LESSON 2-3 The Remainder and Factor Theorems

:: Then	:: Now	:: Why?
● You factored quadratic expressions to solve equations. (Lesson 0–3)	**1** Divide polynomials using long division and synthetic division. **2** Use the Remainder and Factor Theorems.	● The redwood trees of Redwood National Park in California are the oldest living species in the world. The trees can grow up to 350 feet and can live up to 2000 years. Synthetic division can be used to determine the height of one of the trees during a particular year.

NewVocabulary
synthetic division
depressed polynomial
synthetic substitution

1 Divide Polynomials

Consider the polynomial function $f(x) = 6x^3 - 25x^2 + 18x + 9$. If you know that f has a zero at $x = 3$, then you also know that $(x - 3)$ is a factor of $f(x)$. Because $f(x)$ is a third-degree polynomial, you know that there exists a second-degree polynomial $q(x)$ such that

$$f(x) = (x - 3) \cdot q(x).$$

This implies that $q(x)$ can be found by dividing $6x^3 - 25x^2 + 18x + 9$ by $(x - 3)$ because

$$q(x) = \frac{f(x)}{x - 3}, \text{ if } x \neq 3.$$

To divide polynomials, we can use an algorithm similar to that of long division with integers.

Example 1 Use Long Division to Factor Polynomials

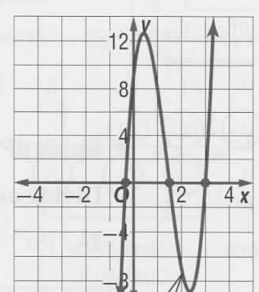
Factor $6x^3 - 25x^2 + 18x + 9$ completely using long division if $(x - 3)$ is a factor.

$$
\begin{array}{rll}
 & 6x^2 - 7x - 3 & \\
x - 3 \,\big) & 6x^3 - 25x^2 + 18x + 9 & \\
(-) & \underline{6x^3 - 18x^2} & \leftarrow \text{Multiply divisor by } 6x^2 \text{ because } \frac{6x^3}{x} = 6x^2. \\
 & -7x^2 + 18x & \leftarrow \text{Subtract and bring down next term.} \\
(-) & \underline{-7x^2 + 21x} & \leftarrow \text{Multiply divisor by } -7x \text{ because } -\frac{7x^2}{x} = -7x. \\
 & -3x + 9 & \leftarrow \text{Subtract and bring down next term.} \\
(-) & \underline{-3x + 9} & \leftarrow \text{Multiply divisor by } -3 \text{ because } \frac{-3x}{x} = -3. \\
 & 0 & \leftarrow \text{Subtract. Notice that the remainder is 0.}
\end{array}
$$

From this division, you can write $6x^3 - 25x^2 + 18x + 9 = (x - 3)(6x^2 - 7x - 3)$.

Factoring the quadratic expression yields $6x^3 - 25x^2 + 18x + 9 = (x - 3)(2x - 3)(3x + 1)$.

So, the zeros of the polynomial function $f(x) = 6x^3 - 25x^2 + 18x + 9$ are 3, $\frac{3}{2}$, and $-\frac{1}{3}$. The x-intercepts of the graph of $f(x)$ shown support this conclusion.



$f(x) = 6x^3 - 25x^2 + 18x + 9$

GuidedPractice

Factor each polynomial completely using the given factor and long division.

1A. $x^3 + 7x^2 + 4x - 12$; $x + 6$

1B. $6x^3 - 2x^2 - 16x - 8$; $2x - 4$

StudyTip

Proper vs. Improper A rational expression is considered improper if the degree of the numerator is greater than or equal to the degree of the denominator. So in the division algorithm, $\frac{f(x)}{d(x)}$ is an *improper* rational expression, while $\frac{r(x)}{d(x)}$ is a *proper* rational expression.

Long division of polynomials can result in a zero remainder, as in Example 1, or a nonzero remainder, as in the example below. Notice that just as with integer long division, the result of polynomial division is expressed using the quotient, remainder, and divisor.

$$\begin{array}{rl} & x + 3 \leftarrow \text{Quotient} \\ \text{Divisor} \rightarrow x + 2 \big) & x^2 + 5x - 4 \leftarrow \text{Dividend} \\ (-) & \underline{x^2 + 2x} \\ & 3x - 4 \\ (-) & \underline{3x + 6} \\ \text{Remainder} \rightarrow & -10 \end{array}$$

$$\underset{\text{Divisor}}{\overset{\text{Dividend}}{\frac{x^2 + 5x - 4}{x + 2}}} = \overset{\text{Quotient}}{x + 3} + \frac{\overset{\text{Remainder}}{-10}}{\underset{\text{Divisor}}{x + 2}}, \underset{\text{Excluded value}}{x \neq -2}$$

Recall that a dividend can be expressed in terms of the divisor, quotient, and remainder.

divisor	·	quotient	+	remainder	=	dividend
$(x + 2)$	·	$(x + 3)$	+	(-10)	=	$x^2 + 5x - 4$

This leads to a definition for polynomial division.

KeyConcept Polynomial Division

Let $f(x)$ and $d(x)$ be polynomials such that the degree of $d(x)$ is less than or equal to the degree of $f(x)$ and $d(x) \neq 0$. Then there exist unique polynomials $q(x)$ and $r(x)$ such that

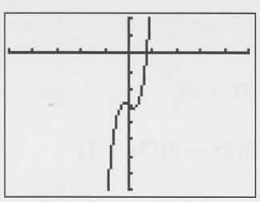
$$\frac{f(x)}{d(x)} = q(x) + \frac{r(x)}{d(x)} \quad \text{or} \quad f(x) = d(x) \cdot q(x) + r(x),$$

where $r(x) = 0$ or the degree of $r(x)$ is less than the degree of $d(x)$. If $r(x) = 0$, then $d(x)$ *divides evenly* into $f(x)$.

Before dividing, be sure that each polynomial is written in standard form and that placeholders with zero coefficients are inserted where needed for missing powers of the variable.

StudyTip

Graphical Check You can also check the result in Example 2 using a graphing calculator. The graphs of $Y_1 = 9x^3 - x - 3$ and $Y_2 = (3x^2 - 2x + 1) \cdot (3x + 2) - 5$ are identical.



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Example 2 Long Division with Nonzero Remainder

Divide $9x^3 - x - 3$ by $3x + 2$.

First rewrite $9x^3 - x - 3$ as $9x^2 + 0x^2 - x - 3$. Then divide.

$$\begin{array}{rl} & 3x^2 - 2x + 1 \\ 3x + 2 \big) & 9x^3 + 0x^2 - x - 3 \\ (-) & \underline{9x^3 + 6x^2} \\ & -6x^2 - x \\ (-) & \underline{-6x^2 - 4x} \\ & 3x - 3 \\ (-) & \underline{3x + 2} \\ & -5 \end{array}$$

You can write this result as

$$\frac{9x^3 - x - 3}{3x + 2} = 3x^2 - 2x + 1 + \frac{-5}{3x + 2}, x \neq -\frac{2}{3}$$
$$= 3x^2 - 2x + 1 - \frac{5}{3x + 2}, x \neq -\frac{2}{3}.$$

CHECK Multiply to check this result.

$$(3x + 2)(3x^2 - 2x + 1) + (-5) \stackrel{?}{=} 9x^3 - x - 3$$
$$9x^3 - 6x^2 + 3x + 6x^2 - 4x + 2 - 5 \stackrel{?}{=} 9x^3 - x - 3$$
$$9x^3 - x - 3 = 9x^3 - x - 3 \checkmark$$

GuidedPractice

Divide using long division.

2A. $(8x^3 - 18x^2 + 21x - 20) \div (2x - 3)$ **2B.** $(-3x^3 + x^2 + 4x - 66) \div (x - 5)$

When dividing polynomials, the divisor can have a degree higher than 1. This can sometimes result in a quotient with missing terms.

StudyTip

Division by Zero In Example 3, this division is not defined for $x^2 - 2x + 7 = 0$. From this point forward in this lesson, you can assume that x cannot take on values for which the indicated division is undefined.

Example 3 Division by Polynomial of Degree 2 or Higher

Divide $2x^4 - 4x^3 + 13x^2 + 3x - 11$ by $x^2 - 2x + 7$.

$$\begin{array}{r}
2x^2 \qquad\qquad\qquad -1 \\
x^2 - 2x + 7\overline{)2x^4 - 4x^3 + 13x^2 + 3x - 11} \\
(-)\ \underline{2x^4 - 4x^3 + 14x^2} \qquad\qquad\quad \\
-x^2 + 3x - 11 \\
(-)\ \underline{-x^2 + 2x - 7} \\
x - 4
\end{array}$$

You can write this result as $\dfrac{2x^4 - 4x^3 + 13x^2 + 3x - 11}{x^2 - 2x + 7} = 2x^2 - 1 + \dfrac{x - 4}{x^2 - 2x + 7}$.

GuidedPractice

Divide using long division.

3A. $(2x^3 + 5x^2 - 7x + 6) \div (x^2 + 3x - 4)$ **3B.** $(6x^5 - x^4 + 12x^2 + 15x) \div (3x^3 - 2x^2 + x)$

Synthetic division is a shortcut for dividing a polynomial by a linear factor of the form $x - c$. Consider the long division from Example 1.

Long Division	Suppress Variables	Collapse Vertically	Synthetic Division
Notice the coefficients highlighted in colored text. $\begin{array}{r} 6x^2 - 7x - 3 \\ x - 3\overline{)6x^3 - 25x^2 + 18x + 9} \\ (-)\ \underline{6x^3 - 18x^2} \qquad\qquad \\ -7x^2 + 18x \quad\ \\ (-)\ \underline{-7x^2 + 21x} \quad\ \\ -3x + 9 \\ (-)\ \underline{-3x + 9} \\ 0 \end{array}$	Suppress x and powers of x. $\begin{array}{r} 6 \quad -7 \quad -3 \\ -3\overline{)6 \quad -25 \quad +18 \quad +9} \\ (-)\ \underline{6 \quad -18} \qquad\qquad \\ -7 \quad +18 \quad\ \\ (-)\ \underline{-7 \quad +21} \quad\ \\ -3 \quad +9 \\ (-)\ \underline{-3 \quad +9} \\ 0 \end{array}$	Collapse the long division vertically, eliminating duplications. $\begin{array}{r\|rrr\|r} -3 & 6 & -25 & 18 & 9 \\ & & -18 & 21 & 9 \\ \hline & 6 & -7 & -3 & 0 \end{array}$	Change the signs of the divisor and the numbers on the second line. $\begin{array}{r\|rrr\|r} 3 & 6 & -25 & 18 & 9 \\ & & 18 & -21 & -9 \\ \hline & 6 & -7 & -3 & 0 \end{array}$ The number now representing the divisor is the related zero of the binomial $x - c$. Also, by changing the signs on the second line, we are now adding instead of subtracting.

We can use the synthetic division shown in the example above to outline a procedure for synthetic division of any polynomial by a binomial.

KeyConcept Synthetic Division Algorithm

To divide a polynomial by the factor $x - c$, complete each step.

Step 1 Write the coefficients of the dividend in standard form. Write the related zero c of the divisor $x - c$ in the box. Bring down the first coefficient.

Step 2 Multiply the first coefficient by c. Write the product under the second coefficient.

Step 3 Add the product and the second coefficient.

Step 4 Repeat Steps 2 and 3 until you reach a sum in the last column. The numbers along the bottom row are the coefficients of the quotient. The power of the first term is one less than the degree of the dividend. The final number is the remainder.

Example

Divide $6x^3 - 25x^2 + 18x + 9$ by $x - 3$.

$$\begin{array}{r|rrrr}
3 & 6 & -25 & 18 & 9 \\
 & & 18 & -21 & -9 \\
\hline
 & \underbrace{6 \quad -7 \quad -3}_{\text{coefficients of quotient}} & & & \underset{\text{remainder}}{0}
\end{array}$$

$\downarrow$ = Add terms. $\nearrow$ = Multiply by c, and write the product.

As with division of polynomials by long division, remember to use zeros as placeholders for any missing terms in the dividend. When a polynomial is divided by one of its binomial factors $x - c$, the quotient is called a **depressed polynomial**.

Example 4 Synthetic Division

Divide using synthetic division.

a. $(2x^4 - 5x^2 + 5x - 2) \div (x + 2)$

Because $x + 2 = x - (-2)$, $c = -2$. Set up the synthetic division as follows, using zero as a placeholder for the missing x^3-term in the dividend. Then follow the synthetic division procedure.

$$\begin{array}{r|rrrrr} -2 & 2 & 0 & -5 & 5 & -2 \\ & & & & & \\ \hline & & & & & \end{array} \quad \Rightarrow \quad \begin{array}{r|rrrrr} -2 & 2 & 0 & -5 & 5 & -2 \\ & & -4 & 8 & -6 & 2 \\ \hline & \underbrace{2 \quad -4 \quad 3 \quad -1}_{\text{coefficients of depressed polynomial}} & & & & \underset{\text{remainder}}{0} \end{array}$$

↓ = Add terms.

↗ = Multiply by c, and write the product.

The quotient has degree one less than that of the dividend, so

$$\frac{2x^4 - 5x^2 + 5x - 2}{x + 2} = 2x^3 - 4x^2 + 3x - 1.$$

Check this result.

b. $(10x^3 - 13x^2 + 5x - 14) \div (2x - 3)$

Rewrite the division expression so that the divisor is of the form $x - c$.

$$\frac{10x^3 - 13x^2 + 5x - 14}{2x - 3} = \frac{(10x^3 - 13x^2 + 5x - 14) \div 2}{(2x - 3) \div 2} \text{ or } \frac{5x^3 - \frac{13}{2}x^2 + \frac{5}{2}x - 7}{x - \frac{3}{2}}.$$

So, $c = \frac{3}{2}$. Perform the synthetic division.

$$\begin{array}{r|rrrr} \frac{3}{2} & 5 & -\frac{13}{2} & \frac{5}{2} & -7 \\ & & & & \\ \hline & & & & \end{array} \quad \Rightarrow \quad \begin{array}{r|rrrr} \frac{3}{2} & 5 & -\frac{13}{2} & \frac{5}{2} & -7 \\ & & \frac{15}{2} & \frac{3}{2} & 6 \\ \hline & 5 & 1 & 4 & -1 \end{array}$$

So, $\dfrac{10x^3 - 13x^2 + 5x - 14}{2x - 3} = 5x^2 + x + 4 - \dfrac{1}{x - \frac{3}{2}}$ or $5x^2 + x + 4 - \dfrac{2}{2x - 3}$. Check this result.

TechnologyTip

Using Graphs To check your division, you can graph the polynomial division expression and the depressed polynomial with the remainder. The graphs should coincide.

GuidedPractice

4A. $(4x^3 + 3x^2 - x + 8) \div (x - 3)$

4B. $(6x^4 + 11x^3 - 15x^2 - 12x + 7) \div (3x + 1)$

2 The Remainder and Factor Theorems

When $d(x)$ is the divisor $(x - c)$ with degree 1, the remainder is the real number r. So, the division algorithm simplifies to

$$f(x) = (x - c) \cdot q(x) + r.$$

Evaluating $f(x)$ for $x = c$, we find that

$$f(c) = (c - c) \cdot q(c) + r = 0 \cdot q(c) + r \text{ or } r.$$

So, $f(c) = r$, which is the remainder. This leads us to the following theorem.

KeyConcept Remainder Theorem

If a polynomial $f(x)$ is divided by $x - c$, the remainder is $r = f(c)$.

The Remainder Theorem indicates that to evaluate a polynomial function $f(x)$ for $x = c$, you can divide $f(x)$ by $x - c$ using synthetic division. The remainder will be $f(c)$. Using synthetic division to evaluate a function is called **synthetic substitution**.

Real-WorldLink

High school football rules are similar to most college and professional football rules. Two major differences are that the quarters are 12 minutes as opposed to 15 minutes and kick-offs take place at the 40-yard line instead of the 30-yard line.

Source: National Federation of State High School Associations

Real-World Example 5 Use the Remainder Theorem

FOOTBALL The number of tickets sold during the Northside High School football season can be modeled by $t(x) = x^3 - 12x^2 + 48x + 74$, where x is the number of games played. Use the Remainder Theorem to find the number of tickets sold during the twelfth game of the Northside High School football season.

To find the number of tickets sold during the twelfth game, use synthetic substitution to evaluate $t(x)$ for $x = 12$.

$$\begin{array}{r|rrrr} 12 & 1 & -12 & 48 & 74 \\ & & 12 & 0 & 576 \\ \hline & 1 & 0 & 48 & \boxed{650} \end{array}$$

The remainder is 650, so $t(12) = 650$. Therefore, 650 tickets were sold during the twelfth game of the season.

CHECK You can check your answer using direct substitution.

$t(x) = x^3 - 12x^2 + 48x + 74$ — Original function

$t(12) = (12)^3 - 12(12)^2 + 48(12) + 74$ or 650 ✔ — Substitute 12 for x and simplify.

GuidedPractice

5. **FOOTBALL** Use the Remainder Theorem to determine the number of tickets sold during the thirteenth game of the season.

If you use the Remainder Theorem to evaluate $f(x)$ at $x = c$ and the result is $f(c) = 0$, then you know that c is a zero of the function and $(x - c)$ is a factor. This leads us to another useful theorem that provides a test to determine whether $(x - c)$ is a factor of $f(x)$.

KeyConcept Factor Theorem

A polynomial $f(x)$ has a factor $(x - c)$ if and only if $f(c) = 0$.

You can use synthetic division to perform this test.

Example 6 Use the Factor Theorem

Use the Factor Theorem to determine if the binomials given are factors of $f(x)$. Use the binomials that are factors to write a factored form of $f(x)$.

a. $f(x) = 4x^4 + 21x^3 + 25x^2 - 5x + 3$; $(x - 1)$, $(x + 3)$

Use synthetic division to test each factor, $(x - 1)$ and $(x + 3)$.

$$\begin{array}{r|rrrrr} 1 & 4 & 21 & 25 & -5 & 3 \\ & & 4 & 25 & 50 & 45 \\ \hline & 4 & 25 & 50 & 45 & \boxed{48} \end{array}$$

$$\begin{array}{r|rrrrr} -3 & 4 & 21 & 25 & -5 & 3 \\ & & -12 & -27 & 6 & -3 \\ \hline & 4 & 9 & -2 & 1 & \boxed{0} \end{array}$$

Because the remainder when $f(x)$ is divided by $(x - 1)$ is 48, $f(1) = 48$ and $(x - 1)$ is not a factor.

Because the remainder when $f(x)$ is divided by $(x + 3)$ is 0, $f(-3) = 0$ and $(x + 3)$ is a factor.

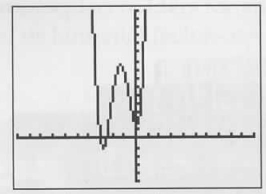
Because $(x + 3)$ is a factor of $f(x)$, we can use the quotient of $f(x) \div (x + 3)$ to write a factored form of $f(x)$.

$$f(x) = (x + 3)(4x^3 + 9x^2 - 2x + 1)$$

TechnologyTip

Zeros You can confirm the zeros on the graph of a function by using the zero feature on the CALC menu of a graphing calculator.

CHECK If $(x + 3)$ is a factor of $f(x) = 4x^4 + 21x^3 + 25x^2 - 5x + 3$, then -3 is a zero of the function and $(-3, 0)$ is an x-intercept of the graph. Graph $f(x)$ using a graphing calculator and confirm that $(-3, 0)$ is a point on the graph. ✔



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b. $f(x) = 2x^3 - x^2 - 41x - 20$; $(x + 4)$, $(x - 5)$

Use synthetic division to test the factor $(x + 4)$.

-4	2	-1	-41	-20
		-8	36	20
	2	-9	-5	0

Because the remainder when $f(x)$ is divided by $(x + 4)$ is 0, $f(-4) = 0$ and $(x + 4)$ is a factor of $f(x)$.

Next, test the second factor, $(x - 5)$, with the depressed polynomial $2x^2 - 9x - 5$.

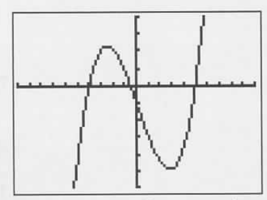
5	2	-9	-5
		10	5
	2	1	0

Because the remainder when the quotient of $f(x) \div (x + 4)$ is divided by $(x - 5)$ is 0, $f(5) = 0$ and $(x - 5)$ is a factor of $f(x)$.

Because $(x + 4)$ and $(x - 5)$ are factors of $f(x)$, we can use the final quotient to write a factored form of $f(x)$.

$$f(x) = (x + 4)(x - 5)(2x + 1)$$

CHECK The graph of $f(x) = 2x^3 - x^2 - 41x - 20$ confirms that $x = -4$, $x = 5$, and $x = -\frac{1}{2}$ are zeros of the function. ✔



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GuidedPractice

Use the Factor Theorem to determine if the binomials given are factors of $f(x)$. Use the binomials that are factors to write a factored form of $f(x)$.

6A. $f(x) = 3x^3 - x^2 - 22x + 24$; $(x - 2)$, $(x + 5)$

6B. $f(x) = 4x^3 - 34x^2 + 54x + 36$; $(x - 6)$, $(x - 3)$

You can see that synthetic division is a useful tool for factoring and finding the zeros of polynomial functions.

ConceptSummary Synthetic Division and Remainders

If r is the remainder obtained after a synthetic division of $f(x)$ by $(x - c)$, then the following statements are true.

- r is the value of $f(c)$.
- If $r = 0$, then $(x - c)$ is a factor of $f(x)$.
- If $r = 0$, then c is an x-intercept of the graph of f.
- If $r = 0$, then $x = c$ is a solution of $f(x) = 0$.

Exercises

= Step-by-Step Solutions begin on page R29.

Factor each polynomial completely using the given factor and long division. (Example 1)

1. $x^3 + 2x^2 - 23x - 60; x + 4$
2. $x^3 + 2x^2 - 21x + 18; x - 3$
3. $x^3 + 3x^2 - 18x - 40; x - 4$
4. $4x^3 + 20x^2 - 8x - 96; x + 3$
5. $-3x^3 + 15x^2 + 108x - 540; x - 6$
6. $6x^3 - 7x^2 - 29x - 12; 3x + 4$
7. $x^4 + 12x^3 + 38x^2 + 12x - 63; x^2 + 6x + 9$
8. $x^4 - 3x^3 - 36x^2 + 68x + 240; x^2 - 4x - 12$

Divide using long division. (Examples 2 and 3)

9. $(5x^4 - 3x^3 + 6x^2 - x + 12) \div (x - 4)$
10. $(x^6 - 2x^5 + x^4 - x^3 + 3x^2 - x + 24) \div (x + 2)$
11. $(4x^4 - 8x^3 + 12x^2 - 6x + 12) \div (2x + 4)$
12. $(2x^4 - 7x^3 - 38x^2 + 103x + 60) \div (x - 3)$
13. $(6x^6 - 3x^5 + 6x^4 - 15x^3 + 2x^2 + 10x - 6) \div (2x - 1)$
14. $(108x^5 - 36x^4 + 75x^2 + 36x + 24) \div (3x + 2)$
15. $(x^4 + x^3 + 6x^2 + 18x - 216) \div (x^3 - 3x^2 + 18x - 54)$
16. $(4x^4 - 14x^3 - 14x^2 + 110x - 84) \div (2x^2 + x - 12)$
17. $\dfrac{6x^5 - 12x^4 + 10x^3 - 2x^2 - 8x + 8}{3x^3 + 2x + 3}$
18. $\dfrac{12x^5 + 5x^4 - 15x^3 + 19x^2 - 4x - 28}{3x^3 + 2x^2 - x + 6}$

Divide using synthetic division. (Example 4)

19. $(x^4 - x^3 + 3x^2 - 6x - 6) \div (x - 2)$
20. $(2x^4 + 4x^3 - 2x^2 + 8x - 4) \div (x + 3)$
21. $(3x^4 - 9x^3 - 24x - 48) \div (x - 4)$
22. $(x^5 - 3x^3 + 6x^2 + 9x + 6) \div (x + 2)$
23. $(12x^5 + 10x^4 - 18x^3 - 12x^2 - 8) \div (2x - 3)$
24. $(36x^4 - 6x^3 + 12x^2 - 30x - 12) \div (3x + 1)$
25. $(45x^5 + 6x^4 + 3x^3 + 8x + 12) \div (3x - 2)$
26. $(48x^5 + 28x^4 + 68x^3 + 11x + 6) \div (4x + 1)$
27. $(60x^6 + 78x^5 + 9x^4 - 12x^3 - 25x - 20) \div (5x + 4)$
28. $\dfrac{16x^6 - 56x^5 - 24x^4 + 96x^3 - 42x^2 - 30x + 105}{2x - 7}$

29. **EDUCATION** The number of U.S. students, in thousands, that graduated with a bachelor's degree from 1970 to 2006 can be modeled by $g(x) = 0.0002x^5 - 0.016x^4 + 0.512x^3 - 7.15x^2 + 47.52x + 800.27$, where x is the number of years since 1970. Use synthetic substitution to find the number of students that graduated in 2005. Round to the nearest thousand. (Example 5)

30. **SKIING** The distance in meters that a person travels on skis can be modeled by $d(t) = 0.2t^2 + 3t$, where t is the time in seconds. Use the Remainder Theorem to find the distance traveled after 45 seconds. (Example 5)

Find each $f(c)$ using synthetic substitution. (Example 5)

31. $f(x) = 4x^5 - 3x^4 + x^3 - 6x^2 + 8x - 15; c = 3$
32. $f(x) = 3x^6 - 2x^5 + 4x^4 - 2x^3 + 8x - 3; c = 4$
33. $f(x) = 2x^6 + 5x^5 - 3x^4 + 6x^3 - 9x^2 + 3x - 4; c = 5$
34. $f(x) = 4x^6 + 8x^5 - 6x^3 - 5x^2 + 6x - 4; c = 6$
35. $f(x) = 10x^5 + 6x^4 - 8x^3 + 7x^2 - 3x + 8; c = -6$
36. $f(x) = -6x^7 + 4x^5 - 8x^4 + 12x^3 - 15x^2 - 9x + 64; c = 2$
37. $f(x) = -2x^8 + 6x^5 - 4x^4 + 12x^3 - 6x + 24; c = 4$

Use the Factor Theorem to determine if the binomials given are factors of $f(x)$. Use the binomials that are factors to write a factored form of $f(x)$. (Example 6)

38. $f(x) = x^4 - 2x^3 - 9x^2 + x + 6; (x + 2), (x - 1)$
39. $f(x) = x^4 + 2x^3 - 5x^2 + 8x + 12; (x - 1), (x + 3)$
40. $f(x) = x^4 - 2x^3 + 24x^2 + 18x + 135; (x - 5), (x + 5)$
41. $f(x) = 3x^4 - 22x^3 + 13x^2 + 118x - 40; (3x - 1), (x - 5)$
42. $f(x) = 4x^4 - x^3 - 36x^2 - 111x + 30; (4x - 1), (x - 6)$
43. $f(x) = 3x^4 - 35x^3 + 38x^2 + 56x + 64; (3x - 2), (x + 2)$
44. $f(x) = 5x^5 + 38x^4 - 68x^2 + 59x + 30; (5x - 2), (x + 8)$
45. $f(x) = 4x^5 - 9x^4 + 39x^3 + 24x^2 + 75x + 63; (4x + 3), (x - 1)$

46. **TREES** The height of a tree in feet at various ages in years is given in the table.

Age	Height	Age	Height
2	3.3	24	73.8
6	13.8	26	82.0
10	23.0	28	91.9
14	42.7	30	101.7
20	60.7	36	111.5

a. Use a graphing calculator to write a quadratic equation to model the growth of the tree.

b. Use synthetic division to evaluate the height of the tree at 15 years.

47. **BICYCLING** Patrick is cycling at an initial speed v_0 of 4 meters per second. When he rides downhill, the bike accelerates at a rate a of 0.4 meter per second squared. The vertical distance from the top of the hill to the bottom of the hill is 25 meters. Use $d(t) = v_0t + \frac{1}{2}at^2$ to find how long it will take Patrick to ride down the hill, where $d(t)$ is distance traveled and t is given in seconds.

Factor each polynomial using the given factor and long division. Assume $n > 0$.

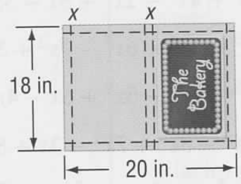
48. $x^{3n} + x^{2n} - 14x^n - 24$; $x^n + 2$

49. $x^{3n} + x^{2n} - 12x^n + 10$; $x^n - 1$

50. $4x^{3n} + 2x^{2n} - 10x^n + 4$; $2x^n + 4$

51. $9x^{3n} + 24x^{2n} - 171x^n + 54$; $3x^n - 1$

52. MANUFACTURING An 18-inch by 20-inch sheet of cardboard is cut and folded into a bakery box.



a. Write a polynomial function to model the volume of the box.

b. Graph the function.

c. The company wants the box to have a volume of 196 cubic inches. Write an equation to model this situation.

d. Find a positive integer for x that satisfies the equation found in part c.

Find the value of k so that each remainder is zero.

53. $\dfrac{x^3 - kx^2 + 2x - 4}{x - 2}$ **54.** $\dfrac{x^3 + 18x^2 + kx + 4}{x + 2}$

55. $\dfrac{x^3 + 4x^2 - kx + 1}{x + 1}$ **56.** $\dfrac{2x^3 - x^2 + x + k}{x - 1}$

57. SCULPTING Esteban will use a block of clay that is 3 feet by 4 feet by 5 feet to make a sculpture. He wants to reduce the volume of the clay by removing the same amount from the length, the width, and the height.

a. Write a polynomial function to model the situation.

b. Graph the function.

c. He wants to reduce the volume of the clay to $\frac{3}{5}$ of the original volume. Write an equation to model the situation.

d. How much should he take from each dimension?

Use the graphs and synthetic division to completely factor each polynomial.

58. $f(x) = 8x^4 + 26x^3 - 103x^2 - 156x + 45$ (Figure 2.3.1)

59. $f(x) = 6x^5 + 13x^4 - 153x^3 + 54x^2 + 724x - 840$ (Figure 2.3.2)

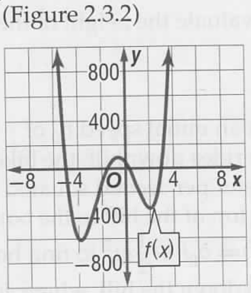


Figure 2.3.1

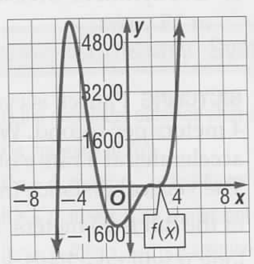


Figure 2.3.2

60. MULTIPLE REPRESENTATIONS In this problem, you will explore the upper and lower bounds of a function.

a. **GRAPHICAL** Graph each related polynomial function, and determine the greatest and least zeros. Then copy and complete the table.

Polynomial	Greatest Zero	Least Zero
$x^3 - 2x^2 - 11x + 12$		
$x^4 + 6x^3 + 3x^2 - 10x$		
$x^5 - x^4 - 2x^3$		

b. **NUMERICAL** Use synthetic division to evaluate each function in part **a** for three integer values greater than the greatest zero.

c. **VERBAL** Make a conjecture about the characteristics of the last row when synthetic division is used to evaluate a function for an integer greater than its greatest zero.

d. **NUMERICAL** Use synthetic division to evaluate each function in part **a** for three integer values less than the least zero.

e. **VERBAL** Make a conjecture about the characteristics of the last row when synthetic division is used to evaluate a function for a number less than its least zero.

H.O.T. Problems Use Higher-Order Thinking Skills

61. CHALLENGE Is $(x - 1)$ a factor of $18x^{165} - 15x^{135} + 8x^{105} - 15x^{55} + 4$? Explain your reasoning.

62. WRITING IN MATH Explain how you can use a graphing calculator, synthetic division, and factoring to completely factor a fifth-degree polynomial with rational coefficients, three integral zeros, and two non-integral, rational zeros.

63. REASONING Determine whether the statement below is *true* or *false*. Explain.

If $h(y) = (y + 2)(3y^2 + 11y - 4) - 1$, then the remainder of $\dfrac{h(y)}{y + 2}$ is -1.

CHALLENGE Find k so that the quotient has a 0 remainder.

64. $\dfrac{x^3 + kx^2 - 34x + 56}{x + 7}$

65. $\dfrac{x^6 + kx^4 - 8x^3 + 173x^2 - 16x - 120}{x - 1}$

66. $\dfrac{kx^3 + 2x^2 - 22x - 4}{x - 2}$

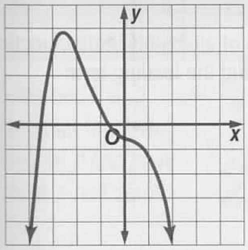
67. CHALLENGE If $2x^2 - dx + (31 - d^2)x + 5$ has a factor $x - d$, what is the value of d if d is an integer?

68. WRITING IN MATH Compare and contrast polynomial division using long division and using synthetic division.

Spiral Review

Determine whether the degree *n* of the polynomial for each graph is *even* or *odd* and whether its leading coefficient a_n is *positive* or *negative*. (Lesson 2–2)

69.



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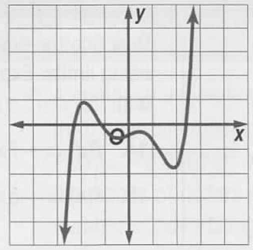

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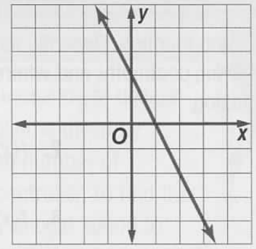


72. SKYDIVING The approximate time t in seconds that it takes an object to fall a distance of d feet is given by $t = \sqrt{\frac{d}{16}}$. Suppose a skydiver falls 11 seconds before the parachute opens. How far does the skydiver fall during this time period? (Lesson 2–1)

73. FIRE FIGHTING The velocity v and maximum height h of water being pumped into the air are related by $v = \sqrt{2gh}$, where g is the acceleration due to gravity (32 feet/second^2). (Lesson 1–7)

a. Determine an equation that will give the maximum height of the water as a function of its velocity.

b. The Mayfield Fire Department must purchase a pump that is powerful enough to propel water 80 feet into the air. Will a pump that is advertised to project water with a velocity of 75 feet/second meet the fire department's needs? Explain.

Solve each system of equations algebraically. (Lesson 0–5)

74. $5x - y = 16$
$2x + 3y = 3$

75. $3x - 5y = -8$
$x + 2y = 1$

76. $y = 6 - x$
$x = 4.5 + y$

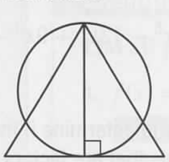
77. $2x + 5y = 4$
$3x + 6y = 5$

78. $7x + 12y = 16$
$5y - 4x = -21$

79. $4x + 5y = -8$
$3x - 7y = 10$

Skills Review for Standardized Tests

80. SAT/ACT In the figure, an equilateral triangle is drawn with an altitude that is also the diameter of the circle. If the perimeter of the triangle is 36, what is the circumference of the circle?



A $6\sqrt{2}\pi$ C $12\sqrt{2}\pi$ E 36π

B $6\sqrt{3}\pi$ D $12\sqrt{3}\pi$

81. REVIEW If (3, −7) is the center of a circle and (8, 5) is on the circle, what is the circumference of the circle?

F 13π H 18π K 26π

G 15π J 25π

82. REVIEW The first term in a sequence is x. Each subsequent term is three less than twice the preceding term. What is the 5th term in the sequence?

A $8x - 21$ C $16x - 39$ E $32x - 43$

B $8x - 15$ D $16x - 45$

83. Use the graph of the polynomial function. Which is not a factor of $x^5 + x^4 - 3x^3 - 3x^2 - 4x - 4$?

F $(x - 2)$

G $(x + 2)$

H $(x - 1)$

J $(x + 1)$