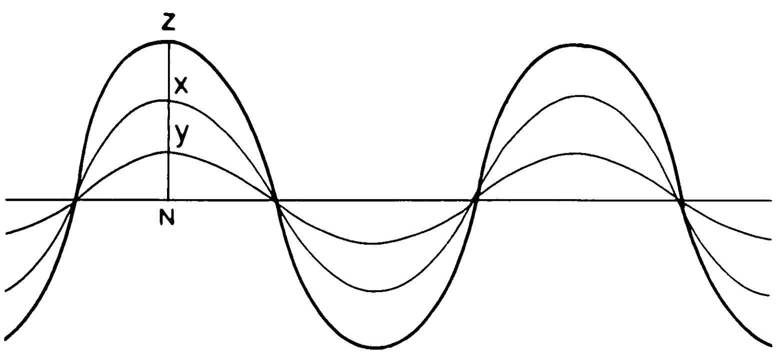
THE PRINCIPLE OF SUPERPOSITION AND ITS APPLICATION IN GROUND-WATER HYDRAULICS



Waves in phase (z = x + y)

U.S. GEOLOGICAL SURVEY
Open-File Report 84-459

THE PRINCIPLE OF SUPERPOSITION AND ITS APPLICATION IN GROUND-WATER HYDRAULICS

By Thomas E. Reilly, O. Lehn Franke, and Gordon D. Bennett

U.S. GEOLOGICAL SURVEY
Open-File Report 84-459

UNITED STATES DEPARTMENT OF THE INTERIOR

DONALD PAUL HODEL, Secretary

GEOLOGICAL SURVEY

Dallas L. Peck, Director

For additional information write to:

U.S. Geological Survey
Branch of Ground Water
Mail Stop 411
Reston, Virginia 22092
Telephone: (703)860-6904

Copies of this report may be purchased from:

Open-File Services Section
Western Distribution Branch
U.S. Geological Survey
Box 25425, Federal Center
Denver, Colorado 80225
Telephone: (303)236-7476

CONTENTS

ILLUSTRATIONS

ILLUSTRATIONS (continued)

WORK SHEETS

COMPLETED WORK SHEETS

CONVERSION FACTORS AND ABBREVIATIONS

Multiply inch-pound units	By	To obtain metric (SI)[1] units
cubic foot per second (ft^3/s)	0.0283	cubic meter per second (m^3/s)
million gallons per day (Mgal/d)	0.0438	cubic meter per second (m^3/s)
mile (mi)	1.609	kilometer (km)
foot (ft)	0.3048	meter (m)

[1] International System of units.

THE PRINCIPLE OF SUPERPOSITION AND ITS APPLICATION IN GROUND-WATER HYDRAULICS

by Thomas E. Reilly, O. Lehn Franke, and Gordon D. Bennett

ABSTRACT

The principle of superposition, a powerful mathematical technique for analyzing certain types of complex problems in many areas of science and technology, has important applications in ground-water hydraulics and modeling of ground-water systems. The principle of superposition states that problem solutions can be added together to obtain composite solutions. This principle applies to linear systems governed by linear differential equations.

This report introduces the principle of superposition as it applies to ground-water hydrology and provides background information, discussion, illustrative problems with solutions, and problems to be solved by the reader.

INTRODUCTION

The principle of superposition in physics is a simple concept that has numerous applications in ground-water hydraulics and modeling[1] of ground-water systems. The theory of superposition, which states that the solutions to individual parts of a problem can be added to solve composite problems, is explained in most books on advanced calculus or differential equations. Several texts on ground water provide some discussion of this topic as well; Bear (1979) gives perhaps the most comprehensive treatment that is readily

---

[1] The word "model" is used in several different ways in this report and in ground-water hydrology. A general definition of model is a representation of some or all of the properties of a system. Developing a "conceptual model" of the ground-water system is the first and critical step in any study, particularly studies involving mathematical-numerical modeling. In this context, a conceptual model is a clear, qualitative, physical picture of how the natural system operates. A "mathematical model" represents the system under study through mathematical equations and procedures. The differential equations that describe a physical process (for example, ground-water flow and solute transport) in approximate terms are a mathematical model of that process. The solution to these differential equations in a specific problem frequently requires numerical procedures (algorithms), although many simpler mathematical models can be solved analytically. Thus, the process of "modeling" usually implies either developing a conceptual model, a mathematical model, or a mathematical-numerical model of the system or problem under study. The context will suggest which meaning of "model" is intended.

available. Yet, many ground-water hydrologists who do not have a strong background in mathematics do not understand the concept of superposition nor its application to ground-water problems, despite the fact that they often use this principle, perhaps unknowingly, in the analysis of pumping tests[1].

Purpose and Scope

The purpose of this report is to introduce the principle of superposition to hydrologists by providing background information, discussion, and four problems to be solved by the reader. (Solutions are included.)

The discussion and problems in this report are directed toward the analysis and computer simulation of flow patterns within ground-water systems through superposition. It is hoped that after serious study of this document, the reader will be prepared for practical application of this concept.

The method of images, which is an important application of superposition in ground-water hydraulics, is not discussed in this report but is discussed in some detail by Walton (1970, p. 157-167).

Acknowledgments

The authors are deeply indebted to the other instructors of the course "Ground-Water Concepts," given at the U.S. Geological Survey's National Training Center. Over the years, the training materials for this course, including this report, have been generated through a melding of the ideas and work of many individuals including Eugene Patten, Ren Jen Sun, Edwin Weeks, and others.

LINEAR SYSTEMS AND LINEAR EQUATIONS

Superposition applies to linear problems or linear systems. A simple diagrammatic representation of a system with its input (or stress acting upon it) and its output (or response) is shown in figure 1. Simply stated, in a linear system, doubling a given stress (input) will double the response, halving the stress (input) will halve the response, and so on. For example, a vertical spring fixed firmly at its upper end represents a close approximation to a linear physical system. If we attach a 1-pound weight (the stress) to the lower end of the spring (the system), the spring will elongate or displace some distance equal to x. (This constitutes the system response.) If we remove the 1-pound weight and substitute a 3-pound weight, the response will be 3x, and so on.

---

[1] The first step in analyzing a pumping test is to convert absolute head measurements to drawdowns, which represent changes in head superposed on the ground-water system in response to the pumping stress. This step is required because the analytical solutions to linear well hydraulic problems are expressed in terms of head changes--that is, these solutions use superposition.

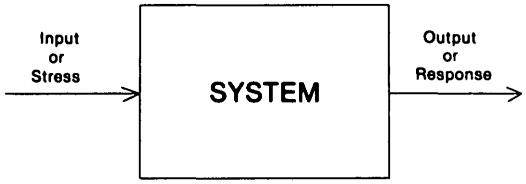


Figure 1.— Diagrammatic representation of a system and its associated stress and response.

Linear systems are described by linear mathematical equations. The previous example relating spring elongation to applied weight is described by the simple linear algebraic equation

$$F = - Kx,$$

where:

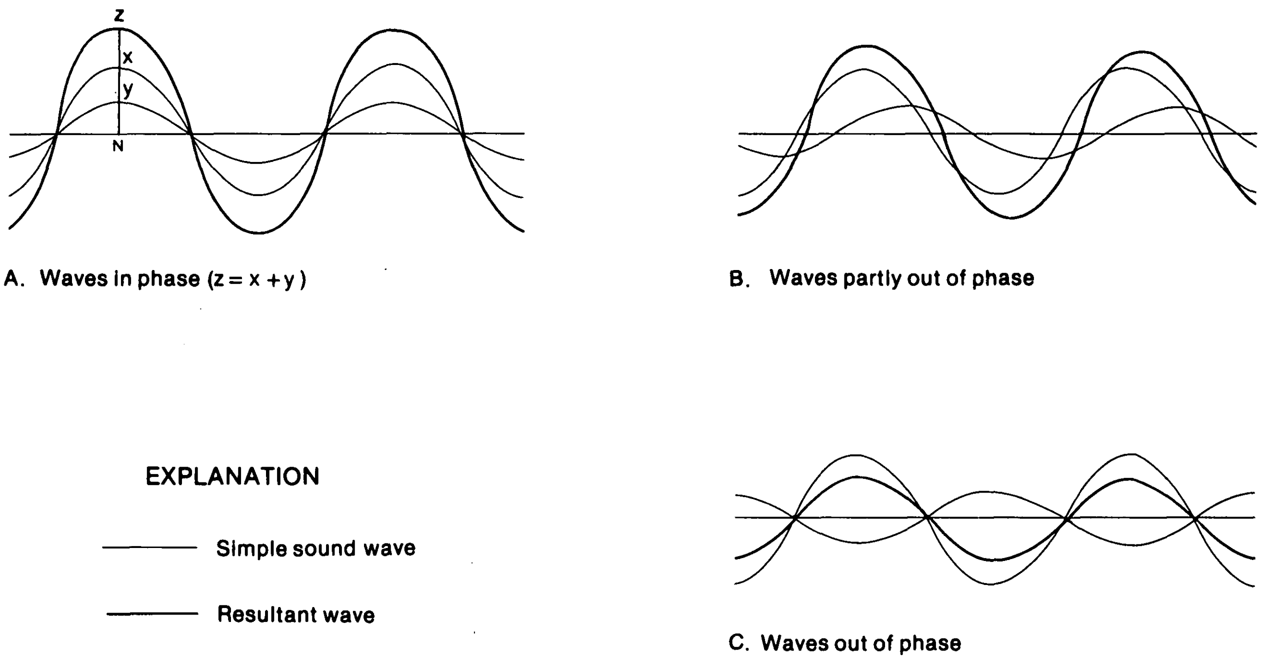
F is the force (weight) acting at the end of spring,
x is the elongation of the spring (a length), and
K is a constant (the spring constant), which is a property of the spring.

The flow of ground water is defined in the general case by partial differential equations. The solution to a ground-water problem entails solving the governing partial differential equation and satisfying the boundary and initial conditions that define the particular problem. (A detailed discussion on boundary and initial conditions is given in Franke, Reilly, and Bennett, 1984, and a discussion of the solution of differential equations and the role of boundary conditions is given in appendix 1.) Some of the differential equations that describe ground-water flow are linear and some nonlinear. Because superposition applies to linear systems that are described by linear equations, the concept of the linearity (or nonlinearity) of a differential equation is important. This topic is briefly reviewed in appendix 2.

DEFINITION OF SUPERPOSITION

The principle of superposition means that for linear systems, the solution to a problem involving multiple inputs (or stresses) is equal to the sum of the solutions to a set of simpler individual problems that form the composite problem. For example, suppose we want to know the shape of the sound wave generated by two sound waves interfering with each other. This shape can be determined by algebraically adding the two simple waves together, as in figure 2, where the darker wave is the resultant wave. Thus, the shape of a composite sound wave can be calculated from only the shape of each component wave. This procedure is valid because the properties of the waves in figure 2 are governed by linear differential equations.

A more formal definition of superposition is that, if Y_1 and Y_2 are two solutions to a linear differential equation with linear boundary conditions, then $C_1Y_1 + C_2Y_2$ is also a solution, where C_1 and C_2 are constants.



A. Waves in phase (z = x + y)

B. Waves partly out of phase

C. Waves out of phase

Figure 2. — Superposition of simple soundwaves. (Modified from Jeans, 1968.)

APPLICATION OF SUPERPOSITION TO A SIMPLE HYDROLOGIC SYSTEM

Just as the amplitudes of two simple waves were added together in figure 2 to obtain the amplitude of the composite wave, two different potentiometric distributions resulting from two separate stresses in a confined aquifer can be added together to obtain the potentiometric distribution resulting from the sum of the two stresses. For example, the one-dimensional aquifer system shown in figure 3 is bounded by a river on one side and a canal on the other.

In example a, the river stage is at datum, and the canal stage is 200 ft above datum; in example b, the river stage is 50 ft above datum, and the canal stage is at datum. If the head distribution in the aquifer is known from field measurements or numerical calculations for examples a and b, then the head distribution in example c, where the river stage is at 50 ft and the canal at 200 ft, can be obtained by adding the heads in examples a and b.

If the heads in examples a and b can be superposed (added together) to give a solution for example c, then the sum of the gradients (the slope of the line describing the head) in examples a and b should also give the gradient in example c. In example a, the gradient is 0.100, and in example b it is -0.025; the gradient in example c is the difference, 0.075. The gradient sign convention as used here is arbitrary; the point is that the gradients in examples a and b are of opposite sign.

Finally, if the gradients can be superposed, so can the flows. Thus, if the flow (Q) from the canal to the aquifer is 4 units in example a and is -1 unit in example b, then the flow in example c will be their sum, 3 units.

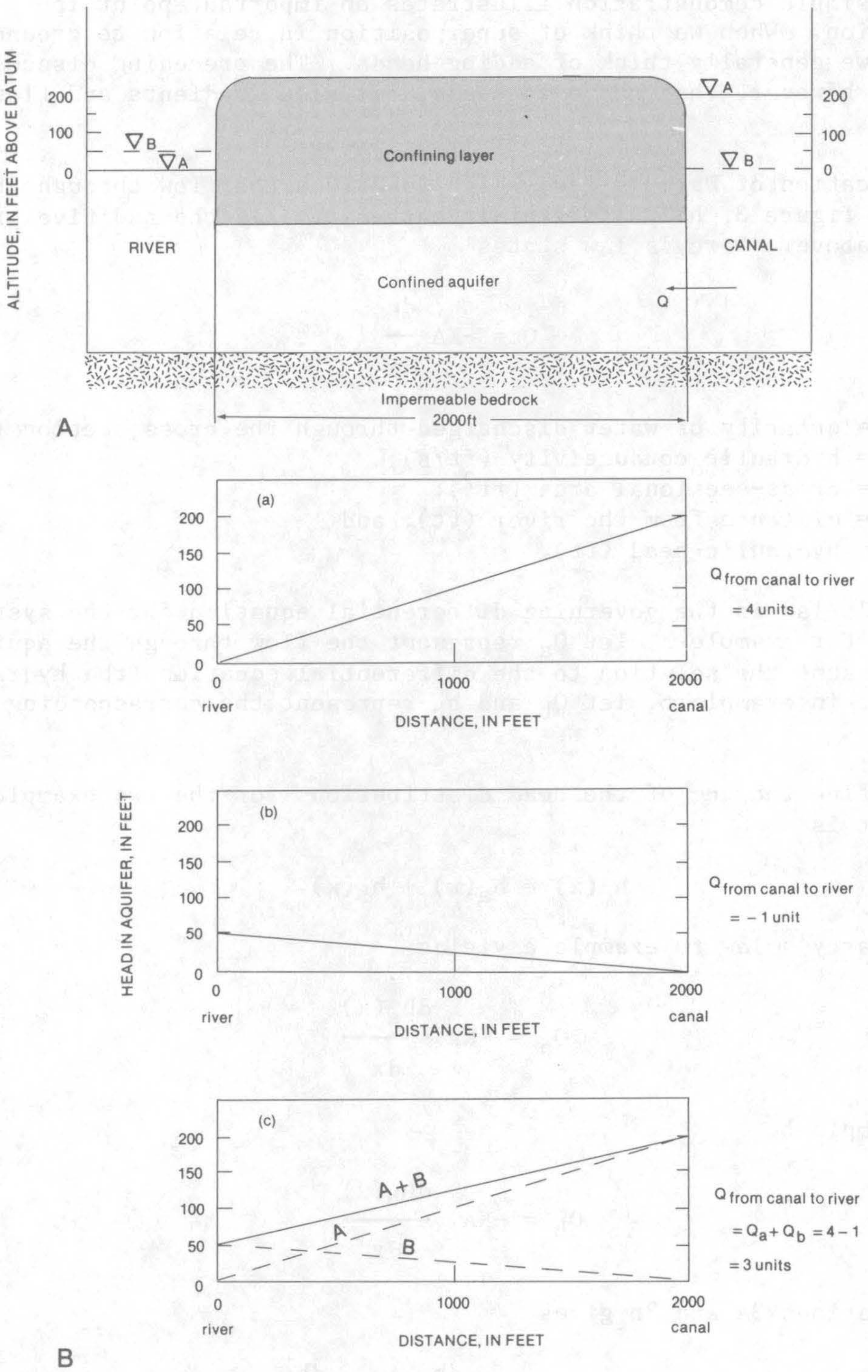


Figure 3. — Superposition of heads and flows in a one-dimensional example— A. Confined aquifer bounded by a river and canal. B. Plots of head distribution under three conditions: (a) with river stage at datum (0 ft.) and canal stage at 200 ft.; (b) with river stage at 50 ft. and canal stage at datum (0 ft.); (c) addition of heads in (a) and (b) to obtain head distribution with river stage at 50 ft. and canal stage at 200 ft.

This simple demonstration illustrates an important point in superposition. When we think of superposition in relation to ground-water problems, we generally think of adding heads. The preceding discussion indicates, however, that not only heads, but also gradients and flows, are additive.

Application of Darcy's law, which describes the flow through the cross section in figure 3, helps to explain mathematically the additive process as described above. Darcy's law states

$$Q = -KA \frac{dh}{dx} \tag{1}$$

where:
- Q = quantity of water discharged through the cross section (ft^3/s);
- K = hydraulic conductivity (ft/s);
- A = cross-sectional area (ft^2);
- x = distance from the river (ft); and
- h = hydraulic head (ft).

Darcy's law is the governing differential equation for the system of figure 3. For example a, let Q_a represent the flow through the aquifer and $h_a(x)$ represent the solution to the differential equation (the hydraulic head), and, in example b, let Q_b and h_b represent the corresponding flow and head.

We define the sum of the head distributions for the two examples as $h_c(x)$; that is

$$h_c(x) = h_a(x) + h_b(x). \tag{2}$$

Applying Darcy's law to example a yields

$$Q_a = -KA \frac{dh_a(x)}{dx} \tag{3a}$$

and to example b

$$Q_b = -KA \frac{dh_b(x)}{dx} \tag{3b}$$

Adding equations 3a and 3b gives

$$Q_a + Q_b = -KA \frac{dh_a}{dx} - KA \frac{dh_b}{dx}. \tag{3c}$$

However, the sum of the two derivatives may be written as the derivative of the sum. Thus,

$$\frac{dh_a}{dx} + \frac{dh_b}{dx} = \frac{d(h_a + h_b)}{dx} = \frac{dh_c}{dx}.$$

Substituting this into equation 3c gives

$$Q_a + Q_b = -KA \frac{dh_c(x)}{dx}. \tag{4a}$$

Equation 4a, like equations 3a and 3b, is a statement of Darcy's law. It tells us that $h_c(x)$, the head distribution defined by $h_a(x) + h_b(x)$, must satisfy Darcy's law provided $h_a(x)$ and $h_b(x)$ satisfy it individually. Equation 4a also shows that the flow corresponding to the combined head distribution is $Q_a + Q_b$, the sum of the flows corresponding to the individual head distribution. That is,

$$Q_a + Q_b = Q_c \tag{4b}$$

where Q_c is the flow for the combined case. Thus, when superposition is used, flows as well as heads must be added.

MATHEMATICAL DEMONSTRATION OF SUPERPOSITION CONCEPT

In ground-water problems involving a linear governing equation (such as two-dimensional confined flow), the effects of individual changes (or stresses) can be evaluated without the need to consider the other concurrent stresses on the system. To demonstrate this, consider a system governed by the following partial differential equation, which describes nonequilibrium flow in two dimensions under confined conditions:

$$\frac{\partial}{\partial x}\left(T_x \frac{\partial h}{\partial x}\right) + \frac{\partial}{\partial y}\left(T_y \frac{\partial h}{\partial y}\right) + W = S \frac{\partial h}{\partial t} \tag{5}$$

In equation 5,

T_x = transmissivity in the x direction (L^2/T);
T_y = transmissivity in the y direction (L^2/T);
W = source or sink term (L/T);
S = storage coefficient (unitless);

and the coordinate directions, x and y, are alined with the principal directions of aquifer transmissivity. Now suppose that a particular set of inputs or stresses, which is designated W_1, prevails in the aquifer (in general, W_1 is a function of position and time). A certain distribution of head in space and time, h_1 (x,y,t), (or for brevity simply h_1) is observed in response to these stresses. This head distribution must be such that when it is substituted into equation 5 together with the stress function W_1, the equation is satisfied. That is, it must be true that

$$\frac{\partial}{\partial x}\left(T_x \frac{\partial h_1}{\partial x}\right) + \frac{\partial}{\partial y}\left(T_y \frac{\partial h_1}{\partial y}\right) + W_1 = S \frac{\partial h_1}{\partial t} \tag{6a}$$

must be valid.

Now suppose the stresses are changed, so that corresponding to the function W_1 in equation 6a there is a new function ΔW, which gives the change or difference in the stress from W_1. The new stress function is then $W_1 + \Delta W$, and corresponding to this new stress pattern there is a new head pattern, $h_1 + \Delta h$, where Δh represents the difference in head, from h_1, which is observed in response to the change in stress ΔW. Like h_1 and W_1, Δh and ΔW are, in general, functions of x,y, and t.

Because the system is still governed by equation 5, the new head distribution and the new stress function must also satisfy this equation; that is, it must be true that

$$\frac{\partial}{\partial x}\left(T_x \frac{\partial(h_1 + \Delta h)}{\partial x}\right) + \frac{\partial}{\partial y}\left(T_y \frac{\partial(h_1 + \Delta h)}{\partial y}\right) + W_1 + \Delta W = S \frac{\partial(h_1 + \Delta h)}{\partial t} \quad . \tag{6b}$$

Again, by the principle that the derivative of a sum is equal to the sum of the individual derivatives, equation 6b can be written

$$\frac{\partial}{\partial x}\left(T_x \frac{\partial h_1}{\partial x}\right) + \frac{\partial}{\partial x}\left(T_x \frac{\partial \Delta h}{\partial x}\right) + \frac{\partial}{\partial y}\left(T_y \frac{\partial h_1}{\partial y}\right) + \frac{\partial}{\partial y}\left(T_y \frac{\partial \Delta h}{\partial y}\right) + W_1 + \Delta W = S \frac{\partial h_1}{\partial t} + S \frac{\partial \Delta h}{\partial t} \tag{7}$$

Subtracting equation 6a from equation 7 gives

$$\frac{\partial}{\partial x}\left(T_x \frac{\partial \Delta h}{\partial x}\right) + \frac{\partial}{\partial y}\left(T_y \frac{\partial \Delta h}{\partial y}\right) + \Delta W = S \frac{\partial \Delta h}{\partial t} \quad . \tag{8}$$

Thus Δh (x,y,t), the function describing the change in head caused by the stress change ΔW, must itself satisfy the governing differential equation of flow when it is substituted into that equation together with ΔW. It follows that: (1) the governing differential equation can be solved only for the head changes Δh corresponding to the stress changes ΔW; (2) head changes, Δh (drawdowns), can be used to solve for aquifer parameters, such as T_x and T_y; and (3) in general, individual solutions to a linear partial differential equation can be added to provide new solutions corresponding to a combined stress.

BOUNDARY CONDITIONS, STRESSES, AND INITIAL CONDITIONS IN MODELS THAT USE SUPERPOSITION

In using superposition to solve ground-water flow problems, we are dealing in terms of changes in head (drawdown) and changes in flow rather than absolute values of head and flow. The natural hydrologic boundaries (namely constant head, constant flow, and leakage boundaries), and the initial conditions, must be represented in models (either conceptual, analytical, analog, or numerical) in terms of changes instead of the actual values observed in the flow system. The key to proper definition of boundary and initial conditions when using superposition in the simulation of ground-water problems is to keep in mind that the model is solving for changes in heads (drawdowns) and flows. Thus, to define the boundary conditions in a model simulation that uses superposition means representing the change in head or flow that will occur at these boundaries.

In ground-water models that use superposition, boundary conditions are usually represented in the following manner (see Franke, Reilly, and Bennett, 1984, for further discussion of boundary conditions):

Constant-head and specified-head boundaries are represented as zero potential boundaries, corresponding to zero drawdown or head change. If the absolute value of head does not change at these boundaries in the natural system, the superposition model represents this boundary as having zero drawdown or buildup of head. If, on the other hand, the purpose of the simulation is to determine the effect on the ground-water system of a change in absolute head at one of these boundaries, then the absolute value of this change in head, Δh (difference between two absolute heads) becomes the new value of constant head in the superposition model, and the other constant head and specified head boundaries that remain unchanged are represented as zero-head potential boundaries.

Constant-flux boundaries are represented by zero change in flow--that is, as zero-flux boundaries in a superposition model because the assumption that flow across these boundaries remains constant implies that change in flow is zero. For example, when we evaluate the response of an aquifer to a pumping well through superposition, we often assume that natural recharge does not change and thus represent the boundary at which recharge occurs as a no-flow boundary. If, on the other hand, we wish to simulate the effect on the ground-water system of a change in flux at one of these boundaries, then the value of this change in flux becomes the new value of constant flux at this boundary in the superposition model, and we again represent the other constant-flux boundaries that remain unchanged as no-flow boundaries.

Leakage across a confining unit from a constant head source (mixed boundary condition) is represented in superposition by maintaining the source at zero drawdown, or zero change in head. As a result, the flow through the confining unit in the superposition model represents the change in flow through the unit in the natural system due to the stress.

These concepts related to definition of boundary conditions in superposition models will be further clarified in a series of examples. A ground-water system with its associated physical boundaries is depicted both in plan view and cross section in figure 4. The values for areal recharge and lateral inflow to the aquifer from the northern bedrock hills represent best estimates based on a water budget, a measured potentiometric surface in the aquifer, and an estimate of aquifer transmissivity. A model of this system based on absolute heads would use the following six boundary conditions: (1) a constant head equal to 400 ft along the western lateral boundary (reservoir) of the aquifer; (2) a constant flux of 3 ft^3/s distributed along the northern lateral aquifer boundary; (3) specified heads along the eastern lateral aquifer boundary corresponding to the river stages in figure 4a; (4) a stream surface or no-flow boundary along the southern impervious bedrock hills; (5) a stream surface or no-flow boundary on the bottom of the aquifer; and (6) a constant areally distributed flux of 2 ft/yr on the top surface of the aquifer. In the following examples, a hypothetical modeling problem for

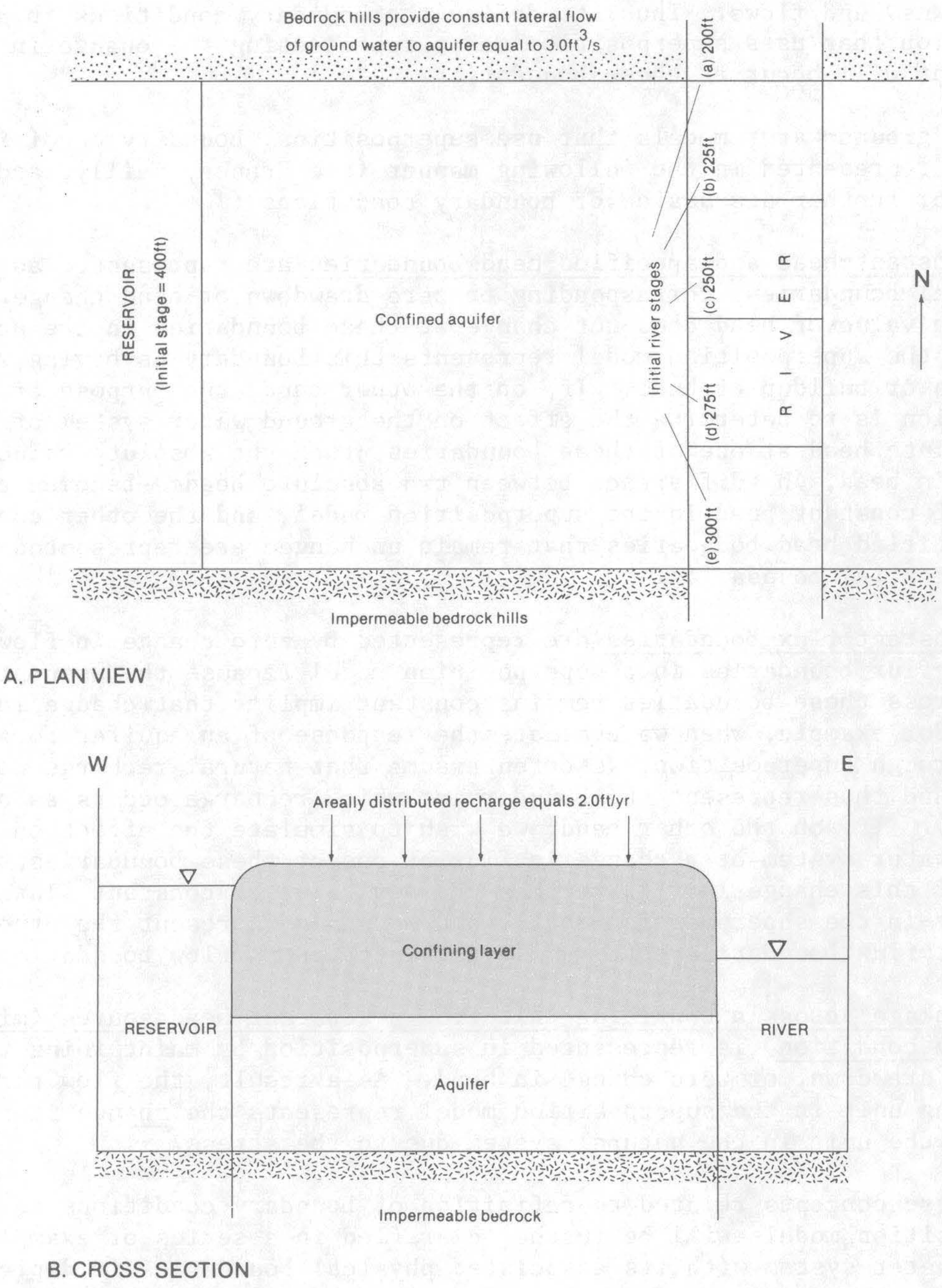


Figure 4. —Plan view and cross section of a hypothetical aquifer and associated boundaries

the system depicted in figure 4 is presented and is followed by the definition of the boundary conditions in a model that simulates the problem through superposition.

Example 1: We wish to investigate the effect of a pumping well on the system. In a superposition model, both the western lateral boundary (reservoir) and eastern lateral boundary (river) are represented as constant-head boundaries of zero potential. All other boundaries are no flow.

Example 2: We wish to investigate the effect on the system of raising the stage of the reservoir from 400 ft to 450 ft. In a superposition model, the western boundary (reservoir) is represented as a constant potential of 50 ft, the eastern lateral boundary (river) as a constant potential of zero, and all other boundaries are no flow.

Example 3: The lateral inflow to the aquifer from the northern bedrock hills is uncertain. We wish to investigate the effect on the system of increasing this lateral inflow from 3 ft^3/s to 4 ft^3/s. In a superposition model, this northern boundary is represented as a constant-flux boundary with areally distributed inflow totaling 1 ft^3/s, the western boundary (reservoir) and eastern boundary (river) are constant head with zero potential, and all other boundaries are no flow.

Example 4: We wish to investigate the effect on the system of changing the stage (and slope) of the river surface along the eastern boundary of the aquifer. The new values of stage at selected points are (a) 220 ft, (b) 240 ft, (c) 260 ft, (d) 280 ft, and (e) 300 ft. In a superposition model investigating these changes in stage, the eastern river boundary is a specified head boundary with potentials of 20 ft at a, 15 ft at b, 10 ft at c, 5 ft at d, and 0 ft at e. The western boundary (reservoir) has a constant potential of zero, and all other boundaries are no flow.

Each of these examples considers a single change in the definition of some aspect of the system boundaries or the stress acting on the system. The total effect on the system of some combination of these individual changes (or other changes) may be obtained by adding algebraically the response of the system (changes in heads and flows in the various parts of the system) to the individual changes. Note that we can investigate not only increases but also decreases in reservoir stage, river stage, lateral inflow, or areal recharge as well as artificial recharge. We must, however, keep track of the reference values of the variables because it is the changes in these values that we are defining and investigating.

Stresses are represented in superposition models through the same logic as the representation of boundaries--that only changes in stress are represented. Suppose, for example, we wish to simulate only the effect of an additional pumping well on a ground-water system that is already heavily stressed. The discharge of water from this pumping well represents the change in stress on the system that will be simulated, and drawdowns in response to only this pumping are determined by the model through superposition.

These same concepts apply also to a pumping-test analysis wherein the natural system may have many stresses acting on it, but we are interested only in the effect of the test well. An early step in the test analysis is to calculate drawdowns at all observation points as a function of time. These measured drawdowns may undergo a series of corrections to account for temporal trends in ground-water levels in the area during the pumping test, barometric effects, tidal effects, and so on. The purpose of these corrections is to obtain, finally, drawdown data that reflect only the effect of pumping at the test well.

In a model that uses superposition, the initial potential distribution is normally taken as zero throughout the system, thus representing zero head change or drawdown. The stresses represented in the model would then be any changes in stress under consideration from the time represented by the initial condition. When comparing drawdowns calculated by a model through superposition with field-measured drawdowns, we must remember that the calculated drawdowns reflect only the changes in stress represented in the model. The field-measured drawdowns, however, may include changes in head resulting from stresses that were affecting the system before the initial time represented in the model. Thus, the drawdowns calculated by the model will be comparable to the actual drawdowns only if the natural system was in equilibrium at the initial time represented by the model. (See Franke, Reilly, and Bennett, 1984, for further discussion of initial conditions.)

The above discussion assumes that the reference head in all models that use superposition is a zero change in head (zero drawdown), and this is almost always true. However, because the principle of superposition states that any solutions to linear differential equations can be added to obtain new solutions, a reference head of zero drawdown is not a requirement--it is simply the most straightforward.

PROBLEM 1

A square confined aquifer with a uniform transmissivity of 1.55×10^{-2} ft^2/s is shown in figure 5. The aquifer is bounded by two impermeable rock walls and two surface-water bodies laterally and by two assumed impermeable boundaries above and below the aquifer. The surface-water bodies are a river and a reservoir whose stages remain constant (fig. 5A, 5B). Thus, a constant head is exerted by the surface-water bodies at their contact surfaces with the aquifer. The natural head distribution with the river stage at zero altitude and the reservoir at 200 ft was calculated by a finite-difference model; the resulting head distribution is an approximate solution to the ground-water flow equation (eq. 5) and is shown in figure 5C.

A pumping rate of 3.1 ft^3/s was then simulated for the well shown in figures 5A and 5B, and the resulting drawdown at equilibrium was calculated by superposition (eq. 8). Because superposition was used, the aquifer-boundary conditions on the surface of contact with the river and reservoir were defined as a zero change in head ($\Delta h = 0$), which corresponds to a constant drawdown of zero. The calculated steady-state drawdowns, Δh, in response to the pumping well are plotted in figure 5D.

From the heads (h) under prestress conditions as shown in figure 5C and the transmissivity of the aquifer, the initial flow of water from the reservoir to the aquifer and from the aquifer to the river can be calculated. From the changes in head (Δh) due to pumping, as shown in figure 5D, and the aquifer transmissivity, the changes in boundary flows due to the pumping can be calculated. These flows are calculated to be:

Figure 5C: (initial state)	Initial flow from reservoir to aquifer	= 3.1 ft^3/s
	Initial flow from aquifer to river	= 3.1 ft^3/s

Figure 5D: Change in flow from reservoir to aquifer = 2.0 ft^3/s
(pumping
response) Change in flow from river to aquifer = 1.1 ft^3/s

It is important to recognize that although we may seem to be calculating flows into the aquifer in figure 5D, these flows actually represent the changes from the initial flow pattern due to the pumping. The sum of these changes in flow must equal the pumpage from the well.

We now wish to determine the actual head distribution and flows in the aquifer as the well is pumped. We could obtain the solution to this problem by simulating simultaneously both the pre-stress system and the pumping well in an additional model run, but we can also obtain the solution by superposition--that is, by adding the heads and flows in figure 5C to those in figure 5D.

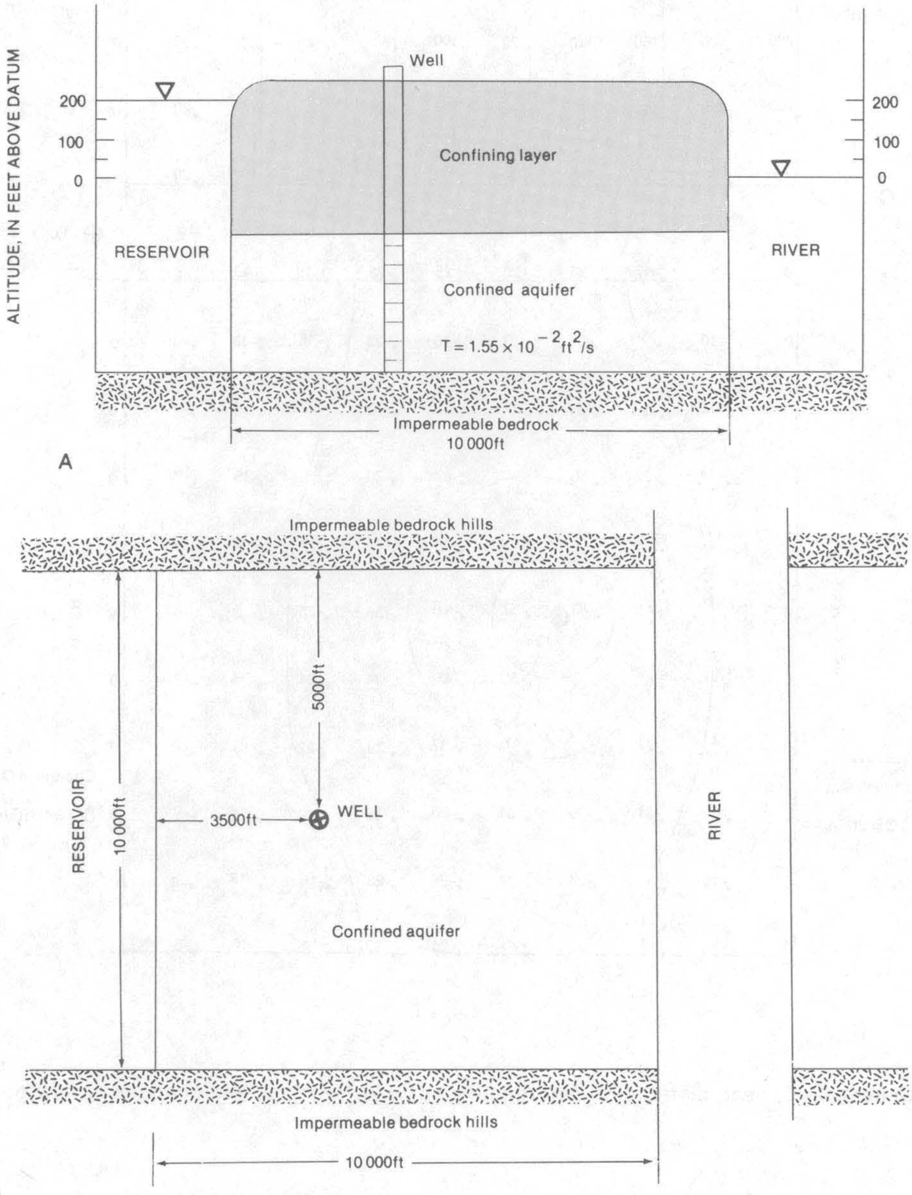


Figure 5. — Aquifer system under study and head distribution in response to stress: A, cross-sectional view of aquifer system and boundaries; B, plan view of aquifer;

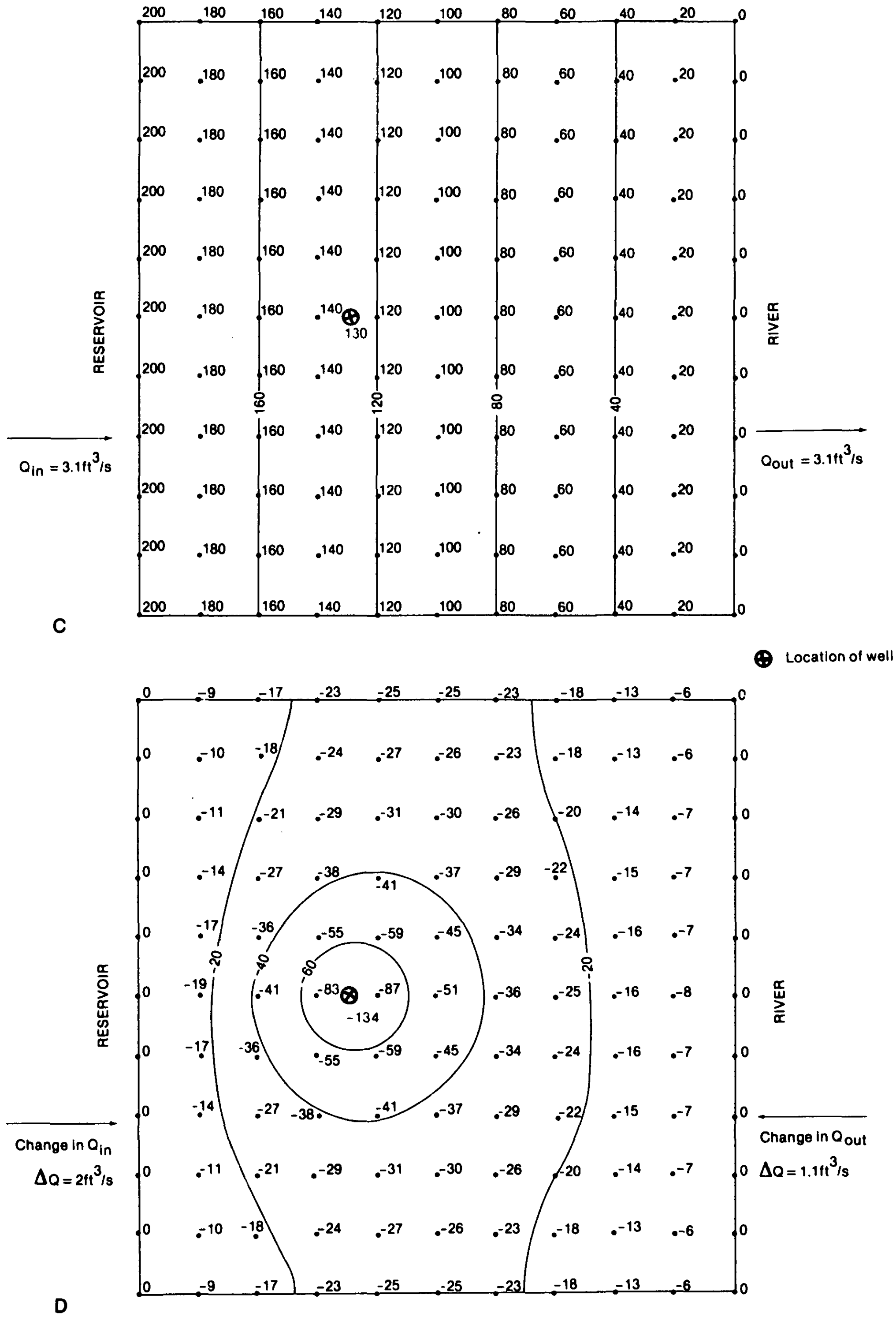


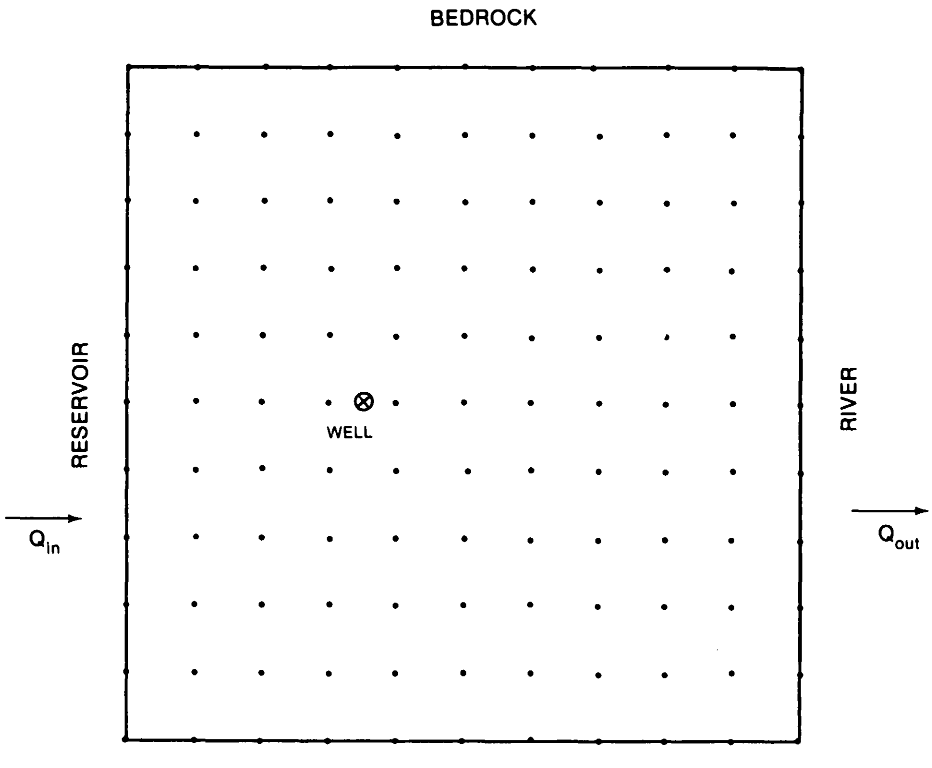
Figure 5 (continued). — C, head distribution before pumping; D, drawdowns and changes in flow due to pumping.

Exercise

Add the heads and flows from figures 5C and 5D and enter the results on worksheet 1. Then contour the head values to decide whether the results are physically reasonable. Answers are given in appendix 3.

WORKSHEET 1

Grid for calculation of new head distribution and flow rates. (Add change in head, fig. 5D, to initial head, fig. 5C, to obtain value for each node.)



$Q_{in} = Q_{in_C} + Q_{in_D} =$ $\qquad$ $Q_{out} = Q_{out_C} + Q_{out_D} =$

Discussion

The flows calculated in the preceding exercise reflect an important point regarding the source of water to the pumping well. The superposition model and the resultant change in boundary flows (fig. 5D) indicate the quantities of water derived from each boundary but do not indicate in what direction the water is flowing in the natural system. The calculated flows in the exercise indicate that the 2 ft^3/s derived from the reservoir is actually increased inflow to the aquifer system, whereas the change of 1.1 ft^3/s at the river boundary actually represents reduced outflow from the aquifer to the river.

APPLICATION OF SUPERPOSITION IN A WELL PROBLEM

An example of adding (superimposing) solutions is the calculation of the drawdown that occurs at a given point in a confined aquifer in response to a sudden change in pumping rate at a well, as shown in figure 6. If the well begins pumping 1.0 ft^3/s at time t_0, and the rate is increased to 1.25 ft^3/s at time t_1, the total drawdown at any time can be calculated by superposition. Because pumpage before t_0 was zero, the initial conditions of the system are zero drawdown everywhere. Figure 6A shows the drawdown (or change in heads) caused by steady pumpage of 1.0 ft^3/s starting at t_0. Figure 6B shows the drawdown caused by pumpage of 0.25 ft^3/s starting at time t_1. The total

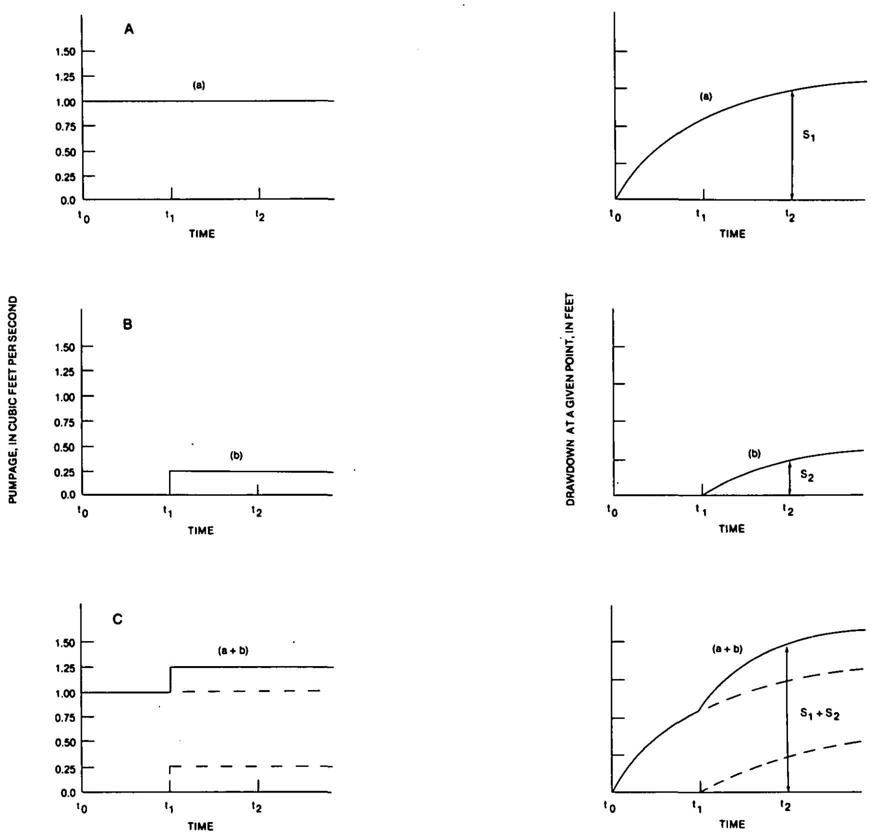


Figure 6. — Superposition of well solutions: A, initial pumpage starting at t_0 and its resulting drawdown s_1 at t_2; B, change in pumpage from the initial rate starting at t_1 and its resulting drawdown s_2 at t_2; C, total pumpage starting at initial rate and increasing at t_1, and its resulting drawdown $s_1 + s_2$ at t_2 as obtained by superposition.

drawdown at some point in the aquifer due to the discharge of the well at 1.0 ft^3/s beginning at t_0, followed by a stepwise increase in pumping rate to 1.25 ft^3/s at t_1, is calculated by adding the drawdowns from figures 6A and 6B. Note that the pumping rate of the well is defined in figure 6C by adding (or superimposing) the two pumping rates from 6A and 6B. Thus, we are adding system stresses (pumping rates) as well as system responses (drawdowns). The solution to the problem, as obtained by superposition, is shown in figure 6C.

ADVANTAGES OF SUPERPOSITION

The principal advantages of using superposition in ground-water studies may be summarized as follows:

1) The effects of a specified stress on the system can be evaluated even if other stresses acting on the system are unknown. For example, the drawdown caused by a pumping well can be calculated even if the recharge rate, the actual heads, gradients, or even the pumping rates of other wells in the aquifer are unknown.

2) The effects of a change in stress on the system can be evaluated even if the original equilibrium conditions or some subsequent period of equilibrium conditions resulting from a long-acting constant stress are unknown. Defining a problem in terms of changes allows the initial conditions to be represented simply by zero drawdown everywhere. In other words, employing superposition as part of the modeling strategy avoids the problem of defining initial conditions (See Franke, Reilly, and Bennett, 1984, on initial conditions).

3) The effect of one stress on the system can be isolated from the effects of all other stresses acting on the system. For example, the sources of water to a pumping well can be determined directly by superposition, as demonstrated in problem 1 of this report.

4) Through superposition, information (parameter identification) on the natural flow system may be obtained through model calibration, even when predevelopment heads and flows in the system are unknown. After the aquifer parameters and boundary conditions of the flow system have been reasonably well established and incorporated into a flow model through superposition, the model may be used to reconstruct an approximate representation of the predevelopment flow system by calculating absolute heads[1]. Such a reconstruction may be of considerable aid in understanding the hydraulics of the aquifer system and the effects of subsequent historical development.

---

[1] Absolute head is water-level elevation above a reference datum, usually mean sea level. Drawdown is the head difference between two water-level surfaces and is thus independent of the elevation datum. It has already been emphasized that superposition uses drawdowns (or changes in head). When we solve a problem by superposition, it is simplest to conceive of the ground-water system as having zero drawdown (or zero change in head) everywhere at the start of the problem. The introduction of a stress will

5) Taking into account the effects of prevailing water-level trends in an aquifer system during analysis of aquifer-test data (trend correction) is a particular case of (3) above that deserves special mention. For example, if the aquifer system has a declining trend due to other stresses on the system, the effects of the pumping may be superimposed on this declining trend. Determination of the aquifer parameters is based only on the drawdown caused by the pumping test, and the effects of the background trend are removed by subtraction.

Common themes in the preceding list are that superposition enables us to simplify complex problems and to obtain useful results despite a lack of certain information describing the ground-water system and the stresses acting on it. Through the use of superposition, the problem can be formulated in simpler terms, which saves effort and reduces data requirements. Thus, if the technique is applicable, it may be advantageous to use superposition in solving many specific problems.

CONSTRAINTS ON THE USE OF SUPERPOSITION

As emphasized in the previous discussion, the most important constraints in the use of superposition in ground-water problems are that the governing differential equation and boundary conditions must be linear. This means that the governing differential equation cannot contain terms such as: $h\dfrac{\partial h}{\partial x}$, $\left(\dfrac{\partial h}{\partial x}\right)^2$, or h^2. (See appendix 2.) In general (disregarding complicated boundary and source or sink terms), flow in confined aquifers is described by linear differential equations, and flow in unconfined aquifers by nonlinear differential equations.

For the system to yield a linear response to stress, not only the governing differential equation, but also the boundary conditions, must be linear. Usually, boundaries that are fixed in space and known as part of the problem definition are characterized by linear boundary conditions. Examples of nonlinear boundary conditions in a ground-water system are a steady-state water table whose position must be calculated as part of the problem solution, a water table whose position is known initially but changes as a function of time, a moving freshwater-saltwater interface, and a stream that changes in length during the course of a transient stress.

PROBLEM 2

A confined aquifer is bounded on one side by a fully penetrating stream and on the other side by an impermeable boundary (fig. 7). Its thickness (b) is 30 ft, and its hydraulic conductivity (K) is 125 ft/d. The distance

---

cause drawdowns (changes in head) relative to this zero change water-level surface. To determine absolute heads in the ground-water system after application of the stress, the drawdowns are added to the absolute system heads, as in problem 1.

between the stream and the impermeable boundary is 14,000 ft. The aquifer and stream extend a great distance perpendicular to the cross section in figure 7; thus, flow in this system approximates one-dimensional flow.

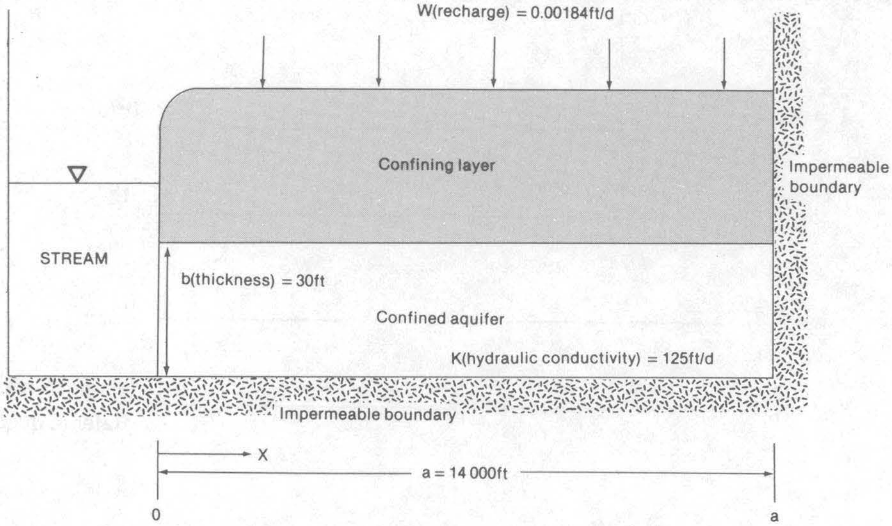


Figure 7. —Cross section of confined aquifer with one-dimensional flow into stream, as described in problem two.

Exercise

Question A:

Assume that natural recharge is areally uniform at the constant rate of 0.00184 ft/d. The analytical solution for the head in this one-dimensional problem (a confined aquifer with a uniform recharge rate, known as Jacob's formula) is:

$$h = \frac{W}{T}\left(ax - \frac{x^2}{2}\right) \qquad (9)$$

where:

h = head (ft) measured above the stream level as datum (that is, the water level in the stream is assumed to be at zero elevation);

W = recharge rate (ft/d);

a = width of profile from stream to impermeable boundary (ft);

T = transmissivity (ft^2/d) (T = hydraulic conductivity, K, times aquifer thickness, b); and

x = distance from constant-head stream (ft).

Assuming a one-dimensional flow system, calculate the head every 2,000 ft by Jacob's formula. Use worksheet 2 for the calculations and enter the answers on worksheet 3. Calculate the total ground-water flow entering the stream per foot of stream perpendicular to the cross section and enter this value in the last column of worksheet 3.

WORKSHEET 2

Table for calculation of head values using Jacob's formula. From data given in problem 2 question A, and figure 7, calculate the following constants for use in subsequent calculations.

(1) $T = Kb =$ ______________________ ft^2/d

(2) $\frac{W}{T} =$ ______________________ ft^{-1}

(3) $a =$ ______________________ ft

(4) $-\frac{Q}{Kb} =$ ______________________ (dimensionless)
Refer to question B.

Fill in the table:

x	ax	$x^2/2$	$(ax - x^2/2)$	h
2,000				
4,000				
6,000				
8,000				
10,000				
12,000				
14,000				

WORKSHEET 3

Table of h and Δ h values calculated in problem 2.
[h = absolute head value;
Δh = change in head due to specific stress]

Question	Condition	Distance from boundary, in feet (x)								Total flow entering the stream
		0	2,000	4,000	6,000	8,000	10,000	12,000	14,000	
A	Original steady-state profile with natural recharge (h)	0								
B	Head build-up with artificial recharge at 14,000 ft (Δ h)	0								
C	Absolute heads with recharge at 14,000 ft and natural recharge (h)	0								
D	Absolute heads with withdrawal at 14,000 ft and natural recharge (h)	0								
E	Absolute heads with three times the original steady-state recharge (h)	0								

Question B:

Suppose the only fluid source is high recharge at the impermeable boundary of the cross section (x = 14,000 ft) at a constant rate of 6.4 (ft^3/d)/ft. No other recharge occurs along the cross section. The head distribution resulting from this point source[1] of recharge can be calculated from Darcy's law:

$$Q = -KA \frac{dh}{dx} \quad (10)$$

where:

K = hydraulic conductivity (ft/d);
A = cross-sectional area of a 1-ft width of aquifer (b x 1 ft = b ft^2);
b = aquifer thickness (ft).

Rearranging the above equation to solve for h gives

$$dh = \frac{-Q}{Kb} dx. \quad (11)$$

Integrating,

$$\int dh = \int \frac{-Q}{Kb} dx \quad (12)$$

gives

$$h + C_1 = \frac{-Qx}{Kb} + C_2 \quad (13)$$

or

$$h = \frac{-Q}{Kb} x + C \quad (14)$$

From the boundary condition that head is zero at x = 0, that is, at the stream, we see that the value of C must be zero, and the solution simplifies to

$$h = \frac{-Q}{Kb} x \quad (15)$$

Calculate the head that occurs at every 2,000-ft interval in response to artificial recharge near the impermeable boundary. Use the above formula, substituting the recharge for Q as a -6.4 (ft^3/d)/ft, and enter the answers on worksheet 3. Calculate the flow that the artificial recharge will contribute to the stream and enter it in the last column of worksheet 3.

From the two independent solutions given in A and B, the theory of superposition can be used to calculate more complex head and flow distributions.

---

[1] This is a point source in cross section. It actually represents a line source in the physical system.

Question C:

Calculate the absolute heads that would occur in the system if the natural recharge rate were 0.00184 ft/d everywhere and an additional recharge of 6.4 (ft^3/d)/ft were added at the impermeable boundary. Calculate the total flow entering the stream. Enter the answers on worksheet 3.

Question D:

Calculate the absolute heads and the flow to the stream that would occur if, instead of a recharge of 6.4 (ft^3/d)/ft at the impermeable boundary, the same amount were withdrawn from the system at this location.

Question E:

What would the absolute heads and the flow to the stream be in the natural system if the natural recharge rate were tripled?

Plot the calculated head values for questions A through D on worksheet 4.

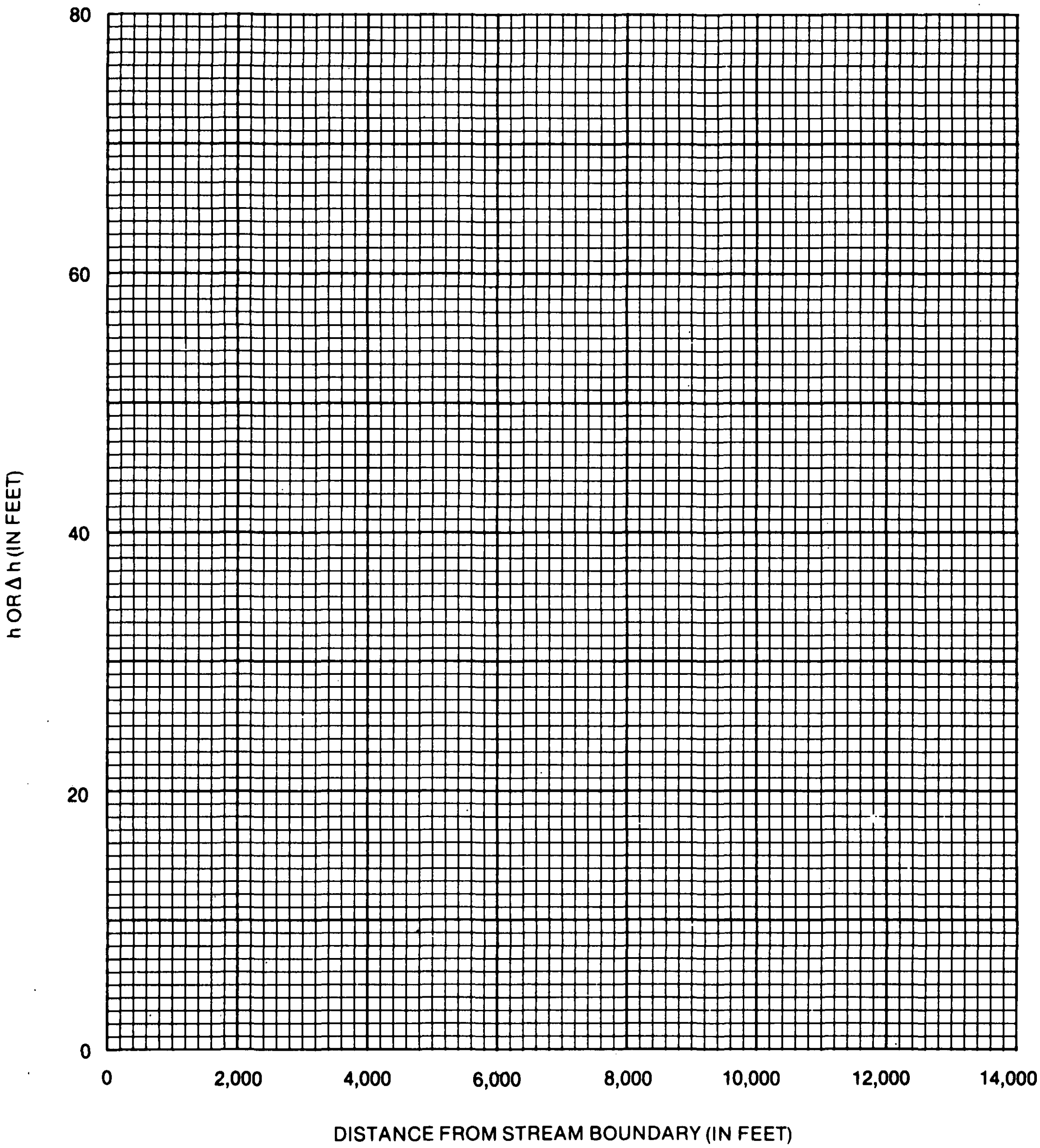
Discussion

With reference to the last column in worksheet 3, it is obvious that flow rates, as well as heads, are superimposed (added and subtracted). In this simple steady-state problem this observation seems almost trivial, but in more complex problems it is often overlooked or confused.

The results for question D (uniform recharge with a withdrawal of water at the boundary) deserve additional comment. Clearly, every quantity of water withdrawn at the impermeable boundary results in a corresponding loss of outflow to the stream. In other words, the artificial withdrawal causes a reduced natural outflow of ground water. A ground-water divide is located at approximately x = 10,000 ft (h = 27.1 ft), but the withdrawal at x = 14,000 ft causes a decline in head through the entire system from the impermeable boundary to the stream. These head declines are numerically equal but opposite in sign to the head values calculated in question B. These numerical results illustrate two distinct concepts that are often confused--the area of <u>diversion</u> of a stress and the area of <u>influence</u> caused by the stress. In problem D, the area of <u>diversion</u> extends from the divide at x = 10,000 ft, to the location of the stress at x = 14,000 ft; this is the area from which all flow is diverted to the source of stress. In contrast, the area of influence--that is, the area within which the stress causes a water-level change--extends all the way to the opposite boundary of the system (the stream). If the stress were increased, the area of diversion would increase until it reached the stream, and ground water would then flow directly from the stream toward the source of stress. The effect of the stress, which had initially been only to decrease the discharge of ground water to the stream, would now capture the entire original ground-water flow to the stream and cause induced infiltration of water from the stream to the aquifer. The concepts of area of influence, area of diversion, reduction in natural discharge, and induced recharge are discussed more thoroughly by Brown (1963).

WORKSHEET 4

Graph of heads and changes in heads at 2,000-ft intervals from stream, as calculated in problem 2.



APPLICATION OF SUPERPOSITION TO NONLINEAR SYSTEMS

The preceding sections have emphasized that superposition applies only to linear systems governed by linear differential equations. In practice, however, because of the power and convenience of superposition, the principle is sometimes applied to mildly nonlinear systems if it can be shown that the resulting error will be acceptably small. For example, if the problem concerns an unconfined aquifer, we might consider using superposition if the regional drawdown in the aquifer is small relative to the full saturated thickness of the aquifer (as a rule of thumb, 10 percent or less). As another example, if the change in position of a freshwater-saltwater interface in a given problem is small relative to the dimensions of the aquifer system, superposition can be used as an approximation. However, if a new distribution of stress is introduced that causes appreciable movement of the interface, the response of the same system could become highly nonlinear.

No set rule can determine whether application of superposition will provide acceptable answers in a given instance; each problem must be judged individually. Usually, preliminary numerical results for the specific problem under study are needed to make this judgment. These preliminary numerical results should include cases in which the maximum stress under consideration is applied to the model. This analysis of the "extreme case" is a useful and time-hononored procedure in scientific and engineering investigations.

SUMMARY AND CONCLUDING REMARKS

The discussion, problems, and references given in this report are designed to give the reader a foundation in the theory of superposition and its application to ground-water problems. Superposition embodies the concept that problem solutions can be added together to obtain new solutions, provided that the system under consideration is linear (that is, governed by linear differential equations and boundary conditions).

When superposition is used to solve ground-water problems, we deal in terms of changes in head (drawdowns) and changes in flows rather than absolute values of heads and flows. These changes are usually calculated from initial conditions of zero change in head everywhere (zero drawdown). When superposition is used in a boundary-value problem (which includes all problems in ground-water flow), it is customary to set constant-head boundaries to zero (representing zero change in head) and to represent specified-flux boundaries as impermeable or no-flow boundaries (zero change in flow).

Walter J. Karplus, in his book Analog Simulation (1958), concisely explains the superposition theorem as follows:

> In linear systems the response due to a number of excitations may be found by adding algebraically...the response due to each excitation taken separately, while the other excitations are reduced to zero.

In terms of ground-water concepts, this statement means that calculated changes in head can then be added to other head distributions to construct solutions corresponding to combined stresses and (or) boundary conditions.

Superposition also allows investigation of the effects of stresses on the ground-water system in isolation from other acting stresses and permits us to obtain results even when we lack certain information describing the ground-water system and the stresses acting on it. Through superposition, problems can often be formulated in simpler terms to save effort and reduce data requirements. Thus, if the technique is applicable, there are many compelling reasons for using superposition in the simulation of ground-water systems.

REFERENCES CITED

Bear, Jacob, 1979, Hydraulics of groundwater: New York, McGraw Hill, 567 p.

Bennett, G. D., 1976, Introduction to ground-water hydraulics: U.S. Geological Survey Techniques of Water-Resource Investigations, Book 3, Chapter B2, 172 p.

Brown, R. H., 1963, The cone of depression and the area of diversion around a discharging well in an infinite strip aquifer subject to uniform recharge, in Shortcuts and special problems in aquifer tests: U.S. Geological Survey Water-Supply Paper 1545-C, p. C69-C85.

Franke, O. L., Reilly, T. E., and Bennett, G. D., 1984, Definition of boundary and initial conditions in the analysis of saturated ground-water flow systems--An introduction: U.S. Geological Survey Open-File Report 84-458, 26 p.

Jeans, Sir James, 1968, Science and music: New York, Dover Publications, 258 p.

Karplus, Walter J., 1958, Analog simulation: New York, McGraw Hill, 434 p.

Walton, W. C., 1970, Groundwater resource evaluation: New York, McGraw-Hill, 664 p.

APPENDIX 1

DISCUSSION ON THE SOLUTION OF DIFFERENTIAL EQUATIONS AND THE ROLE OF BOUNDARY CONDITIONS

The solution of a differential equation describing ground-water flow provides a distribution of hydraulic head over the entire domain of the problem. For simple problems, this distribution of hydraulic head can be expressed formally by a statement giving head as a function of the independent variables. For one independent space variable, we may express this statement in general mathematical notation as

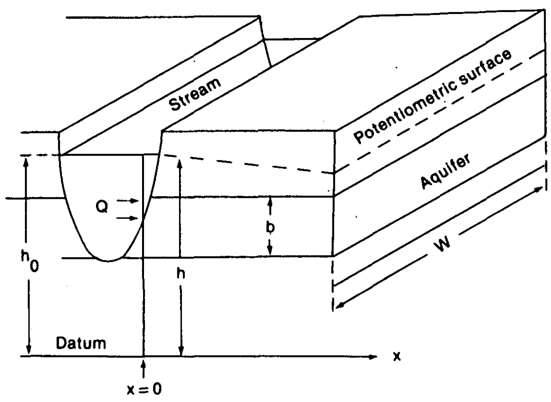
$$h = f(x). \tag{1}$$

This function, f(x), when substituted into the differential equation, must satisfy the equation--that is, the equation must be a true statement. The function f(x) usually contains arbitrary constants and is called the general solution of the differential equation.

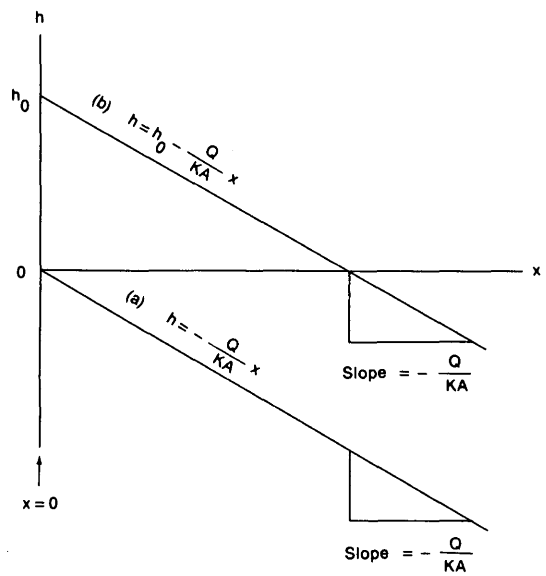
The solution must also satisfy the boundary conditions (and initial conditions for time-dependent problems) that have been specified for the flow region. To satisfy the boundary conditions, the arbitrary constants in the general solution must be defined, resulting in a more specific function, $f_p(x)$, which is called the particular solution to the differential equation. Thus, a particular solution of a differential equation is the solution that solves the particular problem under consideration, and the general solution of a differential equation is the set of all solutions. The following example from Bennett (1976, p. 34-44) helps develop these concepts by using the differential form of Darcy's law as the governing differential equation in a specific problem.

An idealized aquifer system (fig. 8) consists of a confined aquifer of thickness b, which is completely cut by a stream. Water seeps from the stream into the aquifer. The stream level is at elevation h_o above the head datum, which is an arbitrarily chosen level surface. The direction at right angles to the stream axis is denoted as the x direction, and x equals 0 at the edge of the stream. We assume that the system is in steady state, so that no changes occur with time. Along a reach of the stream having length w, the total rate of seepage from the stream (in ft^3/s, for example) is denoted as 2Q. Because only half of this seepage occurs through the right bank of the stream, the amount entering the part of the aquifer shown in our sketch is Q. This seepage moves away from the stream as a steady flow in the x direction. The resulting distribution of hydraulic head within the aquifer is indicated by the dashed line marked "potentiometric surface." This surface, sometimes also referred to as the "piezometric surface," actually traces the static water levels in wells or pipes tapping the aquifer at various points. The differential equation applicable to this problem is obtained by applying Darcy's law to the flow, Q, across the cross-sectional area, bw, and may be written

$$\frac{dh}{dx} = -\frac{Q}{KA}, \tag{2}$$



A



B

Figure 8.— Example of solutions to a differential equation: A, idealized aquifer system; B, two of the family of curves solving the general differential equation for the idealized aquifer system. (Modified from Bennett, 1976.)

where K is the hydraulic conductivity of the aquifer, and A is the cross sectional area perpendicular to the direction of flow; for this problem, A is equal to bw.

Integration of the previous equation gives the general solution f(x) as simply:

$$h = C - \frac{Q}{KA} x \tag{3}$$

where C is an arbitrary constant. Two particular solutions from the family of general solutions are shown in figure 8B, one where the arbitrary constant equals zero (eq. a), and one where the arbitrary constant equals h_o (eq. b). The differential equation (Darcy's law) states that if head is plotted with respect to distance, the slope of the plot will be constant--that is, the graph will be a straight line. Both of the lines in figure 8B are solutions to the differential equation. Each is a straight line having a slope equal to

$$-\frac{Q}{KA}. \tag{4}$$

The intercept of equation a on the h axis is h = 0, whereas the intercept of equation b on the h axis is $h = h_o$. These intercepts give the values of h at x = 0, and thus provide the reference points from which changes in h are measured.

The particular solution for the ground-water system depicted in figure 8 is obtained when the boundary conditions are considered. In this problem, the head in the stream, which is represented at x = 0, is designated as the constant h_o. Thus, the line in figure 8 that has an h axis intercept of h_o is the particular solution to the problem as posed. Therefore, the particular solution, $f_p(x)$, of the governing differential equation in this problem is

$$h = h_o - \frac{Q}{KA} x. \tag{5}$$

This solution satisfies the boundary condition at x = 0.

An accurate description of boundary conditions in obtaining a particular solution to any ground-water problem is of critical importance. In multi-dimensional problems, boundaries are just as important as in the example above, although their effect on the solution may not always be as obvious. Assuming incorrect or inappropriate boundary conditions for a modeling study must inevitably generate an incorrect particular solution to the problem.

In summary, a particular solution to a differential equation is a function that satisfies the differential equation and its boundary conditions. In numerical models that simulate the differential equation by a set of simultaneous algebraic equations, the concepts are analogous, although the solution is not a continuous function.

APPENDIX 2

RECOGNITION OF LINEAR AND NONLINEAR DIFFERENTIAL EQUATIONS

The purpose of this appendix is twofold: first, to enable the reader to determine whether a differential equation is linear, and second, to review some fundamental concepts relating to differential equations, particularly those used in ground-water hydraulics.

The mathematical notation $\frac{df}{dx}$ represents the derivative of f with respect to x. This derivative can be written $\frac{d}{dx}(f)$, which emphasizes the concept that an operator $\frac{d}{dx}$ (signifying differentiation) operates on the variable f.

A differential equation is an equation that involves at least one derivative of an unknown function. Examples are:

$$\text{a)} \qquad \frac{df}{dx} = \tan x, \tag{1a}$$

$$\text{b)} \qquad \frac{df}{dx} = x^2 + 2x + 3, \tag{1b}$$

and

$$\text{c)} \qquad \frac{df}{dx} = e^x . \tag{1c}$$

When an equation involves one or more derivatives with respect to a particular variable, that variable is called an independent variable. If the derivative of a variable occurs, that variable is called a dependent variable. In the examples above, x is the independent variable and f is the dependent variable.

The use of ordinary derivatives, such as $\frac{df}{dx}$, implies that there is only one independent variable, in this case x. The notation $\frac{\partial f}{\partial x}$, however, represents the partial derivative of f with respect to x. The use of partial derivatives implies that the problem contains more than one independent variable.

Ordinary differential equations contain only ordinary derivatives. The three examples above are all ordinary differential equations with the dependent variable f and the independent variable x. Partial differential equations contain partial derivatives with respect to more than one independent variable.

The order of a differential equation is the order of the highest derivative appearing in the equation.

For example,

$$x \frac{d^3f}{dx^3} + x^2 \frac{d^2f}{dx^2} + 4a\left(\frac{df}{dx}\right)^4 + f = 0 \tag{2}$$

is an equation of order 3, or a third-order equation.

The degree of an ordinary differential equation is the algebraic degree of the highest ordered derivative in the equation. For example, the equation

$$x\left(\frac{d^2f}{dx^2}\right)^3 + \left(\frac{df}{dx}\right)^4 + \sin f = 0 \tag{3}$$

is of degree 3 because the equation is cubic with respect to the highest ordered derivative $\frac{d^2f}{dx^2}$. Equation 2 is of degree 1. Note that both the order and degree of a differential equation refer only to the highest ordered derivative.

A differential equation is linear if each term of the differential equation is either linear in all dependent variables and their various derivatives or contains no dependent variables. Otherwise, the equation is nonlinear.

Note that the linearity of a differential equation relates to how the dependent variable occurs in the equation and has nothing to do with the independent variables. Both equations 2 and 3 are nonlinear. In equation 2, the terms $x \frac{d^3f}{dx^3}$, $x^2 \frac{d^2f}{dx^2}$, and f are all linear in f. The term $4a\left(\frac{df}{dx}\right)^4$ is nonlinear in f, however, because it is raised to the fourth power; therefore, the equation must be nonlinear. Every linear equation is of the first degree, but not every equation of the first degree is linear. Equation 2 is of the first degree because the term $x \frac{d^3f}{dx^3}$ is of the first degree, but we have just seen that it is nonlinear.

The equations of ground-water flow are second-order partial differential equations and are usually written in terms of head (h) as the dependent variable or in terms of a head difference (Δh), which is equivalent to a drawdown (s). The independent variables are space coordinates (x, y, z) and time (t). In general, we can write

$$h = f(x, y, z, t) \tag{4}$$

which means that the dependent variable h is a function of the independent variables of space and time. Nonlinear terms in ground-water equations might have the form h^2, $h \frac{\partial h}{\partial x}$, $\left(\frac{\partial h}{\partial x}\right)^2$, and so on.

Examples of linear ground-water flow equations are:

$$\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} = 0 \tag{5}$$

$$K_x \frac{\partial^2 h}{\partial x^2} + K_y \frac{\partial^2 h}{\partial y^2} = 0 \tag{6}$$

$$\frac{d^2 h}{dx^2} = \frac{-W}{T} \tag{7}$$

$$T_x \frac{\partial^2 h}{\partial x^2} + T_y \frac{\partial^2 h}{\partial y^2} + W = S \frac{\partial h}{\partial t} \tag{8}$$

In these equations K, T, and W are parameters and designate hydraulic conductivity, transmissivity, and areal recharge rate, respectively.

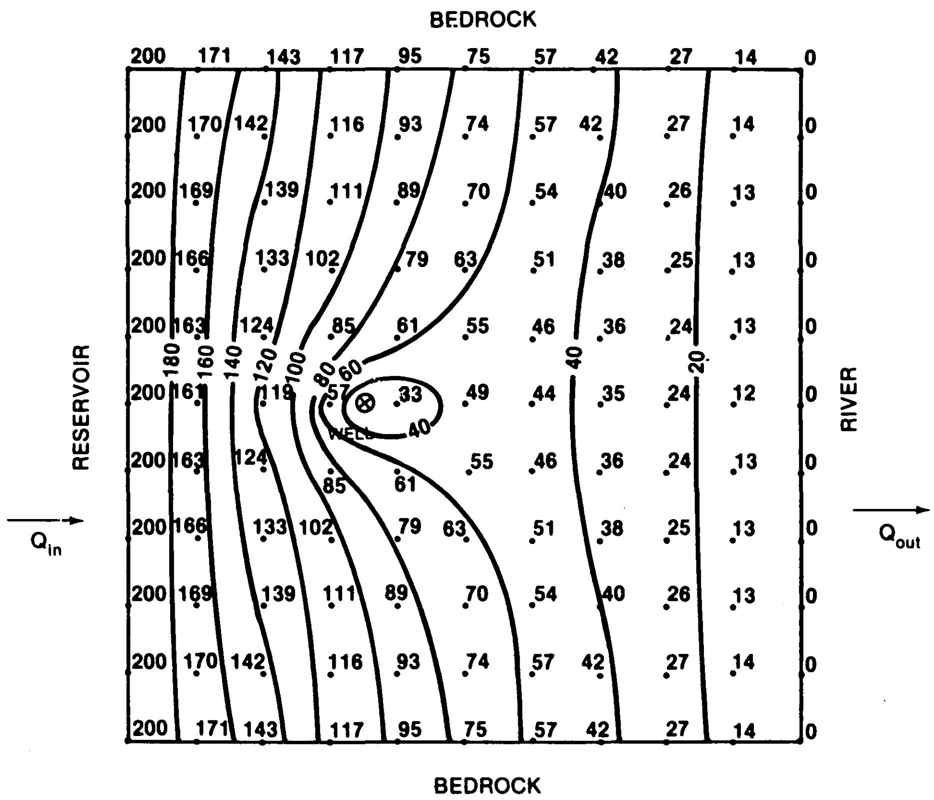
An example of a nonlinear equation is:

$$\frac{\partial}{\partial x}\left(h \frac{\partial h}{\partial x}\right) + \frac{\partial}{\partial y}\left(h \frac{\partial h}{\partial y}\right) = \frac{S}{K}\left(\frac{\partial h}{\partial t}\right) \tag{9}$$

because it is nonlinear in h. This equation describes approximately the transient two-dimensional flow of ground water in a water-table aquifer, incorporating the Dupuit assumptions concerning unconfined flow, and is known as the Boussinesq equation.

APPENDIX 3. Completed Worksheets 1-4.

COMPLETED WORKSHEET 1



$Q_{in} = Q_{in_C} + Q_{in_D} =$

$3.1\ ft^3/s + 2.0\ ft^3/s = 5.1\ ft^3/s$

$Q_{out} = Q_{out_C} + Q_{out_D} =$

$3.1\ ft^3/s + (-1.1)\ ft^3/s = 2.0\ ft^3/s$

COMPLETED WORKSHEET 2

Table for calculation of head values using Jacob's formula. From data given in problem 2 question A, and figure 7, calculate the following constants for use in subsequent calculations.

(1) $T = Kb =$ 125 ft/d × 30 ft = 3750 ft^2/d

(2) $\frac{W}{T} =$ $\frac{.00184 \text{ ft/d}}{3750 \text{ ft}^2\text{/d}} = 4.9 \times 10^{-7}$ ft^{-1}

(3) $a =$ 14,000 ft

(4) $-\frac{Q}{Kb} =$ $\frac{-(-6.4 \text{ ft}^3\text{/d/ft})}{3750 \text{ ft}^2\text{/d}} = 1.7 \times 10^{-3}$ (dimensionless)

Refer to question B.

Fill in the table:

x	ax	$x^2/2$	$(ax - x^2/2)$	h
2,000	2.8×10^7	2.0×10^6	2.6×10^7	12.7
4,000	5.6×10^7	8.0×10^6	4.8×10^7	23.5
6,000	8.4×10^7	1.8×10^7	6.6×10^7	32.3
8,000	1.12×10^8	3.2×10^7	8.0×10^7	39.2
10,000	1.40×10^8	5.0×10^7	9.0×10^7	44.1
12,000	1.68×10^8	7.2×10^7	9.6×10^7	47.0
14,000	1.96×10^8	9.8×10^7	9.8×10^7	48.0

COMPLETED WORKSHEET 3

Table of h and Δh values calculated in problem 2.
[h = absolute head value;
Δh = change in head due to specific stress]

Question	Condition	Distance from boundary, in feet (x)								Total flow entering the stream
		0	2,000	4,000	6,000	8,000	10,000	12,000	14,000	
A	Original steady-state profile with natural recharge (h)	0	12.7	23.5	32.3	39.2	44.1	47.0	48.0	25.76 ft^3/d
B	Head build-up with artificial recharge at 14,000 ft (Δ h)	0	3.4	6.8	10.2	13.6	17.0	20.4	23.8	6.4 ft^2/d
C	Absolute heads with recharge at 14,000 ft and natural recharge (h)	0	16.1	30.3	42.5	52.8	61.1	67.4	71.8	32.16 ft^2/d
D	Absolute heads with withdrawal at 14,000 ft and natural recharge (h)	0	9.3	16.7	22.1	25.6	27.1	26.6	24.2	19.36 ft^2/d
E	Absolute heads with three times the original steady-state recharge (h)	0	38.1	70.5	96.9	117.6	132.3	141.0	144.0	77.28 ft^2/d

COMPLETED WORKSHEET 4

Graph of heads and changes in heads at 2,000-ft intervals from stream, as calculated in problem 2.