

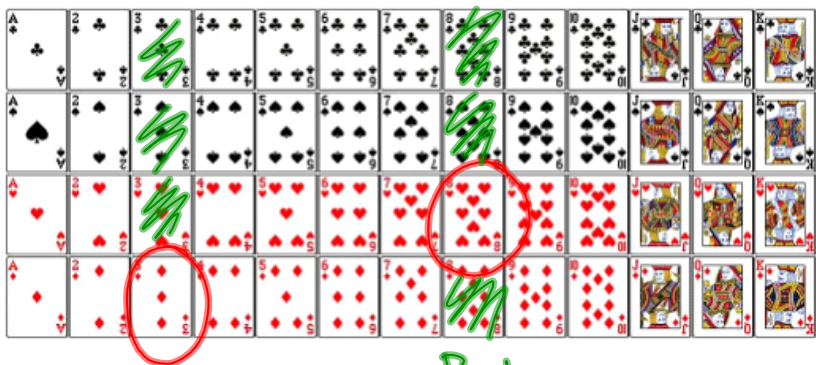
Chapter 1

The Mathematics of Voting

The Paradoxes of Democracy

- Elections involving ballots ranking several candidates.
- Consider many examples of balloting in society.
- Understand several different methods for determining a winner.
- Understand several different fairness criteria.
- Arrows' impossibility theorem.

1. Suppose you are playing poker. You are dealt 5 cards. How many ways can it happen that all five cards are different rank (Ace-throughp-King).



N =

Pr

| | | | | |
|----|----|----|--|--|
| 52 | 48 | 44 | | |
| 52 | 51 | 50 | | |

| | | | | |
|-----------------|-----------------|-----------------|-----------------|-----------------|
| 1st Romney | 1st Paul | 1st Romney | 1st Paul | 1st Obama |
| 2nd Paul | 2nd Obama | 2nd Gingrich | 2nd Romney | 2nd Romney |
| 3rd Gingrich | 3rd Romney | 3rd Paul | 3rd Gingrich | 3rd Gingrich |
| 4th Obama | 4th Gingrich | 4th Obama | 4th Obama | 4th Paul |

|| ||| |||

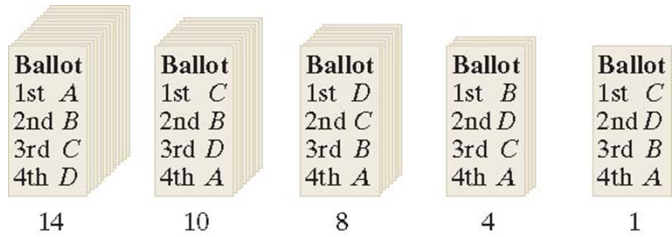
| Romney | | Obama | | Gingrich | | Paul | |
|-----------------|-----------------|-----------------|-----------------|---------------|-----------------|-----------------|-----------------|
| 1st Romney | 1st Obama | 1st Paul | 1st Gingrich | 1st Paul | 1st Paul | 1st Gingrich | 1st Paul |
| 2nd Paul | 2nd Paul | 2nd Romney | 2nd Obama | 2nd Obama | 2nd Obama | 2nd Obama | 2nd Obama |
| 3rd Obama | 3rd Gingrich | 3rd Obama | 3rd Paul | 3rd Paul | 3rd Gingrich | 3rd Paul | 3rd Gingrich |
| 4th Gingrich | 4th Romney | 4th Gingrich | 4th Romney | 4th Romney | 4th Romney | 4th Romney | 4th Romney |

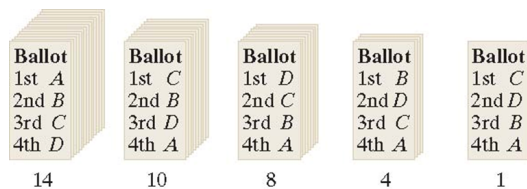
1st place votes Romney 6 Paul 6 Obama 2
Gingrich 1
Runoff R+P 7 Romney 8 Paul
Wins.

- Preference ballots
A ballot in which the voters are asked to rank the candidates in order of preference.
- Linear ballot
A ballot in which ties are not allowed.

| | | | | | | | | | | | | | | | | | | | |
|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|
| Ballot | Ballot | Ballot | Ballot | Ballot | Ballot | Ballot | Ballot | Ballot | Ballot | Ballot | Ballot | Ballot | Ballot | Ballot | Ballot | Ballot | Ballot | Ballot | Ballot |
| 1st A | 1st B | 1st A | 1st C | 1st B | 1st C | 1st A | 1st B | 1st C | 1st A | 1st B | 1st C | 1st A | 1st B | 1st C | 1st A | 1st B | 1st C | 1st A | 1st D |
| 2nd B | 2nd D | 2nd B | 2nd B | 2nd D | 2nd B | 2nd B | 2nd D | 2nd B | 2nd B | 2nd C | 2nd B | 2nd B | 2nd B | 2nd B | 2nd B | 2nd B | 2nd B | 2nd C | 2nd C |
| 3rd C | 3rd C | 3rd C | 3rd D | 3rd C | 3rd D | 3rd C | 3rd C | 3rd D | 3rd C | 3rd D | 3rd B | 3rd C | 3rd C | 3rd D | 3rd C | 3rd B | 3rd C | 3rd B | 3rd B |
| 4th D | 4th A | 4th D | 4th A | 4th A | 4th A | 4th D | 4th A | 4th D | 4th A | 4th A | 4th D | 4th D | 4th D | 4th D | 4th A | 4th D | 4th A | 4th A | 4th D |

| | | | | | | | | | | | | | | | | | | | |
|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|
| Ballot | Ballot | Ballot | Ballot | Ballot | Ballot | Ballot | Ballot | Ballot | Ballot | Ballot | Ballot | Ballot | Ballot | Ballot | Ballot | Ballot | Ballot | Ballot | Ballot |
| 1st A | 1st A | 1st D | 1st C | 1st C | 1st D | 1st A | 1st D | 1st C | 1st A | 1st D | 1st B | 1st A | 1st C | 1st A | 1st A | 1st D | 1st A | 1st A | 1st A |
| 2nd B | 2nd B | 2nd C | 2nd C | 2nd B | 2nd B | 2nd C | 2nd B | 2nd C | 2nd B | 2nd C | 2nd B | 2nd B | 2nd D | 2nd B | 2nd B | 2nd B | 2nd C | 2nd B | 2nd B |
| 3rd D | 3rd C | 3rd B | 3rd D | 3rd D | 3rd B | 3rd C | 3rd B | 3rd D | 3rd C | 3rd B | 3rd C | 3rd C | 3rd D | 3rd B | 3rd C | 3rd C | 3rd B | 3rd C | 3rd C |
| 4th A | 4th D | 4th A | 4th A | 4th A | 4th A | 4th D | 4th A | 4th A | 4th D | 4th A | 4th D | 4th A | 4th D | 4th D | 4th D | 4th D | 4th A | 4th A | 4th D |





Who wins the election?

A wins with 14 votes more than anyone else

Are most people happy about the result?

23 people would prefer anyone else
A has plurality (more than anyone else)
but NOT Majority (more than $\frac{1}{2}$)

Assumptions about Ballots

- The first is that a voter's preference are **transitive**, i.e., that a voter who prefers candidate A over candidate B and prefers candidate B over candidate C automatically prefers candidate A over C (if B were not running).
- Secondly, that the relative preferences of a voter are not affected by the elimination of one or more of the candidates.

Methods of Choosing a Winner based on all the preference ballots

Method

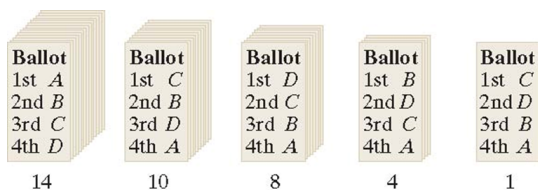
- **Plurality method**
Election of 1st place votes
- **Plurality candidate**
The Candidate with the most 1st place votes

- Rule or Criterion.*
- **Majority rule** *Something that should be True*
The candidate with a more than half the votes should be the winner.
 - **Majority candidate** *(but might not happen)*
The candidate with the majority of 1st place votes .

Criterion for selecting a Winner
(Things that "should" be true about an election)

The Majority Criterion

If candidate X has a *majority* of the 1st place votes, then candidate X should be the *winner* of the election.



Criterion for selecting a Winner
(Things that "should" be true about an election)

The Condorcet Criterion

If candidate X is preferred by the voters over each of the other candidates in a head-to-head comparison, then candidate X should be the winner of the election.

Ballot
1st A
2nd B
3rd C
4th D

14

Ballot
1st C
2nd B
3rd D
4th A

10

Ballot
1st D
2nd C
3rd B
4th A

8

Ballot
1st B
2nd D
3rd C
4th A

4

Ballot
1st C
2nd D
3rd B
4th A

1

Run off

C vs A
23 14

C vs B
10 14
8 4
19 18

C vs D
14 8
18 4
33 12

http://en.wikipedia.org/wiki/Marquis_de_Condorcet

Method is Point system

2008 Heisman Trophy Results

3pts 2pts 1pt

| Player | School | First | Second | Third | Total points |
|------------------|----------------|-------|--------|-------|--------------|
| Sam Bradford | Oklahoma | 300 | 315 | 196 | 1,726 |
| Colt McCoy | Texas | 266 | 288 | 230 | 1,604 |
| Tim Tebow | Florida | 309 | 207 | 234 | 1,575 |
| Graham Harrell | Texas Tech | 13 | 44 | 86 | 213 |
| Michael Crabtree | Texas Tech | 3 | 27 | 53 | 116 |
| Shonn Greene | Iowa | 5 | 9 | 32 | 65 |
| Patrick White | West Virginia | 3 | 1 | 8 | 19 |
| Nate Davis | Ball State | 0 | 1 | 8 | 10 |
| Rey Maualuga | USC | 2 | 1 | 1 | 9 |
| Javon Ringer | Michigan State | 1 | 0 | 5 | 8 |

Most 1st place votes

2000 Presidential Election

| Candidate | Party | Votes | Percent | Electoral College Votes |
|-----------------|--------------|------------|---------|-------------------------|
| George W. Bush | Republican | 50,456,002 | 47.87% | 271 |
| Al Gore | Democratic | 50,999,897 | 48.38% | 266 |
| Ralph Nader | Green | 2,882,955 | 2.7% | 0 |
| Pat Buchanan | Reform | 448,895 | 0.4% | 0 |
| Harry Browne | Libertarian | 384,431 | 0.4% | 0 |
| Howard Phillips | Constitution | 98,020 | 0.1% | 0 |
| John Hagelin | Natural Law | 83,714 | 0.1% | 0 |

Another examples where the plurality method fails to satisfy the Condorcet Criterion

| Number of voters | 49 | 48 | 3 |
|------------------|----|----|---|
| 1st choice | R | H | F |
| 2nd choice | H | S | H |
| 3rd choice | F | O | S |
| 4th choice | O | F | O |
| 5th choice | S | R | R |

Who wins?

R wins with
more 1st place
than anyone else

H vs R H vs anyone
else
51 to 49 by a lot
one to one H wins
against anyone else

Nice Web Page to Compare Several Types of Voting Methods



Insincere Voting (or Strategic Voting)

Three voters decide not to “waste” their vote on F and swing the election over to H in doing so.

| The Real Preferences | | | |
|----------------------|----|----|---|
| Number of voters | 49 | 48 | 3 |
| 1st choice | R | H | F |
| 2nd choice | H | S | H |
| 3rd choice | F | O | S |
| 4th choice | O | F | O |
| 5th choice | S | R | R |

| The Actual Votes | | | |
|------------------|----|----|---|
| Number of voters | 49 | 48 | 3 |
| 1st choice | R | H | H |
| 2nd choice | H | S | F |
| 3rd choice | F | O | S |
| 4th choice | O | F | O |
| 5th choice | S | R | R |

4 might vote for their 2nd choice since 1st place want win.

The "Election Spoiler" Controversy

The extremely close race between the Democratic and Republican presidential candidates, Al Gore and George W. Bush, helped to create some additional controversy around the 2000 campaign. Many Democrats claimed that Nader had no realistic chance of winning in the close election. They felt that those who supported Nader should have instead voted for Gore, and that a victory for Gore would have been preferable to a victory for George W. Bush [3]. Many prominent liberal politicians, activists, and celebrities campaigned for Nader [4]; others made the argument of prominent Democrats to voters in swing states, sometimes using the catch phrase "a vote for Nader is a vote for Bush." The Republican Leadership Council ran pro-Nader ads in a few states in a likely effort to split the "left" vote, a tactic from which the Nader campaign disassociated itself.[20] Nader and many of his supporters, including filmmaker Michael Moore, responded with their own catch phrase: "a vote for Gore is a vote for Bush." [21][22] The Nader campaign proclaimed that while Gore was perhaps marginally preferable to Bush, the differences between the two were not great enough to merit support of Gore.[citation needed]

Results of Bush, Gore, Nader Presidential Vote in 2000



In the **Borda Count Method** each place on a ballot is assigned points. In an election with N candidates we give 1 point for *last* place, 2 points for *second from last* place, and so on.

At the top of the ballot, a *first-place* vote is worth N points. The points are tallied for each candidate separately, and the candidate with the highest total is the winner.

We call such a candidate the *Borda winner*.

5 candidates

1st 5 points
2nd 4 pts
3rd 3 pts
4th 2 pts
5th 1 pt

http://en.wikipedia.org/wiki/Jean-Charles_de_Borda



Same ballots from beginning,

| Number of voters | 14 | 10 | 8 | 4 | 1 |
|----------------------|----|----|---|---|---|
| 1st choice: 4 points | A | C | D | B | C |
| 2nd choice: 3 points | B | B | C | D | D |
| 3rd choice: 2 points | C | D | B | C | B |
| 4th choice: 1 point | D | A | A | A | A |

Plurality Method
A wins

A gets $56 + 10 + 8 + 4 + 1 = 81$ points
B gets $42 + 30 + 16 + 16 + 2 = 106$ points
C gets $28 + 40 + 24 + 8 + 4 = 104$ points
D gets $14 + 20 + 32 + 12 + 3 = 81$ points

C wins head to head with every candidate

B gets most points

Borda Method winner.

What if the point system is different?

| Number of voters | 14 | 10 | 8 | 4 | 1 | |
|----------------------|----|----|---|---|---|---|
| 1st choice: 4 points | 10 | A | C | D | B | C |
| 2nd choice: 3 points | 5 | B | B | C | D | D |
| 3rd choice: 2 points | 2 | C | D | B | C | B |
| 4th choice: 1 point | 1 | D | A | A | A | A |

$$A: 14(10) + 10(1) + 8(1) + 4(1) + 1(1)$$

$$= 163$$

$$B = 14(5) + 10(5) + 8(2) + 4(10) + 1(2)$$

$$= 178$$

$$C = 14(2) + 10(10) + 8(5) + 4(2) + 1(10)$$










$$\text{wins } 186$$

$$D = 14(1) + 10(2) + 8(10) + 4(5) + 1(5)$$

http://en.wikipedia.org/wiki/Voting_system



Attachments

-  Heisman Trophy Winner Selection
-  Alternate Voting Methods for Presidential Primaries
-  Results of Bush, Gore, Nader Presidential Vote in 2000
-  Wikipedia Article on Voting Methods and Criteria
-  Monotonicity Criterion
-  Wikipedia Voting Systems Page
-  wikipedia Arrows Impossibility Theorem
-  Wikipedia Page on Kenneth Arrow
-  Nice Web Page to Compare Several Types of Voting Methods