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THE PERFORMANCE OF A PROJECTILE WHICH USES A BANG-BANG TYPE GUIDANCE LAW: PART 1

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Abstract—A projectile which uses a bang-bang type guidance law is launched, and its goal is to hit a fixed target whose centre is located on the ground. Using stochastic optimal control, the performance of the projectile is compared with cases where an optimal guidance law, or a saturated proportional navigation law, are being applied using the same airframe.

1. INTRODUCTION

Conventional missile systems make use of seekers which are mounted on some sort of gimbal system inside the missile, for detection and tracking of the target. The gimbals allow the seeker to be inertially stabilized, and it is then possible to implement a proportional navigation guidance law. The proportional navigation guidance law provides in general very good guidance accuracies. The alternative to mounting the seeker on gimbals is to fix the seeker to the missile airframe in the so-called strapdown configuration. To implement a proportional navigation guidance law using a strapdown seeker requires the use of an on-board computer and some additional sensors. All the above schemes are complex, costly and difficult to implement in a high-g launch environment. In this work a cheaper option for a guidance scheme, although a less accurate one, is discussed. This scheme uses the concept of a wind stabilized seeker. The concept of a wind stabilized seeker makes use of a seeker mounted on a "sting" which protrudes from the front of the missile. The seeker is mounted on a universal joint, and has a housing which is aerodynamically shaped so that it will align itself with the relative wind vector. The seeker sightline is therefore aligned with the missile flight path, irrespective of airframe attitude. The seeker can then measure the error angle between the seeker's flight path and the seeker-to-target line-of-sight. By using the seeker's outputs to control the flight surfaces on the missile airframe, the guidance loop tries to zero this error angle. Henceforward, it is assumed that the system is to be used against targets on the ground and that the projectile is unpowered after initial launch.

This paper deals with the guidance of a projectile which uses a wind stabilized seeker. Using stochastic optimal control, three guidance laws are considered, that is, an optimal guidance law, a saturated proportional navigation law and a bang-bang guidance law. However, from these three laws only the bang-bang guidance law can be implemented on the above-mentioned projectile, and the study of the first two laws is done for the sake of the evaluation of the performance of the projectile acting under the third guidance law.

This work is to a large extent a continuation of [1], and the methods applied here are the same as those applied in [1]. However, the physical problem dealt with here differs from the one dealt with in [1].

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2. THE EQUATIONS OF MOTION

Consider the motion of a projectile P in the (x, z)-plane. It is assumed here that the equations of motion of P are given by

$$\frac{dx}{dt} = v\cos\gamma, \tag{1}$$

$$\frac{dz}{dt} = v\sin\gamma, \tag{2}$$

$$\frac{d\gamma}{dt} = \frac{\left(\frac{L}{m} - g\cos\gamma\right)}{v}, \tag{3}$$

$$\frac{dv}{dt} = -\frac{D}{m} - g\sin\gamma, \tag{4}$$

where (x, z) denotes the coordinates of the projectile P, v its speed, γ its flight path angle, L the lift force acting on P, D the drag force acting on P and m its mass. It is assumed that

$$L = 0.5\rho(z)v^2 SC_{L_\alpha}\alpha \tag{5}$$

and

$$D = 0.5\rho(z)v^2 S(C_{DO} + KC_{L_\alpha}^2\alpha^2), \tag{6}$$

where $\rho(z)$ denotes the air density, S an appropriate reference area and C_{DO}, K and C_{L_α} are given aerodynamical coefficients. It is assumed in this simplified model that the angle of attack is the control function of the projectile's motion.

In the real system, the purpose of the wind stabilized seeker platform is to keep the seeker sightline aligned with the airframe's flight path angle. The wind stabilized seeker platform is coupled to the airframe by a universal joint, which allows the platform to align itself with the relative wind vector, but not to rotate relative to the airframe roll axis.

In the model used here, the effects of the motion of the seeker platform together with the forces and moments applied on it, are incorporated as additional Gaussian white noise processes. Hence, using the notation $x_1 := x$, $x_2 := z$, $x_3 := \gamma$ and $x_4 := v$, equations (1)–(6) yield

$$\frac{dx_1}{dt} = x_4\cos x_3 + \sigma_1\frac{dW_1}{dt}, \tag{7}$$

$$\frac{dx_2}{dt} = x_4\sin x_3 + \sigma_2\frac{dW_2}{dt} \tag{8}$$

$$\frac{dx_3}{dt} = \frac{0.5\rho(x_2)x_4^2 SC_{L_\alpha}\frac{\alpha}{m} - g\cos x_3}{x_4} + \sigma_3\frac{dW_3}{dt}, \tag{9}$$

$$\frac{dx_4}{dt} = -0.5\rho(x_2)x_4^2 S\frac{(C_{DO} + KC_{L_\alpha}^2\alpha^2)}{m} - g\sin x_3 + \sigma_4\frac{dW_4}{dt}, \tag{10}$$

$t > 0$, where $W = \{W(t) = (W_1(t), W_2(t), W_3(t), W_4(t)),\ t \geq 0\}$ is an $\mathbf{R}^4$-valued standard Wiener process, and σ_i, $i = 1,2,3,4$ are given numbers satisfying $0 < \sigma_i \ll v_1$, $i = 1,2,4$, and $0 < \sigma_3 \ll \pi$. The parameter v_1 is defined later. Note that $\sigma_i\frac{dW_i}{dt}$, $i = 1,2,3,4$, model the kinematical and dynamical effects on P emerging from the motion of the seeker's platorm.

It is assumed here that the projectile P can manoeuvre as long as $v_1 < v < v_2$, where v_1 and v_2 are given positive numbers, and that it has a detection range of radius R_o. Also, it is assumed that the motion of P is confined to the strip $0 < z < H_o$, where H_o is a given positive number. In addition, it is assumed that during its motion, the flight path angle γ is subject to the constraint $-\pi/2 \leq \gamma \leq \gamma_o$, where γ_o is a given positive number. Thus P has an "operation zone" D_{op} determined by: $0 \leq x < R_o$, $0 < z < H_o$, $-\pi/2 \leq \gamma \leq \gamma_o$ and $v_1 < v < v_2$. The projectile P is launched and its goal is to hit a fixed target set whose centre is located in (x_D, z_D), where $z_D > 0$ is a small enough number. Hence, once P is launched its goal is to reach, before leaving the domain D_{op}, a target set T,

$$T = \{(x, z) : (x - x_D)^2 + (z - z_D)^2 \leq r_o^2, z > 0\}, \tag{11}$$

where r_o is a given positive number.

In this work, one may view the time $t = 0$ as the first time during P's flight that the seeker begins to operate.

3. FORMULATION OF THE PROBLEM

In the sequel, the following set of stochastic differential equations will serve as the model for the motion of P

$$dx_1 = I(\mathbf{x})x_4 \cos x_3\, dt + \sigma_1\, dW_1, \tag{12}$$

$$dx_2 = I(\mathbf{x})x_4 \sin x_3\, dt + \sigma_2\, dW_2, \tag{13}$$

$$dx_3 = I(\mathbf{x})x_4^{-1}[0.5\rho(x_2)x_4^2 S C_{L_\alpha}\alpha(\mathbf{x})/m - g\cos x_3]\, dt + \sigma_3\, dW_3, \tag{14}$$

$$dx_4 = I(\mathbf{x})[-0.5\rho(x_2)x_4^2 S(C_{DO} + KC_{L_\alpha}^2\alpha^2(\mathbf{x}))/m - g\sin x_3]\, dt + \sigma_4\, dW_4, \tag{15}$$

$t > 0$, where $I(\mathbf{x}) = 1$ if $\mathbf{x} \in \{\mathbf{x} : v_1 < x_4 < v_2\}$ and $I(\mathbf{x}) = 0$ otherwise, $\mathbf{x} = (x_1, x_2, x_3, x_4)$. The function $I(\cdot)$ is introduced here to guarantee the existence of solutions to equations (12)–(15) over the whole of $\mathbf{R}^4$. In fact, we are interested in these solutions only over a set $D_o, D_o \subset \mathbf{R}^4$, which will be defined later.

Denote by U_o the class of all *feedback strategies* $\alpha(\cdot) = \{\alpha(\mathbf{x}), \mathbf{x} \in \mathbf{R}^4\}$ such that $\alpha(\cdot) : \mathbf{R}^4 \to \mathbf{R}$ is measurable and $|\alpha(\mathbf{x})| \le a_o$ for all $\mathbf{x} \in \mathbf{R}^4$.

Let $\alpha(\cdot) \in U_o$. Then, [2], equations (12)–(15) determine a stochastic process $\zeta_x^\alpha = \{\zeta_x^\alpha(t) = (\zeta_{x1}^\alpha(t), \zeta_{x2}^\alpha(t), \zeta_{x3}^\alpha(t), \zeta_{x4}^\alpha(t)), t \ge 0\}$, $\zeta_x^\alpha(0) = \mathbf{x}$, such that ζ_x^α is a weak solution (in the sense of [2]) to equations (12)–(15) associated with a family $\{P_x^\alpha, \mathbf{x} \in \mathbf{R}^4\}$ of probability measures, and such that $\{(\zeta_x^\alpha, P_x^\alpha), \mathbf{x} \in \mathbf{R}^4\}$ is a family of strong Markov processes. Furthermore, the weak infinitesimal operator of this family is given by

$$\begin{aligned}
\mathcal{L}(\alpha(\cdot))V(\mathbf{x}) =& I(\mathbf{x})x_4 \cos x_3 \frac{\partial V(\mathbf{x})}{\partial x_1} \\
&+ I(\mathbf{x})x_4 \sin x_3 \frac{\partial V(\mathbf{x})}{\partial x_2} \\
&+ I(\mathbf{x})x_4^{-1}[0.5\rho(x_2)x_4^2 S C_{L_\alpha}\alpha(\mathbf{x})/m \\
&- g\cos x_3]\frac{\partial V(\mathbf{x})}{\partial x_3} \\
&+ I(\mathbf{x})[-0.5\rho(x_2)x_4^2 S(C_{DO} + KC_{L_\alpha}^2\alpha^2(\mathbf{x}))/m \\
&- g\sin x_3]\frac{\partial V(\mathbf{x})}{\partial x_4} \\
&+ \frac{1}{2}\sum_{i=1}^{4}\sigma_i^2 \frac{\partial^2 V(\mathbf{x})}{\partial x_i^2},
\end{aligned} \tag{16}$$

for any $V \in C_o^\infty(\mathbf{R}^4)$.

Define the following sets in $\mathbf{R}^4$:

$$\begin{aligned}
D_o := \{\mathbf{x} : -\delta < x_1 < R_o,\ 0 < x_2 < H_o,\ -\delta - \frac{\pi}{2} < x_3 < \gamma_o + \delta, \quad \text{and} \\
v_1 < x_4 < v_2\}, \qquad 0 < \delta \ll 1,
\end{aligned} \tag{17}$$

$$K := \{\mathbf{x} \in D_o : (x_1 - x_D)^2 + (x_2 - z_D)^2 \le r_o^2\}, \tag{18}$$

and

$$D := D_o - K. \tag{19}$$

Note that now D_o and K are respectively P's "operation zone" and the target set.

Denote by $\tau(\mathbf{x}; \alpha(\cdot))$ the first exit time of ζ_x^α from D and define the following class of *admissible feedback strategies*:

$$U := \{\alpha(\cdot) \in U_o : \sup_{\mathbf{x} \in D} E_x^\alpha \tau(\mathbf{x}; \alpha(\cdot)) < \infty\}, \tag{20}$$

where E_x^α denotes the expectation operator with respect to P_x^α. Also, define the following functional:

$$\begin{aligned} V(\mathbf{x};\alpha(\cdot)) &:= P_x^\alpha(\{\zeta_x^\alpha(\tau(\mathbf{x};\alpha(\cdot)))\in K\}) \\ &= P_x^\alpha(\{\text{for some } t,\ 0\le t<\tau_o(\mathbf{x};\alpha(\cdot)): \\ &\qquad \text{the projectile } P \text{ reaches the target set } K\}), \\ &\qquad \alpha(\cdot)\in U, \mathbf{x}\in\mathbf{R}^4, \end{aligned} \tag{21}$$

where $\tau_o(\mathbf{x};\alpha(\cdot))$ is the first time t, $t\ge 0$, that ζ_x^α leaves the domain D_o.

In this work we consider three guidance laws for the projectile P. These laws are: (i) an optimal guidance law, (ii) a saturated proportional navigation guidance law and (iii) a bang-bang guidance law.

Although these guidance laws are feedback laws, only the third one can be implemented by the wind stabilized seeker whereas the first two serve as reference for the evaluation of the performance of P acting under the third guidance law.

4. COMPUTATION OF $V(\cdot;\alpha(\cdot))$

Let $\mathcal{D}$ denote the class of all functions $V:\mathbf{R}^4\to\mathbf{R}$ such that V is continuous on the closure $\bar{D}_o$ of D_o, twice continuously differentiable on D, and such that $\mathcal{L}(\alpha(\cdot))V\in L_2(D)$ for any $\alpha(\cdot)\in U$.

By following the same procedure as in [3], it follows that an optimal guidance law $\alpha^*(\cdot)$ may be found by solving the following problem

$$\mathcal{L}(\alpha(\cdot))V(\mathbf{x})=0,\qquad \mathbf{x}\in D \tag{22}$$

$$\begin{aligned} V(\mathbf{x})&=1, &\mathbf{x}\in K;\\ V(\mathbf{x})&=0, &\mathbf{x}\notin D_o, \end{aligned} \tag{23}$$

where $\alpha(\cdot)$ is given by:

If $\frac{\partial V(\mathbf{x})}{\partial x_4}>0$ then

$$\alpha(\mathbf{x})=\begin{cases} a_0 & \text{if } B_0(\mathbf{x})>a_0\\ B_0(\mathbf{x}) & \text{if } |B_0(\mathbf{x})|\le a_0\\ -a_0 & \text{if } B_0(\mathbf{x})<-a_0, \end{cases} \tag{24}$$

where

$$B_0(\mathbf{x})=\frac{\partial V(\mathbf{x})}{\partial x_3}\left[2x_4KC_{L_\alpha}\frac{\partial V(\mathbf{x})}{\partial x_4}\right]^{-1}. \tag{25}$$

Otherwise, if $\frac{\partial V(\mathbf{x})}{\partial x_4}\le 0$, then

$$\alpha(\mathbf{x})=a_0\,\mathrm{sign}\,\frac{\partial V(\mathbf{x})}{\partial x_3}. \tag{26}$$

Assume that equations (22)–(26) have a solution denoted here by $(\alpha^*(\cdot),V(\cdot;\alpha^*(\cdot)))$ which satisfies $\alpha^*(\cdot)\in U$ and $V(\cdot;\alpha^*(\cdot))\in\mathcal{D}$, then [3],

$$\begin{aligned} V(\mathbf{x};\alpha^*(\cdot)) &= P_x^{\alpha^*}(\{\zeta_x^{\alpha^*}(\tau(\mathbf{x};\alpha^*(\cdot)))\in K\})\\ &\ge P_x^\alpha(\{\zeta_x^\alpha(\tau(\mathbf{x};\alpha(\cdot)))\in K\}) \end{aligned} \tag{27}$$

for any $\alpha(\cdot)\in U$ and all $\mathbf{x}\in D$.

In this work the following saturated proprotional navigation guidance law has been used:

$$\alpha_P(\mathbf{x})=\begin{cases} a_0 & \text{if } B_P(\mathbf{x})>a_0\\ B_P(\mathbf{x}) & \text{if } |B_P(\mathbf{x})|\le a_0\\ -a_0 & \text{if } B_P(\mathbf{x})<-a_0, \end{cases} \tag{28}$$

where

$$B_P(\mathbf{x}):=4x_4\frac{[-(x_D-x_1)\sin x_3+(z_D-x_2)\cos x_3]}{r^2},$$

and

$$r^2 = (x_D - x_1)^2 + (z_D - x_2)^2. \tag{29}$$

Assume that equations (22)–(23) where $\alpha(\cdot) = \alpha_P(\cdot)$ is given by equations (28)–(29) have a solution denoted here by $V(\cdot;\alpha_P(\cdot))$. If $\alpha_P(\cdot) \in U$ and $V(\cdot;\alpha_P(\cdot)) \in \mathcal{D}$, then [3],

$$V(\mathbf{x};\alpha_P(\cdot)) = P_x^{\alpha_P}\left(\{\zeta_x^{\alpha_P}(\tau(\mathbf{x};\alpha_P(\cdot))) \in K\}\right), \qquad \mathbf{x} \in \mathbf{R}^4. \tag{30}$$

The third guidance law, which is proposed here to be used by the wind stabilized seeker, is given by

$$\alpha_B(\mathbf{x}) := -a_0 \operatorname{sign}\left[x_3 - \arctan\left(\frac{(z_D - x_2)}{(x_D - x_1)}\right)\right]. \tag{31}$$

Assume that equations (22)–(23) where $\alpha(\cdot) = \alpha_B(\cdot)$ is given by Equation (31) have a solution denoted here by $V(\cdot;\alpha_B(\cdot))$. If $\alpha_B(\cdot) \in U$ and $V(\cdot;\alpha_B(\cdot)) \in \mathcal{D}$, then [3],

$$V(\mathbf{x};\alpha_B(\cdot)) = P_x^{\alpha_B}\left(\{\zeta_x^{\alpha_B}(\tau(\mathbf{x};\alpha_B(\cdot))) \in K\}\right), \qquad \mathbf{x} \in \mathbf{R}^4. \tag{32}$$

In the next section the computation of $V(\cdot;\alpha^*(\cdot))$, $V(\cdot;\alpha_P(\cdot))$ and $V(\cdot;\alpha_B(\cdot))$ is discussed and a numerical study is conducted.

5. A NUMERICAL STUDY AND CONCLUSIONS

Denote by $\mathbf{R}_h^4$ the following finite-difference grid on $\mathbf{R}^4$

$$\mathbf{R}_h^4 := \{(i_1h_1, i_2h_2, i_3h_3, i_4h_4) : i_1, i_2, i_3, i_4 = 0, \pm 1, \pm 2, \dots\}, \tag{33}$$

Define $D_h := D \cap \mathbf{R}_h^4$. Equations (22)–(26), or Equations (22)–(23) and (28)–(29), or Equations (22)–(23) and (31) have here been solved using a finite-difference scheme on $\mathbf{R}_h^4$, similar to that described in [4].

Denote by $V^h(\cdot;\alpha^*(\cdot))$, $V^h(\cdot;\alpha_P(\cdot))$ and $V^h(\cdot;\alpha_B(\cdot))$ the solutions to the finite-difference equations corresponding to Equations (22)–(26), or Equations (22)–(23) and (28)–(29), or Equations (22)–(23) and (31), respectively. Define

$$P_{0a} := \sum_{\mathbf{x}\in D_h} \frac{V^h(\mathbf{x};\alpha^*(\cdot))}{N(D_h)}, \tag{34}$$

$$P_{Pa} := \sum_{\mathbf{x}\in D_h} \frac{V^h(\mathbf{x};\alpha_P(\cdot))}{N(D_h)}, \tag{35}$$

$$P_{Ba} := \sum_{\mathbf{x}\in D_h} \frac{V^h(\mathbf{x};\alpha_B(\cdot))}{N(D_h)}, \tag{36}$$

$$P_0(x_1) := \max_{\substack{(x_2,x_3,x_4)\\ \mathbf{x}\in D_h}} V^h(\mathbf{x};\alpha^*(\cdot)), \tag{37}$$

$$(x_{2m}^0(x_1),\ x_{3m}^0(x_1),\ x_{4m}^0(x_1)) := \operatorname*{argmax}_{\substack{(x_2,x_3,x_4)\\ \mathbf{x}\in D_h}} V^h(\mathbf{x};\alpha^*(\cdot)), \tag{38}$$

$$P_P(x_1) := \max_{\substack{(x_2,x_3,x_4)\\ \mathbf{x}\in D_h}} V^h(\mathbf{x};\alpha_P(\cdot)), \tag{39}$$

$$(x_{2m}^P(x_1),\ x_{3m}^P(x_1),\ x_{4m}^P(x_1)) := \operatorname*{argmax}_{\substack{(x_2,x_3,x_4)\\ \mathbf{x}\in D_h}} V^h(\mathbf{x};\alpha_P(\cdot)), \tag{40}$$

$$P_B(x_1) := \max_{\substack{(x_2,x_3,x_4)\\ \mathbf{x}\in D_h}} V^h(\mathbf{x};\alpha_B(\cdot)), \tag{41}$$

$$(x_{2m}^B(x_1),\ x_{3m}^B(x_1),\ x_{4m}^B(x_1)) := \operatorname*{argmax}_{\substack{(x_2,x_3,x_4)\\ \mathbf{x}\in D_h}} V^h(\mathbf{x};\alpha_B(\cdot)), \tag{42}$$

where $N(D_h)$ denotes the number of points in D_h.

Table 1. The values of $P_0(x_D - x_1)$, x^0_{2m}, x^0_{3m} and x^0_{4m} as functions of $x_D - x_1$ for the case: $R_o = 1000$, $H_o = 1200$, $x_D = 980$, $z_D = 15$, $h_1 = 20$, $h_2 = 15$, $h_3 = \pi/40$, $h_4 = 16$, $r_o = 15$. In this case $N(D_h) = 1725276$ and $P_{0a} = 0.064165$.

$x_D - x_1$	x^0_{2m}	x^0_{3m}	x^0_{4m}	$P_0(x_D - x_1)$
980	15	0.0	282	0.7258
900	15	0.0	266	0.7484
800	15	0.0	250	0.7778
700	15	0.0	250	0.8095
600	15	0.0	234	0.8430
500	15	0.0	218	0.8770
400	15	0.0	218	0.9105
300	15	0.0	202	0.9444
200	15	0.0	186	0.9723

Table 2. The values of $P_P(x_D - x_1)$, x^P_{2m}, x^P_{3m} and x^P_{4m} as functions of $x_D - x_1$ for the case: $R_o = 1000$, $H_o = 1200$, $x_D = 980$, $z_D = 15$, $h_1 = 20$, $h_2 = 15$, $h_3 = \pi/40$, $h_4 = 16$, $r_o = 15$. In this case $N(D_h) = 1725276$ and $P_{Pa} = 0.060612$.

$x_D - x_1$	x^P_{2m}	x^P_{3m}	x^P_{4m}	$P_P(x_D - x_1)$
980	15	0.0	314	0.4585
900	15	0.0	314	0.4983
800	15	0.0	314	0.5509
700	15	0.0	314	0.6066
600	15	0.0	314	0.6653
500	15	0.0	314	0.7268
400	15	0.0	314	0.7904
300	15	0.0	314	0.8549
200	15	0.0	314	0.9174

Table 3. The values of $P_B(x_D - x_1)$, x^B_{2m}, x^B_{3m} and x^B_{4m} as functions of $x_D - x_1$ for the case: $R_o = 1000$, $H_o = 1200$, $x_D = 980$, $z_D = 15$, $h_1 = 20$, $h_2 = 15$, $h_3 = \pi/40$, $h_4 = 16$, $r_o = 15$. In this case $N(D_h) = 1725276$ and $P_{Ba} = 0.055880$.

$x_D - x_1$	x^B_{2m}	x^B_{3m}	x^B_{4m}	$P_B(x_D - x_1)$
980	60	0.0	314	0.3391
900	60	0.0	314	0.3648
800	60	0.0	314	0.3901
700	45	0.0	314	0.4272
600	30	0.0	314	0.4645
500	30	0.0	314	0.5559
400	30	0.0	314	0.6528
300	30	0.0	314	0.7530
200	30	0.0	314	0.8587

Computations were carried out using the following set of parameters: $\gamma_o = \pi/40$, $v_1 = 10$, $v_2 = 330$, $m = 15$, $K = 1.06$, $C_{L_\alpha} = 145$, $S = 0.0111$, $C_{D_o} = 0.34955$, $a_0 = 0.0058$, $\sigma_1^2 = \sigma_2^2 = 1$, $\sigma_3^2 = (\pi/200)^2$ and $\sigma_4^2 = 0.01$. Some of the results are presented in Tables 1–6.

In order to get further insight into the problem, a simulation study of Equations (7)–(10) for the case $\sigma_i = 0$, $i = 1,2,3,4$ has been carried on. In this study the following set of initial conditions and values for x_D has been used: $x_1(0) = 0$; $x_2(0) = 20 + i \cdot 200$, $i = 0,1,2,\dots,14$; $x_3(0) = 0$; $x_4(0) = j \cdot 15$, $j = 1,\dots,21$, and $x_D = k \cdot 500$, $k = 1,\dots,5$. This set includes 1890 elements $(x_1(0), x_2(0), x_3(0), x_4(0), x_D)$. Two guidance rules were used during the simulation study, that is, $\alpha_P(\cdot)$ (Equations (28)–(29)) and $\alpha_B(\cdot)$ (Equation (31)), and the target set K (Equation (18)) has been used with $r_o = 4$. Denote by N the number of simulation runs, by N_K the number of

Table 4. The values of $P_0(x_D - x_1)$, x^0_{2m}, x^0_{3m} and x^0_{4m} as functions of $x_D - x_1$ for the case: $R_o = 1600$, $H_o = 3600$, $x_D = 1560$, $z_D = 20$, $h_1 = 40$, $h_2 = 20$, $h_3 = \pi/40$, $h_4 = 16$, $r_o = 20$. In this case $N(D_h) = 3128046$ and $P_{0a} = 0.10894$.

$x_D - x_1$	x^0_{2m}	x^0_{3m}	x^0_{4m}	$P_0(x_D - x_1)$
1560	20	0.0	314	0.6336
1400	20	0.0	314	0.6842
1200	20	0.0	314	0.7389
1000	20	0.0	298	0.7840
800	20	0.0	266	0.8311
600	20	0.0	250	0.8822
400	20	0.0	218	0.9349
200	20	0.0	202	0.9797

Table 5. The values of $P_P(x_D - x_1)$, x^P_{2m}, x^P_{3m} and x^P_{4m} as functions of $x_D - x_1$ for the case: $R_o = 1600$, $H_o = 3600$, $x_D = 1560$, $z_D = 20$, $h_1 = 40$, $h_2 = 20$, $h_3 = \pi/40$, $h_4 = 16$, $r_o = 20$. In this case $N(D_h) = 3128046$ and $P_{Pa} = 0.10751$.

$x_D - x_1$	x^P_{2m}	x^P_{3m}	x^P_{4m}	$P_P(x_D - x_1)$
1560	20	0.0	314	0.2873
1400	20	0.0	314	0.3430
1200	20	0.0	314	0.4217
1000	20	0.0	314	0.5101
800	20	0.0	314	0.6078
600	20	0.0	314	0.7137
400	20	0.0	314	0.8249
200	20	0.0	314	0.9322

Table 6. The values of $P_B(x_D - x_1)$, x^B_{2m}, x^B_{3m} and x^B_{4m} as functions of $x_D - x_1$ for the case: $R_o = 1600$, $H_o = 3600$, $x_D = 1560$, $z_D = 20$, $h_1 = 40$, $h_2 = 20$, $h_3 = \pi/40$, $h_4 = 16$, $r_o = 20$. In this case $N(D_h) = 3128046$ and $P_{Ba} = 0.07768$.

$x_D - x_1$	x^B_{2m}	x^B_{3m}	x^B_{4m}	$P_B(x_D - x_1)$
1560	200	0.0	314	0.2421
1400	100	$\pi/40$	314	0.2790
1200	80	$\pi/40$	314	0.3308
1000	60	$\pi/40$	314	0.3850
800	60	0.0	314	0.4526
600	40	0.0	314	0.5571
400	40	0.0	314	0.7078
200	40	0.0	314	0.8872

hits, and by T_K the average flight time in case of a hit, that is,

$$T_K = \frac{\{\text{the sum of all flight times in cases where there was a hit}\}}{N_K}. \tag{43}$$

Thus, the following results have been obtained:

	N	N_K	T_K
$\alpha_P(\cdot)$	1890	500	20.2689
$\alpha_B(\cdot)$	1890	463	22.3017

These results together with those presented in Tables 1–6 might help the designer in the evaluation of the performance of P when the guidance law $\alpha_B(\cdot)$ is being applied.

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