The Principle of Least Action

Jason Gross, December 7, 2010

Introduction

Recall that we defined the *Lagrangian* to be the kinetic energy less potential energy, $L = K - U$, at a point. The action is then defined to be the integral of the Lagrangian along the path,

$$S = \int_{t_0}^{t_1} L \, dt = \int_{t_0}^{t_1} K - U \, dt$$

It is (remarkably!) true that, in any physical system, the path an object actually takes minimizes the action. It can be shown that the extrema of action occur at

$$\frac{\partial L}{\partial q} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} = 0$$

This is called the Euler equation, or the Euler-Lagrange Equation.

■ Derivation

Courtesy of Scott Hughes's Lecture notes for 8.033. (Most of this is copied almost verbatim from that.)

Suppose we have a function $f(x, \dot{x}; t)$ of a variable x and its derivative $\dot{x} = dx/dt$. We want to find an extremum of

$$J = \int_{t_0}^{t_1} f(x(t), \dot{x}(t); t) \, dt$$

Our goal is to compute $x(t)$ such that J is at an extremum. We consider the limits of integration to be fixed. That is, $x(t_1)$ will be the same for any x we care about, as will $x(t_2)$.

Imagine we have some $x(t)$ for which J is at an extremum, and imagine that we have a function which parametrizes how far our current path is from our choice of x:

$$x(t; \alpha) = x(t) + \alpha \, A(t)$$

The function A is totally arbitrary, except that we require it to vanish at the endpoints: $A(t_0) = A(t_1) = 0$. The parameter α allows us to control how the variation $A(t)$ enters into our path $x(t; \alpha)$.

The "correct" path $x(t)$ is unknown; our goal is to figure out how to construct it, or to figure out how f behaves when we are on it.

Our basic idea is to ask how does the integral J behave when we are in the vicinity of the extremum. We know that ordinary functions are flat --- have zero first derivative --- when we are at an extremum. So let us put

$$J(\alpha) = \int_{t_0}^{t_1} f(x(t; \alpha), \dot{x}(t; \alpha); t) \, dt$$

We know that $\alpha = 0$ corresponds to the extremum by definition of α. However, this doesn't teach us anything useful, sine we don't know the path $x(t)$ that corresponds to the extremum.

But we also know We know that $\frac{\partial J}{\partial \alpha}\big|_{\alpha=0} = 0$ since it's an extremum. Using this fact,

$$\frac{\partial J}{\partial \alpha} = \int_{t_0}^{t_1} \left(\frac{\partial f}{\partial x} \frac{\partial x}{\partial \alpha} + \frac{\partial f}{\partial \dot{x}} \frac{\partial \dot{x}}{\partial \alpha} \right) dt$$

$$\frac{\partial x}{\partial \alpha} = \frac{\partial}{\partial \alpha}(x(t) + \alpha\, A(t)) = A(t)$$

$$\frac{\partial \dot{x}}{\partial \alpha} = \frac{\partial}{\partial \alpha} \frac{d}{dt}(x(t) + \alpha\, A(t)) = \frac{dA}{dt}$$

So

$$\frac{\partial J}{\partial \alpha} = \int_{t_0}^{t_1} \left(\frac{\partial f}{\partial x} A(t) + \frac{\partial f}{\partial \dot{x}} \frac{dA}{dt} \right) dt$$

Integration by parts on the section term gives

$$\int_{t_0}^{t_1} \frac{\partial f}{\partial \dot{x}} \frac{dA}{dt}\, dt = A(t) \frac{\partial f}{\partial \dot{x}}\bigg|_{t_0}^{t_1} - \int_{t_0}^{t_1} A(t) \frac{d}{dt} \frac{\partial f}{\partial \dot{x}}\, dt$$

Since $A(t_0) = A(t_1) = 0$, the first term dies, and we get

$$\frac{\partial J}{\partial \alpha} = \int_{t_0}^{t_1} A(t) \left(\frac{\partial f}{\partial x} + \frac{d}{dt} \frac{\partial f}{\partial \dot{x}} \right) dt$$

This must be zero. Since $A(t)$ is arbitrary except at the endpoints, we must have that the integrand is zero at all points:

$$\frac{\partial f}{\partial x} + \frac{d}{dt} \frac{\partial f}{\partial \dot{x}} = 0$$

This is what was to be derived.

Least action: *F* = *m a*

Suppose we have the Newtonian kinetic energy, $K = \frac{1}{2} m\, v^2$, and a potential that depends only on position, $U = U(\vec{r})$. Then the Euler-Lagrange equations tell us the following:

```
Clear[U, m, r]
L = 1/2 m r'[t]^2 - U[r[t]];
∂_r[t] L - Dt[∂_r'[t] L, t, Constants → m] == 0
-U'[r[t]] - m r''[t] == 0
```

Rearrangement gives

$$-\frac{\partial U}{\partial r} = m\, \ddot{r}$$

$$F = m\, a$$

Least action with no potential

Suppose we have no potential, $U = 0$. Then $L = K$, so the Euler-Lagrange equations become

$$\frac{\partial K}{\partial q} - \frac{d}{dt}\frac{\partial K}{\partial \dot{q}} = 0$$

For Newtonian kinetic energy, $K = \frac{1}{2} m \dot{x}^2$, this is just

$$\frac{d}{dt} m \dot{x} = 0$$
$$m \dot{x} = m v$$
$$x = x_0 + v t$$

This is a straight line, as expected.

Least action with gravitational potential

Suppose we have gravitational potential close to the surface of the earth, $U = m g y$, and Newtonian kinetic energy, $K = \frac{1}{2} m \dot{y}^2$. Then the Euler-Lagrange equations become

$$-m g - \frac{d}{dt} m \dot{y} = -m g - m \ddot{y} = 0$$
$$-g = \ddot{y}$$
$$y = y_0 + a_y t - \frac{1}{2} g t^2$$

This is a parabola, as expected.

Constants of motion: Momenta

We may rearrange the Euler-Lagrange equations to obtain

$$\frac{\partial L}{\partial q} = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}}$$

If it happens that $\frac{\partial L}{\partial q} = 0$, then $\frac{d}{dt}\frac{\partial L}{\partial \dot{q}}$ is also zero. This means that $\frac{\partial L}{\partial \dot{q}}$ is a constant (with respect to time). We call $\frac{\partial L}{\partial \dot{q}}$ a (conserved) momentum of the system.

Linear Momentum

By noting that Newtonian kinetic energy, $K = \frac{1}{2} m v^2$, is independent of the time derivatives of position, if potential energy depends only on position, we can infer that $\frac{\partial L}{\partial \dot{x}}$ (and, similarly, $\frac{\partial L}{\partial \dot{y}}$ and $\frac{\partial L}{\partial \dot{z}}$) are constant. Then $\frac{\partial L}{\partial \dot{x}} = \frac{\partial}{\partial \dot{x}}\left(\frac{1}{2} m \dot{x}^2\right) = m \dot{x}$. This is just standard linear momentum, $m v$.

Angular Momentum

Let us change to polar coordinates.

```
x[t_] := r[t] Cos[θ[t]]
y[t_] := r[t] Sin[θ[t]]
K = Expand[FullSimplify[1/2 m (x'[t]^2 + y'[t]^2)]] // TraditionalForm
```

$$\frac{1}{2} m\, r'(t)^2 + \frac{1}{2} m\, r(t)^2\, \theta'(t)^2$$

Using dot notation, this is

```
K /. r_'[t] → OverDot[r] /. r_[t] → r // TraditionalForm
```

$$\frac{1}{2} \dot{\theta}^2 m\, r^2 + \frac{m\, \dot{r}^2}{2}$$

Note that θ does not appear in this expression. If potential energy is not a function of θ (is only a function of r), then $\frac{\partial L}{\partial \dot{\theta}} = m\, r^2\, \dot{\theta}$ is constant. This is standard angular momentum, $m\, r^2\, \omega = r\, m\, r\, \omega = r \times m\, v$.

Classic Problem: Brachistochrone ("shortest time")

Problem

A bead starts at $x = 0$, $y = 0$, and slides down a wire without friction, reaching a lower point (x_f, y_f). What shape should the wire be in order to have the bead reach (x_f, y_f) in as little time as possible.

Solution

Idea

Use the Euler equation to minimize the time it takes to get from (x_i, y_i) to (x_f, y_f).

Implementation

Letting ds be the infinitesimal distance element and v be the travel speed,

$$T = \int_{t_i}^{t_f} \frac{ds}{v}\, dt$$

$$ds = \sqrt{(dx)^2 + (dy)^2} = dy\, \sqrt{1 + (x')^2} \qquad x' = \frac{dx}{dy}$$

$$v = \sqrt{2\, g\, y} \qquad \text{(Assumption: bead starts at rest)}$$

$$T = \int_0^{y_f} \sqrt{\frac{1+(x')^2}{2\,g\,y}}\; dy$$

Now we apply the Euler equation to $f = \sqrt{\frac{1+(x')^2}{2\,g\,y}}$ and change $t \to y$, $\dot{x} \to x'$.

$$\frac{\partial f}{\partial x} - \frac{d}{dy}\frac{\partial f}{\partial \dot{x}} = 0$$

$$\frac{\partial f}{\partial x} = 0$$

$$\frac{\partial f}{\partial \dot{x}} = \frac{1}{\sqrt{2\,g\,y}}\frac{x'}{\sqrt{1+(x')^2}}$$

$$\frac{d}{dy}\frac{\partial f}{\partial \dot{x}} = 0 \;\longrightarrow\; \frac{1}{\sqrt{2\,g\,y}}\frac{x'}{\sqrt{1+(x')^2}} = \text{Constant}$$

Squaring both sides and making a special choice for the constant gives

$$\frac{(x')^2}{2\,g\,y\left(1+(x')^2\right)} = \frac{1}{4\,g\,A}$$

$$\longrightarrow\quad \left(\frac{dx}{dy}\right)^2 = \frac{y/(2\,A)}{1-y/(2\,A)} = \frac{y^2}{2\,A\,y - y^2}$$

$$\longrightarrow\quad x = \int_0^{y_f} \frac{dx}{dy}\,dy = \int_0^{y_f} \frac{y}{\sqrt{2\,A\,y-y^2}}\,dy$$

To solve this, change variables:

$$y = A(1-\cos(\theta)), \qquad dy = A\sin(\theta)\,d\theta$$

```
FullSimplify[2 A y - y^2 /. y → A (1 - Cos[θ])]
```

```
A^2 Sin[θ]^2
```

$$\frac{y}{\sqrt{2\,A\,y-y^2}}\,dy = \frac{A(1-\cos(\theta))}{\sqrt{A^2\sin^2(\theta)}}\,A\sin(\theta)\,d\theta = A(1-\cos(\theta))$$

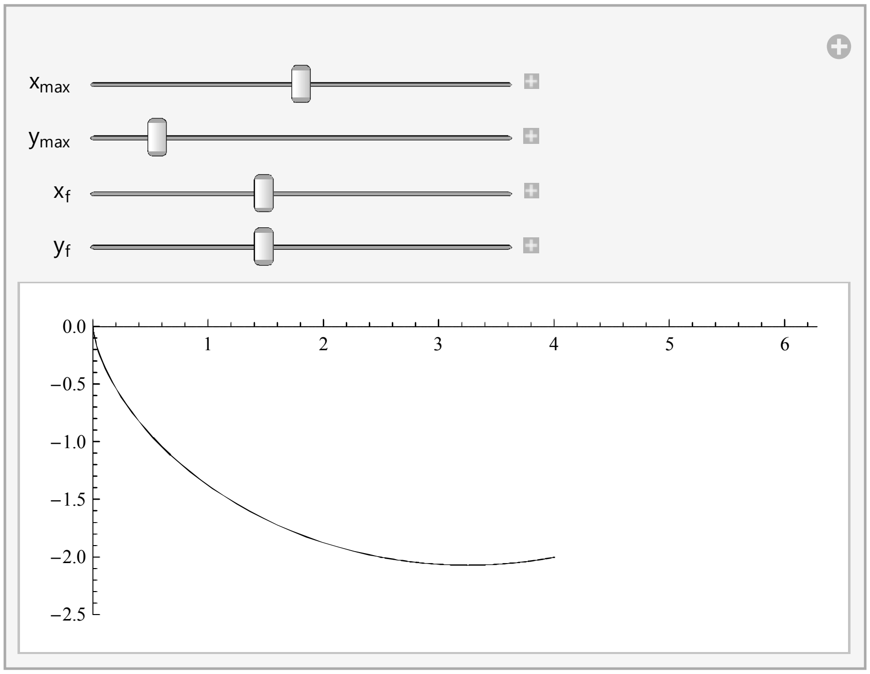
$$x = \int_0^{\theta} A(1-\cos(\theta))\,d\theta = A(\theta - \sin(\theta))$$

Full solution: The brachistochrone is described by

$$\boxed{\begin{array}{l} x = A(\theta - \sin(\theta)) \\ y = A(1-\cos(\theta)) \end{array}}$$

There’s no analytic solution, but we can compute them.

```
Clear[x, y, A, θ, soln, yf, xf, xmax, θmax, Asol, f]; Manipulate[
 Module[{y = Function[{A, θ}, A (1 - Cos[θ])], x = Function[{A, θ}, A (θ - Sin[θ])]},
  Module[{soln = FindRoot[{x[A, θ] == xf, y[A, θ] == yf}, {A, -1}, {θ, π}]}, Module[
    {Asol = A /. soln, θmax = θ /. soln}, ParametricPlot[{x[Asol, θ], y[Asol, θ]},
     {θ, 0, θmax}, PlotRange → {{0, xmax}, {ymax, 0}}, PlotStyle → Black]]]],
 {{xmax, 2 π, x_max}, 0, 4 π}, {{ymax, -2.5, y_max}, 0, -20},
 {{xf, 4, x_f}, 0, 10}, {{yf, -2, y_f}, 0, -5}]
```



Classic Problem: Catenary

■ Problem

Suppose we have a rope of length l and linear mass density λ. Suppose we fix its ends at points (x_0, y_0) and (x_f, y_f). What shape does the rope make, hanging under the influence of gravity?

■ Solution

■ Idea

Calculate the potential energy of the rope as a function of the curve, $y(x)$, and minimize this quantity using the Euler-Lagrange equations.

■ Implementation

Suppose we have curve parameterized by t, $(x(t),\ y(t))$. The potential energy associated with this curve is

$$U = \int_0^l \lambda\, g\, y\, ds$$

$$ds = \sqrt{(dx)^2 + (dy)^2} = dy\,\sqrt{1 + (x')^2} \qquad x' = \frac{dx}{dy}$$

$$U = \int_{y_0}^{y_f} \lambda\, g\, y\, \sqrt{1 + (x')^2}\, dy$$

Note that if we choose to factor ds the other way (for y'), we get a mess.

Now we apply the Euler-Lagrange equation to $f = \lambda\, g\, y\, \sqrt{1 + (x')^2}$ and change $t \to y$, $\dot{x} \to x'$.

$$\frac{\partial f}{\partial x} - \frac{d}{dy}\frac{\partial f}{\partial x'} = 0$$

$$\frac{\partial f}{\partial x} = 0$$

$$\frac{\partial f}{\partial x'} = \frac{\lambda\, g\, y\, x'}{\sqrt{1 + (x')^2}}$$

Since $\frac{\partial f}{\partial x} = 0$, $\frac{\partial f}{\partial x'}$ is constant, say $a = \frac{1}{\lambda\, g}\frac{\partial f}{\partial x'} = \frac{y\, x'}{\sqrt{1+(x')^2}}$. Then

$$x' = \frac{dx}{dy} = \pm\frac{a}{\sqrt{y^2 - a^2}}$$

Using the fact that

$$\int \frac{dy}{\sqrt{y^2 - a^2}} = \cosh^{-1}\left(\frac{y}{a}\right) + b,$$

integration of x' gives

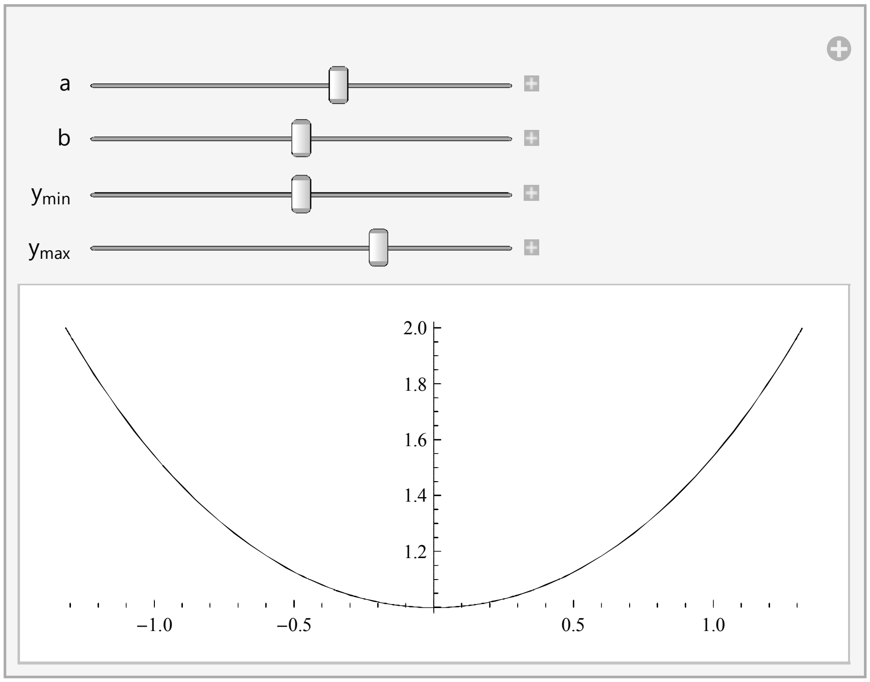
$$x(y) = \pm\, a \cosh^{-1}\left(\frac{y}{a}\right) + b$$

where b is a constant of integration.

Plotting this for $a = 1$, $b = 0$ gives:

In[14]:=
```
Clear[y]; Manipulate[ParametricPlot[{{-a ArcCosh[t/a] + b, t}, {a ArcCosh[t/a] + b, t}},
  {t, ymin, ymax}, PlotStyle → Black], {{a, 1}, -5, 5},
 {{b, 0}, -5, 5}, {{ymin, 0, y_min}, -5, 5}, {{ymax, 2, y_max}, -5, 5}]
```

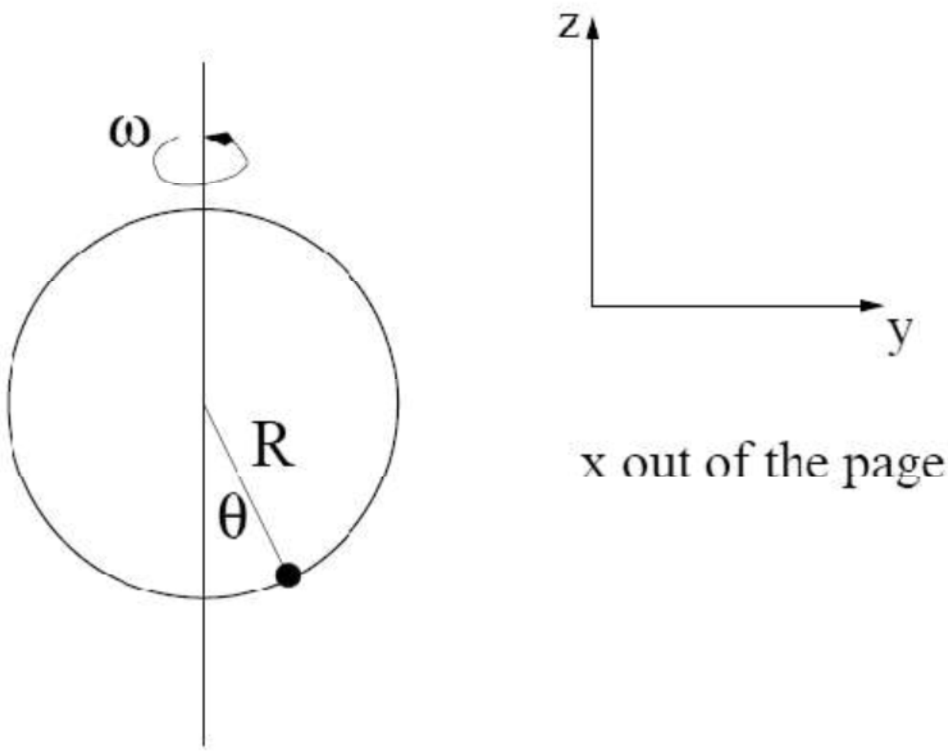
Out[14]=



Problem: Bead on a Ring

From 8.033 Quiz #2

■ Problem



A bead of mass m slides without friction on a circular hoop of radius R. The angle θ is defined so that when the bead is at the bottom of the hoop, $\theta = 0$. The hoop is spun about its vertical axis with angular velocity ω. Gravity acts downward with acceleration g.
Find an equation describing how θ evolves with time.
Find the minimum value of ω for the bead to be in equilibrium at some value of θ other than zero.
("equilibrium" means that $\dot{\theta}$ and $\ddot{\theta}$ are both zero.) How large must ω be in order to make $\theta = \pi/2$?

■ Solution

The general Lagrangian for the object in Cartesian coordinates is

```
Clear[x, y, z, t]; (L = 1/2 m (x'[t]^2 + y'[t]^2 + z'[t]^2) - m g z[t]) // TraditionalForm
```

$$\frac{1}{2} m \left(x'(t)^2 + y'(t)^2 + z'(t)^2\right) - g\, m\, z(t)$$

Converting to polar coordinates, and using the constraints that $\phi = \omega\, t$ and $r = R$, using the conversion

$$x = R \sin(\theta) \cos(\omega\, t)$$
$$y = R \sin(\theta) \sin(\omega\, t)$$
$$z = R - R \cos(\theta)$$

gives

```
Clear[r, θ, ϕ];
Defer[L] == (Lpolar = Expand[FullSimplify[L /. {x → Function[t, R Cos[ω t] Sin[θ[t]]],
            y → Function[t, R Sin[ω t] Sin[θ[t]]], z → Function[t,
              R - R Cos[θ[t]]]}]]) /. θ'[t] → θ̇ /. θ[t] → θ // TraditionalForm
0 == Defer[∂θ L - Dt["", t] ∂θ̇ L] == (EL = Expand[FullSimplify[
        ∂θ[t] Lpolar - ∂t ∂θ'[t] Lpolar]]) /. θ[t] → θ // TraditionalForm
θ''[t] == (θ''[t] /. Solve[EL == 0, θ''[t]][[1]]) // TraditionalForm
```

$$L = g\, m\, R\cos(\theta) - g\, m\, R - \frac{1}{4}\, m\, R^2\, \omega^2 \cos(2\,\theta) + \frac{1}{2}\, \dot{\theta}^2\, m\, R^2 + \frac{1}{4}\, m\, R^2\, \omega^2$$

$$0 = \frac{\partial L}{\partial \theta} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = -g\, m\, R\sin(\theta) + m\, R^2\, \omega^2 \sin(\theta)\cos(\theta) - m\, R^2\, \theta''(t)$$

$$\theta''(t) = \frac{R\,\omega^2 \sin(\theta(t))\cos(\theta(t)) - g\sin(\theta(t))}{R}$$

Finding the minimum value of ω for the bead to be in equilibrium gives

```
(θ''[t] /. Solve[EL == 0, θ''[t]][[1]]) == 0 // TraditionalForm
Refine[Reduce[(θ''[t] /. Solve[EL == 0, θ''[t]][[1]]) == 0, Cos[θ[t]]],
   Sin[θ[t]] ≠ 0 && R Cos[θ[t]] ≠ 0 && g > 0 && R ω ≠ 0] /. θ[t] → θ // TraditionalForm
```

$$\frac{R\,\omega^2 \sin(\theta(t))\cos(\theta(t)) - g\sin(\theta(t))}{R} = 0$$

$$\cos(\theta) = \frac{g}{R\,\omega^2}$$

In order for this to have a solution, we must have

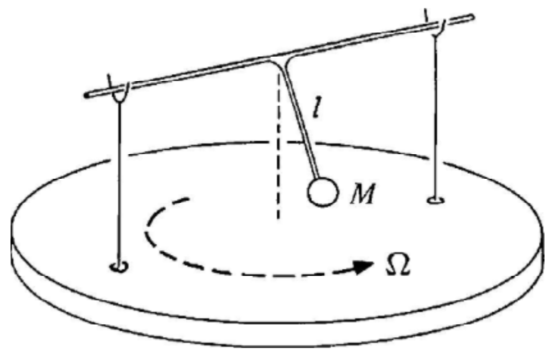
$$\omega \geq \sqrt{\frac{g}{R}}$$

If $\theta = \pi/2$, then $\cos(\theta) = 0$, so $\omega = \infty$.

Problem 11.8: K & K 8.12

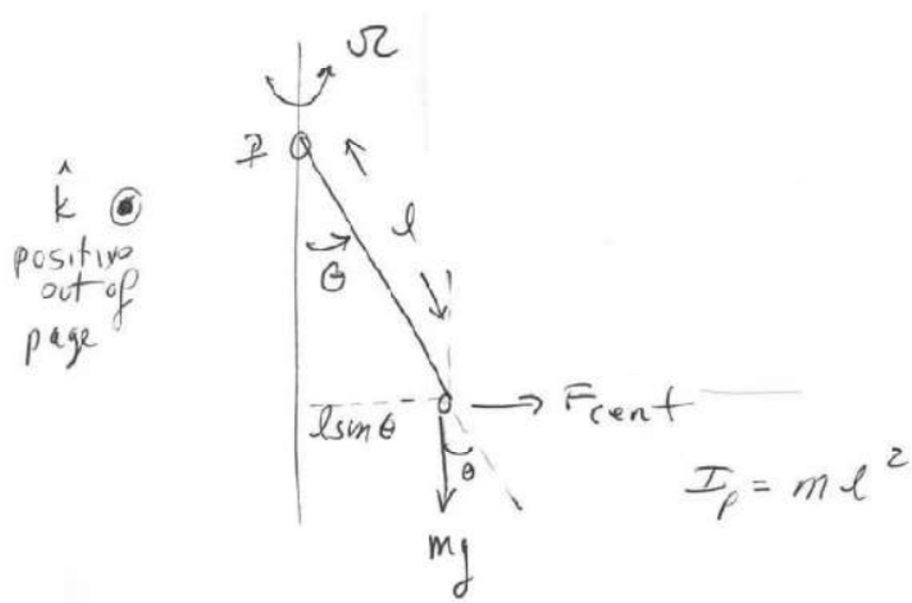
■ Problem

A pendulum is rigidly fixed to an axle held by two supports so that it can only swing in a plane perpendicular to the axle. The pendulum consists of a mass m attached to a massless rod of length l. The supports are mounted on a platform which rotates with constant angular velocity Ω. Find the pendulum's frequency assuming the amplitude is small.



■ Solution by torque

(From the problem set solutions)



The torque about the pivot point is

$$\vec{\tau}_p = \vec{\alpha}\, I_p$$

$$\hat{k}: \qquad -g\,\ell\, m \sin(\theta) + \ell\, F_{\text{cent}} \cos(\theta) = \ddot{\theta}\, I_p \tag{1}$$

The centrifugal effective force is

$$F_{\text{cent}} = m\,(\ell \sin(\theta))\,\Omega^2$$

For small angles, $\sin(\theta) \simeq \theta$, $\cos(\theta) \simeq 1$. Then equation (1) becomes

$$-g\,\ell\, m\,\theta + m\,\ell^2\,\theta\,\Omega^2 \simeq m\,\ell^2\,\ddot{\theta}$$

$$\ddot{\theta} + \left(\frac{g}{\ell} - \Omega^2\right)\theta \simeq 0$$

$$\omega = \sqrt{\frac{g}{\ell} - \Omega^2}$$

If $\Omega^2 > \frac{g}{\ell}$, the motion is no longer harmonic.

■ Solution by least action

The general Lagrangian for the object in Cartesian coordinates is

```
Clear[x, y, z, t]; (L = 1/2 m (x'[t]^2 + y'[t]^2 + z'[t]^2) - m g z[t]) // TraditionalForm
```

$$\frac{1}{2}\, m \left(x'(t)^2 + y'(t)^2 + z'(t)^2\right) - g\, m\, z(t)$$

Converting to polar coordinates, and using the constraints that $\phi = \Omega\, t$ and $r = \ell$, using the conversion

$$x = \ell \sin(\theta) \cos(\Omega\, t)$$
$$y = \ell \sin(\theta) \sin(\Omega\, t)$$
$$z = \ell - \ell \cos(\theta)$$

gives

```
Clear[ℓ, θ, ϕ];
Defer[L] == (Lpolar = Expand[FullSimplify[L /. {x → Function[t, ℓ Cos[Ω t] Sin[θ[t]]],
            y → Function[t, ℓ Sin[Ω t] Sin[θ[t]]], z → Function[t, ℓ - ℓ Cos[θ[t]]]}]]) /.
    θ'[t] → θ̇ /. θ[t] → θ // TraditionalForm
0 == Defer[∂_θ L - Dt["", t] ∂_θ̇ L] ==
    (EL = Expand[FullSimplify[∂_θ[t] Lpolar - ∂_t ∂_θ'[t] Lpolar]]) /.
   θ[t] → θ // TraditionalForm
θ''[t] == (θ''[t] /. Solve[EL == 0, θ''[t]][[1]]) // TraditionalForm
```

$$L = g\, m\, \ell \cos(\theta) - g\, m\, \ell - \frac{1}{4}\, m\, \Omega^2\, \ell^2 \cos(2\,\theta) + \frac{1}{2}\, \dot{\theta}^2\, m\, \ell^2 + \frac{1}{4}\, m\, \Omega^2\, \ell^2$$

$$0 = \frac{\partial L}{\partial \theta} - \frac{d}{dt}\, \frac{\partial L}{\partial \dot{\theta}} = -g\, m\, \ell \sin(\theta) - m\, \ell^2\, \theta''(t) + m\, \Omega^2\, \ell^2 \sin(\theta) \cos(\theta)$$

$$\theta''(t) = \frac{\Omega^2\, \ell \sin(\theta(t)) \cos(\theta(t)) - g \sin(\theta(t))}{\ell}$$

Note that this is, after minor changes of variable, the *exact* same equation that we found in the previous problem. We should('ve) expect(ed) this.
Making the first order approximation that $\theta \approx 0$ (Taylor expanding around $\theta = 0$ to the first order), we get

$$\theta''(t) = -\left(\frac{g}{\ell} - \Omega^2\right) \theta(t)$$

This is the differential equation for a harmonic oscillator, with

$$\omega = \sqrt{\frac{g}{\ell} - \Omega^2}$$

If $\Omega^2 > \frac{g}{\ell}$, the motion is no longer harmonic.