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U.U.D.M. Project Report 2018:13

The Selfinjective Nakayama Algebras and their Complexity

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Examensarbete i matematik, 30 hp
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Juni 2018

A large, faint watermark of the Uppsala University seal is visible in the bottom right corner of the page. The seal features a sun with rays, a figure, and the Latin motto "ALERE FLAMMAM VERITATIS".

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THE SELF-INJECTIVE NAKAYAMA ALGEBRAS AND THEIR COMPLEXITY

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ABSTRACT. For any finite dimensional associative algebra A we define a d -cluster tilting module. We look at the case where A is a selfinjective Nakayama algebra with a d -cluster tilting module and make some observations on how these modules can look like. From this we make some observations in terms of $\Omega^{d+1}\nu$ -orbits and the complexity of the modules.

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1. INTRODUCTION

For an algebra A we have the category $\text{mod } A$ of finitely generated A -modules. O. Iyama introduced d -cluster tilting modules in [1] and defined them as a module $M \in \text{mod } A$ such that the three following conditions are equivalent;

- (i) $X \in \text{add}(M)$
- (ii) $\text{Ext}^i(X, M) = 0$ for all $0 < i < d$
- (iii) $\text{Ext}^i(M, X) = 0$ for all $0 < i < d$

These modules are in some literature referred to as $d - 1$ maximal orthogonal modules and are closely related to tilting modules. They are in the center of, among other things, higher dimensional Auslander-Reiten theory. Nowadays there exist an extended theory about them. The existence of a d -cluster tilting on the other hand are far from obvious and can have consequences for the whole category $\text{mod } A$.

K. Erdmann and T. Holm adressed the problem with finding algebras exhibiting a d -cluster tilting module in their article [2] and, as it turns out, they are somewhat rare. They showed that if the algebra is selfinjective the existence of a d -cluster tilting module bounds the complexity for all modules in $\text{mod } A$. None the less, finding examples are far from impossible. M. Herschend and O. Iyama gives us several examples of selfinjective algebras having 2-cluster tilting modules in [3].

Erdmann and Holm also point out that it seem natural to consider selfinjective algebras when searching for d -cluster tilting modules since there cannot exist any non-split extensions between the injective and projective modules.

E. Darpö and O. Iyama looked closer on selfinjective algebras and categorized which of the selfinjective Nakayama algebras that admits a d -cluster tilting module in [4].

In this paper we will walk though the usual preliminaries, in large following [5] and study selfinjective algebras from a general perspective. Then we will try to expand on both [2] and [4] via considering orbits of indecomposable modules under Ω and $\tau_d = \tau\Omega^{d-1}$. We will study one of the two cases from [4] when the selfinjective Nakayama algebras admits d -cluster tilting modules.

2. BASIC DEFINITIONS

In this section we recollect all basic definitions we will use throughout this paper. For anyone used to work in homological algebra we recommend skipping section 2.1 and possibly the rest of section 2. We will assume that the reader has a brief understanding of category theory and a working knowledge of linear algebra.

2.1. Algebras and Modules. Throughout this paper we will use the notations introduced here. Most of them are quite basic, but we have them here to avoid ambiguity. Here we will largely follow [5], but if anything will become easier via the use of a slightly different definition we have tried to use the latter. Throughout K denotes an algebraically closed field.

Definition 2.1. An **algebra** over a field K (or a K -algebra) is a ring A with identity and a K -vector space structure compatible with the multiplication of the ring. That is:

$$k(ab) = (ak)b = a(kb) = (ab)k$$

for all $k \in K$ and $a, b \in A$. Another way of phrasing this is that the multiplication of A is K -bilinear. We define the dimension of A to be the dimension of A over K , that is $\dim_K(A)$, and we say that A is **finite dimensional** if $\dim_K(A) < \infty$.

The **opposite algebra**, denoted A^{op} is defined by the same underlying set with the the ring multiplication $*$ defined via $a * b = ba$.

Definition 2.2 (Orthogonal Primitive Idempotents). An element $e \in A$ is said to be an **idempotent** if $e^2 = e$. Two idempotents, e_1, e_2 , are said to be **orthogonal** if $e_1 e_2 = 0$. An idempotent, e , is said to be **primitive** if there does not exist two non-zero orthogonal idempotents e_1, e_2 such that $e = e_1 + e_2$.

Clearly 0 and 1 in A_A are orthogonal idempotents, we will call them the **trivial idempotents**.

A **right ideal** (or **left ideal**) I of an algebra A is a K -subspace which is closed under right (or left) multiplication, that is $ia \in I$ (or $ai \in I$) for all $i \in I$ and $a \in A$. If I is both a left and right ideal, we will simply call it an **ideal**.

Definition 2.3 (Module). A **right module** over a K -algebra A (or a right A -module) is a K -vector space M with a right multiplication $\cdot: M \times A \rightarrow M$ that fulfills the following:

- (1) $(x + y)a = xa + ya$
- (2) $x(a + b) = xa + xb$
- (3) $(xa)b = x(ab)$
- (4) $x1 = x$
- (5) $(xk)a = x(ak) = (xa)k$

A **left module** is defined analogously. We will write M_A for a right A -module and ${}_A M$ for a left A -module. If we wish to consider an algebra A as a right or left A -module we will simply write A_A or ${}_A A$, respectively. We define the dimension of M to be the dimension of M over K , that is $\dim_K(M)$, and we say that M is **finite dimensional** if $\dim_K(M) < \infty$.

Definition 2.4. A **homomorphism** between two A -modules M and N is a K -linear map $\varphi: M \rightarrow N$ for which $\varphi(ma) = \varphi(m)a$ for all $m \in M$ and $a \in A$. The class of homomorphisms from M to N is denoted $\text{Hom}_A(M, N)$. If $\varphi \in \text{End}_A(M) := \text{Hom}_A(M, M)$ we say that φ is an **endomorphism**.

A homomorphism is said to be a **monomorphism** if it is injective and an **epimorphism** if it is surjective. A homomorphism which is both is called an **isomorphism**.

For each module M there exists the **identity morphism** denoted $\text{Id}_M \in \text{End}_A(M)$ which sends each element to itself. We say that a homomorphism, $\gamma \in \text{Hom}_A(M, N)$, is a **section** if there exists a homomorphism $\gamma' \in \text{Hom}_A(N, M)$ such that $\gamma' \circ \gamma = \text{Id}_M$. We say that a homomorphism, $\psi \in \text{Hom}_A(M, L)$, is a **retraction** if there exists a homomorphism $\psi' \in \text{Hom}_A(L, M)$ such that $\psi \circ \psi' = \text{Id}_L$.

Definition 2.5. We say that an A -module M is **generated** by the elements $\{m_i\} \subseteq M$ if any element can be written as a finite sum $\sum m_i a_i$ for some elements $a_i \in A$. If there exists a finite set that generates M we say that M is **finitely generated**.

Remark 2.6. It is easily checked that a module M over a finite dimensional algebra A is finitely generated if and only if $\dim_K(M)$ is finite. For more details we refer to [5].

Definition 2.7. The category of all right A -modules, $\text{Mod } A$, is defined as having all A -modules as objects. The morphisms between M and N is $\text{Hom}_A(M, N)$, that is all A -module homomorphisms. The subcategory $\text{mod } A$ is defined to have all finitely generated A -modules as objects and $\text{Hom}_A(M, N)$ as arrows, for some finitely generated A -modules M and N .

With this it is easy to see that $\text{Mod } A^{\text{op}}$ is the category of all left A -modules and $\text{mod } A^{\text{op}}$ is the category of all finitely generated left A -modules. Let A and B be two K -algebras. If we have a K -vector space, M , endowed with both the structure of a left A -module and the structure of a right B -module we say that M is a **A - B -bimodule** if the actions commute, that is $(am)b = a(mb)$ for all $a \in A$, $b \in B$ and $m \in M$.

For any A - B -bimodule ${}_A M_B$ we can define the contravariant functor

$$\text{Hom}_B(-, {}_A M_B): \text{Mod } B \rightarrow \text{Mod } A^{\text{op}}$$

by sending N_B to $\text{Hom}_B(N, M)$, endowed with the left A -module structure from M , that is $(af)(v) = a \cdot f(v)$. For $\varphi \in \text{Hom}_B(N, N')$ we have $\text{Hom}_B(\varphi, M) = - \circ \varphi$. That is, $\text{Hom}_B(-, M)$, works on morphisms via right concatenation.

In the same way we can define the covariant functor

$$\text{Hom}_B({}_A M_B, -): \text{Mod } B \rightarrow \text{Mod } A$$

with the action defined by $(fa)(v) = f(av)$. An important case is when we choose ${}_A M_B = {}_A A_A$ and thus have a contravariant functor

$$(-)^t := \text{Hom}_A(-, A): \text{Mod } A \rightarrow \text{Mod } A^{\text{op}}.$$

It is easy to see that if A is a finite dimensional K -algebra this restricts to a functor $\text{mod } A \rightarrow \text{mod } A^{\text{op}}$.

We can, of course, also consider the vector space of K -linear functions between two K -vector spaces, M and N , which we will denote $\text{Hom}_K(M, N)$. If M is a A -module we can give $\text{Hom}_K(M, N)$ the structure of an A^{op} -module via $(af)(v) := f(va)$. Thus $\text{Hom}_K(-, N)$ is a functor $\text{Mod } A \rightarrow \text{Mod } A^{\text{op}}$.

In a similar way we can define the **standard K -duality**

$$D := \text{Hom}_K(-, K): \text{mod } A \rightarrow \text{mod } A^{\text{op}}.$$

That is we send M_A to the space of K -linear functions from M to K , $\text{Hom}_K(M, K)$. This is a A -module with the action defined by $(a\varphi)(m) = \varphi(ma)$.

A convention

In this paper we will only consider finitely generated modules over a finite dimensional algebra. Thus we introduce the following convention:

Unless specified otherwise an A -module is a finitely generated right A -module for some finitely dimensional K -algebra, where K is an algebraically closed field.

Though the attentive reader can confirm that many results holds anyways.

Definition 2.8 (Submodule). A **submodule** N_A of M_A is a K -subspace which is closed under the right-multiplication in M_A . That is $na \in N$ for all $n \in N$ and $a \in A$. A submodule N_A is said to be **maximal** if there does not exist any submodule L_A such that $N \subsetneq L \subsetneq M$.

With this it is easy to see that the kernel, $\ker \varphi$ and the image, $\operatorname{im} \varphi$, of some homomorphism, $\varphi: M \rightarrow N$, are submodules of M or N , respectively. Moreover we let the **cokernel** be $\operatorname{coker} \varphi := N/\operatorname{im} \varphi$.

Definition 2.9 (Sum of Submodules). Given two right A -submodules, M and N , of some A -module, the sum of them is defined as

$$M + N := \{m + n; m \in M, n \in N\}$$

This can easily be shown to be a submodule. If moreover $M \cap N = \{0\}$ the sum is called a **direct sum** and is written as $M \oplus N$. In the latter case, for each element $a \in M \oplus N$ there exists unique elements $m \in M$ and $n \in N$ such that $a = m + n$.

In the case that $M \in A\text{-mod}$ is generated by $\{m_0, \dots, m_n\}$ we have $M = m_0A + \dots + m_nA$. Moreover we say that a submodule N of M is **superfluous** in M if for any submodule L of M the equality $N + L = M$ implies $L = M$. Clearly, if N is superfluous in M and N' is a submodule of M such that $N' \subseteq N$, then N' is superfluous.

Definition 2.10. A module is said to be **simple** if the only submodules are 0 and the module itself. We say that a module is **semisimple** if it is the direct sum of simple submodules.

Definition 2.11. A module M is said to be **indecomposable** if $M = S \oplus T$ implies that either $S = 0$ or $T = 0$.

An algebra A is said to be **representation finite** if there only exists finitely many indecomposable modules in $\operatorname{mod} A$, up to isomorphism. Similar to the above we can define the direct sum of two modules.

Definition 2.12. The **direct sum** of two A -modules, M and N , is defined as the K -vector space $M \oplus N$ given the right A -module structure via $(m, n)a := (ma, na)$.

Remark 2.13. Notice that both $M \oplus 0$ and $0 \oplus N$ are submodules of $M \oplus N$ and are isomorphic to M and N respectively. Therefore we will often say that M and N are submodules of $M \oplus N$ even though we actually talk about $M \oplus 0$ and $0 \oplus N$.

Remark 2.14. If S is an A -module and M and N are two submodules of S such that $S = M \oplus N$ (with the \oplus here being the internal sum). Then $S \simeq M \oplus N$ if we see M and N as two A -modules. With this in mind we will not distinguish between them.

For any non-trivial idempotent $e \in A$ we have $(1 - e)^2 = 1 - e$ and $(1 - e)e = 0$, thus e and $1 - e$ are two non-trivial orthogonal idempotents. It follows that there exists a direct sum decomposition $A = eA \oplus (1 - e)A$. Indeed, assume $a \in eA \cap (1 - e)A$, then we have $a = ea' = (1 - e)a''$ which gives us $a'' = e(a' + a'') = e^2(a' + a'') = ea''$. Hence $a = (1 - e)ea'' = 0a'' = 0$. Moreover $a = ea + (1 - e)a$.

Now for any finite dimensional K -algebra A we get a decomposition of $A_A = M_0 \oplus \dots \oplus M_n$ where M_i is indecomposable. Notice that by definition there exists unique $e_i \in M_i$ such that $1 = e_0 + \dots + e_n$ is an idempotent. It follows by the previous discussion that the set $\{e_0, \dots, e_n\}$ is a set of primitive orthogonal idempotents such that $M_i = e_iA$. In this case we say that $\{e_i\}$ is a **complete set** of primitive orthogonal idempotents.

Lemma 2.15 (Schur's lemma). *Let M and N be A -modules and $\varphi : M \rightarrow N$ a non-zero homomorphism.*

- (1) *If M is simple then φ is a monomorphism.*
- (2) *If N is simple then φ is an epimorphism.*
- (3) *If both M and N are simple then φ is an isomorphism.*

Proof. Let φ be non-zero. If M is simple then $\ker \varphi = 0$. If N is simple then $\text{im } \varphi = N$. □

Definition 2.16 (The (Jacobson) radical). Let $M \in \text{mod } A$. Then we define $\text{rad}(M)$ to be the intersection of all maximal submodules of M .

We will now list the fundamental properties of the radical. This can be found as proposition (I.3.7) in [5].

Proposition 2.17. *Let M, N and L be A -modules.*

- (1) *An element $m \in M$ is in $\text{rad}(M)$ if and only if $f(m) = 0$ for any $f \in \text{Hom}_A(M, S)$ for all simple A -modules S .*
- (2) *$\text{rad}(M \oplus N) = \text{rad}(M) \oplus \text{rad}(N)$.*
- (3) *For all $f \in \text{Hom}_A(M, N)$ we have $f(\text{rad}(M)) \subseteq \text{rad}(N)$.*
- (4) *$M \text{rad}(A) = \text{rad}(M)$.*
- (5) *$\text{rad}(M)$ is superfluous in M .*

For proof, see [5].

Definition 2.18 (Chain Complex). A **chain complex**, M_\bullet , is a set of A -modules $\{M_n\}_{n \in \mathbb{Z}}$ with homomorphisms $\varphi_n : M_n \rightarrow M_{n-1}$ such that $\varphi_n \circ \varphi_{n+1} = 0$. That is $\text{im } \varphi_{n+1} \subseteq \ker \varphi_n$. We usually visualize this as:

$$\dots \longrightarrow M_2 \xrightarrow{\varphi_2} M_1 \xrightarrow{\varphi_1} M_0 \xrightarrow{\varphi_0} M_{-1} \xrightarrow{\varphi_{-1}} \dots$$

The complex is said to be **exact** if $\text{im } \varphi_{n+1} = \ker \varphi_n$ for all n . It is said to be **bounded above** if all $M_n = 0$ if n is large enough and **bounded below** if all $M_n = 0$ if n is small enough. If a complex is bounded from both above and below, we will simply call it **bounded**.

Remark 2.19. For any exact complex on the form:

$$0 \longrightarrow A \xrightarrow{\varphi} B \longrightarrow \dots$$

we have that φ is a monomorphism and for any exact complex on the form:

$$\dots \longrightarrow M \xrightarrow{\gamma} N \longrightarrow 0$$

we have that γ is an epimorphism.

Definition 2.20 (Short Exact Sequence). A **short exact sequence** is a bounded chain complex on the form:

$$0 \longrightarrow M \xrightarrow{\alpha} N \xrightarrow{\beta} L \longrightarrow 0$$

If there exists either $\alpha' : N \rightarrow M$ or $\beta' : L \rightarrow N$ with $\alpha' \circ \alpha = \text{Id}_M$ or $\beta' \circ \beta = \text{Id}_L$, respectively, it is said to be **split**. If that is the case $B = \text{im } \alpha \oplus \ker \alpha' = \ker \beta \oplus \text{im } \beta' \simeq M \oplus L$ and the sequence is isomorphic to

$$0 \longrightarrow M \xrightarrow{\begin{pmatrix} 1 \\ 0 \end{pmatrix}} M \oplus L \xrightarrow{\begin{pmatrix} 1 & 0 \end{pmatrix}} L \longrightarrow 0.$$

A **diagram** is a directed multi-graph with an A -module in each node and a homomorphism for each vertex such that the homomorphism is from the source of the vertex to the target of the vertex.

The diagram is called **commutative** if the composition of homomorphisms along any to paths with the same start and endpoint are equal.

Example 2.21. One example of a commutative diagram would be:

$$\begin{array}{ccc} \mathbb{Z} & \xrightarrow{\cdot 3} & \mathbb{Z} \\ \downarrow \cdot 2 & & \downarrow \cdot 2 \\ \mathbb{Z} & \xrightarrow{\cdot 3} & \mathbb{Z} \end{array}$$

simply because $-\cdot 2 \cdot 3 = -\cdot 6 = -\cdot 3 \cdot 2$.

Definition 2.22 (Chain Map). A **chain map**, $f_{\bullet}: M_{\bullet} \rightarrow N_{\bullet}$, is a collection of homomorphisms $f_n: M_n \rightarrow N_n$ such that the following diagram commutes:

$$\begin{array}{ccccccc} \dots & \xrightarrow{\varphi_{n+1}} & M_n & \xrightarrow{\varphi_n} & M_{n-1} & \xrightarrow{\varphi_{n-1}} & M_{n-2} & \xrightarrow{\varphi_{n-2}} & \dots \\ & & \downarrow f_n & & \downarrow f_{n-1} & & \downarrow f_{n-2} & & \\ \dots & \xrightarrow{\gamma_{n+1}} & N_n & \xrightarrow{\gamma_n} & N_{n-1} & \xrightarrow{\gamma_{n-1}} & N_{n-2} & \xrightarrow{\gamma_{n-2}} & \dots \end{array}$$

That is $f_{n-1} \circ \varphi_n = \gamma_n \circ f_n$ for all n . The notion of chain map expands the notion of homomorphism to chain complexes and we call a chain map f_{\bullet} a monomorphism, epimorphism or isomorphism if all the f_n are monomorphisms, epimorphisms or isomorphisms respectively.

Remark 2.23. Just because there exists isomorphisms $M_n \simeq N_n$ for all n in two complexes it does not make them isomorphic. It is only a requirement and not a sufficient condition.

Definition 2.24. An A -module I is said to be **injective** if for any monomorphism $f: A \rightarrow B$ and any homomorphism $g: A \rightarrow I$ there exists $g': B \rightarrow I$ with $g' \circ f = g$. Another way to put this is that there exists g' such that the following diagram with exact rows commute:

$$\begin{array}{ccccc} 0 & \longrightarrow & A & \xrightarrow{f} & B \\ & & \downarrow g & \nearrow g' & \\ & & I & & \end{array}$$

Definition 2.25. An A -module P is said to be **projective** if for any epimorphism $f: A \rightarrow B$ and any homomorphism $g: P \rightarrow B$ there exists $g': P \rightarrow A$ with $f \circ g' = g$. Another way to put this is that there exists g' such that the following diagram with exact rows commute:

$$\begin{array}{ccccc} A & \xrightarrow{f} & B & \longrightarrow & 0 \\ & \nearrow g' & \uparrow g & & \\ & & P & & \end{array}$$

Example 2.26. Any algebra A_A , seen as a right A -module, is projective.

Proposition 2.27. *A direct summand of a projective module is projective and the direct sum of two projective modules are projective. The same is true for injectivity of modules.*

Proof. If $M \oplus N$ is projective, then given any epimorphism $\varphi: X \rightarrow Y$ and morphism $\gamma_1: M \rightarrow Y$ there exists $\psi = (\psi_1 \ \psi_2): M \oplus N \rightarrow X$ making the following diagram commute.

$$\begin{array}{ccc} X & \xrightarrow{\varphi} & Y \\ & \searrow \psi & \uparrow (\gamma_1 \ 0) \\ & & M \oplus N \end{array}$$

It is easy to verify that the existence of ψ_1 makes M projective, and by symmetry N must also be.

The other way around, if M and N be projective. Given any epimorphism $\varphi: X \rightarrow Y$ and morphism $\gamma = (\gamma_1 \ \gamma_2): M \oplus N \rightarrow Y$. Then using the projectivity of M and N we have commutative diagrams:

$$\begin{array}{ccc} X & \xrightarrow{\varphi} & Y \\ & \searrow \psi_1 & \uparrow \gamma_1 \\ & & M \end{array} \quad \begin{array}{ccc} X & \xrightarrow{\varphi} & Y \\ & \searrow \psi_2 & \uparrow \gamma_2 \\ & & N \end{array}$$

If we let $\psi = (\psi_1 \ \psi_2)$ we get that $M \oplus N$ is projective via the existence of ψ . \square

Proposition 2.28. *A module P is projective if and only if all short exact sequences on the form*

$$0 \longrightarrow M \longrightarrow N \longrightarrow P \longrightarrow 0$$

splits and a module I is injective if and only if all short exact sequences on the form

$$0 \longrightarrow I \longrightarrow N \longrightarrow L \longrightarrow 0$$

splits.

Proof. A proof for this proposition can be found in [6]. \square

A result similar to the two above results can be found in [5] as lemma (I.5.3).

Definition 2.29. An epimorphism $\varphi: M \rightarrow N$ is called **minimal** if $\ker \varphi$ is superfluous in M . An epimorphism $\varphi: P \rightarrow M$ is called a **projective cover** of M if P is a projective module and φ is minimal. If a projective cover exists we denote the module by $P(M)$.

Proposition 2.30. *For any A -module M there exists a projective cover and it is unique, up to isomorphism. That is, for any two projective covers, $\varphi: P \rightarrow M$ and $\varphi': P' \rightarrow M$, there exists an isomorphism $g: P' \rightarrow P$ that makes the following*

diagram commute:

$$\begin{array}{ccccc}
 & & & & 0 \\
 & & & & \uparrow \\
 & & & & 0 \\
 & & & & \uparrow \\
 P & \xrightarrow{\varphi} & M & \longrightarrow & 0 \\
 & \swarrow g & \uparrow \varphi' & & \\
 & & P' & &
 \end{array}$$

Proof. See (I.5.8) in [5]. \square

Lemma 2.31. *For any two A -modules, M and N , we have $P(M \oplus N) \simeq P(M) \oplus P(N)$.*

Proof. Let φ_M and φ_N be the corresponding projective covers. In view of proposition 2.30 we only need to show that $(\varphi_M, \varphi_N): P(M) \oplus P(N) \rightarrow M \oplus N$ is indeed minimal. We will show something a little more general.

Let M' and N' be superfluous in M and N respectively. Then $M' \oplus N'$ is superfluous in $M \oplus N$.

For this case, assume $M' \oplus N' + C' \oplus D' = M \oplus N$. Indeed, projecting on the first and second coordinate we get $M' + C' = M$ and $N' + D' = N$. Using that N' and M' are superfluous we get the desired result. \square

Definition 2.32. A **projective resolution** of M is an exact chain complex on the form

$$\dots \longrightarrow P_1 \xrightarrow{\varphi_1} P_0 \longrightarrow 0 \longrightarrow \dots$$

With a morphism $\varphi_0: P_0 \rightarrow M$ that makes the following chain complex exact:

$$\dots \longrightarrow P_1 \xrightarrow{\varphi_1} P_0 \xrightarrow{\varphi_0} M \longrightarrow 0 \longrightarrow \dots$$

Where the P_i 's are projective modules. The resolution is said to be **minimal** if $P_0 = P(M)$ and $P_i = P(\ker \varphi_{n-1})$. If P_0 and P_1 is the first two terms of a minimal projective resolution we say that

$$P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

is a **minimal projective presentation**.

Corollary 2.33. *By proposition (2.30) any two minimal projective resolution are isomorphic.*

Example 2.34. For any projective module P we have a projective resolution:

$$\dots \longrightarrow 0 \longrightarrow P \xrightarrow{\text{Id}} P \longrightarrow 0 \longrightarrow \dots$$

Proposition 2.35. *Let D be the standard duality.*

- i) *A sequence $0 \longrightarrow M \longrightarrow N \longrightarrow L \longrightarrow 0$ is exact in $\text{mod } A$ if and only if $0 \longrightarrow D(L) \longrightarrow D(N) \longrightarrow D(M) \longrightarrow 0$ is exact in $\text{mod } A^{\text{op}}$.*
- ii) *P in $\text{mod } A$ is projective if and only if $D(P)$ is injective in $\text{mod } A^{\text{op}}$. E in $\text{mod } A$ is injective if and only if $D(E)$ is projective in $\text{mod } A^{\text{op}}$.*
- iii) *S is simple in $\text{mod } A$ if and only if $D(S)$ is simple in $\text{mod } A^{\text{op}}$*

iv) A morphism $\varphi: P \rightarrow M$ is a projective cover if and only if $D(\varphi): D(M) \rightarrow D(P)$ is an injective envelope. A morphism $\gamma: M \rightarrow E$ is an injective envelope if and only if $D(\gamma): D(E) \rightarrow D(M)$ is a projective cover.

Definition 2.36. The **syzygy** of M , $\Omega(M)$, is defined as the kernel of a projective cover. The **cosyzygy** Ω^{-1} is defined as the cokernel of an injective envelope.

By proposition (2.30) this definition does not depend on the choice of projective cover, up to isomorphism. Moreover, since projective covers respect direct sums, we get $\Omega(M \oplus N) \simeq \Omega(M) \oplus \Omega(N)$.

It is easily checked that $\Omega(M) = 0$ if and only if M is projective. Indeed, we always have an exact sequence

$$0 \longrightarrow \Omega(M) \longrightarrow P(M) \longrightarrow M \longrightarrow 0$$

and that $P(M) \simeq M$ if and only if M projective.

For Ω (or Ω^{-1}) to be well defined functors we would need to consider the projective (injective) stable category. We will discuss this category later but for now we will leave it as it is.

We will say that a module is **Ω -periodic** if there exists an n such that $\Omega^n(M) \simeq M$.

Another type of functor which will be of interest is the following; let $\alpha \in \text{Aut } A$ be an automorphism and define

$$\Phi_\alpha: \text{mod } A \rightarrow \text{mod } A$$

to send every module $M \in \text{mod } A$ to the module with twisted action M_α . That is the module with the same underlying vector space and the action of $a \in A$ in $v \in M_\alpha$ is defined as $v \cdot_{M_\alpha} a := v \cdot_M \alpha(a)$. Moreover, it sends any $\phi \in \text{Hom}_A(M, N)$ to $\Phi_\alpha(\phi) \in \text{Hom}_A(M_\alpha, N_\alpha)$ defined by $\Phi_\alpha(\phi)(v) = \phi(v)$ seen as linear maps between vector spaces. It can be checked that this is an exact functor and an isomorphism $\text{mod } A \rightarrow \text{mod } A$. Thus it sends projective covers to projective covers.

Now a minimal projective resolution will always have the following form:

$$\begin{array}{ccccccc} \dots & \longrightarrow & P(\Omega^2(M)) & \longrightarrow & P(\Omega(M)) & \longrightarrow & P(M) \longrightarrow 0 \\ & & \searrow & & \swarrow & & \swarrow \\ & & \Omega^2(M) & & \Omega(M) & & \end{array}$$

A projective resolution thus encodes how a module looks like in the form of projective modules. This motivates further studies of them, thus let

$$\dots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow 0$$

be a minimal projective resolution for M . The **complexity** of M is then defined as

$$\text{cx}(M) = \inf\{a \in \mathbb{N} \mid \exists C: Cn^{a-1} \geq \dim_K P_n \text{ for all } n\}.$$

Notice that $\text{cx}(M) = 0$ if and only if M has a bounded minimal projective resolution and $\text{cx}(M) = 1$ if the dimension of all P_n are bounded from above. Clearly any Ω -periodic module has complexity 1 and by the construction of minimal projective covers $\text{cx}(M) = \text{cx}(\Omega M)$ and $\text{cx}(M \oplus N) = \max\{\text{cx}(M), \text{cx}(N)\}$. Moreover if P is projective $\text{cx}(P) = 0$ and for any automorphism α we have $\text{cx}(\Phi_\alpha(M)) = \text{cx}(M)$.

Proposition 2.37. *If P is a projective module and the diagram*

$$\begin{array}{ccccccc} M & \xrightarrow{\alpha} & P & \xrightarrow{\beta} & N & & \\ & & & & \downarrow \gamma & & \\ 0 & \longrightarrow & X & \xrightarrow{\varphi} & Y & \xrightarrow{\psi} & Z \longrightarrow 0 \end{array}$$

has exact rows, then there exists γ_0 and γ_1 making the following diagram commute:

$$\begin{array}{ccccccc} M & \xrightarrow{\alpha} & P & \xrightarrow{\beta} & N & & \\ \downarrow \gamma_1 & & \downarrow \gamma_0 & & \downarrow \gamma & & \\ 0 & \longrightarrow & X & \xrightarrow{\varphi} & Y & \xrightarrow{\psi} & Z \longrightarrow 0 \end{array}$$

Proof. The existence of γ_0 follows directly from the definition that P is projective. Since the rows are exact, φ is injective and thus there exists $\varphi^{-1} \in \text{Hom}_A(\text{im } \varphi, X)$, indeed it exists as a linear function by basic linear algebra. We need to show that φ^{-1} commutes with the action of the algebra, then we have

$$\begin{aligned} a\varphi^{-1}(y) = \varphi^{-1}(ay) &\Leftrightarrow \varphi(a\varphi^{-1}(y)) = \varphi(\varphi^{-1}(ay)) \\ &\Leftrightarrow a\varphi(\varphi^{-1}(y)) = \varphi(\varphi^{-1}(ay)) \\ &\Leftrightarrow ay = ay \end{aligned}$$

But we have $\psi \circ \gamma_0 \circ \alpha = \gamma \circ \beta \circ \alpha = \gamma \circ 0 = 0$, thus $\text{im } \gamma_0 \circ \alpha \subseteq \ker \psi = \text{im } \varphi$ and we can choose $\gamma_1 = \varphi^{-1} \circ \gamma_0 \circ \alpha$. \square

Worth noting is that even though γ_0 is not necessarily unique, γ_1 is uniquely determined by γ_0 .

2.2. Extensions and the Stable Category.

Definition 2.38. Let M and N be two A -modules. Then we define $\text{Ext}_A^m(M, N)$ via the following construction: Take any projective resolution for M

$$P_\bullet : \quad \dots \longrightarrow P_3 \xrightarrow{\varphi_3} P_2 \xrightarrow{\varphi_2} P_1 \xrightarrow{\varphi_1} P_0 \longrightarrow 0$$

and apply $\text{Hom}(-, N)$. Then we get a induced complex $\text{Hom}(P_\bullet, N)$ as:

$$0 \longrightarrow \text{Hom}(P_0, N) \xrightarrow{\text{Hom}(\varphi_1, N)} \text{Hom}(P_1, N) \xrightarrow{\text{Hom}(\varphi_2, N)} \text{Hom}(P_2, N) \longrightarrow \dots$$

Then we let

$$\text{Ext}_A^m(M, N) = \ker \text{Hom}(\varphi_{m+1}, N) / \text{im } \text{Hom}(\varphi_m, N).$$

Remark 2.39. For this to be well defined one must show that this does not depend on the choice of P_\bullet , up to isomorphism. This is relatively straightforward and can be found in [5] appendix (A.4).

Any $\gamma \in \text{Hom}(N, N')$ then induces a morphism $\text{Hom}(P_m, \gamma): \text{Ext}_A^m(M, N) \rightarrow \text{Ext}_A^m(M, N')$. Similar to above, this will not depend on the choice of P_\bullet and we can prove that $\text{Ext}_A^m(M, -)$ is a covariant additive functor.

Corollary 2.40. *For any A -modules M and N we have $\text{Ext}_A^0(M, N) \simeq \text{Hom}_A(M, N)$*

Definition 2.41 (The Stable Category). The **projective stable category**, denoted $\underline{\text{Mod}} A$, has the same objects as $A\text{-Mod}$ but morphisms are quotients

$$\underline{\text{Hom}}(M, N) := \text{Hom}_A(M, N) / \mathcal{P}(M, N)$$

where $\mathcal{P}(M, N)$ is the space of all morphisms in $\text{Hom}_A(M, N)$ factoring through a projective module. Similarly the **injective stable category**, denoted $\overline{\text{Mod}} A$ has the same objects as $A\text{-Mod}$ but morphism are quotients

$$\overline{\text{Hom}}(M, N) := \text{Hom}_A(M, N) / \mathcal{I}(M, N)$$

where $\mathcal{I}(M, N)$ is the space of all morphisms in $\text{Hom}_A(M, N)$ factoring through an injective module.

Theorem 2.42. *Given any short exact sequence $0 \longrightarrow M \longrightarrow N \longrightarrow L \longrightarrow 0$ and a module X we have the two induced long exact sequences:*

$$\begin{array}{ccccccc} 0 & \longrightarrow & \text{Hom}_A(X, M) & \longrightarrow & \text{Hom}_A(X, N) & \longrightarrow & \text{Hom}_A(X, L) \\ & & & & & & \\ & & \xrightarrow{\delta_0} & \text{Ext}_A^1(X, M) & \longrightarrow & \text{Ext}_A^1(X, N) & \longrightarrow & \text{Ext}_A^1(X, L) \\ & & & \dots & & \dots & & \dots \\ & & & & & & & \\ & & \xrightarrow{\delta_{n-1}} & \text{Ext}_A^n(X, M) & \longrightarrow & \text{Ext}_A^n(X, N) & \longrightarrow & \text{Ext}_A^n(X, L) \\ & & & \dots & & \dots & & \dots \end{array}$$

and

$$\begin{array}{ccccccc} 0 & \longrightarrow & \text{Hom}_A(L, X) & \longrightarrow & \text{Hom}_A(N, X) & \longrightarrow & \text{Hom}_A(M, X) \\ & & & & & & \\ & & \xrightarrow{\delta_0} & \text{Ext}_A^1(L, X) & \longrightarrow & \text{Ext}_A^1(N, X) & \longrightarrow & \text{Ext}_A^1(M, X) \\ & & & \dots & & \dots & & \dots \\ & & & & & & & \\ & & \xrightarrow{\delta_{n-1}} & \text{Ext}_A^n(L, X) & \longrightarrow & \text{Ext}_A^n(N, X) & \longrightarrow & \text{Ext}_A^n(M, X) \\ & & & \dots & & \dots & & \dots \end{array}$$

Proof. For the construction of δ_0 in the first sequence, consider the following diagram.

$$\begin{array}{ccccccccc} P_2 & \xrightarrow{\varphi_2} & P_1 & \xrightarrow{\varphi_1} & P_0 & \xrightarrow{\varphi_0} & X & \longrightarrow & 0 \\ & & & & & & \downarrow \gamma & & \\ 0 & \longrightarrow & M & \xrightarrow{g} & N & \xrightarrow{f} & L & \longrightarrow & 0 \end{array}$$

Then by proposition (2.37) we can lift $\gamma \in \text{Hom}(M, X)$ to a morphism in $\text{Hom}(P_1, M)$. It can be checked that this defines a morphism to $\text{Ext}_A^1(X, M)$ which fulfills our demands. Then δ_n can be constructed similarly using the definition of Ext_A^n . \square

Proposition 2.43. *A module P in $\text{mod } A$ is projective if and only if $\text{Ext}_A^m(P, -) = 0$ for all $m \geq 1$. A module E in $\text{mod } A$ is injective if and only if $\text{Ext}_A^m(-, E) = 0$ for all $m \geq 1$.*

Proof. We will only prove the first part as the second one is proved similarly. If P is projective we choose the projective resolution as $0 \longrightarrow P \longrightarrow 0$ with $\varphi_0 = \text{Id}_P$, and the rest follows by definition.

Assume $\text{Ext}_A^m(P, -) = 0$, then given any epimorphism $M \longrightarrow N$ we have a short exact sequence

$$0 \longrightarrow \ker f \longrightarrow M \xrightarrow{f} N \longrightarrow 0$$

and thus a long exact sequence

$$0 \longrightarrow \text{Hom}_A(P, \ker f) \longrightarrow \text{Hom}_A(P, M) \longrightarrow \text{Hom}_A(P, N) .$$

$$\xrightarrow{\delta_0} \text{Ext}_A^1(P, \ker f) \longrightarrow \text{Ext}_A^1(P, M) \longrightarrow \text{Ext}_A^1(P, N)$$

...

$$\xrightarrow{\delta_{n-1}} \text{Ext}_A^n(P, \ker f) \longrightarrow \text{Ext}_A^n(P, M) \longrightarrow \text{Ext}_A^n(P, N)$$

...

By assumption, this long exact sequence is in fact the short exact sequence:

$$0 \longrightarrow \text{Hom}_A(P, \ker f) \longrightarrow \text{Hom}_A(P, M) \xrightarrow{f \circ -} \text{Hom}_A(P, N) \longrightarrow 0 .$$

Thus $f \circ -$ is surjective for any epimorphism f . Thus P is projective. \square

Definition 2.44 (Pullback and pushout). Let M , N and L be A -modules.

a) Given homomorphism $M \xleftarrow{\varphi} N \xrightarrow{\gamma} L$ we define the **amalgammed sum** to be the module

$$S = M \oplus L / \{(\varphi(n), -\gamma(n)): n \in N\}$$

with two homomorphisms $M \xrightarrow{\varphi'} S \xleftarrow{\gamma'} L$ given by $\varphi(m) = \overline{(m, 0)}$ and $\gamma(l) = \overline{(0, l)}$.

- b) Given homomorphisms $M \xrightarrow{\varphi} N \xleftarrow{\gamma} L$ we define the **fibered product** to be the submodule

$$P = \{(m, l) \in M \oplus L : \varphi(m) = \gamma(l)\}$$

of $M \oplus L$ together with two homomorphisms $M \xleftarrow{\varphi'} P \xrightarrow{\gamma'} L$ given by $\varphi'(m, l) = m$ and $\gamma'(m, l) = l$.

Theorem 2.45. *Let $0 \longrightarrow M \longrightarrow N \longrightarrow L \longrightarrow 0$ be a short exact sequence.*

- (a) *Given a morphism $\varphi: M \rightarrow X$ there exists a module U and a commutative diagram*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \longrightarrow & N & \longrightarrow & L & \longrightarrow & 0 \\ & & \downarrow \varphi & & \downarrow & & \parallel & & \\ 0 & \longrightarrow & X & \longrightarrow & U & \longrightarrow & L & \longrightarrow & 0 \end{array}$$

with exact rows.

- (b) *Given a morphism $\gamma: X \rightarrow L$ there exists a module V and a commutative diagram*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \longrightarrow & V & \longrightarrow & X & \longrightarrow & 0 \\ & & \parallel & & \downarrow & & \downarrow \gamma & & \\ 0 & \longrightarrow & M & \longrightarrow & N & \longrightarrow & L & \longrightarrow & 0 \end{array}$$

with exact rows.

- (c) *In the above, U and V are unique, up to isomorphism.*

We will not show this, instead we will do a similar construction which we will use later. This construction is used in [2].

Theorem 2.46 (Universal Extention). *Given two modules X and V such that $\text{Ext}_A^1(X, X) = 0$ and $\text{Ext}_A^1(X, V) \neq 0$. Let $n := \dim_K \text{Ext}_A^1(X, V)$. Then there exists a module U and a short exact sequence*

$$0 \longrightarrow V \longrightarrow U \longrightarrow X^n \longrightarrow 0$$

with $\text{Ext}_A^1(X, U) = 0$.

This short exact sequence is referred to as the **universal extention**.

Proof. Let $P_2 \xrightarrow{h_2} P_1 \xrightarrow{h_1} P_0 \xrightarrow{h_0} 0$ be the beginning of a projective resolution of X . In this proof we let h_i^n denote $(h_i, \dots, h_i): P_i^n \rightarrow P_{i-1}^n$, that is n copies of h_i .

We can choose elements $\varphi_i \in \ker(- \circ h_2) \subseteq \text{Hom}(P_1, V)$ so that $\{\overline{\varphi_1}, \dots, \overline{\varphi_n}\}$ is a basis for $\text{Ext}_A^1(X, V)$. Let $\varphi = (\varphi_1, \dots, \varphi_n)$ and let U be the amalgamated sum of $V \xleftarrow{\varphi} P_1^n \xrightarrow{h_1^n} P_0^n$. For convenience we set $H = \{(\varphi(p), -h_1^n(p))\}$, that is $U = V \oplus P_0^n / H$.

We want to define α , β and γ such that the following diagram commute and has exact rows:

$$\begin{array}{ccccccccc} P_2^n & \longrightarrow & P_1^n & \longrightarrow & P_0^n & \longrightarrow & X^n & \longrightarrow & 0 \\ \downarrow & & \downarrow \varphi & & \downarrow \gamma & & \parallel & & \\ 0 & \longrightarrow & V & \xrightarrow{\alpha} & U & \xrightarrow{\beta} & X^n & \longrightarrow & 0 \end{array}$$

Thus define

$$\begin{aligned} \alpha(v) &= (v, 0) + H \\ \beta(v, (p_1, \dots, p_n)) &= h_0^n(p_1, \dots, p_n) = (h_0(p_1), \dots, h_0(p_n)) \\ \gamma(p_1, \dots, p_n) &= (0, h_1^n(p_1, \dots, p_n)) + H = (h_1(p_1), \dots, h_1(p_n)) + H \end{aligned}$$

To check that all of these are well defined, commute and form a short exact sequence as in the above picture is left as an exercise for the reader.

Then by the proof of theorem (2.42) we have that δ_0 in the long exact sequence

$$0 \longrightarrow \text{Hom}_A(X, V) \longrightarrow \text{Hom}_A(X, U) \longrightarrow \text{Hom}_A(X, X^n)$$

$$\xrightarrow{\delta_0} \text{Ext}_A^1(X, V) \longrightarrow \text{Ext}_A^1(X, U) \longrightarrow \text{Ext}_A^1(X, X^n)$$

is surjective. But we assumed that $\text{Ext}_A^1(X, X) = 0$ and it follows that

$$\text{Hom}_A(X, X^n) \xrightarrow{\delta_0} \text{Ext}_A^1(X, V) \longrightarrow \text{Ext}_A^1(X, U) \longrightarrow 0$$

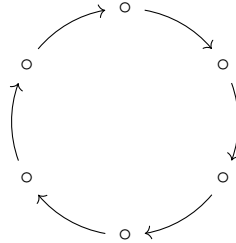
is exact. Thus $\text{Ext}_A^1(X, U) = 0$. \square

2.3. Quiver Algebras. Here we will discuss **quivers** and how we can get algebras from them. We will briefly break our convention that all algebras are finitely generated for this chapter.

Definition 2.47 (Quiver). A **quiver** Q is a quadruple (Q_0, Q_1, s, t) . The set Q_0 are the **vertices**. The set Q_1 are the **arrows**, s and t are functions $Q_1 \rightarrow Q_0$ giving each arrow $v \in Q_1$ a source $s(v)$ and a target $t(v)$. A quiver is said to be **finite** if both Q_0 and Q_1 is.

Remark 2.48. The definition of a quiver is more or less identical with that of a directed multi-graph. We will use this terminology of two reasons, firstly the definition of latter sometimes varies between literature, secondly the first is commonly used in the context we will use it and we hope this will make it easier for the reader.

Example 2.49. One example of a quiver would be:



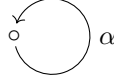
Let $a, b \in Q_0$ for some quiver Q . A **path** from a to b is a sequence of arrows $\alpha_0\alpha_1 \dots \alpha_n$ where $s(\alpha_0) = a$, $t(\alpha_n) = b$ and for all relevant i we have $t(\alpha_i) = s(\alpha_{i+1})$. Such a path is sometimes written as $(a|\alpha_1 \dots \alpha_n|b)$. We also have the stationary path at a as $\epsilon_a = (a||a)$.

Definition 2.50 (Quiver Algebra). We define the **quiver algebra** of Q , usually denoted KQ , to be the algebra generated by all paths and the multiplication defined as

$$(a|\alpha_0 \dots \alpha_n|b)(c|\beta_0 \dots \beta_m|d) = \begin{cases} (a|\alpha_0 \dots \alpha_n\beta_0 \dots \beta_m|d) & \text{if } b = c \\ 0 & \text{otherwise} \end{cases}$$

This is clearly an associative algebra, although it is not always finitely generated. Moreover, for a finite quiver it is easily checked that $1 = \sum_{a \in Q_0} \epsilon_a$.

Example 2.51. This can be found as example [II.1.3] in [5]. Let Q be the quiver:



Then KQ is generated by $\{\epsilon_1, \alpha, \alpha^2, \dots\}$ and is thus isomorphic to $K[X]$, that is, polynomials in one indeterminate over K .

It follows by definition that $\{\epsilon_a | a \in Q_1\}$ is a complete set of primitive orthogonal idempotents.

Theorem 2.52. For any finite quiver Q the quiver algebra KQ is finite dimensional if and only if Q is acyclic.

Proof. The algebra is finite dimensional if and only if there exists a finite number of paths. This is only the case if Q is acyclic. □

Of course we can consider the usual constructions on the path algebra of any quiver.

Definition 2.53. The **arrow ideal** of a quiver Q is the ideal in KQ generated by all arrows in Q . We will denote this ideal R_Q of R whenever it is clear from context.

An **admissible ideal** is an ideal I such that $R_Q^m \subseteq I \subseteq R_Q^2$ for some large enough m .

The study of algebras on the form $A \simeq KQ/I$ for some admissible ideal I is motivated by the following theorem.

Theorem 2.54. If A is a basic connected K -algebra, then there exists a quiver Q and an admissible ideal I such that $A \simeq KQ/I$.

Proof. See [5]. □

Definition 2.55. Given a quiver Q and an ideal I we say that we have a **bounded representation** of KQ/I if for each $a \in Q_0$ we have a vector space V_a over K . Moreover for each $\alpha \in Q_1$ we have a $\varphi_\alpha \in \text{Hom}_K(V_{s(\alpha)}, V_{t(\alpha)})$ such that for each $\sum_i (a^i|\alpha_0^i \dots \alpha_n^i|b^i) \in I$ we have $\sum_i \varphi_{\alpha_n^i} \circ \dots \circ \varphi_{\alpha_0^i} = 0$.

For this reason we have the category $\text{Rep}(Q, I)$ of bounded representations of KQ/I with the subcategory $\text{rep}(Q, I)$ of finite dimensional bounded representations. The reason why we deal in representations is that they are easy to visualize and the following theorem.

Theorem 2.56. *There exists an equivalence of categories $\text{Rep}(Q, I) \simeq \text{Mod } A$, where $A \simeq KQ/I$, that restricts to an equivalence of categories $\text{rep}(Q, I) \simeq \text{mod } A$.*

Proof. See [5]. □

Example 2.57. A example of a representation of the algebra $A \simeq KQ/I$ where Q is the quiver $\circ \overset{\curvearrowright}{\leftarrow} \circ$ and $I = R_Q^2$ would be

$$\begin{array}{ccc}
 & \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} & \\
 & \curvearrowright & \\
 K^2 & & K^2 \\
 & \curvearrowleft & \\
 & \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} &
 \end{array}$$

If we can avoid it we will not write out the whole representation as above and instead focus on the **dimension vectors**. We will simply write out which dimension each vector space has at the position of the node and assume that as many morphisms as possible are equal to the identity morphism. For example if we call the above representation M then $\underline{\dim}(M) = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$. Notice though that the dimension vector does not always uniquely define a representation, even if we want most

of the morphisms to be the identity. For example let $Q = \circ \overset{\circ}{\overset{\leftarrow}{\leftarrow}} \circ$ and $I = R_Q^2$. Then the two representations

$$\begin{array}{ccc}
 & K & \\
 0 \swarrow & & \nwarrow 1 \\
 K & \xleftarrow{1} & K
 \end{array}
 \qquad
 \begin{array}{ccc}
 & K & \\
 1 \swarrow & & \nwarrow 0 \\
 K & \xleftarrow{1} & K
 \end{array}$$

have the same dimension vectors, even though they are not the same or even isomorphic. For the rest of the paper however, we will only use this notation when the dimension vectors for indecomposable modules uniquely defines the indecomposable module.

2.4. Auslander-Reiten Theory.

Definition 2.58. Let M, N and L be right A -modules.

- (a) A homomorphism $\varphi \in \text{Hom}_A(M, N)$ is called **left minimal** if for any $\alpha \in \text{End}(N)$ $\alpha \circ \varphi = \varphi$ implies that α is an automorphism.
- (b) A homomorphism $\gamma \in \text{Hom}_A(N, L)$ is called **right minimal** if for any $\beta \in \text{End}(N)$ $\beta \circ \gamma = \gamma$ implies that β is an automorphism.

- (c) A homomorphism $\varphi \in \text{Hom}_A(M, N)$ is called **left almost split** if φ is not a section and for each non section $\psi \in \text{Hom}_A(M, L)$ there exists $\psi' \in \text{Hom}_A(N, L)$ such that $\psi = \psi' \circ \varphi$. That is, the following diagram commutes:

$$\begin{array}{ccc} M & \xrightarrow{\varphi} & N \\ \downarrow \psi & & \swarrow \psi' \\ L & & \end{array}$$

- (d) A homomorphism $\gamma \in \text{Hom}_A(N, L)$ is called **right almost split** if γ is not a retraction and for each non retraction $\psi \in \text{Hom}_A(M, L)$ there exists $\psi' \in \text{Hom}_A(M, N)$ such that $\psi = \gamma \circ \psi'$. That is, the following diagram commutes:

$$\begin{array}{ccc} & & M \\ & \swarrow \psi' & \downarrow \psi \\ N & \xrightarrow{\gamma} & L \end{array}$$

- (e) A homomorphism $\varphi \in \text{Hom}_A(M, N)$ is called **left minimal almost split** if it is both left minimal and left almost split.
 (f) A homomorphism $\gamma \in \text{Hom}_A(N, L)$ is called **right minimal almost split** if it is both right minimal and right almost split.

Definition 2.59. A short exact sequence

$$0 \longrightarrow M \xrightarrow{\varphi} N \xrightarrow{\gamma} L \longrightarrow 0$$

is said to be an **almost split sequence** if φ is left minimal almost split and γ is right minimal almost split.

Definition 2.60 (Auslander-Reiten Translation). Given a minimal projective presentation $P_1 \xrightarrow{\varphi_1} P_0 \xrightarrow{\varphi_0} M \longrightarrow 0$ of M we can apply $(-)^t$ to get the exact sequence

$$0 \longrightarrow M^t \longrightarrow P_0^t \longrightarrow P_1^t \longrightarrow \text{coker } \varphi_1^t \longrightarrow 0 .$$

Then we define the **transpose** of M to be $\text{coker } \varphi_1^t$ and denote it with $\text{Tr } M$. Then we define the **Auslander-Reiten translation** to be $\tau := D \text{Tr}$ and the inverse $\tau^{-1} := \text{Tr } D$.

The notation of τ^{-1} is a little bit unorthodox since τ^{-1} is not an actual inverse, therefore the notation τ^- is also commonly used. We will begin with a quick summarize of the properties the transpose possess. This next proposition can be found as proposition (IV.2.1) in [5]. Notice that for any $M \in \text{mod } A$ we have $\text{Tr } M \in \text{mod } A^{\text{op}}$.

Proposition 2.61. *The following holds for any indecomposable modules M and N :*

- (a) *The left A -module $\text{Tr } M$ has no nonzero projective direct summands.*
 (b) *If M is non-projective, then*

$$P_0^t \xrightarrow{\varphi_1^t} P_1^t \longrightarrow \text{Tr } M \longrightarrow 0 ,$$

defined as above, is a minimal projective presentation of the left A -module $\text{Tr } M$.

- (c) M is projective if and only if $\text{Tr } M = 0$. If M is not projective, then $\text{Tr } M$ is indecomposable and $\text{Tr}(\text{Tr } M) \simeq M$.
- (d) If M and N are non-projective, then $M \simeq N$ if and only if $\text{Tr } M \simeq \text{Tr } N$.

Proof. See [5]. □

The following theorem can be found as proposition (IV.2.10) in [5].

Theorem 2.62. *Let M and N be indecomposable modules in $\text{mod } A$. Then the following holds:*

- (1) $\tau M = 0$ if and only if M is projective.
- (2) $\tau^{-1}M = 0$ if and only if M is injective.
- (3) If M is non-projective, then τM is non-injective and $\tau^{-1}\tau M \simeq M$.
- (4) If M is non-injective, then $\tau^{-1}M$ is non-projective and $\tau\tau^{-1}M \simeq M$.
- (5) If M and N are non-projective then $M \simeq N$ if and only if $\tau M \simeq \tau N$.
- (6) If M and N are non-injective then $M \simeq N$ if and only if $\tau^{-1}M \simeq \tau^{-1}N$.

Proof. See [5]. □

Theorem 2.63 (Auslander-Reiten Formulas). *If M and N are two modules in $\text{mod } A$, then we have isomorphisms*

$$\text{Ext}_A^1(M, N) \simeq D \underline{\text{Hom}}_A(\tau^{-1}N, M) \simeq D \overline{\text{Hom}}_A(N, \tau M)$$

that are functorial in both arguments.

Proof. See theorem (IV.2.13) in [5]. □

The following theorem due to Auslander and Reiten.

Theorem 2.64 (Existence of almost split sequence). *If A is a finite dimensional K -algebra the following holds in $\text{mod } A$:*

- (a) *For any non-projective indecomposable $L \in \text{mod } A$ there exists a unique, up to isomorphism, almost split sequence*

$$0 \longrightarrow \tau L \longrightarrow N \longrightarrow L \longrightarrow 0 .$$

- (b) *For any non-injective indecomposable $M \in \text{mod } A$ there exists a unique, up to isomorphism, almost split sequence*

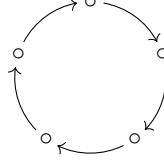
$$0 \longrightarrow M \longrightarrow N \longrightarrow \tau^{-1}M \longrightarrow 0 .$$

Definition 2.65 (AR-quiver). The **Auslander-Reiten quiver** of the module category $\text{mod } A$, denoted $\Gamma(\text{mod } A)$, is defined as follows. The nodes are isoclasses of indecomposable A -modules. The arrows are in bijective correspondence with the dimension of the K -vector space of irreducible morphisms from the source to the target. For a more exact definition of the latter, see [5].

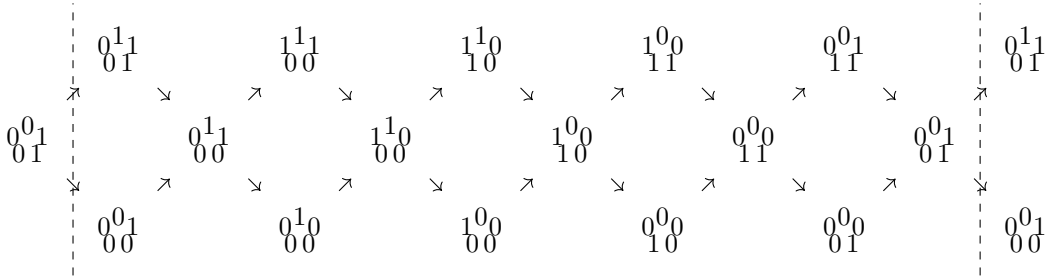
A helpful observation when constructing the Auslander-Reiten quiver for an algebra is the following proposition from [5].

Proposition 2.66. *Assume $0 \longrightarrow M \longrightarrow N \longrightarrow L \longrightarrow 0$ is an almost split sequence. Then $\underline{\dim} N = \underline{\dim} M + \underline{\dim} L$.*

Example 2.67. Let $A = KQ/I$ where Q is the quiver



and let $I = R_Q^3$. Then the Auslander-Reiten quiver of A will be:



The dashed lines are to indicate that the picture repeats itself.

2.5. Selfinjective Algebras. As earlier mentioned, any algebra A seen as an A -module is always projective. In general, the same is not true for injectivity. In this chapter we will take some time and discuss the ones that are injective and see what follows.

Definition 2.68 (Selfinjective Algebra). An algebra A is said to be **selfinjective** (or a **quasi-Frobenius algebra**) if A_A is injective.

Even though this may look rather innocent, the consequences that follows are not.

This next theorem is from [7]. It can be found as proposition (IV.3.1) and the text following it. We will use this result for the rest of the chapter without referring to this theorem, so it would be wise for the reader to remember the result.

Theorem 2.69. *The following are equivalent:*

- a) A is selfinjective.
- b) A module is projective in $\text{mod } A$ if and only if it is injective in $\text{mod } A$.
- c) A^{op} is selfinjective
- d) The functor $(-)^t: \text{mod } A \rightarrow \text{mod } A^{\text{op}}$ is a duality.

Proof. See [7]. □

By the above theorems we clearly have that $\underline{\text{mod}} A$ is the same as $\overline{\text{mod}} A$ for a selfinjective algebra A .

We have seen that in the selfinjective case both D and $(-)^t$ defines dualities between $\text{mod } A$ and $\text{mod } A^{\text{op}}$. This inspires us to consider the **Nakayama functor**:

$$\nu := D(-)^t = \text{Hom}_K(\text{Hom}_A(-, A), K): \text{mod } A \rightarrow \text{mod } A$$

The following theorem explain how ν is related to Ω and τ . Similar results can be found in [7] proposition (IV.3.7).

Theorem 2.70. *If A is a selfinjective algebra then, as functors $\underline{\text{mod}} A \rightarrow \underline{\text{mod}} A$*

- a) $\tau \simeq \Omega^2\nu$,
- b) $\Omega\nu \simeq \nu\Omega$,
- c) $\tau\Omega \simeq \Omega\tau$.

Proof. Let M be in $\text{mod } A$.

- a) Apply the functors $(-)^t$ and D in order to the minimal presentation

$$P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

to get the exact sequence

$$0 \longrightarrow D \text{Tr } M \longrightarrow D(P_1^t) \longrightarrow D(P_0^t) \longrightarrow D(M^t) \longrightarrow 0.$$

This sequence can be rewritten as

$$0 \longrightarrow \tau M \longrightarrow D(P_1^t) \longrightarrow D(P_0^t) \longrightarrow \nu M \longrightarrow 0.$$

Now it follows that $D(P_i^t)$ is injective, thus, since A was selfinjective, it is projective and that the above is a minimal projective presentation of νM follows from 2.35. Thus $\tau \simeq \Omega^2\nu$.

- b) This part is proved similarly but using the exact sequence

$$0 \longrightarrow \Omega M \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

- c) By the above we get $\tau\Omega \simeq \Omega^2\nu\Omega \simeq \Omega\Omega^2\nu \simeq \Omega\tau$.

□

Moreover, we get an easy way to compute $\text{Ext}_A^i(M, N)$.

Theorem 2.71. *Let A be a selfinjective algebra. Then for any M and N in $\text{mod } A$, we have*

$$\text{Ext}_A^m(M, N) \simeq \underline{\text{Hom}}_A(\Omega^m M, N) \simeq \overline{\text{Hom}}_A(M, \Omega^{-m} N)$$

for all $m \geq 1$.

Proof. That there exists an isomorphism $\text{Ext}_A^{m+1}(M, N) \simeq \text{Ext}_A^1(\Omega^m M, N)$ is clear by definition of Ext . Thus we only need to show that $\text{Ext}_A^1(M, N) \simeq \underline{\text{Hom}}(\Omega M, N)$. So let

$$\dots \longrightarrow P_2 \xrightarrow{\varphi_2} P_1 \xrightarrow{\varphi_1} P_0 \xrightarrow{\varphi_0} M \longrightarrow 0$$

be a minimal projective resolution. Then, by definition,

$$\text{Ext}_A^1(M, N) = (\ker - \circ \varphi_2) / (\text{im } - \circ \varphi_1).$$

For each $f \in \ker(- \circ \varphi_2)$ we have a commutative diagram with exact row

$$\begin{array}{ccccccc} P_2 & \xrightarrow{\varphi_2} & P_1 & \xrightarrow{\varphi_1} & P_0 & \longrightarrow & M \longrightarrow 0 \\ \downarrow & & \downarrow f & \searrow h & \uparrow \iota & & \\ 0 & \longrightarrow & N & & \Omega(M) & & \end{array}$$

Then we can consider the map $- \circ h: \text{Hom}(\Omega M, N) \rightarrow \ker(- \circ \varphi_2)$. First of, this is well defined since $- \circ h \circ \varphi_2 = - \circ 0$. Secondly, since h is surjective we have $g \circ h = g' \circ h \iff g = g'$, thus $- \circ h$ is injective. For surjectivity, we want to show that $f(h^{-1}(-))$ is a well defined map, here h^{-1} denotes the preimage. It is

well defined since $\ker h = \ker \varphi_1 = \text{im } \varphi_2 \subseteq \ker f$. The last inclusion follows since $f \in \ker(- \circ \varphi_2)$. Then we get $f(h^{-1}(-)) \mapsto f$ and surjectivity follows. What is left to check is that $- \circ h$ maps $\mathcal{P}(\Omega M, N)$ bijectively to $\text{im}(- \circ \varphi_1)$.

So take $g \in \mathcal{P}(\Omega M, N)$ and let it factor via P as $g = g' \circ g''$. Then, since A was selfinjective, P is injective and there exists a morphism ρ such that $g = g' \circ \rho \circ \iota$ where ι is the inclusion morphism $\Omega M \rightarrow P_0$. Then we have $g \circ h = g' \circ \rho \circ \iota \circ h = g' \circ \rho \circ \varphi_1$ and thus $- \circ h$ maps $\mathcal{P}(\Omega M, N)$ into $\text{im}(- \circ \varphi_1)$. Injectivity is clear, so all that is left is surjectivity. So let $f \in \text{im}(- \circ \varphi_1)$, that is $f = f' \circ \varphi_1$. Then we have $f' \circ \iota \circ h = f' \circ \varphi_1 = f$ and we are done.

A similar construction of Ext using injective envelopes in the second argument shows that $\text{Ext}_A^{m+1}(M, N) \simeq \text{Ext}_A^1(M, \Omega^{-m}N)$, the rest is proved similarly. □

Recall the functor Φ_α from chapter 2.1 which sends each module M to the module M_α which have the same underlying vector space but the action twisted with α .

Proposition 2.72. *For any selfinjective algebra A there exist an automorphism α such that $\nu \simeq \Phi_\alpha$.*

Proof. See [8]. □

We are now in a position to bring up the following theorem. Notice that a formal proof requires the use of triangulated categories, which we will not cover. The theorem can be found as part of lemma (4.1) in [2].

Theorem 2.73. *Let A be a selfinjective algebra and M, N and L be modules in $\text{mod } A$. Moreover let $0 \longrightarrow M \longrightarrow N \longrightarrow L \longrightarrow 0$ be a short exact sequence. If two of the modules has complexity less than or equal to 1 then so does the third.*

Proof. We will prove the case where $\text{cx}(M) \leq 1$ and $\text{cx}(L) \leq 1$. Then, via the use of triangulated categories, there will exist a projective module P and a short exact sequence on the form

$$0 \longrightarrow L \longrightarrow M \oplus P \longrightarrow \Omega^{-1}N \longrightarrow 0$$

thus we will be done. So let $P_{M,i}$ and $P_{L,i}$ be projective resolutions of M and L respectively. In picture we have

$$\begin{array}{ccccccc}
 & & & & 0 & & \\
 & & & & \downarrow & & \\
 \dots & \longrightarrow & P_{M,1} & \longrightarrow & P_{M,0} & \longrightarrow & M \longrightarrow 0 \\
 & & & & \downarrow & & \\
 & & & & N & & \\
 & & & & \downarrow & & \\
 \dots & \longrightarrow & P_{L,1} & \longrightarrow & P_{L,0} & \longrightarrow & L \longrightarrow 0 \\
 & & & & \downarrow & & \\
 & & & & 0 & &
 \end{array}$$

This can be extended to the commuting diagram

$$\begin{array}{ccccccc}
& & 0 & & 0 & & 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
\cdots & \longrightarrow & P_{M,1} & \longrightarrow & P_{M,0} & \longrightarrow & M \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
\cdots & \longrightarrow & P_{M,1} \oplus P_{L,1} & \longrightarrow & P_{M,0} \oplus P_{L,0} & \longrightarrow & N \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
\cdots & \longrightarrow & P_{L,1} & \longrightarrow & P_{L,0} & \longrightarrow & L \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
& & 0 & & 0 & & 0
\end{array}$$

Thus $P_{M,i} \oplus P_{L,i}$ will be a, not necessarily minimal, projective resolution of N and thus $\text{cx}(N) \leq \max\{\text{cx}(M), \text{cx}(L)\} \leq 1$. \square

3. CLUSTER TILTING MODULES

Definition 3.1 (Cluster Tilting Module). A module M is called **d -cluster tilting** if the following three conditions are equivalent:

- (i) $X \in \text{add}(M)$
- (ii) $\text{Ext}^i(X, M) = 0$ for all $0 < i < d$
- (iii) $\text{Ext}^i(M, X) = 0$ for all $0 < i < d$

Here $\text{add}(M)$ denotes the full subcategory of $\text{mod } A$ with objects that are direct summands of some power of M . A d -cluster tilting module is sometimes, for example in [2], referred to as $(d-1)$ -maximal orthogonal. Owing to (ii) and (iii) all indecomposable injectives and projectives of $\text{mod } A$ must be in $\text{add}(M)$.

Theorem 3.2. *If M is d -cluster tilting and $X \in \text{add}(M)$ then $\tau_d(X) \in \text{add}(M)$ where $\tau_d := \tau \circ \Omega^{d-1}$.*

Proof. While this is true in general, we only present a quick proof for when A is selfinjective since we will only use it for this case. We compute $\text{Ext}_A^i(M, \tau_d X)$ for $0 < i < d$, since A was selfinjective we have by theorem 2.71:

$$\begin{aligned}
\text{Ext}_A^i(M, \tau_d X) &\simeq \text{Ext}_A^1(\Omega^{i-1} M, \tau \circ \Omega^{d-1} X) \\
&\simeq D \underline{\text{Hom}}(\Omega^{d-1} X, \Omega^{i-1} M) \\
&\simeq D \underline{\text{Hom}}(\Omega^{d-i} X, M) \\
&\simeq D \text{Ext}_A^{d-i}(X, M) \\
&= 0
\end{aligned}$$

Notice that the subscripts range through $0 < d-i < d$ thus the last equality is valid. Since M was d -cluster tilting $\tau_d X \in \text{add}(M)$. \square

3.1. Cluster Tilting Modules for Selfinjective Algebras. In the case that A is a selfinjective algebra, cluster tilting modules behaves, in some ways, nicer. In this chapter we try to use the most of what we have done and finish with proving the main result from [2]. We then see that the existence of a cluster tilting module affects the whole category $\text{mod } A$ in an interesting way.

Corollary 3.3. *If A is a selfinjective algebra and M is a d -cluster tilting module in $\text{mod } A$, then so is $\Omega^k(M) \oplus A$.*

Proof. Since A is isomorphic to the direct sum of all indecomposable projective modules and $\Omega(M) = 0$ if and only if M is projective it is enough to consider a non-projective indecomposable X . Then we have

$$\begin{aligned} \text{Ext}_A^i(X, \Omega^k(M) \oplus A) &\simeq \text{Ext}_A^i(X, \Omega^k(M)) \\ &\simeq \text{Ext}_A^i(\Omega^{-k}X, M) \end{aligned}$$

and

$$\begin{aligned} \text{Ext}_A^i(\Omega^k(M) \oplus A, X) &\simeq \text{Ext}_A^i(\Omega^k(M), X) \\ &\simeq \text{Ext}_A^i(M, \Omega^{-k}X) \end{aligned}$$

for all $0 < i < d$. Since X was assumed to be non-projective (and hence non-injective) $\Omega^{-k}X \in \text{add } M$ is equivalent with $X \in \text{add } \Omega^k M$. Thus since M was assumed to be d -cluster tilting we get by the above that so is $\Omega^k M \oplus A$. \square

Then we have the following result from [2].

Theorem 3.4. *Let M be a d -cluster tilting module in $\text{mod } A$ for some selfinjective algebra A . If X is in $\text{add } M$, then so is $\Omega^{d+1}\nu X$. Thus $\Omega^{d+1}\nu$ permutes the non-projective indecomposable summands of M .*

Proof. We see that $\Omega^{d+1}\nu \simeq \nu\Omega^2\Omega^{d-1} \simeq \tau\Omega^{d-1} = \tau_d$, thus the first part follows from theorem 3.2. The second part follows easily from the form $\tau\Omega^{d-1}$ and the fact that A was selfinjective. \square

With this we are in a position of proving the following theorem from [2]. We will present the same proof as Erdmann and Holm, though we will expand on some details. First though, a quick lemma.

Lemma 3.5. *Let A be a selfinjective algebra. If $M \in \text{mod } A$ is $\Omega^{d+1}\nu$ -periodic, then $\text{cx}(M) = 1$.*

Proof. Say $(\Omega^{d+1}\nu)^m M \simeq M$ and let $\alpha \in \text{Aut}(A)$ be the automorphism such that $\nu \simeq \Phi_\alpha$. Recall that ν and Ω commute. Then if $\{P_i\}$ is a minimal projective

resolution we have

$$\begin{aligned}
\dim(P_i) &= \dim(P(\Omega^i M)) \\
&= \dim(P(\Omega^i (\Omega^{d+1}\nu)^m M)) \\
&= \dim(P(\nu^m \Omega^{i+m(d+1)} M)) \\
&= \dim(P(\Phi_\alpha^m \Omega^{i+m(d+1)} M)) \\
&= \dim(\Phi_\alpha^m P(\Omega^{i+m(d+1)} M)) \\
&= \dim(P(\Omega^{i+m(d+1)} M)) \\
&= \dim(P_{i+m(d+1)})
\end{aligned}$$

Thus $\sup\{\dim(P_i): i \in \mathbb{N}\} = \sup\{\dim(P_i): 0 \leq i < m(d+1)\}$ and the result follows. \square

Now for the actual theorem.

Theorem 3.6. *If A is a selfinjective algebra which admits a d -cluster tilting module M , then $\text{cx}(X) \leq 1$ for all $X \in A\text{-mod}$.*

Proof. All $\Omega^{d+1}\nu$ -periodic modules have complexity less than or equal to 1, so assume we have a module X in $\text{mod } A$ that is not. Set $U_0 = X$. We will inductively construct modules U_i and short exact sequences

$$(\zeta_k) \quad 0 \longrightarrow U_{k-1} \longrightarrow U_k \longrightarrow (\Omega^{d-1-k} M)^{m_k} \longrightarrow 0$$

such that $\text{Ext}_A^1(\Omega^i M, U_k) = 0$ for $d-1-k \leq i < d-1$. If we do this U_{d-1} will have $\text{Ext}_A^{i+1}(M, U_{d-1}) \simeq \text{Ext}_A^1(\Omega^i M, U_{d-1}) = 0$ for $0 \leq i < d-1$ and since M was an d -cluster tilting module, $U_{d-1} \in \text{add } M$. Then we have $\text{cx}(U_{d-1}) \leq 1$ and $\text{cx}(M^{m_{d-1}}) = \text{cx}(M) \leq 1$ since both are $\Omega^{d+1}\nu$ -periodic. Using backwards induction and the fact that $\text{cx}((\Omega^{d-1-k} M)^{m_k}) = \text{cx}(M) \leq 1$ we will get that $\text{cx}(U_k) \leq 1$ and we will be done.

For the construction of U_k , let U_0, \dots, U_{k-1} be given and let, for easier notation, $n_k = d-1-k$. We have two cases:

(1) If $\text{Ext}_A^1(\Omega^{n_k} M, U_{k-1}) = 0$ let $U_k = U_{k-1} \oplus \Omega^{n_k} M$, $m_k = 1$ and

$$(\zeta_k) \quad 0 \longrightarrow U_{k-1} \xrightarrow{(1 \ 0)} U_k \xrightarrow{(0 \ 1)} \Omega^{n_k} M \longrightarrow 0.$$

Then we clearly have

$$\begin{aligned}
\text{Ext}_A^1(\Omega^i M, U_k) &\simeq \text{Ext}_A^1(\Omega^i M, U_{k-1} \oplus \Omega^{n_k} M) \\
&\simeq \text{Ext}_A^1(\Omega^i M, U_{k-1}) \oplus \text{Ext}_A^1(\Omega^i M, \Omega^{n_k} M) \\
&\simeq \text{Ext}_A^1(\Omega^i M, U_{k-1}) \oplus \text{Ext}_A^{1+i-n_k}(M, M).
\end{aligned}$$

The right summand is zero for all i such that $0 < 2+i+k-d < d$ and the left summand is zero for all $d-k = d-1-(k-1) \leq i < d$ by induction hypothesis and zero for $i = d-1-k$ by assumption. Since $1 \leq k \leq d-1$ the right hand side is zero for all $d-1-k \leq i < d-1$ and thus we are done with this case of the induction step.

- (2) If $\text{Ext}_A^1(\Omega^{n_k} M, U_{k-1}) \neq 0$ we can construct the universal extension and let $m_k = \dim \text{Ext}_A^1(\Omega^{n_k} M, U_{k-1})$. Thus we have

$$(\zeta_k) \quad 0 \longrightarrow U_{k-1} \longrightarrow U_k \longrightarrow (\Omega^{n_k} M)^{m_k} \longrightarrow 0.$$

Then, by construction, $\text{Ext}_A^1(\Omega^{n_k} M, U_k) = 0$. If we now consider the long exact sequence from the construction of the universal extension we have the long exact sequence

$$\begin{array}{ccccccc} 0 & \longrightarrow & \text{Hom}_A(\Omega^{n_k} M, U_{k-1}) & \longrightarrow & \text{Hom}_A(\Omega^{n_k} M, U_k) & \longrightarrow & \text{Hom}_A(\Omega^{n_k} M, (\Omega^{n_k} M)^{m_k}) \\ & & & & & & \\ & & \xrightarrow{\delta_0} & \text{Ext}_A^1(\Omega^{n_k} M, U_{k-1}) & \longrightarrow & \text{Ext}_A^1(\Omega^{n_k} M, U_k) & \longrightarrow & \text{Ext}_A^1(\Omega^{n_k} M, (\Omega^{n_k} M)^{m_k}) \\ & & & \dots & & \dots & & \dots \\ & & \xrightarrow{\delta_{j-1}} & \text{Ext}_A^j(\Omega^{n_k} M, U_{k-1}) & \longrightarrow & \text{Ext}_A^j(\Omega^{n_k} M, U_k) & \longrightarrow & \text{Ext}_A^j(\Omega^{n_k} M, (\Omega^{n_k} M)^{m_k}) \\ & & & \dots & & \dots & & \dots \end{array}$$

The j :th row can be rewritten as:

$$\xrightarrow{\delta_{j-1}} \text{Ext}_A^1(\Omega^{n_k+j-1} M, U_{k-1}) \longrightarrow \text{Ext}_A^1(\Omega^{n_k+j-1} M, U_k) \longrightarrow \text{Ext}_A^1(\Omega^{n_k+j-1} M, (\Omega^{n_k} M)^{m_k}).$$

Since M was d -cluster tilting, the rightmost term is zero for all $0 < j < d$. By our induction hypothesis the leftmost term is zero for all $d-k \leq d-k+j-2 < d-1$. It is easily checked that $0 < j < d$ is the weaker of these two, thus, by exactness, the middle term is zero for all $d-k \leq d-k+j-2 < d-1$. By construction $\text{Ext}_A^1(\Omega^{n_k} M, U_k) = 0$ and thus $\text{Ext}_A^1(\Omega^i M, U_k) = 0$ is true for all $d-1-k \leq i < d-1$, completing our induction step.

By the previous arguments we are done. \square

A similar proof can be done using approximations. Let M once again be a d -cluster tilting algebra. We will get short exact sequences on the form

$$0 \longrightarrow \ker \phi_k \longrightarrow A_k \xrightarrow{\phi_k} \ker \phi_{k-1} \longrightarrow 0$$

where A_k is a right $\text{add}(M)$ -approximation of $\ker \phi_{k-1}$. Then each $A_k \in \text{add } M$ and studying the long exact sequences as above we can conclude that $\ker \phi_d \in \text{add } M$. The rest follows as above. For more information see [1].

The attentive reader will notice that in the above proof we only used two things about M , namely:

- (1) If $X \in \text{add } M$ then $\text{cx}(X) \leq 1$.
- (2) $X \in \text{add } M$ if and only if $\text{Ext}_A^i(M, X) = 0$ for all $0 < i < d$.

By theorem 3.4 both of the above holds if M is d -cluster tilting. Even though this might look weaker than being d -cluster tilting, it is not.

Proposition 3.7. *Let A be a selfinjective algebra and $M \in \text{mod } A$ be a module such that $X \in \text{add } M$ if and only if $\text{Ext}_A^i(M, X) = 0$ for all $0 < i < d$, then M is d -cluster tilting.*

Proof. We will begin by showing that M is closed under the action of τ_d , so assume that it is not. Then there exists an indecomposable module $X \in \text{add } M$ such that $\tau_d X \notin \text{add } M$. Since Ext_A^i is a biadditive functor, this implies that there exists an indecomposable module $Y \in \text{add } M$ such that $\text{Ext}_A^i(Y, \tau_d X) \neq 0$ for some $0 < i < d$. Thus we get

$$\begin{aligned} 0 &\neq D \text{Ext}_A^i(Y, \tau_d X) \\ &\simeq D \underline{\text{Hom}}(Y, \tau \Omega^{d-i-1} X) \\ &\simeq \text{Ext}_A^1(\Omega^{d-i-1} X, Y) \\ &\simeq \text{Ext}_A^{d-i}(X, Y). \end{aligned}$$

Note that $0 < d - i < d$, thus we have a contradiction since $X, Y \in \text{add } M$. Thus M is closed under τ_d .

Clearly all projectives must be in $\text{add } M$. Since A was selfinjective all injectives are also in $\text{add } M$, since they are also projective.

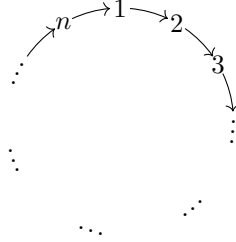
Since A was selfinjective the number of non-projective indecomposable direct summands of M must be the same as in $\tau_d M$. That gives us that $\text{add } M = \text{add}(A \oplus \tau_d M)$ and it follows that $\text{Ext}_A^i(X, M) = 0$ if and only if $\text{Ext}_A^i(X, \tau_d M) = 0$. Then we have

$$\begin{aligned} D \text{Ext}_A^i(X, \tau_d M) &\simeq D \text{Ext}_A^1(X, \tau \Omega^{d-i} M) \\ &\simeq \underline{\text{Hom}}(\Omega^{d-i} M, X) \\ &\simeq \text{Ext}_A^{d-i}(M, X). \end{aligned}$$

The result follows. □

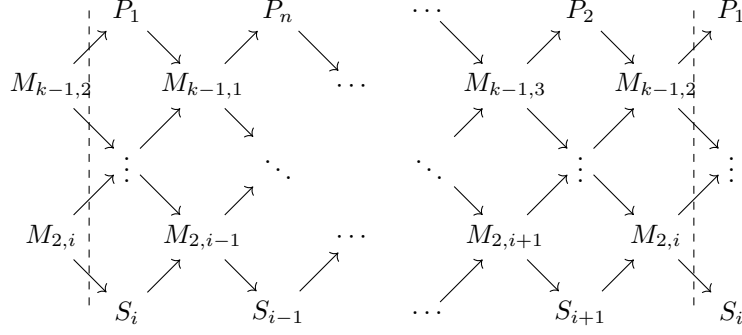
3.2. Selfinjective Nakayama Algebras. Here we will look closer to the case where A is a selfinjective Nakayama algebra. We will use existing theorems from [4] and try to connect this with our earlier results about complexity.

For easier notation we define $\mathcal{N}_{n,k} = KQ/I$ where Q is the quiver



and $I = R_Q^k$. That is, $\mathcal{N}_{n,k}$ is the selfinjective Nakayama algebra with Lowey length k .

Consider the Auslander-Reiten quiver for $\mathcal{N}_{n,k}$. The dashed lines are to indicate that the picture repeats itself.



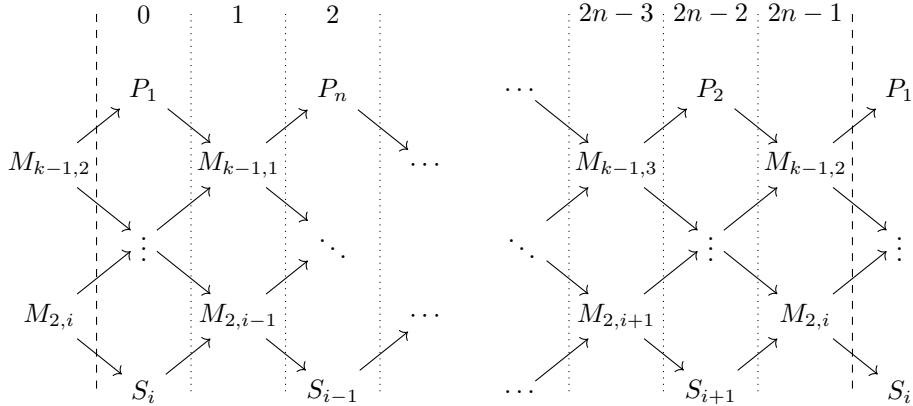
Here we number the modules such that $P(S_j) = P(M_{i,j}) = P_j$ and $\dim_K(M_{i,j}) = i$. In picture we have

$$\underline{\dim}(S_1) = \begin{matrix} & 0 & 1 & 0 \\ & \vdots & & 0 \\ \underline{\dim}(S_1) & \dots & \dots & \dots \end{matrix} \qquad \underline{\dim}(S_2) = \begin{matrix} & 0 & 0 & 1 \\ & \vdots & & 0 \\ \underline{\dim}(S_2) & \dots & \dots & \dots \end{matrix}$$

We will for easier notation let $M_{1,i} = S_i$, $M_{0,i} = 0$ and $M_{k,i} = P_i$. Let us begin by calculating how Ω , τ and $\tau_d \simeq \Omega^{d+1}\nu$ act with this notation. It is easily verified that $\Omega M_{i,j} = M_{k-i,j+i}$ and $\tau M_{i,j} = M_{i,j+1}$ for all $i \neq k$, with the second index calculated modulo n . It follows that $\Omega^2 M_{i,j} = M_{i,j+k}$. Thus we get $\tau_d M_{i,j} = M_{a,b}$ with

$$a = \begin{cases} i & \text{if } d \text{ is odd} \\ i - k & \text{if } d \text{ is even} \end{cases} \qquad b = \begin{cases} j + \left(k \frac{(d-1)}{2} + 1\right) & \text{if } d \text{ is odd} \\ j + \left(k \frac{d-2}{2} + i + 1\right) & \text{if } d \text{ is even} \end{cases}$$

These numbers do not tell us much since it is hard to calculate b modulo n . Although, if we number them a little bit differently, we see something nicer. Therefore, consider the following picture:



Let $c(X)$ be the column which X is in, for example $c(P_i) = 2(n-i+1)$. We can then see that for all non-projective X we have $c(\tau X) = c(X) - 2$ and $c(\Omega X) = c(X) - k$, here counted modulo $2n$. Thus $c(\tau_d X) = c(X) - (k(d-1) + 2)$ for all non-projective indecomposable modules.

From here it is easy to compute the complexity of all $\mathcal{N}_{n,k}$ -modules. Indeed, we only need that $\mathcal{N}_{n,k}$ is representation finite and that for any non-projective indecomposable X the module ΩX will be an indecomposable non-projective module. So take any $Y \in \text{mod } \mathcal{N}_{n,k}$ and let $Y = P \oplus Y'$ where P is projective and Y' does not have a projective direct summand. Then we clearly have $\Omega^i Y' \simeq Y'$ for some i by the pigeon hole principle and thus $\text{cx}(Y) = 1$ if Y' is non-zero. The following proposition follows:

Proposition 3.8. *For any $X \in \text{mod } \mathcal{N}_{n,k}$ we have $\text{cx}(X) = 1$ if and only if X is non-projective. If X is projective $\text{cx}(X) = 0$.*

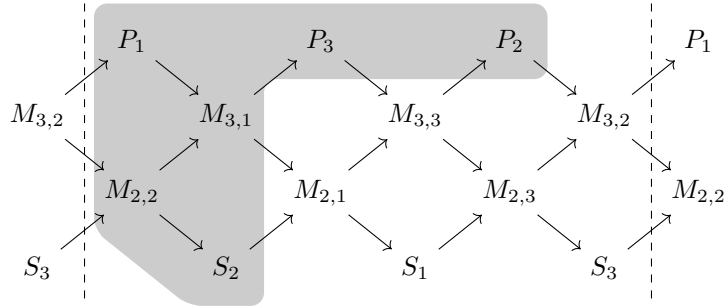
But for completeness, let us study this a little further keeping theorem (3.6) in mind. The following theorem is proved in [4], theorem 5.1.

Theorem 3.9. *There exists a d -cluster tilting module in $\mathcal{N}_{n,k}$ if and only if one of the following conditions are met:*

- i) $k(d-1) + 2 \mid 2n$;
- ii) $k(d-1) + 2 \mid tn$ where $t = \gcd(d+1, 2(k-1))$.

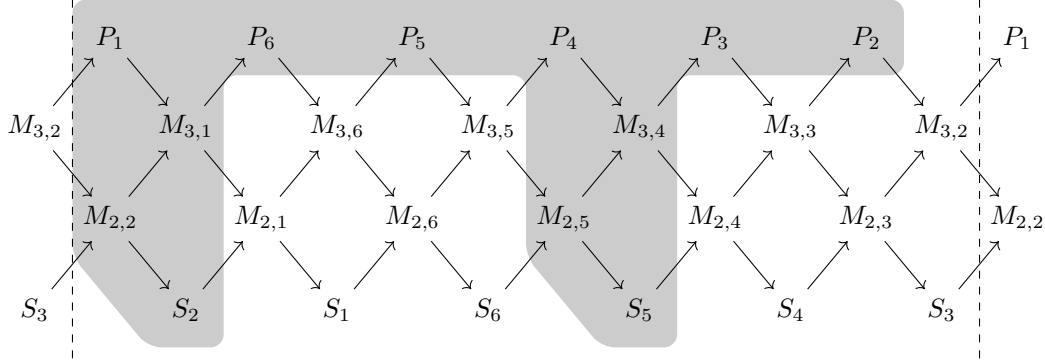
Corollary 3.10. *If there exists a d -cluster tilting module in $\text{mod } \mathcal{N}_{n,k}$ there exists a d -cluster tilting module in $\text{mod } \mathcal{N}_{mn,k}$ for each $m \in \mathbb{Z}^+$.*

A natural question to ask is how the d -cluster tilting module in $\text{mod } \mathcal{N}_{mn,k}$ looks like in relation to the one in $\text{mod } \mathcal{N}_{n,k}$. For this, consider the two cases $\mathcal{N}_{3,4}$ and $\mathcal{N}_{6,4}$. Then we have a 2-cluster tilting module in as follows:



That is, $M = A \oplus S_2 \oplus M_{2,2} \oplus M_{3,1}$ is 2-cluster tilting. This can be easily verified using (2.71) as we will see later. In $\mathcal{N}_{6,4}$ we then have a 2-cluster tilting module as

follows:



Notice how the first picture seems to repeat itself twice. This is really not a coincidence and comes partly from that the Auslander-Reiten quiver have the same structure with a longer period, and partly from theorem (2.71). Notice that a consequence of theorem (2.71) is that $\text{Ext}_A^i(M, N)$ is totally dependant on how M and N lies in the Auslander-Reiten quiver since Ω sends non-projective indecomposables to non-projective indecomposables. Thus we get the following result.

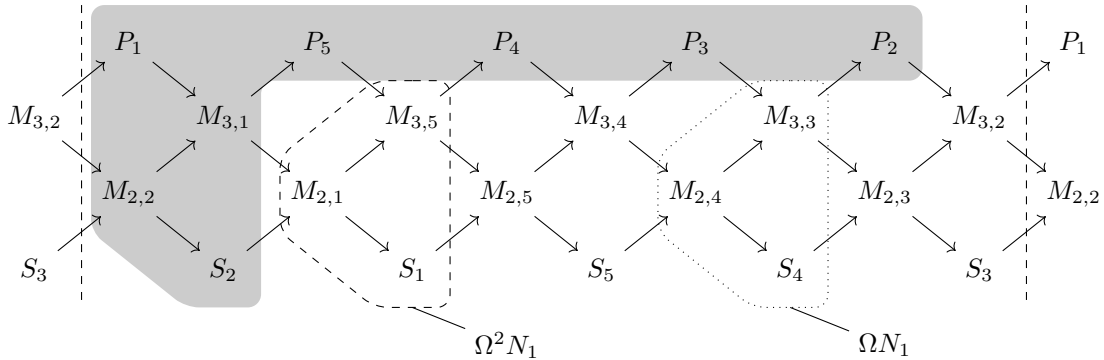
Proposition 3.11. *If $M = \bigoplus_{(i,j) \in I} M_{i,j}$ is a d -cluster tilting module in $\text{mod } \mathcal{N}_{n,k}$ then $M' = \bigoplus_{h=0}^{m-1} \bigoplus_{(i,j) \in I} M_{i,j+nh}$ is a cluster tilting module in $\text{mod } \mathcal{N}_{mn,k}$.*

Proof. This is a consequence of the fact that the Auslander-Reiten quiver for $\mathcal{N}_{n,k}$ is repeating itself and that the Auslander-Reiten quiver for $\mathcal{N}_{mn,k}$ has the same structure, but with a longer period. Using this with theorem (2.71) the result follows. The details are left for the reader. \square

Given (3.9), let us study the case where $2n = k(d - 1) + 2$. We claim that the module

$$N_1 = A \oplus \bigoplus_{c(X) \in \{0,1\}} X$$

is a d -cluster tilting module. To see this, let us take the example with $n = 5$, $k = 4$ and $d = 3$. Then in picture we have that N_1 is



Using theorem (2.71) we see that $\Omega(N_1) \simeq S_4 \oplus M_{2,4} \oplus M_{3,3}$ and $\Omega^2(N_1) \simeq S_1 \oplus M_{2,1} \oplus M_{3,5}$. It is easy to see that all indecomposable modules, except the ones in N_1 , have a non-zero morphism in the projective stable category from at

least one of $S_3, M_{2,3}, M_{2,4}, S_1, M_{2,1}$ or $M_{3,2}$. Thus $X \in \text{add } N_1$ is equivalent to $\text{Ext}_A^i(N_1, X)$ for $0 < i < 3$. By proposition (3.7) it follows that N_1 is 3-cluster tilting.

By symmetry we of course have that $N_j = A \oplus \bigoplus_{c(X) \in \{j-1, j\}} X$ also is a cluster tilting module. This shows the following lemma.

Lemma 3.12. *Let $\mathcal{N}_{n,k}$ be as above. If $2n = k(d-1) + 2$ then for any indecomposable module $X \in \text{mod } \mathcal{N}_{n,k}$ there exists a module $Y \in \text{mod } \mathcal{N}_{n,k}$ such that $X \oplus Y$ is d -cluster tilting.*

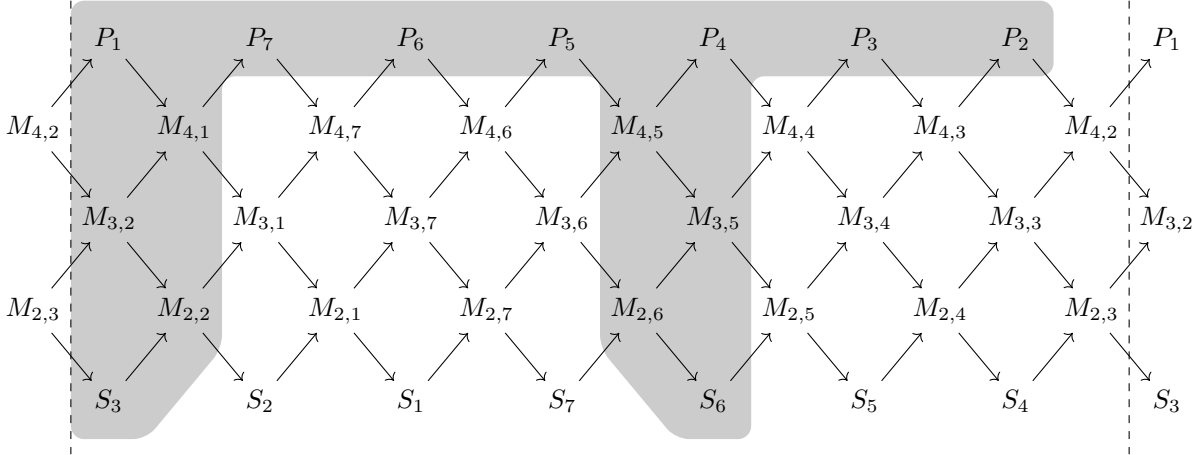
Thus using theorem (3.4) all non-projective modules are $\Omega^{d+1}\nu$ -periodic. In fact, for all non-projective we have, by the proof of (3.4), that $c(\Omega^{d+1}\nu X) \simeq c(\tau_d X) = c(X) - (k(d-1) + 2) = c(X) - 2n \equiv c(X)$ modulo $2n$. Thus, if we collect our above observations, we get that if $2n = k(d-1) + 2$ then for all non-projective indecomposable modules $M_{i,j}$

- $M_{i,j} \simeq \Omega^{d+1}\nu M_{i,j}$ if d is odd.
- $M_{i,j} \simeq (\Omega^{d+1}\nu)^2 M_{i,j}$ if d is even.

Using (3.11) we can expand the above lemma to all cyclic Nakayama algebras such that $k(d-1) + 2 | 2n$, but we first need to clarify one more case. Consider the case when $k(d-1) + 2 = n$. Then we claim that

$$M_1 = A \oplus \bigoplus_{c(X) \in \{0,1,n,n+1\}} X$$

is d -cluster tilting. Indeed, similar to above we will show this via an example. Thus consider $\mathcal{N}_{7,5}$ which has a 2-cluster tilting module on the following form:



Then each "pillar" will have a non-zero Ext_A^1 with all modules to the left of them. Similar to above, we can consider any M_i defined as

$$M_i = A \oplus \bigoplus_{c(X) \in \{i-1, i, n+i-1, n+i\}} X$$

Then we are in a position to prove the following:

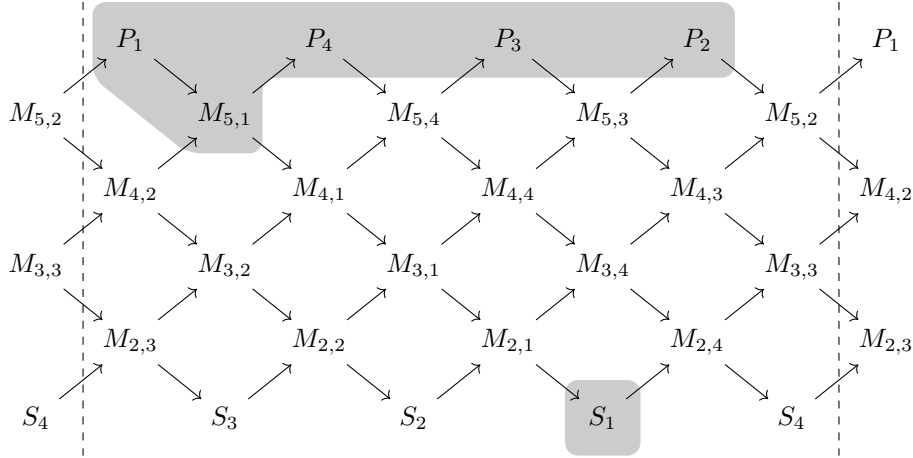
Theorem 3.13. *Let $\mathcal{N}_{n,k}$ be as above. If $k(d-1)+2|2n$ then for any indecomposable module $X \in \text{mod } \mathcal{N}_{n,k}$ there exists a module $Y \in \text{mod } \mathcal{N}_{n,k}$ such that $X \oplus Y$ is d -cluster tilting.*

Proof. Let $m := \frac{2n}{k(d-1)+2}$. If $k(d-1)+2$ is even, this is a direct consequence of proposition (3.11) via considering N_i in $\mathcal{N}_{\frac{n}{m},k}$. If $k(d-1)+2$ is odd it is clear that $k(d-1)+2|n$ and then it follows from proposition (3.11) via considering M_i in $\mathcal{N}_{\frac{2n}{m},k}$. \square

Thus, if $k(d-1)+2|2n$, by theorem (3.4) all non-projectives are $\Omega^{d+1}\nu$ -periodic and by (3.5) they have complexity less than, or equal to, 1. Thus we can calculate the periodicity as

- $M_{i,j} \simeq (\Omega^{d+1}\nu)^m M_{i,j}$ if $(d-1)m$ is even.
- $M_{i,j} \simeq (\Omega^{d+1}\nu)^{2m} M_{i,j}$ if $(d-1)m$ is odd.

The above theorem does not however hold in general. For example, consider $\mathcal{N}_{4,6}$. According to theorem (3.9) there exists at least one 4-cluster tilting module and it can be checked that there does not exist a d -cluster tilting module for any other d . It can be checked that the only 4-cluster tilting modules that do exist are $A \oplus M_{5,i} \oplus S_i$.



Indeed, assume $M_{2,i}$ would be a direct summand of a cluster tilting module. Then so is $\tau_4 M_{2,i} \simeq M_{4,i+1}$, but $\text{Ext}_A^1(M_{2,i}, M_{4,i+1}) \neq 0$. Similarly $\text{Ext}_A^1(\tau_4 M_{4,i}, M_{4,i}) \neq 0$ and $\text{Ext}_A^1(M_{3,i}, \tau M_{3,i}) \neq 0$.

Remark 3.14. We can, as before, conclude that even in this case all non-projective indecomposable modules are both Ω - and $\Omega^{d+1}\nu$ -periodic using our earlier argument with the pigeon hole principle.

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