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THE PARALLEL POSTULATE

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EUCLID'S famous parallel postulate was responsible for an enormous amount of mathematical activity over a period of more than twenty centuries. The failure of mathematicians to prove Euclid's statement from his other postulates contributed to Euclid's fame and eventually led to the invention of non-Euclidean geometries.

Before Euclid's time, various definitions of parallel lines had been considered by the Greeks and then discarded. Among them were "parallel lines are lines everywhere equidistant from one another" and "parallel lines are lines having the same direction from a given line." But these early definitions were sometimes vague or contradictory. Euclid tried to overcome these difficulties by his definition, "Parallel lines are straight lines which, being in the same plane and being produced indefinitely in both directions, do not meet one another in either direction," and by his fifth postulate, "Let it be postulated that, if a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which the angles are less than two right angles."

The statements of Euclid's first four assumptions are: "Let the following be postulated: (1) To draw a straight line from any point to any point. (2) To produce a finite straight line continuously in a straight line. (3) To describe a circle with any center and distance. (4) That all right angles are equal to one another." All of his assumptions fall into one of two categories. The first is the set of "self-evident" facts concerning plane figures. An example of such an assumption is that "a straight line is the shortest distance between two points." The second category deals with concepts beyond the realm of actual experience. For example, Euclid stated that "a straight line must continue undeviatingly in either direction without end and without finite length." Since it is impossible to experience things indefinitely far off, anything that is said about events there is speculation, not self-evident truth. The fifth postulate falls into this latter category.

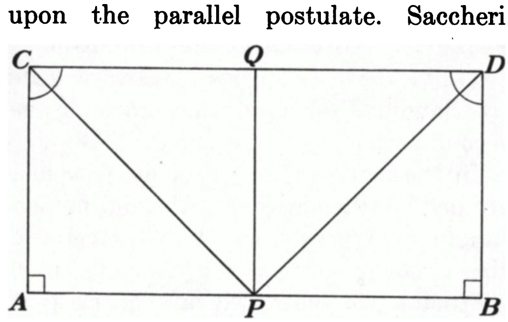
The complicated nature of the fifth postulate led numerous mathematicians to believe that it could be proved using the remaining postulates, and, therefore, ought to be a theorem rather than a postulate. Even Euclid might have supported this viewpoint since he did succeed in proving the converse of the postulate. One of the first geometers who attempted to prove a statement equivalent to Euclid's parallel postulate was Posidonius, in the first century B.C. He had defined parallel lines as lines that are coplanar and equidistant. A second early important attempt to prove the parallel postulate was made by Claudius Ptolemy of Alexandria, in the second century.

In the fifth century, Proclus, who had studied mathematics in Alexandria and taught in Athens, worked extensively on the problem of proving Euclid's fifth postulate. He succeeded in showing that the postulate could be proved if the following statement could be established: If L_1 and L_2 are any two parallel lines and L_3 any line distinct from and intersecting L_1, then L_3 intersects L_2.

In his argument Proclus used the phrase "distance between parallels." Thus, he assumed that parallel lines are everywhere equidistant, and this assumption is logically equivalent to the fifth postulate. In effect, Proclus assumed what he was trying to prove.

The most elaborate attempt to prove the parallel postulate, and the one most significant for the further development of geometry, was made by the Italian priest, Girolamo Saccheri, who taught mathematics at the University of Pavia. His significant work, published in 1733, was entitled *Euclides ab omni naevo vindicatus sive conatus geometricus quo stabiliuntur prima ipsa geometriae principia*. In this treatise, Saccheri tried to free Euclid from all error, including the supposed error of assuming Postulate V.

One of Saccheri's contributions was the introduction of the Saccheri quadrilateral. The construction of this quadrilateral is as follows. At the endpoints of a segment $\overline{AB}$ construct congruent segments $\overline{AC}$ and $\overline{BD}$, each perpendicular to $\overline{AB}$ and draw $\overline{CD}$. Saccheri tried to prove, on the basis of the first four postulates, that $m \measuredangle ACD = m \measuredangle BCD$. He reasoned that if P is the midpoint of $\overline{AB}$ and if Q is the midpoint of $\overline{CD}$ then rt. $\triangle CAP \cong$ rt. $\triangle DBP$, whence $m \measuredangle ACP = m \measuredangle BDP$ and $CP = DP$. Then $\triangle CPQ \cong \triangle DPQ$, so $m \measuredangle PCQ = m \measuredangle PDQ$, and, therefore, $m \measuredangle ACD = m \measuredangle BCD$. This proof does not depend upon the parallel postulate. Saccheri



called $\measuredangle ACD$ and $\measuredangle BDC$ the *summit angles* of the quadrilateral and he formulated the following three possibilities, which are exhaustive and pairwise mutually exclusive: (1) The summit angles are right angles. (2) The summit angles are obtuse angles. (3) The summit angles are acute angles. These possibilities are generally called the *right angle hypothesis*, the *obtuse angle hypothesis*, and the *acute angle hypothesis*, respectively. Saccheri succeeded in proving that if any of these hypotheses is valid for one Saccheri quadrilateral, it is valid for every quadrilateral of the same type. He proved further that the fifth postulate is a consequence of the right angle hypothesis. And, by assuming that a straight line is infinitely long, he showed that the obtuse angle hypothesis is self-contradictory.

Disposing of the acute angle hypothesis presented some difficulty, so Saccheri argued intuitively that the "hypothesis of the acute angle is absolutely false, because it is repugnant to the nature of a straight line." Actually, no logical contradiction can be deduced from the acute angle hypothesis, for it gives rise to a new geometry.

So, while Saccheri was looking for a proof of the parallel postulate, he discovered a new world, the world of "absolute geometry," whose theory is independent of the question of the parallel postulate. Included in "absolute geometry" are the theorems concerning congruent triangles, the inequalities involving the measures of the sides and angles of a triangle, and a set of theorems about the Saccheri quadrilateral. The quadrilateral $ABCD$ is a Saccheri quadrilateral if $\measuredangle A$ and $\measuredangle B$ are right angles and $AC = BD$. Among the theorems that can be proved are: (1) The diagonals of a Saccheri quadrilateral are always congruent. (2) In any Saccheri quadrilateral, the upper base angles are congruent, i.e. $\measuredangle C \cong \measuredangle D$. (3) In any Saccheri quadrilateral, the upper base is congruent to or longer than the lower base, i.e., $CD \geq AB$.

After Saccheri's time, many mathematicians pursued the problem of trying to prove the parallel postulate from the first four postulates. In 1766, J. H. Lambert, a Swiss geometer, showed that Saccheri's obtuse angle hypothesis is consistent with spherical geometry. In many cases those who attacked the problem worked with statements that are logically equivalent to the fifth postulate rather than with the statement of Euclid. Legendre (1752–1833) tried to prove the following alternative to Euclid's postulate: There exists a triangle in which the sum of the measures of the three angles is equal to the sum of the measures of two right angles. He presented a proof of the fact that the sum of the measures of the angles of a triangle cannot be greater than 180°, but he failed to supply a proof of the fact that the sum cannot be less than 180°. At any rate, his alleged proof of the latter rests on assumptions which are equivalent to the theorem he was trying to establish.

In 1809, Bernhard Friedrich Thibaut tried to demonstrate the existence of a triangle with the property that the sum of the measures of the angles is equal to 180°. His argument was based on the assumption that every rigid motion can be resolved into a rotation and an independent translation. This assumption, however, is equivalent to Postulate V.

In 1813, John Playfair copied Thibaut's argument and tried to correct Thibaut's errors. His attempt was unsuccessful, but it is his alternate statement of the fifth postulate that is now best known and most frequently quoted: "Through a given point, not on a given line, only one parallel can be drawn to the given line." Playfair's statement is surely simpler and more direct than Euclid's.

Karl Friedrich Gauss of Göttingen, Germany, studied the theory of parallels for thirty years. His work resulted in the formulation of a non-Euclidean geometry. In a letter to his friend Franz Adolf Taurinus, dated November 8, 1824, he stated: "The assumption that the angle sum [of a triangle] is less than 180° leads to a curious geometry, quite different from ours but thoroughly consistent, which I have developed to my entire satisfaction. The theorems of this geometry appear to be paradoxical, and, to the uninitiated, absurd, but calm, steady reflection reveals that they contain nothing at all impossible." However, Gauss wrote only a short account of his geometry, which was published in 1831. He apparently shrank from the controversy in which a treatise on the new geometry would have involved him. Along with other eminent mathematicians he was influenced by the authority of the German philosopher Immanuel Kant, who had died in 1804. Kant's doctrine emphasized that Euclid's geometry was "inherent in nature." Although Plato had said merely that God geometrizes, Kant asserted, in effect, that God geometrizes according to Euclid's *Elements*.

Gauss was the forerunner of the group of geometers who, instead of trying to prove Euclid's parallel postulate, replaced it by a contradiction of it and thereby invented a new geometry. In 1823, Janos Bolyai replaced Postulate V with the statement: "In a plane two lines can be drawn through a point parallel to a given line and through this point an infinite number of lines may be drawn lying in the angle between the first two and having the property that they will not intersect the given line." Bolyai's first disappointment came when he learned that Gauss had worked on the same problem for thirty years and had achieved the same results. He was disappointed a second time when he discovered that Nikolai Ivanovich Lobachevsky, a professor at the University of Kasan in Russia, had also invented the new geometry and published an account of it in 1829. Bolyai's work was published in 1833 as a twenty-six page appendix to a semiphilosophical two-volume treatise on elementary mathematics written by his father.

Lobachevsky is now accorded most of

the credit for the invention of the new geometry, and his name is the one usually attached to it. His replacement of Euclid's fifth postulate was stated as follows: "Through a point P not on a line there is more than one line which is parallel to the given line." The rest of Euclid's postulates were preserved. A frequently used model of Lobachevskian geometry is the Poincaré model. A detailed description of this model can be found in *Elementary Geometry from an Advanced Standpoint*, by Moise. Of course, any theorems of Euclid's geometry which do not depend on the parallel postulate are valid in Lobachevsky's geometry. On the other hand, the following theorems are examples of statements which are quite different from the corresponding theorems of Euclidean geometry.

1. *No quadrilateral is a rectangle; if a quadrilateral has three right angles, the fourth angle is acute.*

2. *The sum of the measures of the angles of a triangle is always less than 180°.*

3. *If two triangles are similar, they are congruent.*

Thus, through the discoveries of Gauss, Bolyai, and Lobachevsky, mathematicians recognized the existence of more than one consistent geometry. Leonard M. Blumenthal summarized the significance of the work of these three geometers:

> When the culture was ripe for it, three men, Gauss, Bolyai, and Lobachevsky (a German, a Hungarian, and a Russian) arose in widely separated parts of the learned world, and working independently of one another, created a new geometry. It would be difficult to overestimate the importance of their work. A significant milestone in the intellectual progress of mankind had been passed.

The first period in the history of non-Euclidean geometry, in the opinion of Felix Klein, ended with Lobachevsky's research. This period was characterized by the use of synthetic methods. Riemann, Helmholtz, Lie, and Beltrami were the representatives of the second period in the history of non-Euclidean geometry. Their work involved using the tools of differential geometry.

G. F. B. Riemann is credited with the development of another non-Euclidean geometry, in 1854, which can be realized on a sphere. He began by studying Euclid's postulate that a straight line has infinite length. Discarding this assumption, he invented a geometry in which all lines have finite length. Euclid's first postulate was replaced by the statement: "A straight line is restricted in length and without endpoints." And the parallel postulate was replaced by the statement: "through a point in a plane there can be drawn in the plane no line which does not intersect a given line not passing through the given point." The remainder of Euclid's postulates were retained. In Riemannian geometry, the following theorems can be proved:

1. *Two perpendiculars to the same line intersect.*

2. *Two lines enclose an area.*

3. *The sum of the measures of the angles of a triangle is greater than 180°.*

4. *If two sides of a quadrilateral are congruent and perpendicular to a third side, the figure is not a rectangle, since two of the angles are obtuse.*

An important contribution to the study of non-Euclidean geometry was made by the Italian mathematician, Eugenio Beltrami. His paper, published in 1868, gave the final answer to the question of the consistency of the new geometries. Bolyai and Lobachevsky had suspected that extending their investigations to three-dimensional space might reveal inconsistencies. Beltrami's paper gave an interpretation of plane non-Euclidean geometry as the geometry of geodesics on a certain class of surfaces in Euclidean space. Therefore, the new geometries must be as consistent as Euclidean geometry.

Gauss was the first to use the term "non-Euclidean" geometry, and Felix Klein first gave to the new geometries the names currently used to describe them. He called Lobachevsky's geometry *hyper-*

bolic, Riemann's geometry *elliptic*, and Euclid's geometry *parabolic*. This terminology arose from the projective approach to non-Euclidean geometry developed by Klein and Arthur Cayley.

Non-Euclidean geometry not only widened the scope of geometric knowledge; it also stimulated discussions concerning what is a geometry and what is "mathematical truth." As recently as the 1820s, geometry had been thought of as an idealized description of the spatial relations of the world in which we live. Euclid held this viewpoint on the meaning of geometry, and chose for his postulates statements that had their roots in everyday experience. A geometric statement was then regarded as true if it correctly described nature, and false if it did not. But Beltrami's proof that Euclid's geometry was not the only consistent geometry forced mathematicians to abandon the idea that geometric truth involved a description of nature.

David Hilbert called the invention of non-Euclidean geometry "the most suggestive and notable achievement of the last century." And Heath claimed that one of the cornerstones on which Euclid's greatness as a mathematician rests was his parallel postulate, for "when we consider the countless successive attempts made through more than twenty centuries to prove the postulate, many of them by geometers of ability, we cannot but admire the genius of the man who concluded that such a hypothesis, which he found necessary to the validity of his whole system of geometry, was really undemonstrable."

BIBLIOGRAPHY

ARCHIBALD, RAYMOND CLARE. "Outline of the History of Mathematics, Part II," *American Mathematical Monthly*, LVI (January 1949), 21–26.

BELL, E. T. *The Development of Mathematics*. 1st ed. New York: McGraw-Hill Book Co., 1940. Pp. 304–7.

BLUMENTHAL, LEONARD M. *A Modern View of Geometry*. San Francisco: W. H. Freeman & Co., 1961. Pp. 4–17.

CAJORI, FLORIAN. *A History of Mathematics*, 2d ed. New York: Macmillan Co., 1919. Pp. 48, 306.

EVES, HOWARD, and NEWSOM, CARROLL V. *An Introduction to the Foundations and Fundamental Concepts of Mathematics*. Rev. ed. New York: Holt, Rinehart & Winston, 1965. Pp. 58–79.

MERRIMAN, GAYLORD M. *To Discover Mathematics*. New York: John Wiley & Sons, 1942. Pp. 144–51.

MOISE, EDWIN. *Elementary Geometry from an Advanced Standpoint*. Reading, Mass.: Addison-Wesley Publishing Co., 1963. Pp. 115–31.

SANFORD, VERA. *A Short History of Mathematics*. Boston: Houghton Mifflin Co., 1930. Pp. 276–81.

An ANALEMMA CONSTRUCTION for RIGHT AND OBLIQUE ASCENSIONS

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IN CLASSICAL times and during the Middle Ages a variety of spherical-astronomical problems were attacked by graphical methods which we would call descriptive geometry, and to which the name *analemma* was then attached. Historians of mathematics have given the topic little attention, although the close connection between the earliest spherical trigonometry and the analemma techniques has been pointed out.[1]

[1] See Otto Neugebauer, *The Exact Sciences in Antiquity*, 2d ed. (Providence, R.I.: Brown University Press, 1957), pp. 214–20.