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- If we denote $x^2 - 3x - 4$ as $f(x)$, we have $f(4) = 0$ and $f(-1) = 0$.

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Other times it may even give complex numbers that are not real, such as with $x^2 + 1$, which gives $\pm i$.

Higher degree polynomials

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and look for *rational numbers* b/c such that

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Rational solutions to polynomials

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(where b/c is in lowest terms, i.e. b and c have no common factors) then we have

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Since there are finitely many rational b/c such that b divides a_n and c divides a_0, this reduces finding all the rational solutions to $f(x) = 0$ to a simple search problem.

Polynomials in two variables

What if we look instead at polynomials in two variables? Those are polynomials like $x^4y^2 + 5xy^3 + 7x + y + 10$ and $y^2 - x^3 - 2x + 1$.

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Dividing by C, we get

$$\left(\frac{A}{C}\right)^n + \left(\frac{B}{C}\right)^n = 1.$$

Thus, integer solutions to Fermat's equation are the same as rational solutions to the two-variable equation

$$x^n + y^n - 1 = 0.$$

Even older polynomial equations in two variable

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Pythagorean triples $A^2 + B^2 = C^2$, e.g. $3^2 + 4^2 = 5^2$, become solutions to

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Diophantus of Alexandria found that $x = 1/4, y = 9/16$ was a solution in the third century AD. In 1997, Wetherell showed that was the *only* nonzero solution, up to sign (of course $x = \pm 1/4, y = \pm 9/16$ are solutions as well).

Questions about two-variable polynomials

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The degree of $y^2 - x^8 + x^4 + x^2$ is 8, the degree of $y^2x^9 + 7x^5y^3 + x + 3y$ is 11. The degree is the total degree – x-degree plus y-degree – of the term of highest total degree. We'll begin by considering polynomials of various degrees.

Two-variable polynomials of degree 2

Two-variable polynomials of degree 2 may have infinitely many solutions. You may recall that there are infinitely many Pythagorean triples $A^2 + B^2 = C^2$. Dividing through as we saw before gives infinitely many solutions to

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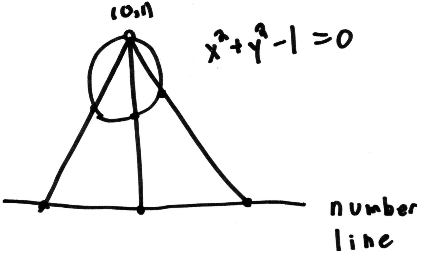
Another way of seeing that there are infinitely many solutions to $x^2 - y^2 - 1 = 0$ is with the following picture, which gives a one-to-one correspondence between the curve $x^2 + y^2 - 1 = 0$ in the Cartesian plane (minus a single point) and the usual number line.



More on two-variable polynomials of degree 2

The one-to-one correspondence on the last page can be written as

$$t \mapsto \left(\frac{t^2 - 1}{t^2 + 1}, \frac{2t}{t^2 + 1} \right)$$

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Using this correspondence, we count the number of rational points on $x^2 + y^2 - 1 = 0$ with numerator and denominator less than some fixed constant M. We see that

$$\# \left\{ \left(\frac{b}{c}, \frac{d}{e} \right) \quad \middle| \quad \left(\frac{b}{c} \right)^2 + \left(\frac{d}{e} \right)^2 = 1 \text{ and } |b|, |c|, |d|, |e| \leq M \right\} \sim M.$$

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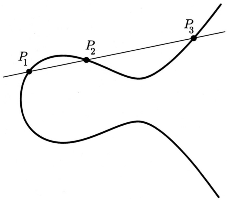
In other words, there are quite a lot of rational points on the curve $x^2 + y^2 - 1 = 0$.

Two-variable polynomials of degree 3

In the case of a two-variable polynomial $f(x, y)$ of degree 3, any straight line intersects our curve $f(x, y) = 0$ in three points. Thus, given two rational points we can "add them together" to get a third as in this picture below (where we have "$P_1 + P_2 = P_3$").



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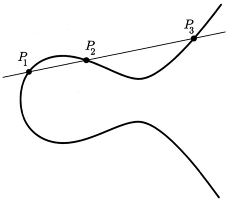

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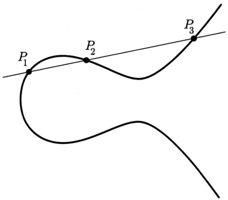

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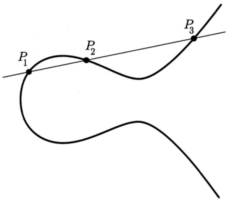

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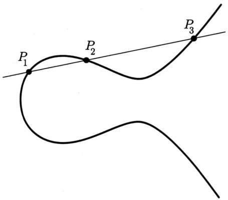

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Two-variable polynomials of degree 4 or more

How about for polynomials of degree 4 or more?

Conjecture

(Mordell conjecture, 1922) If $f(x, y)$ *is a "good" polynomial of degree 4 or greater, then there are finitely many pairs of rational numbers* $(b/c, d/e)$ *such that* $f(b/c, d/e) = 0$.

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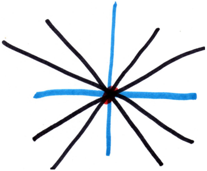
The theorem was finally was proved by Faltings in 1983 and reproved by Faltings and Vojta in a more exact form in 1991.

"Bad polynomial" #1

Here's some polynomials where we clearly do have infinitely many rational solutions despite being of degree 4. Here's a picture of the curve corresponding to the equation $x^4 - 5x^2y^2 + 4y^4 = 0$, which is just the union of four lines, so clearly has infinitely many rational points on it.



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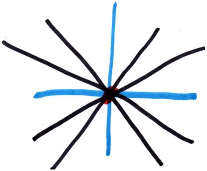

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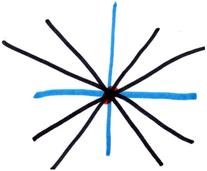

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Notice that all four points meet at the origin so there is no clear "direction" for the curve there.

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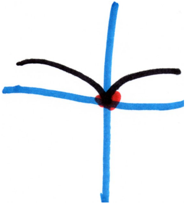

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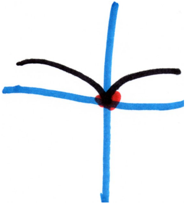

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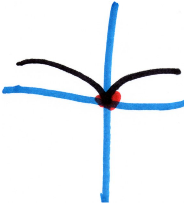

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Notice that here again the curve has no clear “direction” at the origin.

Tangent vectors

The technical term for the direction a curve is moving in at a point (x_0, y_0) is the tangent vector (up to scaling). It can be defined as

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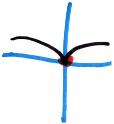

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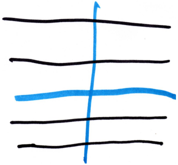

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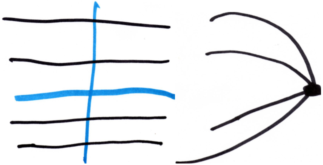

But the lines all meet "at infinity" in the projective plane, which is the natural place to compactify curves in the Cartesian plane.

A proper statement of the Mordell Conjecture

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It turns out that what really matters is what the curves look like over the *complex numbers*.

Curves over the complex numbers

When you take the set of all complex numbers a and b such that $f(a, b) = 0$, you get a two-dimensional object.
Here's what a curve of degree 2 looks like over the complex numbers.

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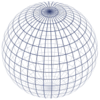

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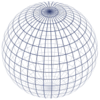

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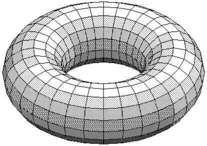

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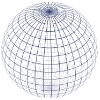

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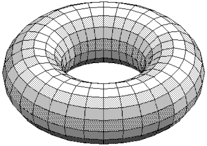

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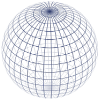

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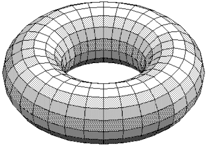

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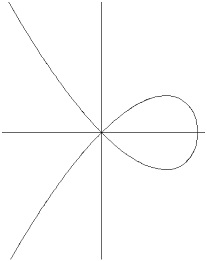
When you have a singularity, it looks like a hole but it is not really one. This is why singular curves are different from nonsingular ones.

A nodal cubic

Look for example at the "nodal cubic" defined by

$$y^2 - x^2(1 - x) = 0$$



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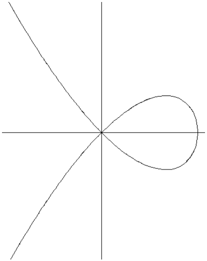

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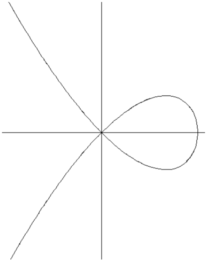

One sees what looks like a hole. However, it can be disentangled (the technical term is *desingularized*) so that the hole disappears.

In this case, what one ends up with is a degree three polynomial that has "as many" rational solutions as a degree two polynomial equation. That is one gets M – rather than rather than $\log M$ – solutions with numerator and denominator bounded by M.

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In the case of two-variable polynomials, the geometric object will be the entire curve. In three or more variables, it becomes more complicated.

Finding all the solutions

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Answer: No one knows. There is an approach, called the method of Coleman-Chabauty which often seems to work but there is no guarantee that it will work in a particular situation. On the negative side there is something called Hilbert's Tenth Problem, solved by Matiyasevich, Robinson, Davis, and Putnam. I'll state it roughly in Hilbert's language.

Hilbert's tenth problem

Theorem

There is no process according to which it can be determined in a finite number of operations whether a polynomial equation $F(x_1, \ldots, x_n) = 0$ with integer coefficients has an integer solution (that is, some $b_1, \ldots, b_n$ such that $F(b_1, \ldots, b_n) = 0$.

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- It is not known whether or not such an algorithm exists for determining whether there is a *rational* solution.
- It is not known whether or not such an algorithm exists when we look at polynomials with only two variables.

How many solutions?

For a one variable equation $F(x) = 0$, we know that if we have solutions $\alpha_1, \ldots, \alpha_n$, then

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So it is a true mystery.

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The proof of the Mordell conjecture is quite difficult, but we would like to give some kind of a proof of something related. So let us look at a simpler question, involving *integer* solutions to a special type of polynomial equation.

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- Suppose that $f(x,y)$ factors over the $\mathbb{C}$ as

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- Suppose the degree n of f is at least 3 and that all the coefficients of f are integers.

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Since the α_i are not equal, they cannot be too close together so we have

$$\left| \frac{b}{c} - \alpha_i \right| \leq \frac{M}{|c|^n}$$

for some constant M (not depending on b and c).

Diophantine approximation

So we are reduced to showing that we cannot have

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infinitely often for any complex α_i that is algebraic of degree $n >= 3$ (that is, a solution to a polynomial equation of degree $n >= 3$ over the integers). This is what Thue showed to prove his theorem. This technique is called *diophantine approximation*.

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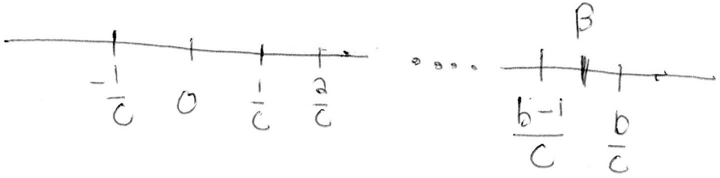

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Thus, it makes sense to think that there is some bound on the number r such that we can get a rational number $\frac{b}{c}$ within $\frac{1}{|c|^r}$ of β. The following is due to Liouville (1844).

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Note that the constant M here is not the same as the one for Thue's theorem, so this does *not* imply the finiteness of solutions to Thue's equation.

Proof of Liouville's theorem

Since $f(\beta) = 0$, we may write

$$f(x) = (x - \beta)g(x) \tag{1}$$

for some polynomial g such that $g(\beta) \neq 0$ (note that after dividing through we may assume that $(x - \beta)$ only divides f once).

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Plugging b/c into (1), and letting $M = 1/D$ in (2) gives

$$\left| \beta - \frac{b}{c} \right| \geq \frac{M}{|c|^n}.$$

Conclusion

The proof of the Mordell conjecture by Faltings-Vojta is simply a much more complicated version of the proof of Liouville's theorem.