THE RIEMANN MAPPING THEOREM

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ABSTRACT. This paper aims to provide all necessary details to give the standard modern proof of the Riemann Mapping Theorem utilizing normal families of functions. From this point the paper will provide a brief introduction to Riemann Surfaces and conclude with stating the generalization of Riemann Mapping Theorem to Riemann Surfaces: The Uniformization Theorem.

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1. INTRODUCTION

The Riemann Mapping theorem is one of the most useful theorems in elementary complex analysis. From a planar topology viewpoint we know that there exists simply connected domains with complicated boundaries. For such domains there are no obvious homeomorphisms between them. However the Riemann mapping theorem states that such simply connected domains are not only homeomorphic but also biholomorphic.

The paper by J.L. Walsh "History of the Riemann Mapping Theorem"[6] presents an outline of how proofs of the Riemann Mapping theorem have evolved over time. A very important theorem in Complex Analysis, Riemann's mapping theorem was first stated, with an incorrect proof, by Bernhard Riemann in his inaugural dissertation in 1851. Since the publication of this "proof" various objections have been made which all in all lead to this proof being labeled as incorrect. While Riemann's proof is incorrect it did provide the general guidelines, via the Dirichlet Principle and Green's function, which would prove vital in future proofs.

It would not be until 1900 that the American mathematician W.F. Osgood would produce a valid proof of the theorem. [1] Osgood's proof utilized the original ideas of Riemann but was made more difficult than today's methods of proof as he did not possess Perron's method of

[1]Osgood's proof was actually of a similar result from which the Riemann Mapping theorem can be derived[4]

solving the Dirichlet problem. Rather he used approximations from the interior of a simply connected region and took limits of the piecewise linear case of the Dirichlet problem which had been solved by Schwarz years earlier. Although this is considered the first correct proof, Osgood did not receive recognition for his achievement with many mathematicians such as Koebe receiving more recognition for giving a proof years later.

As a consequence of many years of study many other successful proofs have been presented over the years using various methods. Such being the proof by Hilbert using the Calculus of Variations and the proof by F. Riesz and L. Fejer using Montel's theory of normal families of functions. The goal of this paper is to provide the canonical proof (both for its simplicity and precision) discovered by F. Riesz and L. Fejer in 1923 with all relevant details.

2. Preliminaries

Before beginning to prove anything a few important results and definitions must be stated. Reader's familiar with automorphisms of the unit disk, normal families of functions, and elementary definitions/results in complex analysis may skip ahead to section 3.

An important distinction that should be understood at the outset of this paper is that between biholomorphic and conformal maps. We take the definition of biholomorphic and conformal as follows,

Definition (Biholomorphic)**.** *Domains U and V are said to be biholomorphically equivalent (or biholomorphic) if there exists a bijective holomorphic function $f : U \to V$ whose inverse is also holomorphic. Such f is called biholomorphic.*

Definition (Conformal)**.** *If $U \subset \mathbb{C}$ is open then $f : U \to \mathbb{C}$ is conformal if f is holomorphic and $f'(z) \neq 0$ for $z \in U$.*

The difference between these definitions is emphasized here as the common statement of the Riemann Mapping theorem utilizes the word conformal with the meaning of the above definition of biholomorphic. Note that without this distinction there are functions that may, depending on the readers background, be called conformal that for the purposes of this paper are not.

Example. *e^z is conformal by the above definition but is not biholomorphic*

This is easily seen by noting that the exponential function maps the complex plane to the punctured complex plane and is its own derivative.

We now proceed with other useful definitions beginning with those pertaining to families of functions.

Definition (Equicontinuous)**.** *Let E be an open and connected subset of the complex plane and $\mathcal{F}$ be a family of complex valued functions on E. We say $\mathcal{F}$ is equicontinuous at $z_0 \in E$ if for $\epsilon > 0$ there exists $\delta > 0$ such that for all $f \in \mathcal{F}$*

$$|z - z_0| < \delta \quad \Rightarrow \quad |f(z) - f(z_0)| < \epsilon$$

This coincides with the standard definition of continuity for each of the f at z_0 except the δ must be valid for all f in the family $\mathcal{F}$. In the obvious way we then also define uniform equicontinuous,

Definition (Uniform Equicontinuous)**.** *A family $\mathcal{F}$ is said to be uniformly equicontinuous on E if for every $\epsilon > 0$ there exists a $\delta > 0$ so that for all $z, w \in E$ and $f \in \mathcal{F}$*

$$|f(z) - f(w)| < \epsilon, \text{ whenever } |z - w| < \delta$$

Definition (Uniformly Bounded)**.** *A family $\mathcal{F}$ is uniformly bounded on E if there is a constant $M > 0$ such that $|f| \leq M$ for all $f \in \mathcal{F}$.*

From here we now present a quick discussion of simply connected domains. While the notion of simply connectedness plays a large role in complex analysis it actually is a topological property of, in our case, the complex plane. Intuitively a simply connected domain can be defined as "a region in the plane which contains no holes." While this general intuition cannot be taken as a mathematical definition, fortunately it does not take very much to make it rigorous.

Definition (Homotopy)**.** *A homotopy between two functions f and g from a space X to a space Y is a continuous map G from $X \times [0,1] \to Y$ such that $G(x,0) = f(x)$ and $G(x,1) = g(y)$. Two mathematical objects are said to be homotopic if there exists a homotopy between them.*

This notion of two functions being homotopic is of great importance examining closed curves in the complex plane. Let $\gamma(t)$ be any closed curve in the complex plane with $a \leq t \leq b$ and choose $z_0 \in \mathbb{C}$. Then the map $G(s,t) = sz_0 + (1-s)\gamma(t)$, $0 \leq s \leq 1$ is a homotopy between the closed curve γ and the point z_0. Since the complex plane is convex we are ensured that for $0 \leq s \leq 1$ that $G(s,t)$ is some closed curve in the complex plane. Thus this shows that every closed curve in $\mathbb{C}$ is homotopic to a single point[2].

This shows that every closed curve in $\mathbb{C}$ is deformable to a point and intuitively implies that the complex plane has no "holes". While we have not yet proven the fact that closed curves in a domain D containing a hole (such as an annulus) are not homotopic to a point we take the case of the complex plane as inspiration for the following definition.

Definition (Simply Connected)**.** *We say that a domain D is simply connected if every closed curve contained in D is homotopic to a point in D.*

Thus by the above reasoning the complex plane (and all star shaped domains) are simply connected. However it still remains to show that domains such as the punctured plane and annuli are not simply connected. To do so we appeal to the following theorem.

Theorem. *A domain D in the complex plane is simply connected if and only if $\mathbb{C}^* \setminus D$ is connected.*

Proof. Let D be a domain in $\mathbb{C}$ and observe that by definition D is open and connected. We first assume that D is simply connected. If $D = \mathbb{C}$ then we are done. If $D \neq \mathbb{C}^*$ then we observe that since D is connected, we must have that $\mathbb{C}^* \setminus D$ is connected. If $\mathbb{C}^* \setminus D$ was not connected then we would be able to construct disjoint open and connected sets A and B such that $\mathbb{C}^* \setminus D = A \cup B$ which implies $\mathbb{C}^* = D \cup A \cup B$. But this implies that the extended complex plane is not connected which is a contradiction.

Thus we have established the first implication and we now proceed with the second by

[2]Moreover this shows that any closed curve in a star shaped domain is homotopic to a point

proving the contrapositive. Assume that D is not simply connected (thus $D \neq \mathbb{C}$). Since D is not simply connected it follows by definition of simply connected that there exists a closed path γ in D that is not homotopic to a point. So there must exist at least one point s in the region enclosed by γ which is not contained in D. So it then follows that s and ∞ lie in different connected components of $\mathbb{C}^* \setminus D$ implying that $\mathbb{C}^* \setminus D$ is not connected. $\square$

This theorem shows that our original definition of simply connected is consistent with the intuition that we originally established. For example if we take a domain $D = \{z : a < |z| < b\}$, an annulus, we observe that $\mathbb{C}^* \setminus D$ is disconnected with one connected component being the disk $\{z : |z| \leq a\}$ and the other being the region $\{z : b \leq |z|\}$. Thus the domain D is not a simply connected domain. In a similar fashion any domain with a "hole" can be seen to not be simply connected.

Concluding the preliminaries we present a few basic definitions and results from elementary complex analysis.

Definition (Univalent)**.** *A function f is said to be univalent on D if it is analytic and injective on D.*

Proposition 1. *If f is a univalent then any function composed of scaling, translating, and/or rotating f is also univalent*

Proof. The proof follows as a consequence of the fact that translation, rotation, and dilation are all injective operations and therefore preserve injectivity. The fact that the resultant function is still analytic is obvious. $\square$

It is then useful to have the closed form representation for self maps on the unit disk.

Proposition 2. *The biholomorphic self-maps of the open unit disk $\mathbb{D}$ are precisely the fractional linear transformations of the form*

$$f(z) = e^{i\phi}\frac{z-a}{1-\bar{a}z}, \qquad |z| = 1$$

where a is complex and $|a| < 1$ and $0 \leq \phi \leq 2\pi$.

The proof of this theorem is standard and can be found in any good introduction to complex analysis.

3. Outline of Riemann Mapping Theorem

It becomes useful now to both state and give an outline of the proof for the Riemann Mapping theorem as that the reader may better anticipate how each of the components below will operate in the proof as a whole. It may be useful for the reader to return to this section as they progress through the paper to fill in details and terms which may be ambiguous after the first reading.

Theorem (Riemann Mapping Theorem)**.** *If D is a simply connected domain in $\mathbb{C}$ and $D \neq \mathbb{C}$ then there exists a biholomorphic map of D onto the unit disk $\mathbb{D}$.*

Sketching the proof, we consider the family $\mathcal{F}$ of univalent functions on D such that $|f(z)| \leq 1$ on D and all elements of $\mathcal{F}$ take a fixed element $z_0 \in D$ to 0. First, we must show that $\mathcal{F}$ is nonempty. Following this we then consider the extremal problem of maximizing $|f'(z_0)|$ over $f \in \mathcal{F}$. It is in this portion of the proof which Montel's theorem will become essential to proving not only proving that the value above is finite but also attained by some function g in $\mathcal{F}$. Finally the proof will be completed in showing that g is biholomorphic, which as a consequence of the inverse function theorem reduces to showing that g is surjective. This will be accomplished as a consequence of Hurwitz's theorem and proof by contradiction.

4. Hurwitz's Theorem

What we will refer to as Hurwitz's theorem in this paper regards the behavior of zeros of a sequence of convergent analytic functions. In this section we seek to show that a sequence of univalent functions converging normally on some domain converges to either a constant function or a univalent function.

Theorem (Hurwitz's Theorem)**.** *Suppose $\{f_k\}$ is a sequence of analytic functions on a domain D that converges normally on D to $f(z)$ and suppose f has a zero of order N at $z_0 \in D$. Then there exists $\rho > 0$ such that for large k, $f_k(z)$ has exactly N zeros in the disk $\{|z - z_0| < \rho\}$ counting multiplicities. Moreover these zeros converge to z_0 as k tends towards infinity.*

The proof of this statement is commonly given as a consequence of the argument principle and is presented in detail in both [1, 2]. However a proof utilizing Rouche's theorem is presented here both for completeness and for the reader who is unfamiliar with the advanced calculus used in both of the above proof.

Proof. It follows since f has a zero of order N at z_0 that for $\rho > 0$ small we can define a disk $S = \{|z - z_0| < \rho\} \subset D$ such that f can be represented as $f(z) = (z - z_0)^N g(z)$ where g is analytic on S and $g(z_0) \neq 0$. Since g is continuous on S it then follows that we can find some $0 < \rho' < \rho$ such that on the circle $\{|z - z_0| = \rho'\}$ we get $|g(z_0)| \leq 2\,|g(z)|$. It then follows that

$$\frac{|g(z_0)|\,(\rho')^N}{2} \leq |g(z)|\,(\rho')^N = |z - z_0|^N\,|g(z)| = |f(z)|$$

Let $0 < c = \frac{|g(z_0)|(\rho')^N}{2}$. Now choosing $0 < \rho'' < \rho'$ it follows that on the disk $\{|z - z_0| < \rho''\}$ we get by the uniform normal convergence of the f_n's that there exists an integer n_0 such that for $n \geq n_0$ we get $|f_n(z) - f(z)|_\infty < c$. Therefore given $0 < \epsilon < \rho''$ it follows that for values z on the circle $\{|z - z_0| = \rho'' - \epsilon\}$ we get the inequality

$$|f_n(z) - f(z)| < c \leq |f(z)|$$

So by Rouche's theorem for $n \geq n_0$, f_n and f have the same number of zeros in the disk $\{|z - z_0| < \rho - \epsilon\}$. The statement that the zeros of the f_n's converge to the zeros of f is seen by letting $\rho'' \to 0$ and observing that the result forces the zeros of f_n to converge to z_0. □

This result is very useful in proving the next necessary result ([2], pg 232).

Theorem. *Suppose $\{f_n\}$ is a sequence of univalent functions on a domain D that converges normally on D to a function f. Then either f is univalent or f is constant.*

Proof. Suppose that the function f is not constant and suppose z_0 and ζ_0 satisfy $f(z_0) = f(\zeta_0) = w_0$. Then it follows that z_0 and ζ_0 are zeros of finite order of $f(z) - w_0$. Now by Hurwitz's theorem we have have the existence of sequences z_k and ζ_k converging to z_0 and ζ_0 respectively such that $f_k(z_k) - w_0 = 0$ and $f_k(\zeta_k) - w_0 = 0$. Since the f_k's are univalent it follows then that either $\zeta_k = z_k$ for all k implying in the limit that $z_0 = \zeta_0$ or we have a contradiction that the f_k's are univalent and thus our assumption that f is not constant must have been false. □

5. Montel's Theorem

The second important component for proving the Riemann Mapping theorem is Montel's theorem. For our purposes it suffices to prove the version of the theorem presented in Montel's thesis

Theorem (Montel's Theorem)**.** *Suppose $\mathcal{F}$ is a family of analytic functions on a domain D such that $\mathcal{F}$ is uniformly bounded on each compact subset of D. Then every sequence in $\mathcal{F}$ has a subsequence that converges normally on D*

We begin our discussion of Montel's theorem by first giving a necessary result in its proof

Theorem (Arzela-Ascoli)**.** *If E is a compact subset of $\mathbb{C}$ and $\mathcal{F}$ is a family of equicontinuous complex valued functions on E that is uniformly bounded then the following are equivalent*

a. The family $\mathcal{F}$ is equicontinuous at each point in E.
b. Each sequence of functions in $\mathcal{F}$ has a subsequence that converges uniformly in E.

This theorem is beyond the scope of this paper and the reader is referred to a good book on analysis for its proof.Note that while this theorem has been stated in the context of complex analysis for the purposes of this paper the result is still valid considering functions in $\mathbb{R}^n$ and even more generally it is true in metric spaces [5].

We now have all the necessary tools to present a proof of Montel's theorem

Proof of Montel's Theorem. Let $r > 0$ be such that for $z_0 \in D$, $K = \{|z - z_0| \leq R\} \subset D$. Since $\mathcal{F}$ is uniformly bounded on compact subsets of D it follows that there exists some constant $M > 0$ which is a uniform bound of $\mathcal{F}$ on K. It then follows by the Cauchy estimates that

$$|f'(z)| \leq \frac{1}{r-\epsilon}M, \quad f \in \mathcal{F}, z \in \{|z - z_0| < r - \epsilon\}, \epsilon > 0$$

We then define $C = \frac{1}{r-\epsilon}M + \epsilon'$ where $\epsilon' > 0$. Then given $z, w \in \{|z - z_0| < r - \epsilon\}$ it follows that

$$\left|\frac{f(z) - f(w)}{z - w}\right| \leq C, \quad \text{when } |z - w| < \delta$$

when $\delta > 0$ is small. This then implies that $|f(z) - f(w)| \leq C\,|z - w|$ where C is a uniform Lipschitz constant for all $f \in \mathcal{F}$. So it follows that given $\epsilon > 0$ for an appropriate choice of δ that $\mathcal{F}$ is uniformly equicontinuous on K.

We now consider sets $E_n = \{z \in D : |z| \leq n, \operatorname{dist}(z, \partial D) \geq 1/n\}$. The E_n's are obviously compact sets that as $n \to \infty$, $E_n \to D$. Thus it follows that any compact subset of D is contained in some E_n. Moreover it is not hard to see that $E_n \subset E_{n+1}$. Now let $\{f_n\}$ be a sequence in $\mathcal{F}$. By the Arzela-Ascoli theorem there is a subsequence $f_{1,1}, f_{1,2}, f_{1,3}, \ldots$ that converges uniformly on E_1. This further has a subsequence $f_{2,1}, f_{2,2}, f_{2,3}, \ldots$ which converges uniformly on E_2. Continuing inductively we get a sequence of sequences

$$\begin{array}{cccc} f_{1,1}, & f_{1,2}, & f_{1,3} & \cdots \\ f_{2,1}, & f_{2,2}, & f_{2,3} & \cdots \\ f_{3,1}, & f_{3,2}, & f_{3,3} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{array}$$

Each of the rows of the above matrix converges uniformly on some E_n. Thus the diagonal sequence converges uniformly on all E_n and therefore on each compact subset of D completing the proof. □

The usefulness of this theorem comes in showing the existence of extremal functions in extremal problems. The basic principle can be illustrated in the problem of maximizing the derivative at a point z_0 over a family $\mathcal{F}$ of functions. This becomes useful in the proof of the Riemann Mapping theorem in that it allows us to show the existence of the function g that will turn out to be the biholomorphic map between the original simply connected domain and the unit disk.

6. The Riemann Mapping Theorem

We now have all the necessary tools to give a proof of the Riemann Mapping theorem using the method of F. Riesz and L. Fejer. The proof given here will be an adaptation of the proof given in Ahlfors *Complex Analysis*

Theorem (Riemann Mapping Theorem)**.** *If D is a simply connected domain in $\mathbb{C}$ and $D \neq \mathbb{C}$ then there exists a biholomorphic map of D into the unit disk $\mathbb{D}$.*

Proof. Fix $z_0 \in D$ and let $\mathcal{F}$ be the family of univalent functions on D such that $|f(z)| \leq 1$ on D and $f(z_0) = 0$ for all $f \in \mathcal{F}$. We proceed with the proof here in three parts,

(1) $\mathcal{F}$ is nonempty
(2) There exists $g \in \mathcal{F}$ with maximal finite derivative
(3) The function g is a biholomorphic map from D to $\mathbb{D}$

Beginning our proof of (1) we note that by assumption there exists $a \notin D$. Then since D is simply connected it follows that we can define a single valued analytic branch of $\sqrt{z-a}$ which we shall denote as $h(z)$. It then follows that h is injective and does not take values in both $h(D)$ and $-h(D)$ (for proof see [2] page 311). Without loss of generality it then follows that the image of D under h covers a disk $\{|h(z) - h(z_0)| < \rho\}$ and does not take values in the disk $\{|h(z) + h(z_0)| < \rho\}$. Thus $|h(z) + h(z_0)| \geq \rho$ for all $z \in D$ and consequently $2\,|h(z_0)| \geq \rho$.

From this point it can then be verified that the function

$$f_0(z) = \frac{\rho}{4} \frac{|h'(z_0)|}{|h(z_0)|^2} \frac{h(z_0)}{h'(z_0)} \frac{h(z) - h(z_0)}{h(z) + h(z_0)}$$

belongs to $\mathcal{F}$. Since h is univalent the fact that f_0 is injective is self evident. To see that f_0 is analytic on D observe that the rightmost fraction is the quotient of two analytic functions where the denominator is nonzero on D (as a consequence of $h(D)$ and $-h(D)$ being disjoint). Thus f_0 is univalent. Moreover the estimate

$$\left|\frac{h(z) - h(z_0)}{h(z) + h(z_0)}\right| = |h(z_0)| \left|\frac{1}{h(z_0)} - \frac{2}{h(z) + h(z_0)}\right| \leq \frac{4\,|h(z_0)|}{\rho}$$

implies that

$$|f_0(z)| \leq \frac{\rho}{4}\frac{|h'(z_0)|}{|h(z_0)|^2}\frac{h(z_0)}{h'(z_0)}\frac{4\,|h(z_0)|}{\rho} = 1$$

proving that $|f_0(z)| \leq 1$ and therefore $f_0 \in \mathcal{F}$.

Proceeding with the proof of (2) we let $A = \sup\{|f'(z_0)| : f \in \mathcal{F}\} > 0$. Note that we have made no assumptions on A and that A could be infinite. We then take a sequence $\{f_n\}$ of functions in $\mathcal{F}$ such that $f_n'(z_0)$ converges to A. Then by Montel's theorem there exists a subsequence of f_n that converges normally on D to an analytic function g. It is clear from the properties originally enforced on the f_n's that $|g(z)| \leq 1$ on D and that $g(z_0) = 0$ and $|g'(z_0)| = A$. This then proves that A is finite completing the proof of (2).

Finally we come to part (3). To prove this we quote our consequence of Hurwitz's theorem that since the f_n's are univalent that g must either be univalent or constant. Since $|g(z_0)| = A > 0$ it follows that g must have nonzero derivative on a neighborhood of z_0 and so g cannot be constant.

It now remains to show that g attains all values in the unit disk. In order to do this we assume for contradiction that $g(z) \neq w_0$ for some w_0 such that $|w_0| < 1$. We now start construction of a function that will give us our contradiction. We begin with the function $\frac{g(z)-w_0}{1-\overline{w_0}g(z)}$. Noting that this function is a biholomorphic self-map of the unit disk and the numerator is never zero it follows that the image of this function is the unit disk minus the origin. Observing that the numerator is always nonzero it follows that we can define an analytic branch of the square root

$$F(z) = \sqrt{\frac{g(z) - w_0}{1 - \overline{w_0}g(z)}}$$

It then follows that F takes the punctured unit disk to the punctured unit disk. We then define the function

$$G(z) = \frac{F(z) - F(z_0)}{1 - \overline{F(z_0)}F(z)}$$

It then follows that this function takes the punctured unit disk to the punctured unit disk taking z_0 to 0. Thus G and g are both maps whose image takes z_0 to 0 and are both elements of $\mathcal{F}$. However it follows by direct computation that

$$|G'(z_0)| = \frac{|g'(z_0)|}{2\sqrt{|w_0|}}(1 - |w_0|^2)\frac{1 - |w_0|}{(1 - |w_0|)^2} = \frac{A}{2\sqrt{|w_0|}}(1 + |w_0|)$$

It then follows that since $|w_0| < 1$ that $|G'(z_0)| > A$ which is a contradiction that g is the function in $\mathcal{F}$ with maximum derivative at z_0. Thus our assumption that g misses a value in the unit disk must be false and we conclude that g must be onto the unit disk completing the proof. □

7. Riemann Surfaces and The Uniformization Theorem

Beyond the scope of studying simply connected regions in the plane, the issue often arises that we may want to study maps between more complicated surfaces. In complex analysis, a Riemann surfaces are one dimensional complex manifolds which can have holomorphic functions defined between them. First introduced by Riemann in his thesis, these surfaces serve as an visual alternative to multivalued functions and play a crucial role in multivariable complex analysis. This section seeks to give the basic definitions necessary to construct a Riemann surface as well as state the abstraction of the Riemann mapping theorem to such surfaces: the uniformization theorem.

These surfaces can be thought to be deformed copies of the complex plane which locally are homeomorphic to the plane while generally their topology could be quite different. We begin our brief examination of Riemann surfaces with some general definitions assuming that the reader is unfamiliar with general topology.

Definition (Topological Space)**.** *A set S of points p becomes a topological space if a family $\mathcal{F}$ of subsets O, to be called open sets, is distinguished satisfying*

(1) the union of any collection of open sets is open
(2) the intersection of finitely many open sets is open

Furthermore a topological space is called $Hausdorff$ if it satisfies for p_1 and p_2 in S where $p_1 \neq p_2$ there exists open sets O_1 and O_2 so that $p_1 \in O_1$ and $p_2 \in O2$ and $O_1 \cap O_2 = \emptyset$.

Remark. *The spaces $\mathbb{R}^n$ and $\mathbb{C}^n$ are Hausdorff spaces*

We now need some fundamental definitions regarding topological manifolds

Definition (Topological Manifold)**.** *An n-dimensional topological manifold is a Hausdorff space M such that every point $p \in M$ has an open neighborhood homeomorphic to an open cell in $\mathbb{R}^n$.*

From here we can give the definition of a Riemann Surface

Definition (Riemann Surface)**.** *A Riemann Surface, R, is a 2-dimensional topological manifold with open sets U_α and complex valued functions $z_\alpha(p)$, $p \in U_\alpha$, such that*

(1) each z_α is injective from U_α onto a domain $z_\alpha(U_\alpha)$ in the complex plane
(2) each composition $z_\beta \circ z_\alpha^{-1}$ is analytic wherever it is defined, that is, from $z_\alpha(U_\alpha \cap U_\beta)$ to $z_\beta(U_\alpha \cap U_\beta)$
(3) R is connected, that is, for any two points p and q in R there is a finite collection of indices $\alpha_1, \ldots, \alpha_m$ such that $p \in U_\alpha$, $q \in U_{\alpha_m}$ and $U_{\alpha_j} \cap U_{\alpha_{j+1}}$ is nonempty for $1 \leq j \leq m$

We refer to U_α as a coordinate patch and to z_α as a coordinate map on U_α.

While this may seem to be a difficult definition to take in at first glance it actually has a very intuitive meaning. If we consider a 2-dimensional topological manifold it follows by definition that it is constructed from open sets U_α. These open sets are each homeomorphic to an open cell in $\mathbb{R}^2$ and are each assigned a coordinate function mapping U_α to $\mathbb{C}$ (which can be interpreted here as $\mathbb{R}^2$). Requirement (1) simply states that we assign to each coordinate patch a bijective function z_α mapping U_α to some domain in $\mathbb{C}$. The existence of such functions is implied by the fact that every Riemann surface is actually a 2-dimensional topological manifold.

Condition (2) states that through composition of coordinate maps we can move from one coordinate patch to the other. This becomes important in defining holomorphic functions on the Riemann surface since the coordinates between patches may vary greatly.

Example. *The simplest nontrivial Riemann surface is the extended complex plane* $\mathbb{C}^* = \mathbb{C} \cup \{\infty\}$*. This can be coordinatized by two coordinate patches* $U_0 = \mathbb{C}$ *and* $U_1 = \mathbb{C}^* \setminus \{0\}$*, with coordinate maps* $z_0(z) = z$ *and* $z_1(z) = 1/z$*. It then follows that* $z_0(U_0 \cap U_1) = \mathbb{C} \setminus \{0\}$ *and the change of coordinates map between coordinate patches is given by the analytic function*

$$(z_1 \circ z_0)(z) = 1/z, \qquad z \in \mathbb{C} \setminus \{0\}$$

Just as the Riemann Mapping theorem was able to generalize the study of arbitrary simply connected subsets of the complex plane to the study of the unit disk there is an analog for Riemann Surfaces.

Theorem (Uniformization Theorem)**.** *Each simply connected Riemann surface is biholomorphically equivalent to either the open unit disk* $\mathbb{D}$*, the complex plane* $\mathbb{C}$*, or the Riemann sphere* $\mathbb{C}^*$

It is important to note here that the proof given above showing that a domain $D \subset \mathbb{C}$ is simply connected if and only if $\mathbb{C}^* \setminus D$ is connected does not apply here. This proof functions only in the case of the plane and in order to understand what it means to have a simply connected Riemann surface we must go back to the definition. The proof of this theorem is beyond the scope of this paper but the reader is encouraged to think about the implications of this theorem. Similar to the effect of the Riemann Mapping theorem on the complex plane, this theorem reduces all complicated Riemann surfaces into three already well understood domains in which it is often easier to prove results.

Although proof of this theorem will not be given here we will end with showing that the conclusion of the Uniformization theorem is valid by proving the following theorem.

Theorem. *No two of the complex plane* $\mathbb{C}$*, unit disk* $\mathbb{D}$*, or extended complex plane* $\mathbb{C}^*$ *are biholomorphically equivalent.*

Proof. We assume first that such a map $\phi : \mathbb{C} \to \mathbb{D}$ exists. It then follows by Liouville's theorem that ϕ is constant and therefore is not biholomorphic. Note also that the proof showing no biholomorphic function exists between $\mathbb{C}^*$ and $\mathbb{D}$ is identical.

So it remains to show that there does not exist a biholomorphic map between $\mathbb{C}^*$ and $\mathbb{C}$. To show this we first prove that $\mathbb{C}^*$ is compact. Let z_n be a sequence in the extended complex plane. If there exists infinitely many indices n such that $z_n = \infty$ then there is an obvious choose of subsequence $z_{n_k} \to \infty$. If there exists only finitely many indices such that $z_n = \infty$ but it removing those indices z_n remains unbounded then we can choose a subsequence by removing the terms equal to infinity that converges to infinity. If there does not exist any index n such that $z_n = \infty$ it follows that z_n is bounded and therefore converges in some compact subset of $\mathbb{C}$. Thus by Bolzano-Weierstrass we get that $\mathbb{C}^*$ is compact.

Now it follows that topologically there does not exist a homeomorphism between $\mathbb{C}^*$ and $\mathbb{C}$ since continuous functions map compact sets to compact sets. Since $\mathbb{C}$ is not compact it follows that there can exist no homeomorphism between the two spaces and therefore no biholomorphic function between them can exist. ∎

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