The seesaw mechanism

This section deals with a model to explain the triflingly small neutrino mass (< 2 eV), which is much smaller than that of other fermions such as the next lightest one, the electron (0.5 MeV). In the standard model neutrinos and antineutrinos are different (Dirac) particles. The chirality of the neutrino is negative, that of the antineutrino positive (see *15.79*). The neutrinos and antineutrinos involved in the weak interaction with other particles are represented by the spinors

$$\psi_L = \left(\frac{1-\gamma^5}{2}\right)\psi \quad \text{and} \quad (\psi^c)_R = \left(\frac{1+\gamma^5}{2}\right)\psi^c, \tag{1}$$

respectively, where ψ and ψ^c are solution of the Dirac equation (chapter *15*). The spinors

$$\psi_R = \left(\frac{1+\gamma^5}{2}\right)\psi \quad \text{and} \quad (\psi^c)_L = \left(\frac{1-\gamma^5}{2}\right)\psi^c \tag{2}$$

correspond to sterile neutrinos and antineutrinos.

The following discussion deals with only one flavour of neutrinos, say ν_e, but is applicable to ν_μ and ν_τ as well. We shall denote the neutrino spinor by ψ and that of the charge conjugated antineutrino by ψ^c. We have shown in section *15.5* that

$$\psi^c = C\overline{\psi}^T = i\gamma^2\gamma^0\overline{\psi}^T. \tag{3}$$

Table 1 lists a few useful relations satisfied by the charge conjugation C, which are easily verified by using the properties of γ matrices derived in chapter *15*.

Table 1: Some properties of the charge conjugation.

$$C^\dagger = C^T = -C$$
$$C^2 = -1$$
$$CC^\dagger = 1$$
$$CC^T = 1$$
$$C\gamma^0 C = \gamma^0$$
$$\gamma^0 C \gamma^0 = -C$$

We have seen e.g. in section *14.4* that the neutrino is observed to be left-handed and the corresponding antineutrino right-handed. As discussed in section *15.3* the spinors (1) and (2) are eigenstates of the chirality operator γ^5, but chirality is equivalent to handedness in the limit of vanishing masses, hence the subscripts L and R in (1) and (2). However, a massive neutrino can be brought to rest and become right-handed by a suitable Lorentz boost, thus interacting nevertheless with other fermions. This is the mechanism at work in the hypothetical neutrinoless double β-decay in which neutrinos and antineutrinos are identical (Majorana) particles (section *14.5*).

Let us now interpret the spinor ψ as a field and write for the energy of a Dirac neutrino with mass M_D in its rest frame, following the Dirac equation,

$$H = \psi^\dagger \beta M_D \psi = \psi^\dagger \gamma^0 M_D \psi = M_D \overline{\psi}\psi = M_D \overline{\psi}_R \psi_L + M_D \overline{\psi}_L \psi_R, \tag{4}$$

because only terms with opposite chiralities contribute to H: indeed, ψ can be decomposed into L and R fields

$$\psi = \psi_L + \psi_R = \left(\frac{1 - \gamma^5}{2}\right)\psi + \left(\frac{1 + \gamma^5}{2}\right)\psi. \tag{5}$$

The products $\overline{\psi}_L\psi_L$ and $\overline{\psi}_R\psi_R$ vanish,

$$\overline{\psi}_L\psi_L = \overline{\psi}\frac{1}{4}\overbrace{(1 + \gamma^5)(1 - \gamma^5)}^{1 - (\gamma^5)^2 = 0}\psi = 0, \tag{6}$$

and likewise for $\overline{\psi}_R\psi_R$. Furthermore,

$$\overline{\psi}_L\psi_R = \psi^\dagger\gamma^0\left[\frac{1 + \gamma^5}{2}\right]\psi_R = \psi^\dagger\left[\frac{1 - \gamma^5}{2}\right]\gamma^0\psi_R = [\psi_R^\dagger\gamma^0\left[\frac{1 - \gamma^5}{2}\right]\psi]^\dagger = [\overline{\psi}_R\psi_L]^\dagger, \tag{7}$$

so that H becomes simply

$$H = M_D\overline{\psi}_R\psi_L + h.c., \tag{8}$$

where $h.c.$ stands for "hermitian conjugate".

Further terms can be added to the Hamiltonian when including charge conjugated L and R fields. The following fields

$$\psi_L, \ \psi_R \ (\psi^c)_L, \ (\psi^c)_R \ \text{ and } \ \overline{\psi}_L, \ \overline{\psi}_R, \ \overline{(\psi^c)}_L, \ \overline{(\psi^c)}_R \tag{9}$$

are available. The charge conjugated L and R fields satisfy the useful relations

$$\boxed{(\psi_L)^c = (\psi^c)_R} \ \text{ and } \ \boxed{(\psi_R)^c = (\psi^c)_L.} \tag{10}$$

The proof uses the properties of γ matrices listed in chapter *15*. One gets, e.g. for the L field,

$$\begin{aligned}
(\psi_L)^c &= \left[\frac{1 - \gamma^5}{2}\psi\right]^c = C\overline{\left[\frac{1 - \gamma^5}{2}\psi\right]}^T \\
&= C\left[\left(\frac{1 - \gamma^5}{2}\psi\right)^\dagger\gamma^0\right]^T = C\left[\psi^\dagger\left(\frac{1 - \gamma^5}{2}\right)\gamma^0\right]^T \\
&= C\left[\gamma^0\left(\frac{1 - \gamma^5}{2}\right)\psi^{\dagger T}\right] = C\left[\left(\frac{1 + \gamma^5}{2}\right)\gamma^0\psi^{\dagger T}\right] \\
&= C\left[\left(\frac{1 + \gamma^5}{2}\right)(\psi^\dagger\gamma^0)^T\right] = \left(\frac{1 + \gamma^5}{2}\right)C\overline{\psi}^T = \left(\frac{1 + \gamma^5}{2}\right)\psi^c = (\psi^c)_R,
\end{aligned} \tag{11}$$

and similarly for $(\psi_R)^c$. Keeping only the pairs of fields in (9) with opposite chiralities leaves the contributions to H

$$\overline{\psi}_R\psi_L, \ \overline{(\psi^c)}_R\psi_L, \ \overline{\psi}_R(\psi^c)_L, \ \overline{(\psi^c)}_R(\psi^c)_L \tag{12}$$

and their corresponding hermitian conjugates. However, the fourth term is not independent. By using the relations listed in table 1 one finds that

$$
\begin{aligned}
\overline{(\psi^c)}_L(\psi^c)_R &= \overline{(\psi_R)^c}(\psi_L)^c = \overline{(C\bar\psi_R^T)}C\bar\psi_L^T \\
&= (C\bar\psi_R^T)^\dagger\gamma^0 C\bar\psi_L^T = (C\gamma^0\psi_R^*)^\dagger\gamma^0 C\bar\psi_L^T \\
&= \psi_R^{*\dagger}\gamma^0 \overbrace{C^\dagger\gamma^0 C}^{-\gamma^0}\bar\psi_L^T = -\psi_R^T\bar\psi_L^T = \bar\psi_L\psi_R,
\end{aligned}
\tag{13}
$$

since fermion fields anticommute.

The most general expression for the Hamiltonian is therefore

$$
\begin{aligned}
H &= M_D\bar\psi_R\psi_L + \frac{1}{2}M_L\overline{(\psi^c)}_R\psi_L + \frac{1}{2}M_R\bar\psi_R(\psi^c)_L + h.c. \tag{14} \\
&= M_D\bar\psi_R\psi_L + \frac{1}{2}M_L\overline{(\psi_L)^c}\psi_L + \frac{1}{2}M_R\bar\psi_R(\psi_R)^c + h.c. \tag{15}
\end{aligned}
$$

where we have used the equalities (10). The first term conserves lepton number and endows the neutrino with a Dirac mass M_D. The second and last terms violate lepton number conservation by $\Delta L = \pm 2$ and contribute Majorana masses M_L and M_R, as illustrated in figure 1.
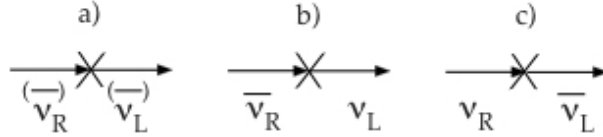

Figure 1: Contributions to the Hamiltonian from Dirac neutrino fields (a) and Majorana fields (b), (c). (The mass M_D is due to direct coupling ($\times$) to the Higgs field, in contrast to M_L and M_R, which require more complicated mass generation mechanisms, such as loops.)

So far, we have listed the contributions to the Hamiltonian with the couplings M_D, M_L and M_R, which are not the physical neutrino masses. Let us therefore identify the eigenstates of the Hamiltonian (the physically observed states) and the corresponding eigenvalues (the observable neutrino masses). Let us define the fields

$$
f \equiv \frac{\psi_L + (\psi_L)^c}{\sqrt{2}} = \frac{\psi_L + (\psi^c)_R}{\sqrt{2}} \quad \text{and} \quad F \equiv \frac{\psi_R + (\psi_R)^c}{\sqrt{2}} = \frac{\psi_R + (\psi^c)_L}{\sqrt{2}}. \tag{16}
$$

These fields are of the Majorana type because $f^c = f$ and $F^c = F$. This can be seen by conjugating ψ_L and ψ_R twice, e.g. for ψ_L:

$$
\begin{aligned}
(\psi_L^c)^c &= C\overline{\psi_L^c}^T = C\overline{C\bar\psi_L^T}^T = C[\overline{C\gamma^0\psi_L^*}]^T = C[(C\gamma^0\psi_L^*)^\dagger\gamma^0]^T \\
&= C[\psi_L^T\overbrace{\gamma^0 C^\dagger\gamma^0}^{C}]^T C = CC^T\psi_L = \psi_L
\end{aligned}
\tag{17}
$$

(see table 1). By keeping only terms with opposite chiralities we can rewrite the Hamiltonian (14) as

$$\begin{aligned} H &= M_L \bar{f}f + M_R \bar{F}F + M_D(\bar{f}F + \bar{F}f) \\ &= (\bar{f}, \bar{F}) \begin{pmatrix} M_L & M_D \\ M_D & M_R \end{pmatrix} \begin{pmatrix} f \\ F \end{pmatrix}. \end{aligned} \tag{18}$$

The eigenvalues m and eigenstates ϕ are found by diagonalizing the matrix, that is by and solving

$$\begin{pmatrix} M_L - m & M_D \\ M_D & M_R - m \end{pmatrix} \phi = 0. \tag{19}$$

The secular equation

$$(M_L - m)(M_R - m) - M_D^2 = 0 \tag{20}$$

leads to

$$m_{1,2} = \frac{M_R + M_L}{2} \mp \sqrt{\frac{(M_R - M_L)^2}{4} + M_D^2}. \tag{21}$$

Consider first the purely Dirac neutrino scenario with $M_R = M_L = 0$, hence $m_1 = -M_D$ and $m_2 = M_D$. The normalized eigenfunctions in the (f, F) basis are

$$\phi_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \equiv \nu = \frac{f + F}{\sqrt{2}}, \quad \text{and} \quad \phi_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \equiv N = \frac{F - f}{\sqrt{2}}. \tag{22}$$

The Hamiltonian now reads in terms of ν and N:

$$H = -M_D \bar{\nu}\nu + M_D \bar{N}N = -M_D \bar{\nu} \overbrace{(\gamma^5)^2}^{=1} \nu + M_D \bar{N}N = M_D \bar{\nu}'\nu' + M_D \bar{N}N, \tag{23}$$

where we have redefined the field ν as $\nu' = \gamma^5 \nu$ to obtain a positive mass. Our Majorana fields ν' and N are superpositions of ψ_L, $(\psi^c)_R$, $(\psi^c)_L$ and ψ_R with degenerate masses M_D. The first two fields occur in the weak interaction where they couple to the known 80 GeV $W^\pm$ boson (section *16.7*), while the last two ones correspond to sterile neutrinos.

In another hypothetical scenario, a gauge boson $W_R^\pm$ might exist, which couples to R fermions or L antifermions. So far W_R has not been observed, being too massive and therefore beyond reach of present accelerators. Let us assume that this boson decays into heavy neutrinos and set

$$M_L = 0, \quad M_R >> M_D. \tag{24}$$

Then from (21)

$$m_1 = \frac{M_R}{2} - \sqrt{\frac{M_R^2}{4} + M_D^2} \simeq \frac{M_R}{2} - \frac{M_R}{2}\left(1 + 2\frac{M_D^2}{M_R^2}\right) = -M_D^2/M_R. \tag{25}$$

The corresponding eigenstate in the (f, F) basis is found by solving the equation

$$\begin{pmatrix} -m_1 & M_D \\ M_D & M_R - m_1 \end{pmatrix} \begin{pmatrix} \sqrt{1 - b^2} \\ b \end{pmatrix} = 0, \tag{26}$$

4

hence

$$b \simeq \frac{M_D}{M_R},\tag{27}$$

and therefore

$$\nu \simeq f + \frac{M_D}{M_R}F \simeq f.\tag{28}$$

We get a **light Majorana** neutrino ν with mass M_D^2/M_R. (A positive mass is again obtained by the substitution $\nu \to \gamma^5\nu$ in the Hamiltonian.) The neutrino ν is essentially the superposition of a Dirac L neutrino and Dirac R antineutrino (see (16)) which couple to the 80 GeV $W^{\pm}$ boson. This is would be our familiar neutrino, a Majorana particle.

The second solution of (21) is

$$m_2 = \frac{M_R}{2} + \sqrt{\frac{M_R^2}{4} + M_D^2} \simeq \frac{M_R}{2} + \frac{M_R}{2}\left(1 + 2\frac{M_D^2}{M_R^2}\right) \simeq M_R.\tag{29}$$

The corresponding eigenstate is orthogonal to ν:

$$N = F - \frac{M_D}{M_R}f \simeq F,\tag{30}$$

a **heavy** Majorana neutrino with mass M_R. According to (16) N is essentially a superposition of a Dirac R neutrino and Dirac L antineutrino which couple to the very massive W_R boson.

Figure 2: The seesaw mechanism would explain the tiny mass of the left-handed neutrino by the existence of a heavy specie with opposite chirality.

From (28) and (30) one finds that

$$\boxed{m_1 m_2 = M_\nu M_R \simeq M_D^2}.\tag{31}$$

This is the essence of the seesaw mechanism originally proposed by [1]. The tiny mass of the neutrino is compensated by the very high mass of a neutrino with opposite chirality (figure 2). For example, with a typical mass of Dirac fermions of $M_D \simeq 1$ MeV and a neutrino mass $M_\nu = 0.1$ eV one finds with (31) a neutrino mass $M_R \simeq 10$ TeV. This

heavy neutrino would decay e.g. into a lepton ℓ and the charged Higgs introduced in section *7.4*, that is $N \to \ell^- H^+$ or $N \to \ell^+ H^-$. A difference between the two partial decay widths would induce CP violation, leading to the observed asymmetry of about 10^{-6} between antimatter and matter in the universe. This **leptogenesis**[2] mechanism of matter-antimatter asymmetry is an alternative to the CP violating **baryogenesis** [3] mechanism of Grand Unified Theories (section *7.4*).

The seesaw mechanism will gain credence once the Majorana nature of neutrinos has been established experimentally by the observation of neutrinoless double β-decay or the discovery of a heavy W_R boson.

References

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[2] Fukugita M and Yanagida T 1986 *Phys. Lett.* **174** 1
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