The Physics of Neutron Stars

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The Problem

Describe how a white dwarf evolves into a neutron star. Compute the neutron degeneracy pressure and balance the gravitational pressure with the degeneracy pressure. Use the Saha equation to determine where the $n \leftrightarrow p^+ + e^-$ equilibrium is below the 'Fermi Sea.' Provide experimental evidence for "starquakes." Explain how the universe would be a different place if $M_n - M_p < m_e$.

Description of Solution

The collapse of a white dwarf core will be described qualitatively. In order to calculate the neutron degeneracy pressure following the collapse, I will:

1. Find the highest filled neutron state in the star (n_F).
2. Compute the energy of this state, which is the Fermi energy ε_F.
3. Compute the internal energy of the star (U), in terms of the Fermi energy.
4. Compute the degeneracy pressure using $P = -\frac{\partial U}{\partial V}$
5. Compute the star's gravitational self-energy.
6. From the gravitational self-energy, calculate the star's gravitational pressure, using the same method as above.
7. Balance the two pressures by requiring that they sum to zero at equilibrium.
8. From the equilibrium equation, find the equilibrium radius.
9. Derive a "danger of collapse" condition.

I will find the relative neutron abundance by first deriving a partition function for the neutrons and then applying the Saha equation.

An example of observational evidence for a starquake will be briefly described, and a discussion of the changes to the universe if the neutron and proton were closer in mass will then occur.

The Birth of a Neutron Star

In very massive stars, the core's nuclear fuel will eventually become sufficiently depleted by fusion so that radiative pressure is no longer capable of supporting the upper layers of the star. Collapse will begin. In cores large than the Chandrasekhar limit (typically quoted as $1.4M_{Sun}$), electron degeneracy pressure will be generated as in a white dwarf, but the electrons will become ultra-relativistic. Ultra-relativistic electrons provide a pressure that has the same scaling as the gravitational pressure of the star's collapse, and are unable to reach an equilibrium. This marks the beginning of a neutron star.

As the core continues to collapse past the white dwarf state, the matter within it will continue to heat up due to the release of gravitational potential energy. Enough free energy is available that the following inverse beta decay reaction can occur:

$$p^+ + e^- + 1.36\text{MeV} \rightarrow n + \bar{\nu_e}$$

Ordinarily neutrons generated in this fashion would be unstable, and the neutron would turn back into an electron and a proton within ~10 minutes via beta decay. This is not possible, however, as the degenerate electron gas in the star has filled all of the available electron states in the core. No electrons (of energies ≤ 1.36MeV can be formed, which makes the neutrons stable. As the energy further increases, inverse beta decay takes place within an surviving atomic nuclei, reaching a peak with iron:

$$^{56}Fe + 3.7\text{MeV} \rightarrow ^{56}Mn + \bar{\nu_e}$$

In this manner, a very large number of neutrons is generated within the core. The outflow of neutrinos is of sufficient flux and the surrounding material of sufficient density that the collapse of the star is slowed from a free-fall time by neutrino pressure.

At some point, enough neutrons have been created that they become degenerate. The neutron degeneracy pressure is enough to immediately halt the collapse and to establish an equilibrium state (unless the core's mass exceeds the limits of neutron degeneracy pressure, in which case it further collapses to a black hole or possibly a quark star). The transition from collapse to equilibrium is very sudden, and the infalling material experiences a "bouce" against the degenerate core, which creates an outward-propagating shock wave. The shock wave is further boosted by neutrino pressure from the core. This shock is called a supernova, and can have a power output on the order of the Sun's integrated luminosity over its lifetime.

Calculation of Degeneracy Pressure

We begin by calculating the quantum number magnitude ($n_F = \sqrt{n_x^2 + n_y^2 + n_z^2}$) of the neutron at the very topmost filled energy level. This assumes zero temperature of the neutrons, which is not entirely accurate, but is a reasonable approximation. The formula relating the total number of neutrons N to n_F is:

$$N = 2\left(\frac{1}{8}\right)\left(\frac{4}{3}\pi n_F^3\right) \tag{1}$$

The factor of 2 represents the fact that both spin up and spin down are available in each quantum state. The remainder expresses the fact that the neutron is able to occupy any state in the positive octent of the quantum configuration space sphere. Rearranging, we find:

$$n_F = \left(\frac{3N}{\pi}\right)^{\frac{1}{3}} \tag{2}$$

At a given energy level, a neutron has a momentum $|p| = \frac{hn}{2V^{\frac{1}{3}}}$, corresponding to the nth wave solution in quantum mechanics. We can use this to compute the Fermi Energy:

$$\varepsilon(n) = \frac{|p|^2}{2m} = \frac{h^2}{8mL^2}|n|^2 \tag{3}$$

$$\varepsilon_F = \varepsilon(n_F) = \frac{h^2}{8m}\left(\frac{3N}{\pi V}\right)^{\frac{2}{3}} \tag{4}$$

Note that we have assumed that the neutrons are behaving non-relativistically. This assumption will be examined again shortly. The symbol m denotes the neutron mass, not the electron mass.

We now compute the internal energy of the core, excluding gravitational energy. This is done by integrating the energies of all of the filled energy levels. Since the core is degenerate, these levels begin at the ground state and are continuously filled up to n_F.

$$\begin{aligned} U &= 2\int\int\int \epsilon(n)\mathrm{d}^3n = 2\int_0^{n_F}\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} \frac{h^2}{8mV^{\frac{2}{3}}}n^2n^2\sin\theta \mathrm{d}n\mathrm{d}\theta\mathrm{d}\phi & (5)\\ &= \left(\frac{\pi h^2}{8mV^{\frac{2}{3}}}\right)\frac{n_F^5}{5} = \left(\frac{h^2}{8mV^{\frac{2}{3}}}n_F^2\right)\left(\frac{\pi}{3}n_F^3\right)\left(\frac{3}{5}\right) & (6)\\ &= \frac{3}{5}N\varepsilon_F & (7)\end{aligned}$$

The factor of 2 exists to account for the spin up and spin down available to each state. The integration limits restrict n to the positive octent of the sphere.

From this expression for the internal energy, we can compute the degeneracy pressure in the usual thermodynamic way:

$$P_d = -\frac{\partial U}{\partial V} = \frac{2}{3}\frac{U}{V} = \frac{2}{5}\frac{h^2}{8m}\left(\frac{3}{\pi}\right)^{\frac{2}{3}}\left(\frac{N}{V}\right)^{\frac{5}{3}} \tag{8}$$

Next we turn our attention to the gravitational pull of the star upon itself. Its gravitational self-energy is given, in the Newtonian Theory, by:

$$U_g = -\frac{3}{5}\frac{GM^2}{R} = -\frac{3}{5}GM^2\left(\frac{4\pi}{3V}\right)^{\frac{1}{3}} \tag{9}$$

The assumption that Newtonian Gravity is an adequate description is suspect and will be revisited shortly.

The gravitational pressure is calculated as before:

$$P_g = -\frac{\partial U_g}{\partial V} = -\frac{GM^2}{5}\left(\frac{4\pi}{3}\right)^{\frac{1}{3}} V^{-\frac{4}{3}} \tag{10}$$

The negative value indicates that the pressure is attempting to compress the star instead of to expand it.

The pressures are balanced simply:

$$P_d + P_g = 0 \tag{11}$$

By making two substitutions, we can now derive a theoretical value for the core's radius at equilibrium:

$$\begin{aligned} M &= mN \qquad (12) \\ V &= \frac{4}{3}\pi R^3 \qquad (13) \\ R &= \frac{3}{8}\frac{\left(\frac{3}{2}\right)^{\frac{1}{3}} h^2}{G\pi^{\frac{4}{3}} m^{\frac{8}{3}}} M^{-\frac{1}{3}} \qquad (14) \end{aligned}$$

Assuming that our core has a mass of $M = 1.5 M_{Sun} = 3 \times 10^{30}$kg, we arrive at an estimate of a neutron star's radius and degeneracy pressure:

$$\begin{aligned} R &= 10.75\text{km} \\ P_d &= 2 \times 10^{33}\text{Pa} \end{aligned}$$

This gives a density of:

$$\rho = \frac{M}{\frac{4}{3}\pi R^3} = 5.75 \times 10^{17}\frac{\text{kg}}{\text{m}^3}$$

This density is similar to that within an atomic nucleus.

An examination of the Fermi energy of the particles (using equation 4) shows that it is 97 MeV. This is around 10% of the neutron rest mass, so we are still in the non-relativistic regime. The correction for ignoring relativity when computing the Fermi energy is of the order of a few percent.

As seen in the white dwarf case, ultra-relativistic neutrons will also exhibit a crisis in that their degeneracy pressure will scale with the same power as gravitational pressure (see, for example, [1]). We can estimate when this occurs by assuming that neutrons become relativistic when their average energy is similar to their rest mass. This gives an estimate of the maximum mass as so:

$$\begin{aligned} \frac{U}{N} &> mc^2 \qquad (15) \\ \frac{3}{5}\frac{h^2}{8m}\left(\frac{3N}{\pi V}\right)^{\frac{2}{3}} &> mc^2 \qquad (16) \\ M_{\text{unstable}} &> \frac{3^{\frac{1}{4}} 5^{\frac{3}{4}} c^{\frac{3}{2}} h^{\frac{3}{2}}}{2^{\frac{7}{4}} G^{\frac{3}{2}} m^2 \pi} = 2.41 \times 10^{31}\text{kg} \approx 12 M_{Sun} \qquad (17) \end{aligned}$$

These results are within an order of magnitude of the published values for typical neutron stars. These are typically given (for example, on Wikipedia) as R ~ 6 km and a maximum mass of ~3 solar masses. I suspect that this difference occurs because of the expression used for the gravitational self-energy. It would seem that, at these extremely high densities, the effects of General Relativity are non-negligible.

The Neutron Abundance

We now look for the location of the equilibrium in the reaction $p^+ + e^- \leftrightarrow n + \bar{\nu_e}$ as a function of temperature. The basic Saha equation is given as equation 18 (taken from [2]).

$$\frac{[n_p]\,[n_e]}{[n_n]} = \frac{Z_p Z_e}{Z_n} \tag{18}$$

We then break the partition function for each particle into the product of a kinetic factor and an internal energy factor. We can compute the kinetic factor by allowing particle momenta to take on all absolute values above the Fermi momentum. The Fermi momentum is:

$$p_F = \frac{hn_F}{2L} = \frac{h}{2}\left(\frac{3N}{\pi V}\right)^{\frac{1}{3}} \tag{19}$$

The integral over the allowed momentum space gives:

$$\frac{2}{h^3}\int_{p_F}^{\infty} \exp\left(\frac{-p^2}{2mkT}\right) p^2 \mathrm{d}p\,(4\pi) = \frac{4\pi mkT}{h^3}\left(2p_F \exp\left(\frac{-p_F^2}{2mkT}\right) + \sqrt{2\pi mkT}\,\mathrm{erfc}\left(\frac{p_F}{\sqrt{2mkT}}\right)\right) \tag{20}$$

The erfc function is related to the better-known error function by $\mathrm{erfc}(x) = 1 - \mathrm{erf}(x)$.

We use the internal partition function "2" for both the protons and the electrons, as they can be in either a spin up or spin down state. For the neutrons, we use $2e^{-\beta\Delta E}$ with $\Delta E = -1.36\mathrm{MeV}$, since this corresponds to the "binding energy" of a neutron in beta decay. The factor of two accounts for the neutron's spin possibilities.

This provides:

$$\frac{[n_p]\,[n_e]}{[n_n]} = \frac{8\pi m^* kT}{h^3}\left(2p_F \exp\left(\frac{-p_F^2}{2m^*kT}\right) + \sqrt{2\pi m^* kT}\,\mathrm{erfc}\left(\frac{p_F}{\sqrt{2m^*kT}}\right)\right)\exp(\beta\Delta E) \tag{21}$$

where $m^* = \frac{m_p m_e}{m_n} = 9.098 \times 10^{-31}\mathrm{kg}$.

We parameterize the concentrations using a reaction coordinate: $[n_p] = xn$, $[n_e] = xn$, and $[n_n] = (1-x)n$. We now have:

$$\frac{x^2}{1-x} = \frac{8\pi m^* kT}{nh^3}\left(2p_F \exp\left(\frac{-p_F^2}{2m^*kT}\right) + \sqrt{2\pi m^* kT}\,\mathrm{erfc}\left(\frac{p_F}{\sqrt{2m^*kT}}\right)\right)\exp(\beta\Delta E) \tag{22}$$

Using the density 3.4×10^{44} from the previous calculation, I attempted to plot this function and to find roots numerically. Unfortunately, Mathematica experiences numerical underflows even at very low temperatures. By inspecting the numerical relationships being solved, I find it clear that at the temperatures present in a neutron star core that neutrons will dominate the species present.

Starquakes

One particularly good piece of evidence for starquakes is the outburst observed from the object SGR 1806-20 on December 27, 2004. The details of this observation are given in [3].

The equation of state of neutron star matter is not well-understood, but it is thought that the core consists of a fluid of primarily neutrons supported by neutron degeneracy pressure. Surrounding this core is a solid crust of neutron matter, often called "neutronium" by science fiction authors. The magnetic field of the neutron star is threaded through this solid surface, in a manner similar to the way in which the Sun's magnetic field permeates its photosphere.

A starquake occurs when the crust, which is ordinarily rigid, undergoes sudden motion. This motion is usually triggered by the neutron star approaching a more spherical shape as it spins down from rapid rotation (the energy of rotation is lost to gravitational waves). Due to the fact that the crust material is extremely stiff, the adjustments are abrupt, like an earthquake.

SGR 1806-20 is a magnetar, which is a neutron star possessing a very large (teragauss) magnetic field. On December 27, 2004, it was observed to undergo a sudden very large emission of gamma rays. This flare was detected by the International Gamma-Ray Astrophysics Laboratory (INTEGRAL), as well as by four other missions, including Swift. The 142s flare emitted energy of order 10^{42}erg.

The physical model which best matches this flare is a starquake on the neutron star being observed. The large release of energy is due to magnetic reconnection as the crust shifts, similar to how solar flares are generated. Several timing features of the light curve for this event are consistent with these magnetic and dynamical timescales. Indeed, starquakes are a possible explanation for at least some of the short-duration gamma ray bursts.

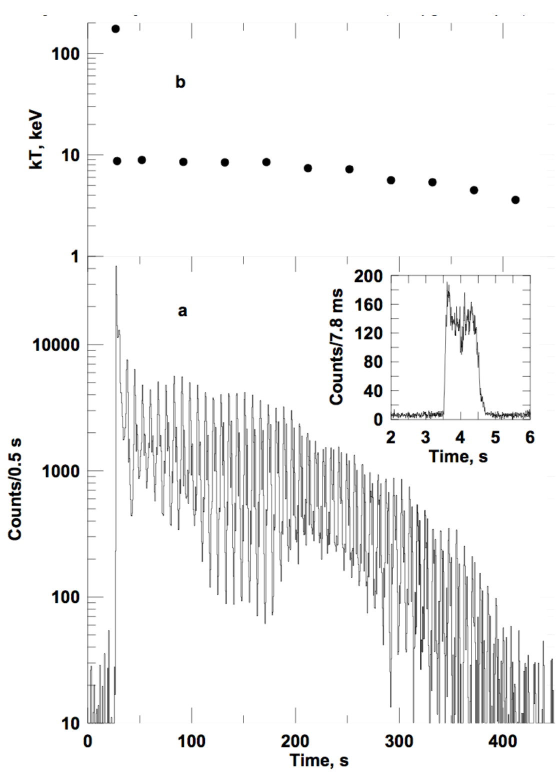

Figure 1: Starquake Light Curve (SGR 1806-20 from RHESSI Detector) [3]

Changing the Universe

If $M_n - M_p < m_e$, then the inverse beta decay reaction $p^+ + e^- \rightarrow n + \bar{\nu}_e$ would be exothermic, and the more conventional beta decay process would be endothermic. This would have the following effects:

1. Neutrons would be stable at low temperatures and pressures.
2. Atoms would be unstable. Whenever a proton and electron approached one another, they would combine to form a neutron.
3. Protons and electrons would only be found in areas of high temperature and density, as there would have to be a degenerate sea of neutrons in order to support stable protons and electrons.

The universe would essentially become "neutrons, neutrons, everywhere."

Conclusion

We have shown that neutrons stars with masses of order $1.5M_{Sun}$ are able to support themselves using neutron degeneracy pressure with radii on order of $R = 10$km. We have also demonstrated that in these stellar cores, neutrons are the dominant species present. Furthermore, a description has been given both of the process of neutron star formation and of starquakes on their surfaces. Some experimental evidence has been offered to support the existence of starquakes. Finally, we have considered what would happen if the difference in masses between the proton and the neutron were a bit smaller.

References

[1] A.C. Phillips, *The Physics of Stars*, Wiley, 2nd edition, 1999.

[2] R Gilmore, "Saha's equation and the partition function".

[3] K Hurley, S. E Boggs, D. M Smith, R. C Duncan, R Lin, A Zoglauer, S Krucker, G Hurford, H Hudson, C Wigger, W Hajdas, C Thompson, I Mitrofanov, A Sanin, W Boynton, C Fellows, A von Kienlin, G Lichti, A Rau, and T Cline, "An exceptionally bright flare from sgr 1806-20 and the origins of short-duration gamma-ray bursts", *Nature*, vol. 434, pp. 1098, Apr 2005, (c) 2005: Nature.