The Roots of Constructivism and Intuitionism in Mathematics

Part II

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Outline

Having come thus far in his speech, Ippolit Kirillovich, who had evidently chosen a strictly historical method of accounting, a favorite resort of all nervous orators who purposely seek a strict framework in order to restrain their own impatient zeal – Ippolit Kirillovich expanded particularly on the "former" and "indisputable" one, and on this topic expressed several rather amusing thoughts.

– The Brothers Karamazov, Book XII, Chapter IX

Logicism

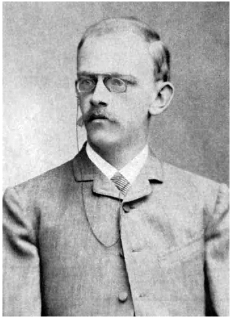
The Classical Period

- **1879 – 1931**: The "classical" period (aka the *Logicist* period) in the foundations of mathematics.
- Extremely fruitful era in philosophy and foundations of mathematics and logic. Compared to our time, the striking fact about this era is the dominant participation of high-profile mathematicians such as Frege, Peano, Dedekind, Klein, Dedekind, Hilbert, Poincare, Weyl, Einstein, whitehead, Brouwer, and Gödel as the deriving force in conceiving the philosophies of mathematics of this period.
- Starts with the publication of Frege's `Begriffsschrift` (concept-script).
- Ends with Gödel's `Über formal unentscheidbare Sätze der Principia Mathematica und verwandter Systeme`.

The Classical Period


(a) Frege


(b) Peano


(c) Hilbert


(d) Gödel

September 1930, Königsberg

- Conference on Epistemology of the Exact Sciences in Königsberg.
- Carnap, von Neumann, and Heyting.
- Gödel announces his First Incompleteness Theorem.
- “It is all over!" says von Neumann.

Frege

- Numerical statements are objectively true or false.
- Moreover, these statements are literally about abstract mathematical objects that do not exist in space or time. Numbers are not properties of objects or their collections; they have independent existence.
- Knowledge about numbers is possible only if it is conceptual and apriori, rather than based on experience or intuition.
- Leibniz: logic as a calculus of thoughts. Frege took this idea to a new level.

Frege's Impossible Dream

- Q: How, then are numbers given to us, if we cannot have any ideas or intuitions of them?
- A: Frege tries to show that arithmetic can be reduced to logic, and the axioms of logic are self-evident (they are law of thought!).
- Frege's impossible dream: showing that all arithmetic truths are formally provable in logic.
- Vienna circle impossible dream: showing that all empirical sciences reduce to logic.

Brouwer's Intuitionism

The origins of more than a century of debate over intuitionism

- In 1907 Luitzen Egbertus Jan Brouwer defended his doctoral dissertation on the foundations of mathematics (Brouwer. Over de Grondslagen der Wiskunde. PhD thesis, University of Amsterdam, 1907).
- With this event mathematical intuitionism as a foundation of mathematics came into being.
- Brouwer attacked the main currents of the philosophy of mathematics of his time: the formalists and the Platonists.
- In turn, both these schools began viewing intuitionism as the most harmful party among all known philosophies of mathematics.

Precursors: Kronecker & Poincaré

In the late 19th century, Kronecker and Poincaré had expressed doubts about, or even disapproval of, the idealistic, nonconstructive methods in mathematics used by some of their contemporaries.

Precursors: Kronecker & Poincaré

> *The definition of irreducibility drawn up in section 1 lacks a secure grounding as long as no method has been indicated by which it can be decided whether a definite given function is irreducible according to that definition or not. (Kronecker, Grundzüge einer arithmetischen Theorie der algebraischen Größen, 1882)*

His student Jules Molk reiterated these doubts in his dissertation (Berlin, 1885):

> *The definitions should be algebraic and not only logical. It does not suffice to say: 'A thing exists or it does not exist'. One has to show what being and not being mean, in the particular domain in which we are moving. Only thus do we make a step forward.*

Brouwer's program (I)

Brouwer's intuitionism holds that

- mathematics is a *free creation* of the human mind (contra Platonism).
- mathematics is prior to logic in reasoning (contra logicism/formalism):

> *Logic is the study of patterns in linguistic records of mathematical acts of construction, and, as such, a form of applied mathematics. Mathematical constructions out of the intuition of time are themselves not of a linguistic nature. Language cannot play a creative role in mathematics (Brouwer. Over de Grondslagen der Wiskunde. PhD thesis, Universiteit van Amsterdam, 1907).*

> *To secure the reliability of mathematical reasoning one cannot succeed solely by starting from some sharply formulated axioms and further strictly adhering to the laws of theoretical logic.*

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- These constructions are based on the *intuition of time*.

A Well-Known Illustration Of The Problematic

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Either $\sqrt{2}^{\sqrt{2}}$ is rational, in which case we take $x = y = \sqrt{2}$; or else $\sqrt{2}^{\sqrt{2}}$ is irrational, in which case we take $x = \sqrt{2}^{\sqrt{2}}$ and $y = \sqrt{2}$. $\square$

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⚠ However, this proof is unsatisfactory!

It does not enable us to *decide* which of the two choices of the pair (x, y) satisfies the required property. This proof does not enable us to construct x as a real number in our mind, that is to say we cannot compute x with any desired degree of accuracy, as required by our description of what it means to be given a real number.

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In Hermann Weyl's words:

> *[the nonconstructive existence proof] informs the world that a treasure exists without disclosing its location.*

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- To prove $\exists x : A.\, P(x)$ we must construct an object $a \in A$ and prove that $P(a)$ holds.
- A proof of $\forall x \in A.\, P(x)$ is an algorithm that, applied to any object x and to the data proving that $x \in A$, proves that $P(x)$ holds.

Excluded Middle

- Problematic excluded middle

$$\forall P.\, P \vee \neg P$$

- Equivalent form: double negation

$$\forall P.\, (\neg\neg P \Rightarrow P)$$

The Fate of Intuitionism?

Continuum

- Aristotle
- Leibniz, Newton
- Cauchy, Weierstrass, Dedekind
- Brouwer, Weyl

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- Witten: I just wonder if you can offer any thoughts about how Analysis, our concepts of real numbers and so on might be different if the usual foundations are inconsistent?

- Voevodsky: so first of all for that we don't need them to be inconsistent ... it is entirely possible that our understanding of real numbers is not an adequate formalization of the notion of continuity and that possibility is quite real even without any inconsistency ... I think as many different observations kind of show, the notion of real number seems to be over-idealized in a sense ... so it's an over-idealized object and it's clear that this over idealization was necessary in order to make reasoning about real numbers simple enough for it to be humanly practical. And it's quite possible that in years to come, because of the development of the computer assisted thinking, maybe we'll be able to explore other possibilities in which the notion of continuity will be formalized in an less idealized way and and so will avoid some of the physical paradoxes, so to speak, one encounters when one tries to use the usual notion of continuity to describe the physical reality.

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- Voevodsky continues: And together with the notion of locality when one gets into this stupid situation where in order for something at all to happen to real numbers have to be precisely equal at some point and which of course can never happen.

- Witten: Even if the usual foundations are inconsistent or are consistent there might be a better theory of real numbers is that what you're saying?

- Voevodsky: There might be better foundations first of all, and second even in a given foundations there might be a better way of formalizing the concept of continuity than the current one but I would rather go for new foundations and new formalization of continuity in the new foundations.

Axiom of Choice

$$(\forall i : I)(\exists x : S) : P(i, x) \Rightarrow (\exists f : I \to S)(\forall i : I) : P(i, f(i))$$

Accepting the axiom of choice and the standard view of the continuum leads to the the famous Banach-Tarski Paradox.

Figure: Banach-Tarski Paradox

Brouwer vs Wittgenstein on the infinite

- Wittgenstein and Brouwer had identified a similar problem in the foundation of mathematics of their time: that mathematics of their day rested upon a projection into infinite domains of methods that were only legitimate in the finite domain.
- However, they have huge differences on excluded middle: Wittgenstein treats excluded middle as a tautology like classical logicians.

Constructivism Meets Structuralism

- Category theory as a new foundation of mathematics manifests the structuralist view of mathematics.
- Category theory has a very close relationship with type theory.

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- The connection between this theory and model theory of intuitionistic logical systems became gradually clearer with the introduction of the elementary topos by Lawvere and Tierney 1969–1970 and Joyal's generalization of Kripke and Beth semantics (possible worlds semantics).
- Category theory, the basis of topos theory, reduces logic— just as type theory does—to very simple and basic mathematical constructions.

Constructivism & Type Theory

Proof assistants are based on the constructive nature of type theory.

Types	Logic	Sets	Homotopy
A	proposition	set	space
$a : A$	proof	element	point
$B(x)$	predicate	family of sets	fibration
$b(x) : B(x)$	conditional proof	family of elements	section
$\mathbf{0}, \mathbf{1}$	$\bot, \top$	$\emptyset, \{\emptyset\}$	$\emptyset, *$
$A + B$	$A \vee B$	disjoint union	coproduct
$A \times B$	$A \wedge B$	set of pairs	product space
$A \to B$	$A \Rightarrow B$	set of functions	function space
$\sum_{(x:A)} B(x)$	$\exists_{x:A} B(x)$	disjoint sum	total space
$\prod_{(x:A)} B(x)$	$\forall_{x:A} B(x)$	product	space of sections
Id_A	equality $=$	$\{ (x, x) \mid x \in A \}$	path space A^I

Table 1: Comparing points of view on type-theoretic operations

Intuitionism & Modern Physics

It has been recently claimed in various places that an intuitionistic based physics can help solving the long-standing problem of unifying general relativity and quantum mechanics.

- Does Time Really Flow? New Clues Come From a Century-Old Approach to Math.
- "Mathematical languages shape our understanding of time in physics" by Gisin
- Bohr Topos

Brouwer was a mystic and his writings are regarded to be notoriously obscure.

In religieuze waarheid, in wijsheid, die de splitsing opheft in subject en iets anders, is geen wiskundig intelligeeren, daar de verschijning van den tijd niet langer wordt aanvaard, nog minder dus betrouwbaarheid van logica. Integendeel, de taal der inkeerende wijsheid verschijnt ordeloos, onlogisch, omdat ze nooit kan voeren langs in het leven gedrukte systemen van gesteldheden, slechts hun breking kan begeleiden, en zoo misschien de wijsheid, die die breking doet, kan laten opengaan.
(De onbetrouwbaarheid der logische principes)

In religious truth, in wisdom, which suspends the splitting into subject and something separate, there is no mathematical intellect, as the appearance of time is no longer accepted, even less thus the reliability of logic. On the contrary, the language of inward-turning wisdom appears without order, illogical, because it can never carry along systems of posits pressed upon life, but can only accompany their breakdown, and thus perhaps unveil the wisdom that effects the break.
(The unreliability of the logical principles)

The End

Thanks for your attention!