \\ \frac{1}{2} & \frac{1}{2} & -1 \end{pmatrix}^2 \right).$$

From this one can derive the parabolic rule of Paar

$$\Delta E(J) = -\frac{3}{4} \frac{(u^2(p)-v^2(p))(u^2(n)-v^2(n))}{2j_p 2j_n(2j_p+2)(2j_n+2)} (J(J+1)-j_p(j_p+1)-j_n(j_n+1) + (J(J+1)-j_p(j_p+1)-j_n(j_n+1))^2)$$

where v^2 is the occupation probability of a given orbital and

$$u^2+v^2 = 1.$$

The parabola is concave down for particle-particle and hole-hole configurations but it is concave up otherwise.

Summary: large-scale shell-model calculations

- One determines the nearby closed shells so as to fix the number of valence protons n_p and neutrons n_n.
- The active particles (or holes) define the single-particle orbitals j_{p_1}, ... and j_{n_1}, ... that determine the properties at low energy when constructing the model space.
- One sets up the model space and the configurations that span the space for each J^π value. The basis configurations are denoted $|(j_{p_1}j_{p_2}...)^{n_p}J_p, (j_{n_1}j_{n_2}...)^{n_n}J_n; JM\rangle$, meaning that one constructs the proton n_p particle state J_p and multiplies it by the neutron n_n particle state J_n, coupling both to total spin J. The basis has $n(J)$ basis configurations.
- Starting from the single-particle energies $\varepsilon_{j_{p_i}}$, $\varepsilon_{j_{n_i}}$ and the two-body matrix elements for identical and non-identical nucleons, one builds up the energy matrix $[H_{ij}]$ and diagonalises the $n(J) \times n(J)$ energy matrix. Thus one obtains the $n(J)$ energy eigenvalues and $n(J)$ corresponding eigenfunctions.
- With the wave functions $\psi_i(J_i^{\pi_i})$ and $\psi_f(J_f^{\pi_f})$ we calculate the physical observables that we compare with data and so improve iteratively upon the procedure above.

ELECTROMAGNETIC PROPERTIES IN THE SHELL MODEL

4 The Electromagnetic Field

4.1 Introduction

For a photon described by a plane wave and energy relation

$$\frac{1}{\sqrt{(2\pi)^3}} e^{i\mathbf{k}\cdot\mathbf{r}}$$
$$E = \hbar ck$$

the density of states in an elementary volume $V = (2\pi)^3$ is given by

$$\begin{aligned} \frac{dN}{dE} &= \frac{1}{(2\pi\hbar)^3} \frac{d}{dE} \int d^3x \int d^3p \\ &= \frac{1}{\hbar^3} \frac{dp}{dE} \frac{d}{dp} \int p^2 dp d\Omega \\ &= \frac{k^2}{\hbar c} \int d\Omega. \end{aligned}$$

Fermi's Golden Rule

$$w = \frac{2\pi}{\hbar} \frac{dN}{dE} |M_{fi}|^2$$

can be written

$$dw = \frac{2\pi}{\hbar} \frac{k^2}{\hbar c} |M_{fi}|^2 d\Omega.$$

The interaction Hamiltonian is given by[1]

$$H = -\frac{1}{c} \int \mathbf{j} \cdot \mathbf{A} \, d\mathbf{r}.$$

We build the electrical field from the helicity unit vectors $\hat{\mathbf{e}}_{k,\lambda=\pm 1}$

$$\begin{aligned} \boldsymbol{\mathcal{E}} &= -\frac{1}{c} \frac{\partial \mathbf{A}}{\partial t} \\ &= \frac{i\omega}{c} \sum_\lambda A_0 \frac{e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)}}{\sqrt{2}} \hat{\mathbf{e}}_{k\lambda} + \text{ cc} \end{aligned}$$

such that

$$\mathcal{E}^2 = \frac{\omega^2}{c^2} A_0^2$$

while for the energy density of the photon

$$\frac{\mathcal{E}^2}{4\pi} = \frac{\hbar\omega}{(2\pi)^3}.$$

Therefore

$$\mathbf{A}(\mathbf{r}, t) = \sqrt{\frac{\hbar c^2}{2\pi^2 \omega}} \sum_\lambda \frac{e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)}}{\sqrt{2}} \hat{\mathbf{e}}_{k\lambda} + \text{ cc}.$$

[1] We use Gaussian or cgs units. For the conversion SI $\leftrightarrow$ cgs, use $\varepsilon_0 \leftrightarrow 1/4\pi$ and $B \leftrightarrow B/c$

4.2 Second Quantisation

In second quantisation the previous formula becomes

$$\mathbf{A}(\mathbf{r},t) = \sqrt{\frac{\hbar c}{2\pi^2}} \sum_\lambda \int \frac{d\mathbf{k}}{\sqrt{2k}} \, (b_\lambda(\mathbf{k}) \, \hat{\mathbf{e}}_{k\lambda} e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} + b^\dagger_\lambda(\mathbf{k}) \, \hat{\mathbf{e}}^*_{k\lambda} e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega t)}).$$

For the creation and annihilation operators it holds

$$\begin{aligned} [b_\lambda(\mathbf{k}), b^\dagger_{\lambda'}(\mathbf{k}')] &= \delta(\mathbf{k}-\mathbf{k}')\delta_{\lambda\lambda'} \\ &= \frac{\delta(k-k')}{kk'}\delta(\hat{\mathbf{k}}-\hat{\mathbf{k}}')\delta_{\lambda\lambda'}. \end{aligned}$$

If the photon takes an angular momentum, we need to project it onto the axis that defines the components m, l by means of the Wigner D matrices. For a photon moving along $\mathbf{k}$ with helicity λ we write

$$\begin{aligned} |\mathbf{k}\lambda\rangle &= \sum_l |kl\lambda\rangle\langle kl\lambda|\mathbf{k}\lambda\rangle \\ &= \sum_{lm} D^l_{m\lambda}(\hat{\mathbf{k}})|klm\rangle\langle kl\lambda|\mathbf{k}\lambda\rangle. \end{aligned}$$

We use the orthogonality relation

$$\int D^{l*}_{m\lambda}(\hat{\mathbf{k}}) D^{l'}_{m'\lambda}(\hat{\mathbf{k}}) \, d\Omega_k = \frac{4\pi}{2l+1}\delta_{ll'}\delta_{mm'}$$

to find that

$$|klm\rangle\langle kl\lambda|\mathbf{k}\lambda\rangle = \frac{2l+1}{4\pi} \int D^{l*}_{m\lambda}(\hat{\mathbf{k}})|\mathbf{k}\lambda\rangle \, d\Omega_k$$

and thus

$$\langle klm|klm\rangle\langle kl\lambda|\mathbf{k}\lambda\rangle^2 = \left(\frac{2l+1}{4\pi}\right)^2 \int D^{l*}_{m\lambda}(\hat{\mathbf{k}}) D^l_{m\lambda}(\hat{\mathbf{k}}')\langle\mathbf{k}'\lambda|\mathbf{k}\lambda\rangle \, d\Omega_k d\Omega_{k'}$$

$$\langle kl\lambda|\mathbf{k}\lambda\rangle = \sqrt{\frac{2l+1}{4\pi}}.$$

Because of

$$|\mathbf{k}\lambda\rangle = \sum_{lm} \sqrt{\frac{2l+1}{4\pi}} \, D^l_{m\lambda}(\hat{\mathbf{k}}) \, |klm;\lambda\rangle$$

we define by analogy

$$b^\dagger_\lambda(\mathbf{k}) = \sum_{lm} \sqrt{\frac{2l+1}{4\pi}} \, D^l_{m\lambda}(\hat{\mathbf{k}}) \, b^\dagger_{lm;\lambda}(k).$$

The electromagnetic field now appears like

$$\mathbf{A}(\mathbf{r},t) = \sqrt{\frac{\hbar c}{8\pi^3}} \sum_{lm\lambda} \sqrt{2l+1} \int \frac{k^2 dk}{\sqrt{2k}} \, (b_{lm;\lambda} \, \mathbf{f}_{lm;\lambda} e^{-i\omega t} + b^\dagger_{lm;\lambda} \, \mathbf{f}^*_{lm;\lambda} e^{i\omega t})$$

with

$$\mathbf{f}_{lm;\lambda} = \int D^{l*}_{m\lambda}(\hat{\mathbf{k}})\ \hat{\mathbf{e}}_{k\lambda} e^{i\mathbf{k}\cdot\mathbf{r}} d\Omega_k.$$

We would like to decompose the field in electric and magnetic multipoles. We consider the reflection operator P

$$P|\mathbf{k}\lambda\rangle = |-\mathbf{k},-\lambda\rangle$$

$$P|klm;\lambda\rangle = (-)^l|klm;-\lambda\rangle$$

and therefore

$$Pb^\dagger_{lm;\lambda}P^{-1} = (-)^l b^\dagger_{lm;-\lambda}.$$

Introducing the polarisation variable $\sigma = 0, 1$, the creation operator that takes parity properly into account is expressed by

$$b^{(\sigma)\dagger}_{lm} = \frac{1}{\sqrt{2}}(b^\dagger_{lm;1} + (-)^\sigma b^\dagger_{lm;-1})$$

such that

$$Pb^{(\sigma)\dagger}_{lm}P^{-1} = (-)^{l+\sigma} b^{(\sigma)\dagger}_{lm}.$$

For electric multipoles of order 2^l, $\sigma = 0$ and the parity is $(-)^l$ while for magnetic multipoles of order 2^l, $\sigma = 1$ and the parity is $(-)^{l+1}$. The photon is written

$$|\mathbf{k}\sigma\rangle = \frac{1}{\sqrt{2}}(|\mathbf{k},1\rangle + (-)^\sigma|\mathbf{k},-1\rangle).$$

Defining

$$\mathbf{f}^{(\sigma)}_{lm} = \frac{1}{\sqrt{2}}(\mathbf{f}_{lm;1} + (-)^\sigma \mathbf{f}_{lm;-1})$$

we obtain

$$\begin{aligned}
\sum_\sigma b^{(\sigma)}_{lm}\mathbf{f}^{(\sigma)}_{lm} &= \frac{1}{2}\sum_\sigma (b_{lm;1} + (-)^\sigma b_{lm;-1})(\mathbf{f}_{lm;1} + (-)^\sigma \mathbf{f}_{lm;-1}) \\
&= \sum_\lambda b_{lm;\lambda}\mathbf{f}_{lm;\lambda}
\end{aligned}$$

such that for the electromagnetic field

$$\mathbf{A}(\mathbf{r},t) = \sqrt{\frac{\hbar c}{8\pi^3}} \sum_{lm\sigma} \sqrt{2l+1} \int \frac{k^2 dk}{\sqrt{2k}}\ (b^{(\sigma)}_{lm}\ \mathbf{f}^{(\sigma)}_{lm} e^{-i\omega t} + b^{(\sigma)\dagger}_{lm}\ \mathbf{f}^{(\sigma)*}_{lm} e^{i\omega t}).$$

We note

$$[b^{(\sigma)}_{lm}(k), b^{(\sigma')\dagger}_{l'm'}(k')] = \frac{\delta(k-k')}{kk'}\delta_{ll'}\delta_{mm'}\delta_{\sigma\sigma'}.$$

4.3 Matrix Elements

The matrix element can be expanded as

$$\begin{aligned}
M_{fi}(\mathbf{k}\sigma) &= \langle f;\mathbf{k}\sigma|H|i;00\rangle \\
&= \sum_{lm\sigma'} \langle \mathbf{k}\sigma|lm\sigma'\rangle\langle f;lm\sigma'|H|i;000\rangle \\
&= \sum_{lm\sigma'} \langle \mathbf{k}\sigma|lm\sigma'\rangle\langle f;000|[b^{(\sigma')}_{lm},H]|i;000\rangle \\
&= -\frac{1}{c}\sum_{lm\sigma'} \langle \mathbf{k}\sigma|lm\sigma'\rangle \int d\mathbf{r}\ \langle f|\mathbf{j}|i\rangle \cdot \langle 000|[b^{(\sigma')}_{lm},\mathbf{A}]|000\rangle.
\end{aligned}$$

We insert the expression for the field

$$\begin{aligned}
&\langle 000|[b^{(\sigma')}_{lm},\mathbf{A}]|000\rangle \\
&\qquad = \langle 000|\sqrt{\frac{\hbar c}{8\pi^3}}\sum_{l'm'\sigma}\sqrt{2l'+1}\int \frac{k^2 dk}{\sqrt{2k}}\ [b^{(\sigma')}_{lm}, b^{(\sigma)\dagger}_{l'm'}]\ \mathbf{f}^{(\sigma)*}_{l'm'} e^{i\omega t}|000\rangle \\
&\qquad = \langle 000|\sqrt{\frac{\hbar c}{16\pi^3 k}}\sqrt{2l+1}\ \mathbf{f}^{(\sigma')*}_{lm} e^{i\omega t}|000\rangle
\end{aligned}$$

and we find

$$M_{fi}(\mathbf{k}\sigma) = -\sqrt{\frac{\hbar}{16\pi^3 kc}}\sum_{lm\sigma'}\langle \mathbf{k}\sigma|lm\sigma'\rangle\sqrt{2l+1}\ \langle f|\int \mathbf{f}^{(\sigma')*}_{lm}\cdot \mathbf{j}\ d\mathbf{r}|i\rangle.$$

We decompose according to helicity

$$\begin{aligned}
\langle \mathbf{k}\sigma|lm\sigma'\rangle &= \frac{1}{2}(\langle \mathbf{k},1| + (-)^{\sigma}\langle \mathbf{k},-1|)(|klm;1\rangle + (-)^{\sigma'}|klm;-1\rangle) \\
&= \frac{1}{2}(\langle \mathbf{k},1|klm;1\rangle + (-)^{\sigma+\sigma'}\langle \mathbf{k},-1|klm;-1\rangle)
\end{aligned}$$

and recalling the earlier result

$$|klm;\lambda\rangle = \sqrt{\frac{2l+1}{4\pi}}\int D^{l*}_{m\lambda}(\hat{\mathbf{k}})|\mathbf{k}\lambda\rangle\ d\Omega_k$$

we calculate

$$\begin{aligned}
&\langle \mathbf{k}\sigma|lm\sigma'\rangle \\
&\quad = \sqrt{\frac{2l+1}{16\pi}}\int (D^{l*}_{m,1}\langle \mathbf{k},1|\mathbf{k},1\rangle + (-)^{\sigma+\sigma'}D^{l*}_{m,-1}\langle \mathbf{k},-1|\mathbf{k},-1\rangle)k^2 dk d\Omega_k \\
&\quad = \sqrt{\frac{2l+1}{16\pi}}(D^{l*}_{m,1} + (-)^{\sigma+\sigma'}D^{l*}_{m,-1}).
\end{aligned}$$

The matrix element becomes

$$M_{fi}(\mathbf{k}\sigma) = -\frac{1}{16\pi^2}\sqrt{\frac{\hbar}{kc}}\sum_{lm\sigma'}(2l+1)(D^{l*}_{m,1} + (-)^{\sigma+\sigma'}D^{l*}_{m,-1})\langle f|M^{(\sigma')}_{lm}|i\rangle$$

with the nuclear structure in the transition determined by

$$M_{lm}^{(\sigma')} = \int \mathbf{f}_{lm}^{(\sigma')*} \cdot \mathbf{j} \, d\mathbf{r}.$$

We apply the Wigner-Eckart theorem

$$\langle f | M_{lm}^{(\sigma')} | i \rangle = (-)^{J_f - M_f} \begin{pmatrix} J_f & l & J_i \\ -M_f & m & M_i \end{pmatrix} \langle J_f \| M_l^{(\sigma')} \| J_i \rangle$$

and observe the conditions

$$|J_f - J_i| \leq l \leq J_f + J_i$$

$$M_f = M_i + m$$

to obtain the transition rate

$$\begin{aligned} \frac{dw}{d\Omega} = \frac{k}{128\pi^3 \hbar c^2} \sum_{ll'\sigma'\sigma''} (2l+1)(2l'+1) \begin{pmatrix} J_f & l & J_i \\ -M_f & m & M_i \end{pmatrix} \begin{pmatrix} J_f & l' & J_i \\ -M_f & m & M_i \end{pmatrix} \\ \times (D_{m,1}^{l*} + (-)^{\sigma+\sigma'} D_{m,-1}^{l*})(D_{m,1}^{l'} + (-)^{\sigma+\sigma''} D_{m,-1}^{l'}) \\ \times \langle J_f \| M_l^{(\sigma')} \| J_i \rangle \langle J_f \| M_{l'}^{(\sigma'')} \| J_i \rangle. \end{aligned}$$

If the initial system is equally populated in all its magnetic substates and the final magnetic substate is not detected, we must average over M_i and sum over M_f. We rely on the unitarity of the Wigner symbols to write

$$\begin{aligned} \frac{dw}{d\Omega} = \frac{k}{128\pi^3 \hbar c^2} \sum_{l\sigma'\sigma''} \frac{2l+1}{2J_i+1} (D_{m,1}^{l*} + (-)^{\sigma+\sigma'} D_{m,-1}^{l*})(D_{m,1}^{l} + (-)^{\sigma+\sigma''} D_{m,-1}^{l}) \\ \times \langle J_f \| M_l^{(\sigma')} \| J_i \rangle \langle J_f \| M_l^{(\sigma'')} \| J_i \rangle. \end{aligned}$$

Conservation of parity $l + \sigma'$ implies that only one σ' survives for every l, as determined by the final polarisation σ

$$\frac{dw}{d\Omega} = \frac{k}{128\pi^3 \hbar c^2} \sum_{l} \frac{2l+1}{2J_i+1} |D_{m,1}^{l} + (-)^{\sigma+\sigma'} D_{m,-1}^{l}|^2 \langle J_f \| M_l^{(\sigma')} \| J_i \rangle^2.$$

If we do not detect the polarisation nor the magnetic quantum number of the emitted photon we sum over σ and m. We use the relation

$$\sum_m D_{m\mu}^l D_{m\mu'}^{l*} = \delta_{\mu\mu'}$$

to arrive at

$$\begin{aligned} \frac{dw}{d\Omega} &= \frac{k}{128\pi^3 \hbar c^2} \sum_{lm\sigma} \frac{2l+1}{2J_i+1} |D_{m,1}^{l} + (-)^{\sigma+\sigma'} D_{m,-1}^{l}|^2 \langle J_f \| M_l^{(\sigma')} \| J_i \rangle^2 \\ &= \frac{k}{64\pi^3 \hbar c^2} \sum_{lm} \frac{2l+1}{2J_i+1} (|D_{m,1}^{l}|^2 + |D_{m,-1}^{l}|^2) \langle J_f \| M_l^{(\sigma')} \| J_i \rangle^2 \\ &= \frac{k}{32\pi^3 \hbar c^2} \sum_{l} \frac{2l+1}{2J_i+1} \langle J_f \| M_l^{(\sigma')} \| J_i \rangle^2. \end{aligned}$$

5 Electromagnetic Transitions and Moments

5.1 Transition Probability

In the expression for the nuclear matrix element

$$M_{lm}^{(\sigma)} = \int \mathbf{f}_{lm}^{(\sigma)*} \cdot \mathbf{j}\, d\mathbf{r}$$

we can develop the plane wave $\mathbf{f}_{lm}^{(\sigma)*}$ in spherical Bessel functions

$$e^{i\mathbf{k}\cdot\mathbf{r}} = 4\pi \sum_{\lambda\mu} i^{\lambda}\, Y_{\lambda\mu}(\hat{\mathbf{r}}) Y_{\lambda\mu}^{*}(\hat{\mathbf{k}}) j_{\lambda}(kr).$$

For the long wavelength limit $kr \ll 1$ (with r the nuclear radius, the transition energy should be $E < 3$ MeV) and

$$j_{\lambda}(kr) \approx \frac{(kr)^{\lambda}}{(2\lambda+1)!!}.$$

Working through it with some application one can show that

$$M_{lm}^{(\sigma)} \approx \frac{8\pi^2 c(-ik)^l}{(2l+1)!!}\sqrt{\frac{l+1}{\pi l(2l+1)}}\, Q_{lm}^{(\sigma)}$$

with

$$Q_{lm}^{(E)} = \int r^l Y_{lm}^{*}\, \rho\, d\mathbf{r}$$

$$Q_{lm}^{(M)} = -\frac{1}{c}\int r^l Y_{lm}^{*}\left(\frac{1}{l+1}\boldsymbol{\nabla}\cdot(\mathbf{r}\times\mathbf{j}) + c\boldsymbol{\nabla}\cdot\mathbf{m}\right) d\mathbf{r}.$$

The transition rate becomes

$$\frac{dw}{d\Omega} = \frac{2}{\hbar}\sum_{l} \frac{l+1}{l((2l+1)!!)^2}\frac{k^{2l+1}}{2J_i+1}\langle J_f \| Q_l^{(\sigma')} \| J_i\rangle^2$$

and if we introduce the reduced transition probability

$$B(\sigma l) = \frac{1}{(2J_i+1)e^2}\langle J_f \| Q_l^{(\sigma)} \| J_i\rangle^2$$

then the partial transition rates are given by

$$w_{\sigma l} = \frac{8\pi e^2}{\hbar}\frac{l+1}{l((2l+1)!!)^2} k^{2l+1} B(\sigma l).$$

The transition rates thus show a $2l+1$ dependence on the energy that is removed from the reduced rates.

5.2 Single-Particle Estimates

If we consider the nucleus as a system of point charges then

$$\rho(\mathbf{r}) = \sum_{i=1}^{A} \tilde{e}_i \delta(\mathbf{r} - \mathbf{r}_i)$$

$$\mathbf{j}(\mathbf{r}) = \sum_{i=1}^{A} \frac{\tilde{e}_i}{m} \mathbf{p}_i \delta(\mathbf{r} - \mathbf{r}_i)$$

$$\mathbf{m}(\mathbf{r}) = \sum_{i=1}^{A} \mu_i \mathbf{s}_i \delta(\mathbf{r} - \mathbf{r}_i).$$

For electric transitions

$$\begin{aligned} B(El) &= \frac{1}{(2J_i+1)e^2} \langle J_f \| \int \rho(\mathbf{r}) r^l \mathbf{Y}_l^* \, d\mathbf{r} \| J_i \rangle^2 \\ &= \frac{1}{2J_i+1} \langle J_f \| \sum_i \frac{\tilde{e}_i}{e} r_i^l \mathbf{Y}_l^*(\theta_i, \varphi_i) \| J_i \rangle^2 . \end{aligned}$$

For magnetic transitions we use integration by parts

$$\begin{aligned} B(Ml) &= \frac{1}{(2J_i+1)e^2} \langle J_f \| -\frac{1}{c} \int r^l \mathbf{Y}_l^* \left(\frac{\boldsymbol{\nabla} \cdot (\mathbf{r} \times \mathbf{j})}{l+1} + c \boldsymbol{\nabla} \cdot \mathbf{m} \right) d\mathbf{r} \| J_i \rangle^2 \\ &= \frac{1}{(2J_i+1)e^2} \langle J_f \| \frac{1}{c} \int \boldsymbol{\nabla}(r^l \mathbf{Y}_l^*) \cdot \left(\frac{\mathbf{r} \times \mathbf{j}}{l+1} + c\mathbf{m} \right) d\mathbf{r} \| J_i \rangle^2 \\ &= \frac{1}{(2J_i+1)e^2} \langle J_f \| \sum_i \boldsymbol{\nabla}_i (r_i^l \mathbf{Y}_l^*) \cdot \left(\frac{\tilde{e}_i \hbar}{(l+1)mc} \mathbf{l}_i + \mu_i \mathbf{s}_i \right) \| J_i \rangle^2 \\ &= \frac{1}{2J_i+1} \langle J_f \| \sum_i \frac{\hbar}{mc} \boldsymbol{\nabla}_i (r_i^l \mathbf{Y}_l^*) \cdot \left(\frac{\tilde{e}_i}{e(l+1)} \mathbf{l}_i + \frac{1}{2} g_i^s \mathbf{s}_i \right) \| J_i \rangle^2 \end{aligned}$$

where we have introduced the spin gyromagnetic ratio g^s, expressing the magnetic moment in units of nuclear magneton μ_N

$$\mu = \mu_N g^s = \frac{e\hbar}{2mc} g^s.$$

For one particle outside the core the electric transition rate becomes

$$\begin{aligned} B(El) &= \frac{1}{2J_i+1} \langle J_f L_f \| \frac{\tilde{e}}{e} r^l \mathbf{Y}_l^* \| J_i L_i \rangle^2 \\ &= \left(\frac{\tilde{e}}{e} \right)^2 \frac{1}{2J_i+1} \langle J_f L_f \| \mathbf{Y}_l^* \| J_i L_i \rangle^2 \langle r^l \rangle^2 \\ &= \frac{1}{4\pi} \left(\frac{\tilde{e}}{e} \right)^2 (2J_f+1)(2l+1) \begin{pmatrix} J_f & l & J_i \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}^2 \langle r^l \rangle^2 . \end{aligned}$$

We choose a radial wave function given by

$$R(r) = \begin{cases} C & \text{for } 0 < r < R \\ 0 & \text{for } r \geq R \end{cases}$$

for which the normalisation condition

$$C^2 \int_0^R r^2 dr = 1$$

yields

$$C = \sqrt{\frac{3}{R^3}}.$$

Therefore

$$\begin{aligned}
\langle r^l \rangle &= \frac{3}{R^3} \int_0^R r^{l+2} dr \\
&= \frac{3}{l+3} R^l.
\end{aligned}$$

If we consider a transition $L + 1/2 \to 1/2$ then $L = l$ and the single-particle estimate is called the Weisskopf estimate. Assuming $\tilde{e} = e$,

$$\begin{aligned}
W_E(L) &= \frac{1}{2\pi}(2L+1) \begin{pmatrix} \frac{1}{2} & L & L+\frac{1}{2} \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}^2 \left(\frac{3}{L+3}\right)^2 R^{2L} \\
&= \frac{1.2^{2L}}{4\pi} \left(\frac{3}{L+3}\right)^2 A^{2L/3} \, (fm)^{2L}
\end{aligned}$$

with the approximation

$$R = 1.2 \, A^{1/3} fm.$$

The nabla operator reduces the angular momentum rank of the magnetic transitions with one unit such that the photon carries away $\Delta L = L - 1$. A transition $L + 1/2 \to 1/2$ is thus a spin-flip event and one can show that the Weisskopf estimate

$$W_M(L) = \frac{10}{\pi}(1.2)^{2L-2} \left(\frac{3}{L+3}\right)^2 A^{(2L-2)/3} \left(\frac{\hbar}{2mc}\right)^2 (fm)^{2L-2}.$$

While for free nucleons

$$\tilde{e}_p = e, \; g_p^s = 5.58$$

$$\tilde{e}_n = 0, \; g_n^s = -3.82$$

it appears that experimental values are better reproduced by introducing effective charges, for instance

$$\tilde{e}_p \approx (1 - \frac{Z}{A})e$$

$$\tilde{e}_n \approx -\frac{Z}{A}e.$$

5.3 Quadrupole Moment

The electrostatic quadrupole moment is defined by

$$Q = \int (3z^2 - r^2)\rho(\mathbf{x}) d^3x$$

or for a quantummechanical system of point charges, where by convention it is divided by the electrical charge and evaluated for the maximally aligned state,

$$Q = \langle J, M = J | \sqrt{\frac{16\pi}{5}} \sum_i \frac{\tilde{e}_i}{e} r_i^2 Y_{20}(\theta_i, \varphi_i) | J, M = J \rangle .$$

The Wigner-Eckart theorem yields

$$\begin{aligned}
Q &= \sqrt{\frac{16\pi}{5}} \begin{pmatrix} J & 2 & J \\ -J & 0 & J \end{pmatrix} \langle J \| \sum_i \frac{\tilde{e}_i}{e} r_i^2 \mathbf{Y}_2 \| J \rangle \\
&= \sqrt{\frac{16\pi}{5}} \frac{2(3J^2 - J(J+1))}{\sqrt{(2J+3)(2J+2)(2J+1)2J(2J-1)}} \langle J \| \sum_i \frac{\tilde{e}_i}{e} r_i^2 \mathbf{Y}_2 \| J \rangle \\
&= \sqrt{\frac{16\pi}{5}} \left(\frac{J(2J-1)}{(2J+3)(2J+1)(J+1)} \right)^{1/2} \langle J \| \sum_i \frac{\tilde{e}_i}{e} r_i^2 \mathbf{Y}_2 \| J \rangle .
\end{aligned}$$

We insert the reduced matrix element

$$\langle J \| \mathbf{Y}_2 \| J \rangle = \sqrt{\frac{5}{4\pi}} \frac{(\frac{3}{4} - J(J+1))(2J+1)}{\sqrt{J(J+1)(2J-1)(2J+1)(2J+3)}}$$

to see that, for one particle outside the core,

$$\begin{aligned}
Q &= 2 \frac{(\frac{3}{4} - J(J+1))}{(J+1)(2J+3)} \frac{\tilde{e}}{e} \langle r^2 \rangle \\
&= -\frac{J - \frac{1}{2}}{J+1} \frac{\tilde{e}}{e} \langle r^2 \rangle .
\end{aligned}$$

A charge distribution for which the intrinsic quadrupole moment $Q < 0$ we call oblate, while if $Q > 0$ we call it prolate.

5.4 Magnetic Dipole Moment

We define, in units of nuclear magneton and for a maximally aligned state,

$$\mu = \langle J, M = J | \sum_i g_i^l l_{z,i} + \sum_i g_i^s s_{z,i} | J, M = J \rangle.$$

We rewrite and apply the Wigner-Eckart theorem

$$\begin{aligned}
\mu &= \langle J, J | \sum_i g_i^l j_{z,i} + \sum_i (g_i^s - g_i^l) s_{z,i} | J, J \rangle \\
&= \begin{pmatrix} J & 1 & J \\ -J & 0 & J \end{pmatrix} \langle J \| \sum_i g_i^l \mathbf{j}_i + \sum_i (g_i^s - g_i^l) \mathbf{s}_i \| J \rangle \\
&= \frac{J}{\sqrt{(2J+1)(J+1)J}} \langle J \| \sum_i g_i^l \mathbf{j}_i + \sum_i (g_i^s - g_i^l) \mathbf{s}_i \| J \rangle.
\end{aligned}$$

It can be shown that

$$\langle J \| \mathbf{j} \| J \rangle = \sqrt{(2J+1)(J+1)J}$$

$$\langle J \| \mathbf{s} \| J \rangle = \frac{1}{2} \sqrt{\frac{2J+1}{(J+1)J}} \left(J(J+1) + \frac{3}{4} - l(l+1) \right)$$

such that the single-particle Schmidt limits become

$$\begin{aligned}
\mu &= Jg^l + \frac{1}{2(J+1)} \left(J(J+1) + \frac{3}{4} - l(l+1) \right) (g^s - g^l) \\
&= \begin{cases} Jg^l + \frac{1}{2}(g^s - g^l) & \text{for } J = l + \frac{1}{2} \\ Jg^l - \frac{J}{2(J+1)}(g^s - g^l) & \text{for } J = l - \frac{1}{2}. \end{cases}
\end{aligned}$$

For free nucleons, one sets $g_n^l = 0$ and $g_p^l = 1$.

SUPPLEMENTARY MATERIALS

Selected Experimental Techniques

Beam purification

Nowadays much of the experimental research that is related to the shell model concerns nuclear structure at low energy of exotic nuclei far from stability. The nuclei of interest are often produced at low rates within an overwhelming background of contaminants. The purity of the extracted beam becomes of utmost importance and techniques for its purification are among the central topics of instrumentation. As examples we cite lasers, ion traps, coincidence measurements.

Resonant laser ionisation

The atomic emission or absorption spectrum, *ie* the position of the electronic levels in the atomic shell structure, is unique for a every chemical element. Resonant laser ionisation therefore offers an opportunity to select, among the many residues of a nuclear reaction, one given element of interest. It is the idea behind the laser ion source, that when coupled to an electromagnetic mass separator, allows to extract a beam that is isotopically pure.

The hyperfine interaction between the nucleus and the electron that is excited by the laser light leads to a splitting of the energy levels. The effect is visible with lasers the bandwidth of which is sufficiently narrow and it opens up the possibility of intrasource laser spectroscopy, yielding access to nuclear charge radii and magnetic moments.

Ion traps

The Laplace equation shows that any constraining movement in one direction is always accompanied by a diverging movement in another direction. In a Paul trap, however, a quadrupolar electrical RF potential is successfully employed to confine charged particles of a given mass. In a Penning trap confinement happens by means of an axial magnetic field combined with a quadrupolar electrostatic potential.

When exciting the eigenmovement of ions in a trap, in particular the one given by the cyclotron frequency $\omega_c = qB/m$, the Penning trap becomes a mass filter. A series of several Penning traps then allows for mass measurements with precisions of up to 10^{-8} for radioisotopes, 10^{-11} for stable nuclei.

Coincidence measurements

The occurrence of simultaneous events that are physically correlated is a clever and efficient tool to reduce experimental background. For example, the γ transitions from a β emitter will come out more clearly if one requires that a

β particle was detected together with the γ ray, or nuclear levels populated in a (d,p) transfer reaction will be better visible if one imposes a time coincidence between the escaping proton and the heavy ejectile.

Spectroscopy at rest

Spectroscopy at rest may pertain to those techniques that are commonly known as isotope separation on-line or to reaction products from in-flight techniques that are subsequently stopped. They include α, β, γ decay spectroscopy, high-precision mass measurements through trap spectroscopy, high-precision moment measurements through collinear laser spectroscopy.

In-flight spectroscopy

Under this heading we place the study of ejectiles that emerge from nuclear reactions at energies of tens to hundreds of MeV/A as well as reaction products from isotope separation on-line that are post-accelerated. According to impact parameter and energy regime, one may distinguish between Coulomb excitation; transfer and knock-out reactions; deep inelastic and fragmentation reactions; fusion-evaporation, fusion-fission and multifragmentation. Multi-detectors such as staged silicon arrays, BaF_2, CsI, or $LaBr_3$ scintillator arrays, or germanium semi-conductors grow increasingly complex. Half-lives of nuclear levels can be measured with the plunger technique.

Nuclear forensics

These days the applications of nuclear physics are numerous. Energy, aerospace, medicine, datation, detection (airport or container security) are examples but we can also mention non-proliferation and verification of international treaties, known as nuclear forensics or sometimes nuclear archeology.

Emerging over the last few years as a new discipline in its own right, the purpose of nuclear forensics is to determine the origin and history of environmental contaminations or confiscated nuclear materials. Nuclear spectroscopy by means of α, β, γ ray detection is capable of establishing a fingerprint with a precision that goes down to a picogram. With Resonance Ionisation Mass Spectroscopy (Rims), resonant laser ionisation is used to select the element of interest while a time-of-flight measurement will determine the mass of the isotope. This technique is directly related to the physics of superheavy or exotic nuclei and can detect traces of radioactivity down to 10^6-10^7 atoms, *ie* femtograms.

Selected articles from scientific journals

Persistence of magic numbers (in-flight γ spectroscopy)

- R Bengtsson and P Möller, Nature 449, 411 (2007)
- D Seweryniak *et al*, Physical Review Letters 99, 22504 (2007)

Persistence of magic numbers (post-acceleration & particle spectroscopy)

- P Cottle, Nature 465, 430 (2010)
- K Jones *et al*, Nature 465, 454 (2010)

Disappearance of magic numbers (in-flight γ spectroscopy)

- B Bastin *et al*, Physical Review Letters 99, 22503 (2007)

Superheavy elements (coincidence technique)

- M Stoyer, Nature 442, 876 (2006)
- R Herzberg *et al*, Nature 442, 896 (2006)

Superheavy elements (ion trap)

- G Bollen, Nature 463, 740 (2010)
- M Block *et al*, Nature 463, 785 (2010)

Isospin (in-flight γ spectroscopy)

- B Cederwall *et al*, Nature 469, 68 (2011)

Paar parabola (ion traps & intrasource spectroscopy)

- J Van Roosbroeck *et al*, Physical Review Letters 92, 112501 (2004)

Electromagnetic transitions (Coulomb excitation)

- J Van de Walle *et al*, Physical Review Letters 99, 142501 (2007)

Electromagnetic moments (intrasource spectroscopy)

- T Cocolios *et al*, Physical Review Letters 103, 102501 (2009)

Nuclear forensics ($\beta\gamma$ spectroscopy & resonant laser ionisation)

- M Kalinowski *et al*, Complexity 14, 89 (2008)
- M Nunnemann *et al*, Journal of Alloys and Compounds 271, 45 (1998)

Appendix

§1. The associated Laguerre polynomials

$$L_n^a(x) = \sum_{k=0}^{n} \frac{(-)^k}{k!} \binom{a+n}{n-k} x^k$$

are the solutions of the Laguerre equation

$$xy'' + (a+1-x)y' + ny = 0$$
$$y = L_n^a(x).$$

If now

$$y = x^{\frac{a-1}{2}} e^{-x/2} L_n^a(x)$$

then one verifies that

$$xy'' + 2y' + \left(\frac{2n+a+1}{2} - \frac{x}{4} - \frac{a^2-1}{4x}\right) y = 0.$$

The equation

$$xy'' + 2y' + \left(\frac{2n+a+1}{2}\nu x - \frac{\nu^2 x^3}{4} - \frac{a^2 - \frac{1}{4}}{4x}\right) 4y = 0$$

is solved by

$$y = x^{\frac{2a-1}{2}} e^{-\nu x^2/2} L_n^a(\nu x^2).$$

It can be rewritten

$$y = \frac{u}{x}$$
$$a = \ell + \frac{1}{2}, \quad n = k$$
$$u'' + \left((4k + 2\ell + 3)\nu - \nu^2 x^2 - \frac{\ell(\ell+1)}{x^2}\right) u = 0$$

and its solution becomes

$$u = x^{\ell+1} e^{-\nu x^2/2} L_k^{\ell+1/2}(\nu x^2).$$

Here the Laguerre functions are defined through the gamma function instead of factorials. So we introduce the Pochhammer symbol

$$(x)_n = \frac{\Gamma(x+n)}{\Gamma(x)}$$

and we express the Laguerre functions for non-integer a by

$$L_n^a(x) = \frac{(a+1)_n}{n!} \sum_{k=0}^{\infty} \frac{(-n)_k}{(a+1)_k} \frac{x^k}{k!}.$$

Indeed, for integer a this reduces as follows

$$\begin{aligned}
L_n^a(x) &= \sum_{k=0}^{\infty} \frac{\Gamma(a+n+1)\Gamma(k-n)}{n!\Gamma(-n)\Gamma(a+k+1)} \frac{x^k}{k!} \\
&= \sum_{k=0}^{\infty} \frac{(a+n)!}{n!(a+k)!}(-n)(-n+1)\ldots(-n+k-1)\frac{x^k}{k!} \\
&= \sum_{k=0}^{\infty} \frac{(a+n)!}{n!(a+k)!} \frac{(-)^k n!}{(n-k)!} \frac{x^k}{k!} \\
&= \sum_{k=0}^{\infty} \frac{(-)^k}{k!} \binom{a+n}{n-k} x^k.
\end{aligned}$$

§2. Two angular momentum states can be coupled to a total angular momentum of zero. Normalised and antisymmetrised we write

$$|\Phi_0\rangle = \sum_m \frac{(-)^{j-m}}{\sqrt{2j+1}} |j,m\rangle|j,-m\rangle.$$

Introducing the Wigner $1j$-symbol

$$\begin{pmatrix} j \\ mm' \end{pmatrix} = (-)^{j+m}\delta_{m,-m'} = (-)^{j-m'}\delta_{m,-m'}$$

this becomes

$$|\Phi_0\rangle = -\sum_{m_1,m_2} \frac{1}{\sqrt{2j+1}} \begin{pmatrix} j \\ m_1 m_2 \end{pmatrix} |jm_1\rangle|jm_2\rangle.$$

For non-zero couplings we define the Clebsch-Gordan coefficients,

$$|j_1 j_2; jm\rangle = \sum_{m_1,m_2} \langle j_1 m_1, j_2 m_2 | jm\rangle |j_1 m_1\rangle|j_2 m_2\rangle.$$

There exist many symmetry relations, such as

$$\langle j_1 m_1, j_2 m_2 | jm\rangle = (-)^{j_1+j_2-j} \langle j_2 m_2, j_1 m_1 | jm\rangle.$$

When we couple three angular momentum states to a total momentum of zero, we may write

$$|\Psi_0\rangle = -\sum_{m_1,m_2,m_3} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} |j_1 m_1\rangle|j_2 m_2\rangle|j_3 m_3\rangle$$

where the Wigner $3j$-symbols are given by

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = \frac{(-)^{j_1-j_2-m_3}}{\sqrt{2j_3+1}} \langle j_1 m_1, j_2 m_2 | j_3 - m_3\rangle.$$

For $j_2 = m_2 = 0$ one indeed verifies that

$$\begin{aligned} |\Psi_0\rangle &= -\sum_{m_1,m_3} \frac{(-)^{j_1-m_3}}{\sqrt{2j_3+1}} \langle j_1 m_1, 00 | j_3 - m_3\rangle |j_1 m_1\rangle|00\rangle|j_3 m_3\rangle \\ &= -\sum_{m_1,m_3} \frac{(-)^{j_1-m_3}}{\sqrt{2j_3+1}} \delta_{j_1,j_3}\delta_{m_1,-m_3} |j_1 m_1\rangle|j_3 m_3\rangle \\ &= |\Phi_0\rangle. \end{aligned}$$

It can be seen that swapping two columns in the $3j$-symbol introduces a phase factor $(-)^{j_1+j_2+j_3}$. Among the many other symmetry properties we quote the unitarity relation

$$\sum_{m_1,m_3} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \begin{pmatrix} j_1 & j_2' & j_3 \\ m_1 & m_2' & m_3 \end{pmatrix} = \frac{1}{2j_2+1} \delta_{j_2 j_2'} \delta_{m_2 m_2'}.$$

When coupling three angular momenta, several choices can be made for the intermediate state. The transformation between the different sets of eigenvectors is given by (we use the shorthand notation $\hat{J} = \sqrt{2J+1}$)

$$\begin{aligned} &|j_1(j_2 j_3)J_{23}; JM\rangle \\ &\quad = \sum_{J_{12}} \langle (j_1 j_2)J_{12} j_3; J | j_1 (j_2 j_3) J_{23}; J\rangle |(j_1 j_2) J_{12} j_3; JM\rangle \end{aligned}$$

$$= \sum_{J_{12}} (-)^{j_1+j_2+j_3+J} \hat{J}_{12}\hat{J}_{23} \begin{Bmatrix} j_1 & j_2 & J_{12} \\ j_3 & J & J_{23} \end{Bmatrix} |(j_1j_2)J_{12}j_3; JM\rangle.$$

The $6j$-symbol can be written as a contraction over products of $3j$-symbols. It holds

$$\sum_k (2k+1) \begin{Bmatrix} j_1 & j_2 & J \\ j_4 & j_3 & k \end{Bmatrix} \begin{pmatrix} j_1 & k & j_3 \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} j_2 & k & j_4 \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}$$
$$= (-)^{j_2+j_3+2j_4} \begin{pmatrix} j_2 & j_1 & J \\ -\frac{1}{2} & -\frac{1}{2} & 1 \end{pmatrix} \begin{pmatrix} j_3 & j_4 & J \\ -\frac{1}{2} & -\frac{1}{2} & 1 \end{pmatrix},$$

$$\sum_k (2k+1)(-)^k \begin{Bmatrix} j_1 & j_2 & J \\ j_4 & j_3 & k \end{Bmatrix} \begin{pmatrix} j_1 & k & j_3 \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} j_2 & k & j_4 \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}$$
$$= (-)^{j_1+j_2-2j_4} \begin{pmatrix} j_2 & j_1 & J \\ -\frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix} \begin{pmatrix} j_3 & j_4 & J \\ \frac{1}{2} & -\frac{1}{2} & 0 \end{pmatrix}.$$

When coupling four angular momenta, one defines the $9j$-symbol,

$$|(j_1j_3)J_{13}(j_2j_4)J_{24}; JM\rangle$$
$$= \sum_{J_{12},J_{34}} \hat{J}_{13}\hat{J}_{24}\hat{J}_{12}\hat{J}_{34} \begin{Bmatrix} j_1 & j_2 & J_{12} \\ j_3 & j_4 & J_{34} \\ J_{13} & J_{24} & J \end{Bmatrix} |(j_1j_2)J_{12}(j_3j_4)J_{34}; JM\rangle.$$

The $9j$-symbol also can be written as a contraction over products of $3j$-symbols. We note the following property

$$\begin{Bmatrix} j_1 & j_2 & J \\ j_4 & j_3 & k \end{Bmatrix} = (-)^{j_2+J+j_3+k} \hat{J}\hat{k} \begin{Bmatrix} j_1 & j_2 & J \\ j_3 & j_4 & J \\ k & k & 0 \end{Bmatrix}.$$

The Wigner-Eckart theorem allows to separate matrix elements of spherical tensor operators in a geometrical and a physical part, the latter called the reduced matrix element,

$$\langle \alpha j m | T^{(k)}_\kappa | \alpha' j' m' \rangle = (-)^{j-m} \begin{pmatrix} j & k & j' \\ -m & \kappa & m' \end{pmatrix} \langle \alpha j \| \mathbf{T}^{(k)} \| \alpha' j' \rangle.$$

If now the tensor operator is a spherical tensor product

$$\begin{aligned} T^{(k)}_\kappa(1,2) &= \left[\mathbf{T}^{(k_1)}(1) \otimes \mathbf{T}^{(k_2)}(2)\right]^{(k)}_\kappa \\ &= \sum_{\kappa_1,\kappa_2} \langle k_1\kappa_1, k_2\kappa_2 | k\kappa \rangle T^{(k_1)}_{\kappa_1}(1) T^{(k_2)}_{\kappa_2}(2) \end{aligned}$$

then one can show that

$$\langle \alpha_1 j_1, \alpha_2 j_2; J \| \mathbf{T}^{(k)}(1,2) \| \alpha_1' j_1', \alpha_2' j_2'; J' \rangle$$
$$= \hat{J}\hat{J}'\hat{k} \begin{Bmatrix} j_1 & j_2 & J \\ j_1' & j_2' & J' \\ k_1 & k_2 & k \end{Bmatrix} \langle \alpha_1 j_1 \| \mathbf{T}^{(k_1)} \| \alpha_1' j_1' \rangle \langle \alpha_2 j_2 \| \mathbf{T}^{(k_2)} \| \alpha_2' j_2' \rangle.$$

If one of both operators is the unit operator then this simplifies to

$$\langle \alpha_1 j_1, \alpha_2 j_2; J \| \mathbf{T}^{(k)}(1) \| \alpha_1' j_1', \alpha_2' j_2'; J' \rangle$$
$$= \hat{J}\hat{J}'(-)^{j_1+j_2+J'+k} \begin{Bmatrix} j_1 & j_2 & J \\ J' & k & j_1' \end{Bmatrix} \langle \alpha_1 j_1 \| \mathbf{T}^{(k)} \| \alpha_1' j_1' \rangle \delta_{j_2 j_2'} \delta_{\alpha_2 \alpha_2'}.$$

We define the scalar product as a spherical tensor product of rank zero

$$\begin{aligned}\mathbf{T}^{(k)} \cdot \mathbf{U}^{(k)} &= (-)^k \hat{k} \left[\mathbf{T}^{(k)} \otimes \mathbf{U}^{(k)}\right]^{(0)}_0 \\ &= (-)^k \hat{k} \sum_\kappa \langle k\kappa, k-\kappa|00\rangle T^{(k)}_\kappa U^{(k)}_{-\kappa} \\ &= \sum_\kappa (-)^\kappa T^{(k)}_\kappa U^{(k)}_{-\kappa}\end{aligned}$$

for which case the reduced matrix element becomes

$$\begin{aligned}&\langle \alpha_1 j_1, \alpha_2 j_2; J \| \mathbf{T}^{(k)}(1) \cdot \mathbf{U}^{(k)}(2) \| \alpha_1' j_1', \alpha_2' j_2'; J' \rangle \\ &= (-)^k \hat{k} \hat{J} \hat{J}' \begin{Bmatrix} j_1 & j_2 & J \\ j_1' & j_2' & J' \\ k & k & 0 \end{Bmatrix} \langle \alpha_1 j_1 \| \mathbf{T}^{(k)} \| \alpha_1' j_1' \rangle \langle \alpha_2 j_2 \| \mathbf{U}^{(k)} \| \alpha_2' j_2' \rangle \\ &= (-)^{j_2 + J + j_1'} \hat{J} \begin{Bmatrix} j_1 & j_2 & J \\ j_2' & j_1' & k \end{Bmatrix} \langle \alpha_1 j_1 \| \mathbf{T}^{(k)} \| \alpha_1' j_1' \rangle \langle \alpha_2 j_2 \| \mathbf{U}^{(k)} \| \alpha_2' j_2' \rangle \delta_{JJ'}.\end{aligned}$$

§3. Any vector $\mathbf{a}(x, y, z)$ is a spherical tensor of rank 1 with components

$$a_{+1} = -\frac{1}{\sqrt{2}}(x + iy)$$

$$a_0 = z$$

$$a_{-1} = \frac{1}{\sqrt{2}}(x - iy).$$

Knowing that

$$\begin{pmatrix} j & 1 & j \\ -m & 0 & m \end{pmatrix} = (-)^{j-m} \frac{m}{\sqrt{(2j+1)(j+1)j}}$$

we can apply the Wigner-Eckart theorem to find that

$$\langle jm|z|jm\rangle = (-)^{2j} \frac{m}{\sqrt{(2j+1)(j+1)j}} \langle j\|\mathbf{a}\|j\rangle.$$

We note furthermore the results

$$\langle j\|\mathbf{j}\|j\rangle = \sqrt{(2j+1)(j+1)j}$$

$$\langle j\|\boldsymbol{\sigma}\|j\rangle = \sqrt{\frac{2j+1}{(j+1)j}}(j(j+1) + \tfrac{3}{4} - l(l+1)).$$

From the reduction rules for spherical tensor operators one obtains

$$\langle l\tfrac{1}{2}; j\|\mathbf{Y}_k\|l'\tfrac{1}{2}; j'\rangle = \hat{j}\hat{j}'(-)^{l+1/2+j'+k} \begin{Bmatrix} l & \frac{1}{2} & j \\ j' & k & l' \end{Bmatrix} \langle l\|\mathbf{Y}_k\|l'\rangle.$$

Calculating

$$\langle l\|\mathbf{Y}_k\|l'\rangle = (-)^l \frac{\hat{l}\hat{l}'\hat{k}}{\sqrt{4\pi}} \begin{pmatrix} l & k & l' \\ 0 & 0 & 0 \end{pmatrix}$$

it can be shown that

$$\langle l\tfrac{1}{2}; j\|\mathbf{Y}_k\|l'\tfrac{1}{2}; j'\rangle = (-)^{l+l'+1/2-j'} \frac{\hat{j}\hat{j}'\hat{k}}{\sqrt{4\pi}} \begin{pmatrix} j & k & j' \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix} \frac{1}{2}(1 + (-)^{l+l'+k}).$$

Using the relation

$$\begin{pmatrix} j & 2 & j \\ -m & 0 & m \end{pmatrix} = (-)^{j-m} \frac{2(3m^2 - j(j+1))}{\sqrt{(2j+3)(2j+2)(2j+1)2j(2j-1)}}$$

we find in particular

$$\langle l\tfrac{1}{2}; j \| \mathbf{Y}_2 \| l\tfrac{1}{2}; j \rangle = \sqrt{\frac{5}{4\pi}} \frac{2(\frac{3}{4} - j(j+1))(2j+1)}{\sqrt{(2j+3)(2j+2)(2j+1)2j(2j-1)}}.$$

§4. Combining the results of the previous sections we can now show that

$$\begin{aligned}
\sum_k f_k F^k &= \sum_k 4\pi F^0 (-)^{j_1+j_2+J} \begin{Bmatrix} j_1 & j_2 & J \\ j_2 & j_1 & k \end{Bmatrix} \langle j_1 \| \mathbf{Y}_k \| j_1 \rangle \langle j_2 \| \mathbf{Y}_k \| j_2 \rangle \\
&= \sum_k F^0 (-)^{J+1} (2j_1+1)(2j_2+1)(2k+1) \\
&\quad \times \begin{Bmatrix} j_1 & j_2 & J \\ j_2 & j_1 & k \end{Bmatrix} \begin{pmatrix} j_1 & k & j_1 \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} j_2 & k & j_2 \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix} \frac{1}{2}(1+(-)^k) \\
&= \frac{F^0}{2}(2j_1+1)(2j_2+1) \left[(-)^{1+j_1-j_2-J} \begin{pmatrix} j_2 & j_1 & J \\ -\frac{1}{2} & -\frac{1}{2} & 1 \end{pmatrix} \begin{pmatrix} j_1 & j_2 & J \\ -\frac{1}{2} & -\frac{1}{2} & 1 \end{pmatrix} \right. \\
&\quad \left. + (-)^{1+j_1-j_2-J} \begin{pmatrix} j_2 & j_1 & J \\ -\frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix} \begin{pmatrix} j_1 & j_2 & J \\ \frac{1}{2} & -\frac{1}{2} & 0 \end{pmatrix} \right] \\
&= \frac{F^0}{2}(2j_1+1)(2j_2+1) \left[\begin{pmatrix} j_1 & j_2 & J \\ -\frac{1}{2} & -\frac{1}{2} & 1 \end{pmatrix}^2 + \begin{pmatrix} j_1 & j_2 & J \\ \frac{1}{2} & -\frac{1}{2} & 0 \end{pmatrix}^2 \right]
\end{aligned}$$

and in a similar way

$$\begin{aligned}
\sum_k g_k G^k &= \frac{F^0}{2}(2j_1+1)(2j_2+1)(-)^{-j_1-j_2-J} \\
&\quad \times \left[\begin{pmatrix} j_1 & j_2 & J \\ -\frac{1}{2} & -\frac{1}{2} & 1 \end{pmatrix}^2 - (-)^{l_1+l_2+J} \begin{pmatrix} j_1 & j_2 & J \\ \frac{1}{2} & -\frac{1}{2} & 0 \end{pmatrix}^2 \right]
\end{aligned}$$

such that

$$\begin{aligned}
\Delta E(j_1 j_2; J) &= \sum_k f_k F^k - (-)^{j_1+j_2-J} \sum_k g_k G^k \\
&= \frac{F^0}{2}(2j_1+1)(2j_2+1) \left[\begin{pmatrix} j_1 & j_2 & J \\ -\frac{1}{2} & -\frac{1}{2} & 1 \end{pmatrix}^2 + \begin{pmatrix} j_1 & j_2 & J \\ \frac{1}{2} & -\frac{1}{2} & 0 \end{pmatrix}^2 \right. \\
&\quad \left. - \begin{pmatrix} j_1 & j_2 & J \\ -\frac{1}{2} & -\frac{1}{2} & 1 \end{pmatrix}^2 + (-)^{l_1+l_2+J} \begin{pmatrix} j_1 & j_2 & J \\ \frac{1}{2} & -\frac{1}{2} & 0 \end{pmatrix}^2 \right] \\
&= \frac{F^0}{2}(2j_1+1)(2j_2+1)(1+(-)^{l_1+l_2+J}) \begin{pmatrix} j_1 & j_2 & J \\ \frac{1}{2} & -\frac{1}{2} & 0 \end{pmatrix}^2.
\end{aligned}$$

§5. Starting from the definition

$$\mathbf{f}_{lm;\lambda} = \int D^{l*}_{m\lambda}(\hat{\mathbf{k}})\ \hat{\mathbf{e}}_{k\lambda} e^{i\mathbf{k}\cdot\mathbf{r}} d\Omega_k$$

developing a plane wave in spherical Bessel functions

$$e^{i\mathbf{k}\cdot\mathbf{r}} = 4\pi \sum_{\lambda\mu} i^\lambda\ Y_{\lambda\mu}(\hat{\mathbf{r}}) Y^*_{\lambda\mu}(\hat{\mathbf{k}}) j_\lambda(kr)$$

and rotating the helicity vectors to spherical unit tensors of first rank

$$\hat{\mathbf{e}}^*_{k1} = -\sum_\nu D^1_{\nu,-1}(\hat{\mathbf{k}})\ \hat{\mathbf{u}}_\nu$$

we can write

$$\begin{aligned}
\mathbf{f}^{(\sigma)*}_{lm} &= \frac{1}{\sqrt{2}}(\mathbf{f}^*_{lm;1} + (-)^\sigma \mathbf{f}^*_{lm;-1}) \\
&= \frac{1}{\sqrt{2}} \int D^l_{m1}(\hat{\mathbf{k}})\ \hat{\mathbf{e}}^*_{k1} e^{-i\mathbf{k}\cdot\mathbf{r}} + (-)^\sigma D^l_{m,-1}(\hat{\mathbf{k}})\ \hat{\mathbf{e}}^*_{k,-1} e^{-i\mathbf{k}\cdot\mathbf{r}} d\Omega_k \\
&= \frac{1}{\sqrt{2}} \int D^l_{m1}(\hat{\mathbf{k}})\ \hat{\mathbf{e}}^*_{k1} e^{-i\mathbf{k}\cdot\mathbf{r}} - (-)^{\sigma+l} D^l_{m1}(\hat{\mathbf{k}})\ \hat{\mathbf{e}}^*_{k1} e^{i\mathbf{k}\cdot\mathbf{r}} d\Omega_k \\
&= \frac{1}{\sqrt{2}} \int D^l_{m1}(\hat{\mathbf{k}})\ \hat{\mathbf{e}}^*_{k1} (e^{-i\mathbf{k}\cdot\mathbf{r}} - (-)^{\sigma+l} e^{i\mathbf{k}\cdot\mathbf{r}})\ d\Omega_k \\
&= \frac{4\pi}{\sqrt{2}} \sum_{\lambda\mu} i^\lambda((-)^\lambda - (-)^{\sigma+l}) j_\lambda(kr) Y^*_{\lambda\mu}(\hat{\mathbf{r}}) \int D^l_{m1}(\hat{\mathbf{k}})\ \hat{\mathbf{e}}^*_{k1} Y_{\lambda\mu}(\hat{\mathbf{k}})\ d\Omega_k \\
&= -\frac{4\pi}{\sqrt{2}} \sum_{\lambda\mu\nu} i^\lambda((-)^\lambda - (-)^{\sigma+l}) j_\lambda(kr) Y^*_{\lambda\mu}(\hat{\mathbf{r}}) \int D^l_{m1}(\hat{\mathbf{k}}) D^1_{\nu,-1}(\hat{\mathbf{k}}) Y_{\lambda\mu}(\hat{\mathbf{k}})\ \hat{\mathbf{u}}_\nu\ d\Omega_k.
\end{aligned}$$

We now use the identities

$$D^\lambda_{\mu 0}(\hat{\mathbf{k}}) = \sqrt{\frac{4\pi}{2\lambda+1}} Y^*_{\lambda\mu}(\hat{\mathbf{k}})$$

$$D^{l_1}_{m_1 m_1'} D^{l_2}_{m_2 m_2'} = \sum_{l_3} (2l_3+1) \begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \begin{pmatrix} l_1 & l_2 & l_3 \\ m_1' & m_2' & m_3' \end{pmatrix} D^{l_3}_{m_3 m_3'}$$

$$\int D^l_{m\mu}(\hat{\mathbf{k}}) D^{l'*}_{m'\mu}(\hat{\mathbf{k}})\ d\Omega_k = \frac{4\pi}{2l+1} \delta_{ll'} \delta_{mm'}$$

to pursue

$$\begin{aligned}
\mathbf{f}^{(\sigma)*}_{lm} &= -\sqrt{2\pi} \sum_{\lambda\mu\nu} i^\lambda((-)^\lambda - (-)^{\sigma+l}) \sqrt{2\lambda+1}\ j_\lambda(kr) Y^*_{\lambda\mu}(\hat{\mathbf{r}}) \int D^l_{m1}(\hat{\mathbf{k}}) D^1_{\nu,-1}(\hat{\mathbf{k}}) D^{\lambda*}_{\mu 0}(\hat{\mathbf{k}})\ \hat{\mathbf{u}}_\nu\ d\Omega_k \\
&= -\sqrt{2\pi} \sum_{\lambda\lambda'\mu\nu} i^\lambda((-)^\lambda - (-)^{\sigma+l}) \sqrt{2\lambda+1}\ (2\lambda'+1) \begin{pmatrix} l & 1 & \lambda' \\ m & \nu & \mu \end{pmatrix} \begin{pmatrix} l & 1 & \lambda' \\ 1 & -1 & 0 \end{pmatrix} j_\lambda(kr) Y^*_{\lambda\mu}(\hat{\mathbf{r}}) \\
&\quad \times \int D^{\lambda'}_{\mu 0}(\hat{\mathbf{k}}) D^{\lambda*}_{\mu 0}(\hat{\mathbf{k}})\ \hat{\mathbf{u}}_\nu\ d\Omega_k \\
&= -\sqrt{32\pi^3} \sum_{\lambda\mu\nu} i^\lambda((-)^\lambda - (-)^{\sigma+l}) \sqrt{2\lambda+1} \begin{pmatrix} l & 1 & \lambda \\ m & \nu & \mu \end{pmatrix} \begin{pmatrix} l & 1 & \lambda \\ 1 & -1 & 0 \end{pmatrix} j_\lambda(kr) Y^*_{\lambda\mu}(\hat{\mathbf{r}})\ \hat{\mathbf{u}}_\nu.
\end{aligned}$$

For electric multipoles, $\sigma = 0$ so only if $\lambda = l \pm 1$ the $3j$-symbols survive. Then

$$\begin{aligned}
\mathbf{f}^{(E)*}_{lm} = -\sqrt{128\pi^3} \sum_{\mu\nu} \Big((-i)^{l-1} \sqrt{2l-1} \begin{pmatrix} l & 1 & l-1 \\ m & \nu & \mu \end{pmatrix} \begin{pmatrix} l & 1 & l-1 \\ 1 & -1 & 0 \end{pmatrix} j_{l-1}(kr) Y^*_{l-1,\mu}(\hat{\mathbf{r}})\ \hat{\mathbf{u}}_\nu \\
+ (-i)^{l+1} \sqrt{2l+3} \begin{pmatrix} l & 1 & l+1 \\ m & \nu & \mu \end{pmatrix} \begin{pmatrix} l & 1 & l+1 \\ 1 & -1 & 0 \end{pmatrix} j_{l+1}(kr) Y^*_{l+1,\mu}(\hat{\mathbf{r}})\ \hat{\mathbf{u}}_\nu \Big).
\end{aligned}$$

It holds

$$\begin{pmatrix} l & 1 & l-1 \\ 1 & -1 & 0 \end{pmatrix} = (-)^{l-1}\sqrt{\frac{l+1}{2(2l-1)(2l+1)}}$$

$$\begin{pmatrix} l & 1 & l+1 \\ 1 & -1 & 0 \end{pmatrix} = (-)^{l+1}\sqrt{\frac{l}{2(2l+1)(2l+3)}}$$

therefore

$$\begin{aligned}\mathbf{f}_{lm}^{(E)*} = & -\sqrt{64\pi^3}\sum_{\mu\nu}\Big(i^{l-1}\sqrt{\frac{l+1}{2l+1}}\begin{pmatrix} l & 1 & l-1 \\ m & \nu & \mu \end{pmatrix} j_{l-1}(kr)Y^*_{l-1,\mu}(\hat{\mathbf{r}})\;\hat{\mathbf{u}}_\nu \\ & + i^{l+1}\sqrt{\frac{l}{2l+1}}\begin{pmatrix} l & 1 & l+1 \\ m & \nu & \mu \end{pmatrix} j_{l+1}(kr)Y^*_{l+1,\mu}(\hat{\mathbf{r}})\;\hat{\mathbf{u}}_\nu\Big).\end{aligned}$$

In classical electrodynamics one can show that

$$\mathbf{A}_{lm}^{(E)*} = (-)^l\sum_{\mu\nu}\left(\sqrt{l+1}\begin{pmatrix} l & 1 & l-1 \\ m & \nu & \mu \end{pmatrix} j_{l-1}Y^*_{l-1,\mu}\hat{\mathbf{u}}_\nu - \sqrt{l}\begin{pmatrix} l & 1 & l+1 \\ m & \nu & \mu \end{pmatrix} j_{l+1}Y^*_{l+1,\mu}\hat{\mathbf{u}}_\nu\right)$$

such that we may identify

$$\mathbf{f}_{lm}^{(E)*} = 8\pi^2\frac{(-i)^{l-1}}{\sqrt{\pi(2l+1)}}\mathbf{A}_{lm}^{(E)*}.$$

If $\sigma = 1$ then $\lambda = l$ and a similar result can be obtained for the magnetic case. We use

$$\begin{pmatrix} l & 1 & l \\ 1 & -1 & 0 \end{pmatrix} = \frac{(-)^{l+1}}{\sqrt{2(2l+1)}}$$

$$\mathbf{A}_{lm}^{(M)*} = (-)^l\sum_{\mu\nu}\sqrt{2l+1}\begin{pmatrix} l & 1 & l \\ m & \nu & \mu \end{pmatrix} j_l Y^*_{l\mu}\hat{\mathbf{u}}_\nu$$

and find

$$\begin{aligned}\mathbf{f}_{lm}^{(M)*} &= -\sqrt{128\pi^3}\sum_{\mu\nu}(-i)^l\sqrt{2l+1}\begin{pmatrix} l & 1 & l \\ m & \nu & \mu \end{pmatrix}\begin{pmatrix} l & 1 & l \\ 1 & -1 & 0 \end{pmatrix} j_l(kr)Y^*_{l\mu}(\hat{\mathbf{r}})\;\hat{\mathbf{u}}_\nu \\ &= \sqrt{64\pi^3}\sum_{\mu\nu} i^l\begin{pmatrix} l & 1 & l \\ m & \nu & \mu \end{pmatrix} j_l(kr)Y^*_{l\mu}(\hat{\mathbf{r}})\;\hat{\mathbf{u}}_\nu \\ &= 8\pi^2\frac{(-i)^l}{\sqrt{\pi(2l+1)}}\mathbf{A}_{lm}^{(M)*}.\end{aligned}$$

§6. For a free field the Maxwell equations of the vector potential reduce to the wave equations

$$\begin{aligned} \nabla^2 \mathbf{A} - \frac{1}{c}\frac{\partial^2 \mathbf{A}}{\partial t^2} &= 0 \\ \boldsymbol{\nabla} \cdot \mathbf{A} &= 0 \end{aligned}$$

the general solution of which is, for magnetic multipole radiation,

$$\begin{aligned} \mathbf{A}_{lm}^{(M)} &= -\frac{1}{\sqrt{l(l+1)}}(\mathbf{r} \times \boldsymbol{\nabla})(j_l(kr)Y_{lm}(\hat{\mathbf{r}})) \\ &\approx -\frac{k^l}{(2l+1)!!\sqrt{l(l+1)}}(\mathbf{r} \times \boldsymbol{\nabla})(r^l Y_{lm}(\hat{\mathbf{r}})). \end{aligned}$$

The electric field that is free of sources is defined by the Maxwell equation

$$\boldsymbol{\mathcal{E}} = \frac{i}{k}\boldsymbol{\nabla} \times \boldsymbol{\mathcal{B}}.$$

Above we have seen that there exists a phase factor $-i$ when rotating from $\mathbf{f}_{lm}^{(E)*}$ to $\mathbf{f}_{lm}^{(M)*}$ hence there also is a phase $-i$ from $\mathbf{f}_{lm}^{(M)}$ to $\mathbf{f}_{lm}^{(E)}$ and we only need a factor -1 from $\mathbf{A}_{lm}^{(M)}$ to $\mathbf{A}_{lm}^{(E)}$. The relation

$$\boldsymbol{\nabla} \times (\mathbf{r} \times \boldsymbol{\nabla}) = \mathbf{r}\nabla^2 - \boldsymbol{\nabla}(1 + r\frac{\partial}{\partial r})$$

then allows to obtain the electric solution

$$\begin{aligned} \mathbf{A}_{lm}^{(E)} &= \frac{1}{k\sqrt{l(l+1)}}\boldsymbol{\nabla} \times (\mathbf{r} \times \boldsymbol{\nabla})(j_l(kr)Y_{lm}(\hat{\mathbf{r}})) \\ &= \frac{1}{k\sqrt{l(l+1)}}\left(\mathbf{r}\nabla^2 - \boldsymbol{\nabla}(1 + r\frac{\partial}{\partial r})\right)(j_l(kr)Y_{lm}(\hat{\mathbf{r}})) \\ &= -\frac{1}{k\sqrt{l(l+1)}}\boldsymbol{\nabla}\left(\frac{\partial}{\partial r}(r j_l(kr)Y_{lm}(\hat{\mathbf{r}}))\right) \\ &\approx -\frac{k^{l-1}}{(2l+1)!!}\sqrt{\frac{l+1}{l}}\boldsymbol{\nabla}(r^l Y_{lm}(\hat{\mathbf{r}})). \end{aligned}$$

§7. Applying the continuity equation to the emission of a photon

$$\begin{aligned}\langle f|\boldsymbol{\nabla}\cdot\mathbf{j}|i\rangle &= -\langle f|\frac{\partial\rho}{\partial t}|i\rangle \\ &= -\frac{i}{\hbar}\langle f|[H,\rho]|i\rangle \\ &= -\frac{i}{\hbar}(E_f-E_i)\langle f|\rho|i\rangle \\ &= -ikc\,\langle f|\rho|i\rangle\end{aligned}$$

and recalling our earlier result

$$M_{lm}^{(\sigma)} = \int \mathbf{f}_{lm}^{(\sigma)*}\cdot\mathbf{j}\,d\mathbf{r}$$

the electric matrix element becomes

$$\begin{aligned}M_{lm}^{(E)} &= 8\pi^2\frac{(-i)^{l-1}}{\sqrt{\pi(2l+1)}}\int \mathbf{A}_{lm}^{(E)*}\cdot\mathbf{j}\,d\mathbf{r} \\ &= -8\pi^2\frac{(-ik)^{l-1}}{(2l+1)!!}\sqrt{\frac{l+1}{\pi l(2l+1)}}\int \boldsymbol{\nabla}(r^lY_{lm}^*(\hat{\mathbf{r}}))\cdot\mathbf{j}\,d\mathbf{r} \\ &= 8\pi^2\frac{(-ik)^{l-1}}{(2l+1)!!}\sqrt{\frac{l+1}{\pi l(2l+1)}}\int r^lY_{lm}^*(\hat{\mathbf{r}})\,\boldsymbol{\nabla}\cdot\mathbf{j}\,d\mathbf{r} \\ &= 8\pi^2c\frac{(-ik)^{l}}{(2l+1)!!}\sqrt{\frac{l+1}{\pi l(2l+1)}}\int r^lY_{lm}^*(\hat{\mathbf{r}})\,\rho\,d\mathbf{r}.\end{aligned}$$

The magnetic matrix element is written[2]

$$\begin{aligned}M_{lm}^{(M)} &= -8\pi^2\frac{(-ik)^{l}}{(2l+1)!!\sqrt{\pi l(l+1)(2l+1)}}\int (\mathbf{r}\times\boldsymbol{\nabla})(r^lY_{lm}^*(\hat{\mathbf{r}}))\cdot\mathbf{j}\,d\mathbf{r} \\ &= -8\pi^2\frac{(-ik)^{l}}{(2l+1)!!\sqrt{\pi l(l+1)(2l+1)}}\int \boldsymbol{\nabla}(r^lY_{lm}^*(\hat{\mathbf{r}}))\cdot(\mathbf{j}\times\mathbf{r})\,d\mathbf{r} \\ &= 8\pi^2\frac{(-ik)^{l}}{(2l+1)!!\sqrt{\pi l(l+1)(2l+1)}}\int r^lY_{lm}^*(\hat{\mathbf{r}})\,\boldsymbol{\nabla}\cdot(\mathbf{j}\times\mathbf{r})\,d\mathbf{r}.\end{aligned}$$

At this point it is appropriate to introduce spin as a manifestation of intrinsic magnetisation (we use a shortcut notation for the numerical factor)

$$M_{lm}^{(M)} = \kappa(l)k^l\int r^lY_{lm}^*(\hat{\mathbf{r}})\,\boldsymbol{\nabla}\cdot((\mathbf{j}+c\boldsymbol{\nabla}\times\mathbf{m})\times\mathbf{r})\,d\mathbf{r}.$$

Knowing that

$$\boldsymbol{\nabla}\times\mathbf{r}=0$$

while for any vector field $\mathbf{a}(\mathbf{r})$

$$(\mathbf{r}\times\boldsymbol{\nabla})\cdot(\boldsymbol{\nabla}\times\mathbf{a}) = -\nabla^2(\mathbf{r}\cdot\mathbf{a})+\frac{1}{r}\frac{\partial}{\partial r}(r^2\boldsymbol{\nabla}\cdot\mathbf{a})$$

$$\nabla^2(\mathbf{r}\cdot\mathbf{a}) = \mathbf{r}\cdot(\nabla^2\mathbf{a})+2\boldsymbol{\nabla}\cdot\mathbf{a}$$

and using the Helmholtz equation for a spatial wave

$$(\nabla^2+k^2)\mathbf{m}=0$$

[2]We use the relation $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \mathbf{c}\cdot(\mathbf{a}\times\mathbf{b})$

we transform the part of $M_{lm}^{(M)}$ that expresses intrinsic magnetisation[3]

$$\begin{aligned}
M_{lm}^{(\mathfrak{M})} &= c\kappa(l)k^l \int r^l Y_{lm}^*(\hat{\mathbf{r}})\, \boldsymbol{\nabla} \cdot ((\boldsymbol{\nabla} \times \mathbf{m}) \times \mathbf{r})\, d\mathbf{r} \\
&= c\kappa(l)k^l \int r^l Y_{lm}^*(\hat{\mathbf{r}})\, \mathbf{r} \cdot (\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{m}))\, d\mathbf{r} \\
&= c\kappa(l)k^l \int r^l Y_{lm}^*(\hat{\mathbf{r}})\, (\mathbf{r} \times \boldsymbol{\nabla}) \cdot (\boldsymbol{\nabla} \times \mathbf{m})\, d\mathbf{r} \\
&= c\kappa(l)k^l \int r^l Y_{lm}^*(\hat{\mathbf{r}})\, \left(-\nabla^2(\mathbf{r} \cdot \mathbf{m}) + \frac{1}{r}\frac{\partial}{\partial r}(r^2 \boldsymbol{\nabla} \cdot \mathbf{m})\right) d\mathbf{r} \\
&= c\kappa(l)k^l \int r^l Y_{lm}^*(\hat{\mathbf{r}})\, \left(-\mathbf{r} \cdot (\nabla^2 \mathbf{m}) + r\frac{\partial}{\partial r}(\boldsymbol{\nabla} \cdot \mathbf{m})\right) d\mathbf{r} \\
&= c\kappa(l)k^l \int r^l Y_{lm}^*(\hat{\mathbf{r}})\, \left(k^2 \mathbf{r} \cdot \mathbf{m} - (l+1)(\boldsymbol{\nabla} \cdot \mathbf{m})\right) d\mathbf{r}.
\end{aligned}$$

In the long wavelength limit $kr \ll 1$ the first term disappears so we find

$$M_{lm}^{(\mathfrak{M})} \approx -8\pi^2 c \frac{(-ik)^l}{(2l+1)!!}\sqrt{\frac{l+1}{\pi l(2l+1)}} \int r^l Y_{lm}^*(\hat{\mathbf{r}})\, \boldsymbol{\nabla} \cdot \mathbf{m}\, d\mathbf{r}$$

and therefore

$$M_{lm}^{(M)} \approx -8\pi^2 \frac{(-ik)^l}{(2l+1)!!}\sqrt{\frac{l+1}{\pi l(2l+1)}} \int r^l Y_{lm}^*(\hat{\mathbf{r}}) \left(\frac{1}{l+1}\boldsymbol{\nabla} \cdot (\mathbf{r} \times \mathbf{j}) + c\boldsymbol{\nabla} \cdot \mathbf{m}\right) d\mathbf{r}.$$

If now we hark back to the classical definitions of the electromagnetic moments that tell us that the electric moments arise from $\boldsymbol{\nabla} \cdot \mathbf{j}$ whereas the magnetic moments from $\boldsymbol{\nabla} \cdot (\mathbf{r} \times \mathbf{j})$, to which one adds the intrinsic magnetisation current $c\boldsymbol{\nabla} \times \mathbf{m}$

$$Q_{lm}^{(E)} = \int \rho\, r^l Y_{lm}^*\, d\mathbf{r}$$

$$Q_{lm}^{(M)} = -\frac{1}{c} \int r^l Y_{lm}^* \left(\frac{1}{l+1}\boldsymbol{\nabla} \cdot (\mathbf{r} \times \mathbf{j}) + c\boldsymbol{\nabla} \cdot \mathbf{m}\right) d\mathbf{r}$$

then we may write the relation

$$M_{lm}^{(\sigma)} \approx \frac{8\pi^2 c(-ik)^l}{(2l+1)!!}\sqrt{\frac{l+1}{\pi l(2l+1)}}\, Q_{lm}^{(\sigma)}.$$

[3]Recall $\boldsymbol{\nabla} \cdot (\mathbf{a} \times \mathbf{b}) = \mathbf{b} \cdot (\boldsymbol{\nabla} \times \mathbf{a}) - \mathbf{a} \cdot (\boldsymbol{\nabla} \times \mathbf{b})$

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Exercises

1.

$$\langle j^2; JT|V|j^2; JT\rangle = \frac{A_0^{(T)}}{4}(2j+1)^2$$
$$\times \left((1-(-)^{J+T}) \begin{pmatrix} j & j & J \\ \frac{1}{2} & -\frac{1}{2} & 0 \end{pmatrix}^2 + (1+(-)^T) \begin{pmatrix} j & j & J \\ \frac{1}{2} & \frac{1}{2} & -1 \end{pmatrix}^2 \right)$$

We apply this for a $\pi f_{7/2} \nu f_{7/2}$ configuration. For $T=0$, J is odd:

$$\langle j^2; J0|V|j^2; J0\rangle = \frac{A_0^{(0)}}{2}(2j+1)^2 \left(\begin{pmatrix} j & j & J \\ \frac{1}{2} & -\frac{1}{2} & 0 \end{pmatrix}^2 + \begin{pmatrix} j & j & J \\ \frac{1}{2} & \frac{1}{2} & -1 \end{pmatrix}^2 \right)$$

$$\begin{pmatrix} 7/2 & 7/2 & 1 \\ 1/2 & -1/2 & 0 \end{pmatrix} = -\tfrac{1}{6}\sqrt{\frac{1}{14}} \qquad \begin{pmatrix} 7/2 & 7/2 & 1 \\ 1/2 & 1/2 & -1 \end{pmatrix} = \tfrac{2}{3}\sqrt{\frac{1}{7}}$$
$$\begin{pmatrix} 7/2 & 7/2 & 3 \\ 1/2 & -1/2 & 0 \end{pmatrix} = \tfrac{1}{2}\sqrt{\frac{3}{154}} \qquad \begin{pmatrix} 7/2 & 7/2 & 3 \\ 1/2 & 1/2 & -1 \end{pmatrix} = -\sqrt{\frac{2}{77}}$$
$$\begin{pmatrix} 7/2 & 7/2 & 5 \\ 1/2 & -1/2 & 0 \end{pmatrix} = -\tfrac{5}{2}\sqrt{\frac{3}{2002}} \qquad \begin{pmatrix} 7/2 & 7/2 & 5 \\ 1/2 & 1/2 & -1 \end{pmatrix} = 2\sqrt{\frac{5}{1001}}$$
$$\begin{pmatrix} 7/2 & 7/2 & 7 \\ 1/2 & -1/2 & 0 \end{pmatrix} = \tfrac{7}{6}\sqrt{\frac{5}{286}} \qquad \begin{pmatrix} 7/2 & 7/2 & 7 \\ 1/2 & 1/2 & -1 \end{pmatrix} = -\tfrac{1}{3}\sqrt{\frac{35}{143}}$$

$$\langle 10|V|10\rangle = 2.095\ A_0^{(0)}$$
$$\langle 30|V|30\rangle = 0.987\ A_0^{(0)}$$
$$\langle 50|V|50\rangle = 0.939\ A_0^{(0)}$$
$$\langle 70|V|70\rangle = 1.632\ A_0^{(0)}$$

For $T=1$, J is even:

$$\langle j^2; J1|V|j^2; J1\rangle = \frac{A_0^{(1)}}{2}(2j+1)^2 \begin{pmatrix} j & j & J \\ \frac{1}{2} & -\frac{1}{2} & 0 \end{pmatrix}^2$$

$$\begin{pmatrix} 7/2 & 7/2 & 0 \\ 1/2 & -1/2 & 0 \end{pmatrix} = -\sqrt{\frac{1}{8}}$$
$$\begin{pmatrix} 7/2 & 7/2 & 2 \\ 1/2 & -1/2 & 0 \end{pmatrix} = \tfrac{1}{2}\sqrt{\frac{5}{42}}$$
$$\begin{pmatrix} 7/2 & 7/2 & 4 \\ 1/2 & -1/2 & 0 \end{pmatrix} = -\tfrac{3}{2}\sqrt{\frac{1}{154}}$$
$$\begin{pmatrix} 7/2 & 7/2 & 6 \\ 1/2 & -1/2 & 0 \end{pmatrix} = \tfrac{5}{2}\sqrt{\frac{1}{858}}$$

$$\langle 01|V|01\rangle = 4.000\ A_0^{(1)}$$
$$\langle 21|V|21\rangle = 0.952\ A_0^{(1)}$$
$$\langle 41|V|41\rangle = 0.468\ A_0^{(1)}$$
$$\langle 61|V|61\rangle = 0.233\ A_0^{(1)}$$

Given an interaction strength $A_0^{(T)} = (1+\alpha \boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2)A_0$ and $\alpha = -0.25$, which will be the ground-state spin and parity of this nucleus ^{42}Sc? Considering figure 3.58, what would rather be the experimental constraint on α?

2. The Weisskopf estimate for an $E2$ transition in ^{17}O is given by

$$\begin{aligned} W_E(2\downarrow) &= \frac{1.2^4}{4\pi}\,\frac{9}{25}\,17^{4/3}\,fm^4 \\ &= 2.60\,fm^4. \end{aligned}$$

However, this assumes a $L+1/2 \rightarrow 1/2$ transition instead of the actual $1/2 \rightarrow 5/2$ case so we need a statistical factor of 6 instead of 2. Then

$$B(E2\downarrow) = 7.79\,fm^4.$$

The experimental value is 6.3 fm^4. Therefore the effective charge is

$$\begin{aligned} \tilde{e} &= \sqrt{\frac{6.3}{7.79}}\,e \\ &= 0.90\,e. \end{aligned}$$

and the neutron would almost behave as if it were a proton. It turns out that more accurate calculations of $\langle r^2\rangle$ yield a value that is twice as large, hence an effective charge that is only half as big.

3. We consider the $0^+ \rightarrow 2^+$ transition at 1492 keV in ^{80}Zn (J Van de Walle *et al*, Physical Review Letters 99, 142501). For this nucleus, a Weisskopf unit equals

$$\begin{aligned} W_E(2\downarrow) &= \frac{1.2^4}{4\pi}\,\frac{9}{25}\,80^{4/3}\,fm^4 \\ &= 20.48\,fm^4. \end{aligned}$$

Experimentally, $B(E2\uparrow) = 0.073\ e^2b^2$ which is 730 e^2fm^4 ($1b = 10^{-28}\ m^2$). Therefore $B(E2\downarrow) = 146\ e^2fm^4$ which is 7.1 Wu so the transition is collective. The partial transition rate becomes

$$\begin{aligned} w_{E2} &= \frac{8\pi e^2}{\hbar^6c^5}\,\frac{3}{2\cdot 9\cdot 25}\,E^5\,B(E2) \\ &= \frac{4\pi}{75}\,\frac{(4.803\times10^{-10})^2}{(6.582\times10^{-22})^6(2.997\times10^8)^5}\,1.492^5\,146\times10^{-60}\,\frac{esu^2}{MeVms} \\ &= 2.12\times10^8\,\frac{1}{1.602\times10^{-13}}\,\frac{g\,cm^3}{kg\,m^3s} \\ &= 1.32\times10^{12}\,s^{-1}. \end{aligned}$$

Remember that we work in cgs units, for which charge is expressed in esu. In the cgs system, for two point charges that are spaced 1 centimetre apart the electrostatic force is equal to 1 dyne. Therefore $1\ g\ cm/s^2 = 1\ esu^2/cm^2$ and $esu = \sqrt{g\ cm^3}/s$. Also $1\ C = 2.997\times10^9\ esu$ resulting in an elementary charge of $4.803\times10^{-10}\ esu$. For the half-life of the excited state we deduce

$$\begin{aligned} t_{1/2} &= \frac{\ln 2}{w_{E2}} \\ &= 0.523\,ps. \end{aligned}$$

2011 Examination

1. Describe the mechanism(s) that lift the degeneracies shown in Fig 3.59. How is this different from the case of Fig. 3.42?

2. In Fig. 2 of the article "Unambiguous identification of three β-decaying isomers in ^{70}Cu", describe the single-particle structure of the three isomers that were observed in the β decay of ^{70}Ni to ^{70}Cu (these are the ground state, one state at 101 keV, and one state at 242 keV). Explain how the parabolic rule of Paar is used to interpret the spin values and therefore the existence of these isomers.

3. The article "Trace analysis of plutonium in environmental samples by resonance ionization mass spectroscopy" explains how lasers can be used to detect plutonium in the environment with extremely high sensitivity. If you would want to use this technique to study nuclear structure, what are the nuclear observables that you would get access to? Explain briefly.

2012 Examination

1. Describe the difference between the interaction of identical and non-identical particles. What is the formal mechanism to express this difference and why is it justified? How do the matrix elements behave?

2. What is the importance of the isobaric multiplet mass equation from the theoretical point of view? What is its use from the experimental point of view? Imagine that you have good knowledge of ^{36}Ar, ^{36}Cl and ^{36}S and that in each of these you identify a $T = 2$ state such that you deduce $a = -19.4$ MeV, $b = -6.0$ MeV and $c = 0.2$ MeV. What are the missing multiplet members? For which multiplet members will this $T = 2$ state also be the ground state? What information do you get from applying the equation? The mass excess of ^{36}S is -30.7 MeV, which you can use as a reference value for your statements.

3. What are the two most likely values for the spin of ^{209}Bi from the simple shell model? What is the single-particle configuration for these two possibilities? What would be the effective charge of the unpaired nucleon in either case, knowing that experimentally the quadrupole moment is -37(3) fm^2? Does your result for the effective charge favour one of the spin values? Compare with Table 4.2 for typical values of the effective charge.

2013 Examination

1. Describe why configuration mixing of two nuclear levels leads to a minimum energy with which the levels are shifted (in what direction?) and deduce the corresponding formula for this minimum energy shift. What is the condition for mixing to occur?

2. Describe what you learn from figure 3.50. Which are the operators that allow you to move through the scheme? What is the range (that is, the extreme values) of the operators? What are the nuclear transitions that these operators correspond to? Is the level ordering that is shown in this figure consistent with what you observe in figure 3.58? If it is consistent, what fundamental principle is this the proof of; but if it is not, what mechanism do you need to explain the difference?

3. What are the spins and parities that you expect for the ground state and the first excited state of ^{54}Co ($Z = 27$)? What are their configurations according to the shell model? What will be the decay mode of the first excited state (α, β^+, β^-, γ, ...)?

4. The measured magnetic moment of the ground state of ^{17}F ($Z = 9$) is $+4.72\ \mu_N$. What is the shell-model configuration that you expect for this state and what does the shell model predict for this magnetic moment? What value do you extract for the effective g-factor? What is the ground-state configuration of ^{19}F, knowing that the magnetic moment is $+2.63$ and that you can use, as a first hypothesis, the same effective g-factors as for ^{17}F?