The Senior Maths Challenge

Advice, tips and tricks on getting the most out of the SMC

Contents

Introduction

The Senior Maths Challenge, or *SMC*, has no defined syllabus, so preparing for it isn't as easy as reading through a textbook. However, there are several topics which are useful to look at and to read around which can help. There are also some tips and tricks which can be useful when tackling SMC problems.

As with all mathematics, the best preparation by far is to practise. Throughout this booklet you will find lots of SMC questions from past papers. The more of them you do, the more used you will be to the style of question you will meet in this year's SMC. At the back of the booklet there is also the whole paper taken from 2010. A set of worked solutions to all of the questions in this booklet will be available from Ms Copin after half term.

Finally, remember that the SMC is not an exam – it's not for any kind of credit – so relax a bit and enjoy the challenge of the problems. After all, how often is it that someone gives you three pages of interesting maths puzzles and promises to make sure nobody disturbs you for the next hour and a half...?

The structure of the SMC

The SMC comprises 25 questions which are designed to be easier at the start and get progressively more challenging as you work through the paper. Each correct answer you give is worth four marks, and for every incorrect answer you give you lose one mark. You have 90 minutes to answer as much of the paper as you can.

The questions under each heading in this booklet are also arranged (approximately) in ascending difficulty.

How to use this booklet

Read the advice for each section, reasoning through any worked examples, and then have a go at one or two of the practice questions. Once you've gone through the whole booklet, then go back over it and try the remaining questions.

There may be some questions which are relevant to more than one section, which rely on you knowing something covered later on in the book. If you think you have hit one of these then try leaving it and read on to the later sections before coming back to it.

At various points, there are extra comments– these are in *[square brackets and italics]*.

GENERAL ADVICE

Be well equipped

Take a couple of spare pens and pencils, a decent eraser and lots of paper. Scrap paper is absolutely fine – you're going to be writing, scribbling and doodling all over it anyway. If you're not that comfortable drawing your diagrams freehand, it might be a good idea to take a ruler or straight edge and a pair of compasses.

Some people also like to jot down a record of their answers on a piece of scrap paper so that they can compare and discuss after they've finished the paper and handed in their answer sheet. Other people can't think of anything worse, so it's up to you whether to do it or not!

Take your time

Very few people complete the SMC in the time allowed (normally about 10 or 15 people manage it in the whole country each year). Don't rush through in an attempt to answer all the questions. The first few questions generally pose little problem to candidates, but take your time over the rest. There is a lot more to be gained from doing a smaller number of questions well rather than bodging solutions to lots of them.

Use all of your time

There is no point in sitting there not doing anything for the last 20 minutes. You aren't going anywhere, so it's worth checking all that you have done and going back to any questions that you skipped earlier. Pick a question you haven't finished yet and have a real go at it.

If you get stuck

It is a sound piece of advice to start at the beginning of the paper and work through (seeing as the questions are designed to get harder as you progress). It is very frustrating to get stuck on a maths problem, especially if it feels like you should be able to do it. If you do get stuck then move on and come back to the one you're stuck on later (if you have time). A startling number of entrants get stuck on (for example) question 19 when they would have no problem with a couple of the later questions. Even if you're not focusing on a question, your subconscious will still be working on it as you work on the others.

That said, don't give up too easily! These problems are designed to be testing. Just because you can't instantly see how to tackle a problem doesn't mean you should immediately give up on it.

Lastly, don't be disheartened if you can't solve a question. Some solutions involve an essential step which you may not see. You can always come back later, and don't worry if you can't work it out.

Is it worth guessing?

Unlike the Junior and Intermediate challenges, **any** incorrect answer in the SMC **will lose you a mark**. Randomly guessing is not recommended. That said, if you are certain that you have managed to exclude three of the given possible answers for a particular question, you will have a 50-50 chance of either gaining four marks or losing one, so you might want to randomly pick one of the two remaining answers.

Know the year

There is almost invariably at least one question involving the year in some way. This may be some interesting property of its digits, or its factors (2002 was quite an interesting one because $2002 = 2 \times 7 \times 11 \times 13$). Often, the first question is a piece of arithmetic involving the year. This first question can be done the "long way round", but will have a much simpler shortcut.

Leap years

Leap years may well feature if you're counting backwards or forwards from a particular date. A year is a leap year if it is divisible by four but not by 100 **unless** it is also divisible by 400. So 2000 was a leap year (divisible by 400) but 1900 was not and 2100 will not be (they are both divisible by 100 but not by 400).

Most months have 31 days in except for April, June, September and November, which have 30, and February, which has 28 in non-leap years and 29 in leap years. You probably know that already (and have a more catchy way of remembering it!).

There are 365 days in a year. $364 = 52 \times 7$, so from one non-leap year to the next, the day moves on by one (e.g. 1 January 2011 was a Saturday, so 1 January 2012 was a Sunday). In a leap year, this moves on two places (due to the extra day in February), so although 1 January 2012 was a Sunday, 1 January 2013 will be a Tuesday.

2011

There isn't a lot to be found that is particularly interesting about 2011! Here are a couple of things I've found – see if you can find anything yourself:

- (2+0)=(1+1). The last time this happened was in 2002 and will next happen in 2020. The last time it happened in a non-palindromic year was in 1982. The next time it will happen in a palindromic date is 2112. *[A number is **palindromic** if when you write it backwards you get the same number (e.g. 4, 2662, 9173719)]*
- ((2*0)+1)=1. This last happened in 2000, and will next happen in 2022.
- There are 501 zeroes on the end of the number 2011!* (this style of question is quite common and is covered in the factorials section on page 22)
- 2011 started and will finish on a Saturday, making Saturday the most common day in the year.
- 2011 is (quite obviously, seeing as its odd) not a leap year.
- 2011 is prime, coming in the sequence …1997, 1999, 2003, 2011, 2017…
  - The difference between the primes leading up to 2011 form the powers of two in ascending order:
    (1997 → 1999): **2**
    (1999 → 2003): **4**
    (2003 → 2011): **8**
    Quite surprisingly, 2011 is the first year ever to display this property.
- It is also the sum of 11 consecutive prime numbers: 2011=157 + 163 + 167 + 173 + 179 + 181 + 191 + 193 + 197 + 199 + 211.

(These last two bullet points are a bit advanced for the SMC. But they looked interesting.)

*[*n factorial, written $n!$, is the product of all the integers up to and including n: $n! = n \times (n-1) \times \ldots \times 3 \times 2 \times 1$]*

Here are a couple of worked examples:

E1

What is the value of $\dfrac{2007}{9} + \dfrac{7002}{9}$?

A 500.5   B 545   C 1001   D 1655   E 2007

Answer: You could, in theory, work out $\frac{2007}{9}$ and $\frac{7002}{9}$ and add them together. But it's much easier to notice that:

$$\frac{2007}{9} + \frac{7002}{9} = \frac{9009}{9} = 1001$$

So the answer is **C**.

E2

What is the remainder when the 2008-digit number 222 ... 22 is divided by 9?

A 8   B 6   C 4   D 2   E 0

Answer: A number is divisible by 9 if (and only if) its digit sum is also divisible by 9.The number 222 … 22 (call it X) has a digit sum of (2008×2). If you replaced the last digit with a 0 then the digit sum would be (2007×2), which is divisible by 9 (the digit sum of 2007 is $2 + 0 + 0 + 7 = 9$, so 2007 is divisible by 9) and so 222 … 220 is also divisible by 9). We can therefore write the number 222 … 220 as $9k$ for some integer k. So it follows immediately that $(= 9k + 2)$ has a remainder of 2 when divided by 9.

So the answer is **D**.

Practice questions

Q1

What is the value of $2 \times 2008 + 2008 \times 8$?

A 4016 B 16064 C 20080 D 64256 E 80020

Q2

Given that January 1st, 2006 fell on a Sunday, which day of the week will occur most frequently in 2007?

A Monday B Tuesday C Wednesday D Thursday E Friday

Q3

Consider the arithmetic sequences 1998, 2005, 2012, ... and 1996, 2005, 2014, ... Which is the next number after 2005 that appears in both sequences?

A 2054 B 2059 C 2061 D 2063 E 2068

Q4

Three consecutive even numbers are such that the sum of four times the smallest and twice the largest exceeds three times the second by 2006. What is the sum of the digits of the smallest number?

A 8 B 11 C 14 D 17 E 20

Try it with numbers

If you're faced with a question that contains some algebra that's starting to get a bit messy, try plugging in a couple of numbers to examine some cases of what you've been given. Keep your eyes open for a pattern which might point you in the right direction.

This trick can also be employed when thinking about geometry questions – If you have an unknown length, assign it a value (temporarily). Pick your value carefully so that it's helpful and not a hindrance (e.g. if you're working with a length you have to both bisect and trisect, a length of 6 might be useful, whereas a length of 7 wouldn't). 6, 12, 24 and 60 can all often be useful numbers in this context.

This is best demonstrated with a couple of worked examples:

E3

What is the greatest number of the following five statements about numbers a, b which can be true at the same time?

$\frac{1}{a} < \frac{1}{b}$ $\quad a^2 > b^2$ $\quad a < b$ $\quad a < 0$ $\quad b < 0$

A 1 B 2 C 3 D 4 E 5

Answer: It is obvious that you can have at least three of these simultaneously true: if a and b are both less than 0 and $a < b$. What happens if you plug in a few numbers for which these three inequalities hold?

If we try $a = -2$ and $b = -1$, then we also notice that $a^2 = 4 > 1 = b^2$, so now we have four of the five inequalities holding. What about the first inequality? With these values, $\frac{1}{a} = -\frac{1}{2} > -1 = \frac{1}{b}$, so it doesn't hold. But how about in general? Looking at our special case it looks like it's not going to be true – we just need to *prove* it. If we look again at the first inequality, it can be restated as $\frac{1}{a} - \frac{1}{b} < 0$. But $\frac{1}{a} - \frac{1}{b} = \frac{b-a}{ab}$, which we know to be positive because we defined b to be greater than a and ab is positive. Hence $\frac{1}{a} - \frac{1}{b} > 0$ for all such a and b, meaning that all five inequalities cannot hold at the same time.

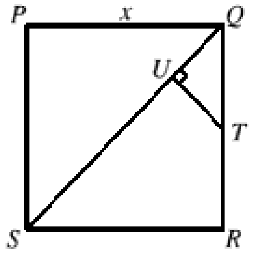
So the answer is **D**.

[Note: Plugging in numbers to give an example did not ***prove*** *our ideas – once we'd looked at our example and decided that the first inequality couldn't hold, we still needed to prove that this was the case. Playing around with some values did, however, make it easier to see what was going on and get our heads round the problem.]*

E4

A square $PQRS$ has sides of length x. T is the midpoint of QR and U is the foot of the perpendicular from T to QS. What is the length of TU?

A $\frac{x}{2}$ B $\frac{x}{3}$ C $\frac{x}{\sqrt{2}}$ D $\frac{x}{2\sqrt{2}}$ E $\frac{x}{4}$



Answer: Let's give x a value and see where it gets us. Firstly, notice that in the triangle QUT, angles $\mathrm{Q\hat{T}U}$ and $\mathrm{U\hat{Q}T}$ are both $45°$ and $\frac{x}{2}$ is going to be the hypotenuse. So let's make x equal to $2\sqrt{2}$. Now look what happens:

Seeing as $\mathrm{Q\hat{T}U}$ and $\mathrm{U\hat{Q}T}$ are both 45°, triangle QUT is isosceles. So, by Pythagoras' Theorem, we have:

$$\mathrm{QT}^2 = \left(\frac{2\sqrt{2}}{2}\right)^2 = 2 = \mathrm{QU}^2 + \mathrm{TU}^2 = 2\mathrm{TU}^2$$

So $\mathrm{TU} = 1$ (seeing as TU is a length it can never be negative). If we then consider TU to be equal to yx for some positive number y, we quickly reach $y = \frac{\mathrm{TU}}{x} = \frac{1}{2\sqrt{2}}$ i.e. $\mathrm{TU} = \frac{x}{2\sqrt{2}}$.

Notice that y doesn't change even if we start with a different value for x, but the value we chose made it easy.

So the answer is **D**.

[Why choose $2\sqrt{2}$ for x? If you're struggling to see why, have a look at the 1,1,√2 triangle on page 27 for inspiration.]

Practice questions

Q5

A sequence of positive integers $t_1, t_2, t_3, t_4, \ldots$ is defined by:
$t_1 = 13;\ t_{n+1} = \frac{1}{2}t_n$ if t_n is even; $t_{n+1} = 3t_n + 1$ if t_n is odd.
What is the value of t_{2008}?

A 1 B 2 C 4 D 8 E None of these.

Q6

An engineer is directed to a faulty signal, one quarter of the way into a tunnel. Whilst there, he is warned of a train heading towards the tunnel entrance. The engineer can run at 12 mph and can either run back to the tunnel entrance or forward to the exit. In either case, the engineer and the front of the train would reach the entrance or exit together. What is the speed in mph of the train?

A 16 B 20 C 24 D 32 E more information needed

Look at the options

Often, instead of directly working out the answer, you can deduce what the answer is by eliminating the other four possibilities or by testing the available answers to see whether they have a given property.

Practice questions

Q7

The factorial of n, written $n!$, is defined by $n! = 1 \times 2 \times 3 \times \ \dots \ \times (n-2) \times (n-1) \times n$.
Which of the following values of n provides a counterexample to the statement:
"If n is a prime number, then $n! + 1$ is also a prime number"?

A 1 B 2 C 3 D 4 E 5

Q8

Which symbol should replace $\oplus$ to make the following equation true?

$$1 \times 2 \times (3 \oplus 4 + 5) \times (6 \times 7 + 8 + 9) = 2006.$$

A + B − C ÷ D × E none of these

Q9

Which one of the following rational numbers *cannot* be expressed as $\frac{1}{m} + \frac{1}{n}$ where m, n are different positive integers?

A $\frac{3}{4}$ B $\frac{3}{5}$ C $\frac{3}{6}$ D $\frac{3}{7}$ E $\frac{3}{8}$

Is your answer sensible?

This principle is often forgotten in maths. Pay attention to your answers. It is sometimes obvious you've slipped up somewhere if you use some common sense. Varieties of this kind of error include giving a triangle one side longer than the sum of the other two, or not checking that the internal angles of a quadrilateral add up to 360 degrees.

A couple of examples demonstrate this idea:

E5

What is 20% of 30%?

A 6% B 10% C 15% D 50% E 60%

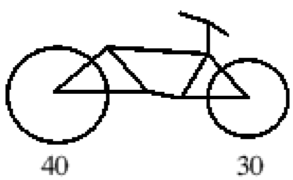
Answer: 20% of $30\% = 30\% \div 5 = (30 \div 5)\% = 6\%$.

So the answer is **A**.

[More important than the answer here is to look at the other options. Two of them (D and E) are more than 30%, and yet the question asks you to find an answer which is less than 100% of 30% (i.e. less than 30%). So before even attempting the question, you should be able to eliminate those D and E as possible answers.]

E6

Boris Biker entered the Tour de Transylvania with an unusual bicycle whose back wheel is larger than the front. The radius of the back wheel is 40 cm, and the radius of the front wheel is 30 cm. On the first stage of the race the smaller wheel made 120000 revolutions. How many revolutions did the larger wheel make?



A 90000 B 90000π C 160000 D $\frac{160000}{\pi}$ E 120000

Answer: Seeing as the larger wheel is has a larger circumference than the smaller one, it will need to complete fewer complete revolutions to cover the same distance. So we can eliminate B, C and E from our calculations, leaving just two possibilities. The circumferences of the wheels are in the ratio 3:4 (the same as the ratio of the radii), so for every three complete revolutions the larger wheel makes, the smaller wheel makes four. Hence if the smaller wheel made 120000 revolutions, the larger one made $(120000 \div 4) \times 3 = 90000$ revolutions.

So the answer is **A**.

[Notice here that if you get the 3 and the 4 the wrong way round, you reach the answer of 160000 (which is option C). Thinking physically about the problem for a moment at the start meant that we avoided making that mistake because we would have noticed that our answer didn't make sense.]

"More information needed"

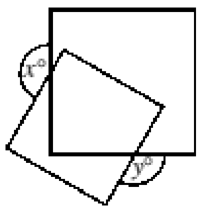
At least a couple of SMC questions every year have one possible answer listed as "More information needed", or "It depends on …", or some other similar phrase. In 12 years' experience of Maths Challenges, I have NEVER seen this be the correct answer. That's not to say it won't be, but if you do think that's the answer then check your reasoning very carefully!

Practice questions

Q10

The diagram shows overlapping squares.
What is the value of $x + y$?

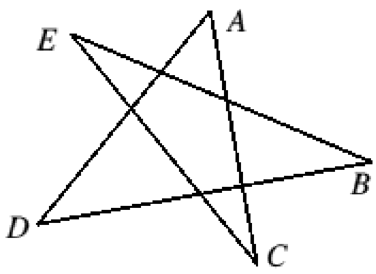
A 270 B 300 C 330
D 360 E more information needed



Q11

In the figure shown, what is the sum of the interior angles at A, B, C, D, E?

A 90° B 135° C 150° D 180°
E more information required.



Be methodical!

This is probably the single most important piece of advice in this booklet. A lot of the questions in the SMC aren't particularly difficult, nor do they require any especially advanced or unusual mathematics or complex logical thinking. They simply require you to be *methodical*. If you can be methodical, working logically and in a sensible order, then you should succeed in the SMC. In particular, when working through sets of things, try to do this in a logical order to make sure you don't miss something. If it helps, draw a diagram to get your head around what you're doing (even if it's not a geometry question, a little sketch picture can be helpful).

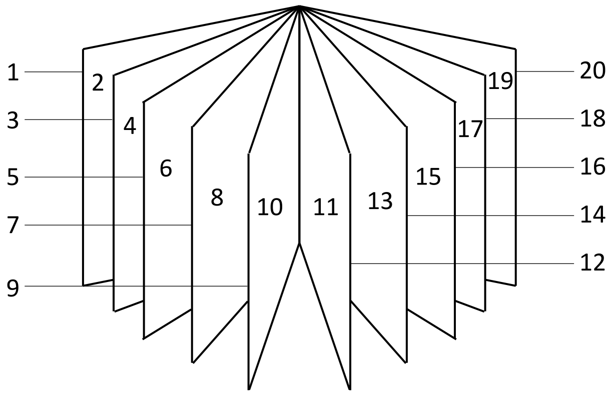
Let's have a look at a couple of examples:

E7

An examination paper is made by taking 5 large sheets of paper, folding the pile in half and stapling it. The pages are then numbered in order from 1 to 20. What is the sum of the three page numbers that are on the same sheet of paper as page number 5?

A 13 B 21 C 33 D 37 E 41

Answer: Draw yourself a picture, or, even better, get five sheets of scrap paper and fold them in half into a booklet shape. Then number the pages before taking it apart to check which other pages are on the same sheet as page 5:



Either way, you should quickly find that page 5 is on the same sheet as pages 6, 15 and 16, which sum to 37.

So the answer is **D**.

E8

The year 1789 (when the French Revolution started) has three and no more than three adjacent digits (7, 8 and 9) which are consecutive integers in increasing order. How many years between 1000 and 9999 have this property?

A 130 B 142 C 151 D 169 E 180

Answer: Working through in a methodical fashion will give us the answer we need:

All years are of the form ABCD where A, B, C and D may be any digit except that $\mathrm{A} \neq 0$. Then we can either have $\mathrm{B} = \mathrm{A} + 1$ and $\mathrm{C} = \mathrm{A} + 2$ (with the condition that $\mathrm{D} \neq \mathrm{A} + 3$, else we'll have more than three adjacent digits in order), **or** $\mathrm{C} = \mathrm{D} - 1$ and $\mathrm{B} = \mathrm{D} - 2$ (with the condition that $\mathrm{A} \neq \mathrm{D} - 3$, for the same reason as before).

Looking at the first type, if we have $\mathrm{A} = 1,2,3,4,5$ or 6, there are 9 possibilities for D. If $A = 7$, then there are no restrictions on D (D can't be 10!), so we have 10 possibilities for D. This gives a total of $(6 \times 9 + 10) = \mathbf{64}$ **years**.

Looking at the second type, if we have $\mathrm{D} = 9,8,7,6,5$ or 4, there are 8 possibilities for A. If, however, we have $\mathrm{D} = 3$ or 2, our condition $A \neq D - 3$ no longer restricts the possible values for A (seeing as A can't be 0 or -1 anyway), so there are 9 possibilities for A. This gives a total of $(6 \times 8 + 2 \times 9) = \mathbf{66}$ **years**. Adding these together gives a total of **130 years**.

[We know that there is no crossover between these two sets of possible years because we restricted $D \neq A + 3$ and $A \neq D - 3$ at the start.]

So the answer is **A**.

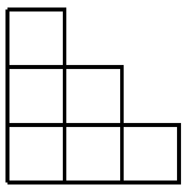
[There are several approaches to this question, with no one better or worse than any other. The method above does demonstrate how a methodical ensures that no years are missed or counted twice.]

Practice questions

Q12

It is required to shade at least one of the six small squares in the diagram on the right so that the resulting figure has exactly one axis of symmetry. In how many different ways can this be done?

A 6 B 9 C 10 D 12 E 15



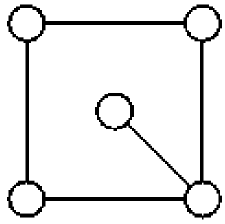
Q13

How many whole numbers between 1 and 2007 are divisible by 2 but not by 7?

A 857 B 858 C 859 D 860 E 861

Q14

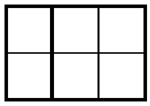
The diagram shows five discs connected by five line segments.
Three colours are available to colour these discs.
In how many different ways is it possible to colour all five discs if discs which are connected by a line segment are to have different colours?



A 6 B 12 C 30 D 36 E 48

Q15

How many hexagons can be found in the diagram on the right if each side of a hexagon must consist of all or part of one of the straight lines in the diagram?



A 4 B 8 C 12 D 16 E 20

Q16

A hockey team consists of 1 goalkeeper, 4 defenders, 4 midfielders and 2 forwards. There are 4 substitutes: 1 goalkeeper, 1 defender, 1 midfielder and 1 forward. A substitute may only replace a player of the same category eg: midfielder for midfielder. Given that a maximum of 3 substitutes may be used and that there are still 11 players on the pitch at the end, how many different teams could finish the game?

A 110 B 118 C 121 D 125 E 132

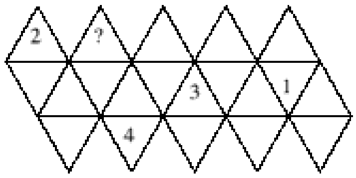
Q17

A bracelet is to be made by threading four identical red beads and four identical yellow beads onto a hoop. How many different bracelets can be made?

A 4 B 8 C 12 D 18 E 24

Q18

The net shown is folded into an icosahedron and the remaining faces are numbered such that at each vertex the numbers 1 to 5 all appear. What number must go on the face with a question mark?



A 1 B 2 C 3 D 4 E 5

Read the question and answer the question

There are so many mistakes people make in all walks of life because they haven't read the question they've been asked properly! This sounds like such an obvious piece of advice, but one that is worth listening to. Remember that, if you're given a piece of information in the question, you will probably need to use it at some point in arriving at answer (the SMC question setters aren't *that* devious!); If you've only used half of the information you were given, the chances are you've missed something.

Also, *answer the question that is asked*. If you are asked to work out what $a + b + c$ is, you may not need (or even be able) to work out distinct values for a, b and c. Look at this example:

E9

The numbers x, y and z satisfy the equations

$$x + y + 2z = 850, \qquad x + 2y + z = 950, \qquad 2x + y + z = 1200.$$

What is their mean?

A 250 B $\frac{1000}{3}$ C 750 D 1000 E More information is needed.

Answer: If we simply add the three equations, we get:

$$4x + 4y + 4z = 3000$$

i.e. $x + y + z = 750$, which means that their average is 250.

So the answer is **A**.

*[It is possible, **but not necessary**, to solve these three equations to get x=450, y=200 and z=100.]*

Practice questions

Q19

Sam and Pat were counting their money. They discovered that if Sam gave Pat £5, then Pat would have 5 times as much as Sam, but if Pat gave Sam £5, then Sam would have 5 times as much as Pat. How much did they have altogether?

A £10 B £15 C £20 D £25 E £30

Q20

If $6x - y = 21$ and $6y - x = 14$, what is the value of $x - y$?

A 1 B 2 C 3 D 4 E 5

Q21

Which of the five expressions shown has a different value from the other four?

A 2^8 B 4^4 C $8^{8/3}$ D 16^2 E $32^{6/5}$

Q22

Suppose that $x - \frac{1}{x} = y - \frac{1}{y}$ and $x \neq y$. What is the value of xy?

A 4 B 1 C -1 D -4 E more information is needed

Q23

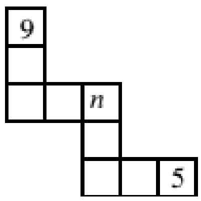
The four statements in the box on the right refer to a mother and her four daughters. One statement is true, three statements are false. Who is the mother?

Alice is the mother.
Carol and Ella are both daughters.
Beth is the mother.
One of Alice, Diane or Ella is the mother.

A Alice B Beth C Carol D Diane E Ella

Q24

The digits 1, 2, 3, 4, 5, 6, 7, 8, and 9 are to be written in the squares so that every row and every column of three squares has a total of 13. Two numbers have already been entered. What is the value of n?



A 2 B 4 C 6 D 7 E 8

Q25

A positive number $a = [a] + \{a\}$ where $[a]$ is the integer part of a and $\{a\}$ is the fractional part of a.
Given that $x + [y] + \{z\} = 4.2$, $y + [z] + \{x\} = 3.6$, $z + [x] + \{y\} = 2.0$, and $x, y, z > 0$, what is the value of $\{y\}$?

A 0.1 B 0.3 C 0.5 D 0.7 E 0.9

TIPS AND TRICKS FOR SUCCESS

There are some things that are useful to notice or to know for different questions.

Geometry questions

Geometry questions are common in the SMC. Let's look a few tips for tackling them *[Practice questions start on p16]*:

Diagrams

The diagrams on the paper are tiny! There is no way you will be able to mark on everything you need, chase angles etc. on those things. Get plenty of paper (as mentioned already) and draw yourself a nice big diagram which you can doodle and jot down angles on, look at upside down, add extra lines to, and mark properties you notice on. A pair of compasses here might be useful if you aren't very comfortable drawing freehand circles, as might a straight edge.

In some questions, you aren't given a diagram and instead you have something described to you. If this is the case, it is definitely worth drawing yourself a picture – it will help you see what is going on in the question.

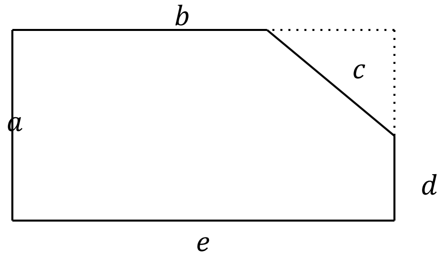
Try to avoid giving things properties may not have (e.g. parallel lines, isosceles triangles etc.) in your diagrams.

Let's look at a couple of examples:

E10 A triangle is cut from the corner of a rectangle. The resulting pentagon has sides of length 8, 10, 13, 15 and 20 units, though not necessarily in that order. What is the area of the pentagon?

A 252.5 B 260 C 270 D 275.5 E 282.5

Answer: They don't give us a picture in this question, so let's draw one:



Now we can visualise what we're dealing with. You can see that $\mathrm{e} = 20$ (seeing as it's the longest side*) leaving us to find a, b, c and d. *[*There is a chance that* c *is longer than* e, *but a quick play with the numbers shows that it isn't.]*

By Pythagoras' Theorem:

$$c^2 = (a-d)^2 + (20-b)^2$$

Considering the different possible values for c one at a time:

- $c = 8$. This is not possible: no Pythagorean triple* exists with a hypotenuse of 8.
- $c = 10$. Then $(a-d)$ and $(20-b)$ are 6 and 8 in some order. But using only 8,13 and 15 it is not possible to make 6 or 8 from $(20-b)$.
- $c = 13$, giving us $(a-d)$ and $(20-b)$ equal to 5 and 12 in some order. This is true when (and only when) $a = 15$, $b = 8$ and $d = 10$.
- Even though we think we've found an answer, we should check the case $c = 15$. Then $(a-d)$ and $(20-b)$ are 9 and 12 in some order. Now setting $b = 8$ gives us $(20-b) = 12$, but then we cannot make $(a-d) = 9$ using just 10 and 13. So, as expected, there is only the one solution when $c = 13$.

Then we can calculate the total area of the pentagon by subtracting the area of the triangle from the area of the whole rectangle:

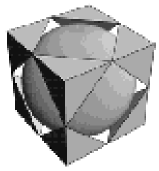
$$\mathrm{A} = 20 \times 15 - \frac{1}{2} \times (20-8) \times (15-10) = 300 - 30 = 270$$

So the answer is **C**.

*[*A* ***Pythagorean triple*** *is a set of three integers,* $\mathrm{a,b}$ *and* c, *that satisfy* $\mathrm{a}^2 + \mathrm{b}^2 = \mathrm{c}^2$. *See page 14 for more details.]*

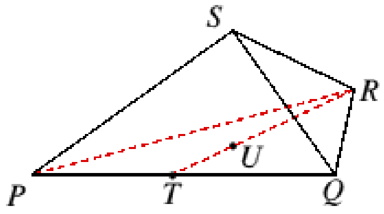
E11

A paperweight is made from a glass cube of side 2 units by first shearing off the eight tetrahedral corners which touch at the midpoints of the edges of the cube. The remaining inner core of the cube is discarded and replaced by a sphere. The eight corner pieces are now stuck onto the sphere so that they have the same positions relative to each other as they did originally. What is the diameter of the sphere?



A $\sqrt{8}-1$ B $\sqrt{8}+1$ C $\frac{1}{3}(6+\sqrt{3})$ D $\frac{4}{3}\sqrt{3}$ E $2\sqrt{3}$

Answer: The sphere has diameter $l-2h$ where l is the length of the diagonal of the cube and h is the height of one of the eight tetrahedral corners. These corners look like the diagram to the right. In this diagram, S is the corner of the cube, and sits directly over U, which is the centroid* of equilateral triangle PQR so U lies a third of the way from T to R (T is the midpoint of PQ) *[*see p14]*. The height of these tetrahedrons, SU, is equal to h. $\mathrm{PS}=\mathrm{QS}=\mathrm{RS}=1$.



By Pythagoras' Theorem, the sides of triangle PQR have length $\sqrt{2}$. Since $\mathrm{R\hat{T}P}$ is a right angle, we can use Pythagoras to deduce that $\mathrm{RT}^2=\mathrm{RP}^2-\mathrm{TP}^2=\sqrt{2}^2-\left(\frac{\sqrt{2}}{2}\right)^2=\frac{3}{2}$. We know that $\mathrm{UT}=\frac{2}{3}\mathrm{RT}$, so $\mathrm{RU}^2=\left(\frac{2}{3}\mathrm{RT}\right)^2=\frac{4}{9}\times\frac{3}{2}=\frac{2}{3}$.

So $h^2=\mathrm{SU}^2=\mathrm{SR}^2-\mathrm{RU}^2=1-\frac{2}{3}=\frac{1}{3}$ i.e. $h=\frac{1}{\sqrt{3}}=\frac{\sqrt{3}}{3}$.

By Pythagoras in 3D *[Covered on p14]*, $l^2=2^2+2^2+2^2=12$ so $l=\sqrt{12}=2\sqrt{3}$

So the diameter of the sphere $=l-2h=2\sqrt{3}-\frac{2}{3}\sqrt{3}=\frac{4}{3}\sqrt{3}$

So the answer is **D**.

[See also the section on Surds on page 25]

Chasing angles

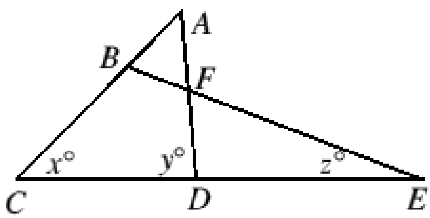
Sometimes you will have to do a bit of angle chasing. When doing this, keep a look out for right angles (including those between a tangent and a radius), pairs of parallel lines and isosceles triangles because they may help you out.

It's also often useful to know the internal angles of a regular polygon: [(n–2) × 180/n]: 60, 90, 108, 120, 128⁴⁄₇, 145…

Here's an example question to have a look at:

E12

In the figure shown, $AB = AF$ and ABC, AFD, BFE and CDE are all straight lines.
Which of the following expressions gives z in terms of x and y?



A $\frac{y-x}{2}$ B $y-\frac{x}{2}$ C $\frac{y-x}{3}$ D $y-\frac{x}{3}$ E $y-x$

Answer: From triangle ACD, angle $\mathrm{C\hat{A}D}=(180-\mathrm{x}-\mathrm{y})°$. Since triangle ABF is isosceles, $\mathrm{A\hat{B}F}=\mathrm{A\hat{F}B}=180-(180-\mathrm{x}-\mathrm{y})=\frac{1}{2}(\mathrm{x}+\mathrm{y})°$. $\mathrm{D\hat{F}E}=\mathrm{A\hat{F}B}=\frac{1}{2}(\mathrm{x}+\mathrm{y})°$ because they are vertically opposite. $\mathrm{F\hat{D}E}=180-\mathrm{y}$. So we now have $z=180-(\mathrm{F\hat{D}E}+\mathrm{D\hat{F}E})=180-\left[(180-\mathrm{y})+\frac{1}{2}(x+y)\right]°=y-\frac{1}{2}(x+y)=\frac{1}{2}(y-x)$

So the answer is **A**.

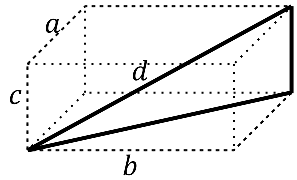
[This is also another question where you may find it helpful to try it with some numbers and see what happens.]

Triangles, Pythagoras and trigonometry

If you can see a right-angled triangle, try whacking it with Pythagoras' Theorem and see what you get.

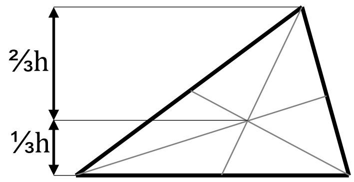
Remember that you can apply Pythagoras in three dimensions:

$$d^2 = a^2 + b^2 + c^2$$



Some sets of Pythagorean triples (integers which are the edges of right-angled triangles) include: [3,4,5]; [5,12,13]; [6,8,10] (which you will notice is just a [3,4,5] triangle but with doubled edges) and [7,24,25].

If you join the midpoints of the sides of a triangle to the opposite vertices (see right), the three lines you have drawn (called the *medians*) will meet at a single point, called the **centroid**. It can be useful to know that this is always a third of the way up the triangle.



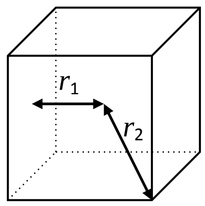
Some basic trigonometry is also useful, including the properties we see in the $[1,\sqrt{3},2]$ triangle and the $[1,1,\sqrt{2}]$ triangle (more on these is covered on page 27).

Let's look at an example question:

E13

A cube exactly fits inside a sphere and another sphere exactly fits inside this cube. What is the ratio of the volume of the smaller sphere to the volume of the larger sphere? A $1:3\sqrt{3}$ B $1:4$ C $1:3$ D $2:3$ E $1:2$

Answer: Drawing the whole lot might get a bit messy, so let's just draw the cube in the middle. Then the radius of the smaller sphere (call it r_1) is half the width and the radius of the large sphere (r_2) is half of the long diagonal:



By Pythagoras' Theorem (in 3 dimensions):

$$r_2^2 = r_1^2 + r_1^2 + r_1^2$$

Giving $r_2 = \sqrt{3}r_1$. Since the volume of a sphere is proportional to the cube of its radius (i.e. $\mathrm{V} = kr^3$ for some fixed k), we have:

$$\mathrm{V}_2 = k{r_2}^3 = k\left(\sqrt{3}r_1\right)^3 = \sqrt{3}^3 kr_1^3 = 3\sqrt{3}\mathrm{V}_1$$

i.e. the ratio of the areas is $1:3\sqrt{3}$. *[Note that we know that* k *is equal to* $\frac{4\pi}{3}$*, but its value isn't important]*

So the answer is **A**.

Circles and pi

Circles are round. You'll have encountered them before. There are a couple of things to say about circles:

Circle Theorems

Remember these? They can be handy when tackling some SMC problems. The ones most commonly used are:

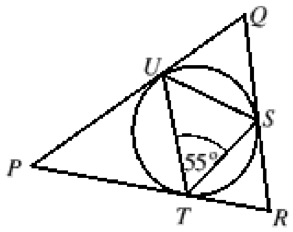
- Angle at the centre are twice the angle at the circumference
- The Alternate Segment Theorem
- A radius and tangent that meet at a point on the circumference are perpendicular
- Two tangents to the same circle from the same external point are equal in length

Here is a worked example to demonstrate:

E14

The largest circle which it is possible to draw inside triangle PQR touches the triangle at S, T and U, as shown in the diagram.
The size of $\angle STU = 55°$. What is the size of $\angle PQR$?

A $55°$ B $60°$ C $65°$ D $70°$ E $75°$



Answer: By the Alternate Segment Theorem, angle $\mathrm{Q\hat{U}S} = 55°$. Since QU and QS are the same length (tangents from the same point), triangle QUS is isosceles, so $\mathrm{Q\hat{S}U} = \mathrm{Q\hat{U}S}$. $\mathrm{P\hat{Q}R} = \mathrm{U\hat{Q}S} = 180 - (2 \times 55) = 70°$.

So the answer is **D**.

Circles and Triangles

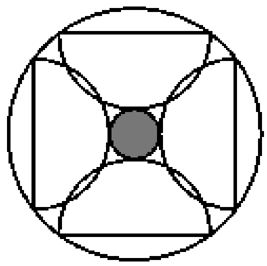
Many questions involving circles in the SMC also involve triangles – often you'll need to draw one or more radii of or tangents to one or more circles (often to the points where these circles touch other things, including other circles). As in the previous section, some basic trig knowledge can be useful here.

Worked example:

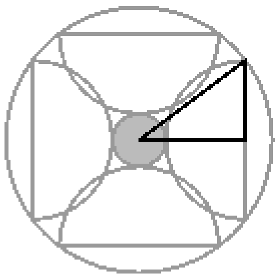
E15

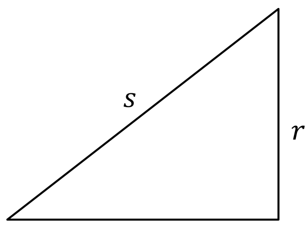
The diagram shows four semicircles symmetrically placed between two circles. The shaded circle has area 4 and each semicircle has area 18. What is the area of the outer circle?

A $72\sqrt{2}$ B 100 C 98 D 96 E $32\sqrt{3}$



Answer: This question can seem a little inaccessible, at first, but in fact you only need to draw one triangle to make it much clearer:





If we let q, r and s be the radii of the small circle, the semicircles and the large circle respectively, then the sides of this right-angled triangle are $(\mathrm{q} + \mathrm{r})$, r and s, as above. The area of the outer circle is πs^2.

Using Pythagoras' Theorem:

$$s^2 = (q + r)^2 + r^2$$

Given that $\pi q^2 = 4$, $q = \sqrt{\frac{4}{\pi}} = \frac{2}{\sqrt{\pi}}$. Similarly, $\pi r^2 = 2 \times 18 = 36$, so $r = \sqrt{\frac{36}{\pi}} = \frac{6}{\sqrt{\pi}} = 3q$.

So we have:

$$s^2 = (q + 3q)^2 + 3q^2 = (4q)^2 + (3q)^2 = 25q^2$$

Which gives us:

$$\text{Area} = \pi s^2 = \pi \times 25q^2 = 25(\pi q^2) = 25 \times 4 = 100$$

So the answer is **B**.

Geometry practice questions

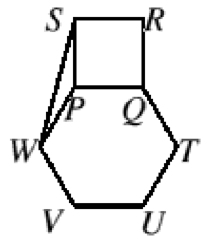
[A note on these questions: Geometry questions often take up a significant chunk of the end of the paper because of the time-consuming nature of their solutions. Some of these practice questions are consequently relatively tough and there are quite a few of them to get your teeth into. As with all the sets of questions in this booklet, they are designed to get harder as you progress through them.]

Q26

The diagram shows square $PQRS$ and regular hexagon $PQTUVW$.

What is the size of $\angle PSW$?

A 10° B 12° C 15° D 24° E 30°



Q27

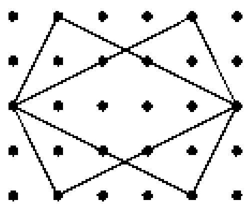
How many differently shaped triangles exist in which no two sides are the same length, each side is of integral unit length and the perimeter of the triangle is less than 13 units?

A 2 B 3 C 4 D 5 E 6

Q28

The distance between two neighbouring dots in the dot lattice is 1 unit. What, in square units, is the area of the region where the two rectangles overlap?

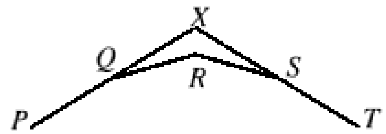
A 6 B $6\frac{1}{4}$ C $6\frac{1}{2}$ D 7 E $7\frac{1}{2}$



Q29

P, Q, R, S, T are vertices of a regular polygon. The sides PQ and TS are produced to meet at X, as shown in the diagram, and $\angle QXS = 140°$. How many sides does the polygon have?

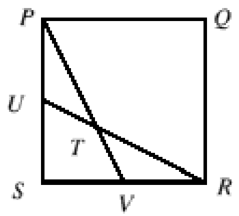
A 9 B 18 C 24 D 27 E 40



Q30

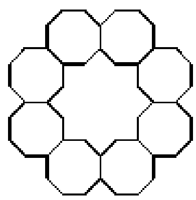
$PQRS$ is a square with U and V the mid-points of the sides PS and SR respectively. Line segments PV and UR meet at T.
What fraction of the area of the square $PQRS$ is the area of the quadrilateral $PQRT$?

A $\frac{1}{2}$ B $\frac{5}{8}$ C $\frac{2}{3}$ D $\frac{3}{4}$ E $\frac{5}{9}$



Q31

Eight identical regular octagons are placed edge to edge in a ring in such a way that a symmetrical star shape is formed by the interior edges.
If each octagon has sides of length 1, what is the area of the star?

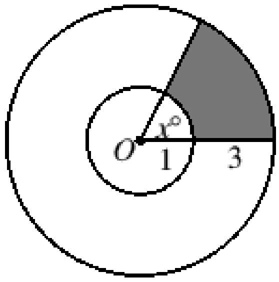


A $5+10\sqrt{2}$ B $8\sqrt{2}$ C $9+4\sqrt{2}$ D $16-4\sqrt{2}$ E $8+4\sqrt{2}$

Q32

The point O is the centre of both circles and the shaded area is one-sixth of the area of the outer circle.
What is the value of x?

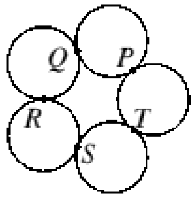
A 60 B 64 C 72 D 80 E 84



Q33

Five touching circles each have radius 1 and their centres are at the vertices of a regular pentagon. What is the radius of the circle through the points of contact P, Q, R, S and T?

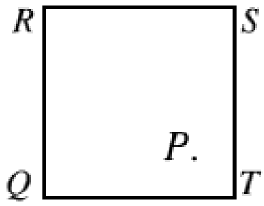
A $\tan 18°$ B $\tan 36°$ C $\tan 45°$ D $\tan 54°$ E $\tan 72°$



Q34

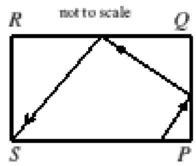
A point P is chosen at random inside a square $QRST$. What is the probability that $\angle RPQ$ is acute?

A $\frac{3}{4}$ B $\sqrt{2}-1$ C $\frac{1}{2}$ D $\frac{\pi}{4}$ E $1-\frac{\pi}{8}$



Q35

A toy pool table is 6 feet long and 3 feet wide. It has pockets at each of the four corners P, Q, R and S. When a ball hits a side of the table, it bounces off the side at the same angle as it hit that side. A ball, initially 1 foot to the left of pocket P, is hit from the side SP towards the side PQ as shown.



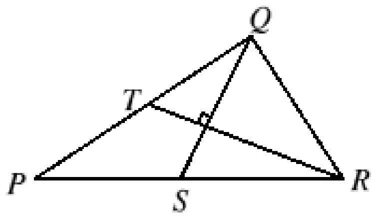
How many feet from P does the ball hit side PQ if it lands in pocket S after two bounces?

A 1 B $\frac{6}{7}$ C $\frac{3}{4}$ D $\frac{2}{3}$ E $\frac{3}{5}$

Q36

In triangle PQR, S and T are the midpoints of PR and PQ respectively; QS is perpendicular to RT; $QS = 8$; $RT = 12$.
What is the area of triangle PQR?

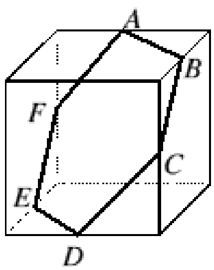
A 24 B 32 C 48 D 64 E 96



Q37

A solid red plastic cube, volume 1 cm^3, is painted white on its outside. The cube is cut by a plane passing through the mid-points of various edges, as shown.
What, in cm^2, is the *total* red area exposed by the cut?

A $\frac{3\sqrt{3}}{2}$ B 2 C $\frac{9\sqrt{2}}{5}$ D 3 E $\frac{3(\sqrt{3}+\sqrt{2})}{4}$



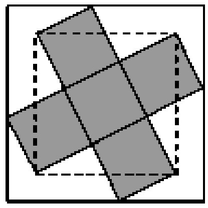
Q38

The sum of the lengths of the 12 edges of a cuboid is x cm. The distance from one corner of the cuboid to the furthest corner is y cm. What, in cm^2, is the total surface area of the cuboid?

A $\frac{x^2 - 2y^2}{2}$ B $x^2 + y^2$ C $\frac{x^2 - 4y^2}{4}$ D $\frac{xy}{6}$ E $\frac{x^2 - 16y^2}{16}$

Q39

A figure in the shape of a cross is made from five 1×1 squares, as shown. The cross is inscribed in a large square whose sides are parallel to the dashed square, formed by four of the vertices of the cross. What is the area of the large outer square?



A 9 B $\frac{49}{5}$ C 10 D $\frac{81}{8}$ E $\frac{32}{3}$

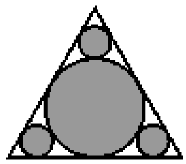
Q40

The length of the hypotenuse of a particular right-angled triangle is given by $\sqrt{1+3+5+7+\ \ldots\ +25}$. The lengths of the other two sides are given by $\sqrt{1+3+5+\ \ldots\ +x}$ and $\sqrt{1+3+5+\ \ldots\ +y}$ where x and y are positive integers. What is the value of $x + y$?

A 12 B 17 C 24 D 28 E 32

Q41

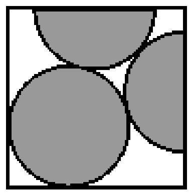
The diagram shows four touching circles, each of which also touches the sides of an equilateral triangle with sides of length 3. What is the area of the shaded region?



A $\frac{11\pi}{12}$ B π C $\frac{(4+\sqrt{3})\pi}{6}$ D $\frac{(3+\sqrt{3})\pi}{4}$ E $\frac{37\pi}{36}$

Q42

In the diagram, the circle and the two semicircles have radius 1. What is the perimeter of the square?



A $6 + 4\sqrt{2}$ B $2 + 4\sqrt{2} + 2\sqrt{3}$ C $3\sqrt{2} + 4\sqrt{3}$ D $4 + 2\sqrt{2} + 2\sqrt{6}$ E 12

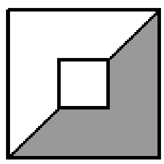
Fractions

You should be used to working with fractions, but remember that things that are the same can look very different. If you see a fraction in one form, see if you can rearrange it to make it look like something else – this might be much more convenient for what you're trying to do with it! For example, $1 - \frac{1}{n} = \frac{n-1}{n}$.

Practice questions

Q43

The diagram shows two squares, with sides of length 1 and 3, which have the same centre and corresponding sides parallel. What fraction of the larger square is shaded?



A $\frac{4}{9}$ B $\frac{4}{11}$ C $\frac{2}{5}$ D $\frac{2}{7}$ E $\frac{6}{11}$

Q44

What is the value of the expression: $(1+\frac{1}{2})(1+\frac{1}{3})(1+\frac{1}{4})\dots(1+\frac{1}{2004})(1+\frac{1}{2005})$?

A 1001 B 1002 C 1003 D 1004 E 1005

Q45

What is the value of $\sqrt{\frac{1}{2^6}+\frac{1}{6^2}}$?

A $\frac{1}{10}$ B $\frac{1}{9}$ C $\frac{1}{3}$ D $\frac{5}{24}$ E $\frac{7}{24}$

Q46

For how many integers n is $\frac{n}{100-n}$ also an integer?

A 1 B 6 C 10 D 18 E 100

Q47

Which positive integer n satisfies the equation

$$\frac{3}{n^3}+\frac{4}{n^3}+\frac{5}{n^3}+\dots+\frac{n^3-5}{n^3}+\frac{n^3-4}{n^3}+\frac{n^3-3}{n^3}=60\ ?$$

A 5 B 11 C 31 D 60 E 2006

[For this last question, see also the section on arithmetic progressions on page 28]

Difference of two squares

$(a^2-b^2)=(a+b)(a-b)$ is one of the most commonly used things in the SMC. You should get to a point where if you see something in one of these forms you can hardly resist writing it down in the other form too. This can be particularly useful if confronted with x^2-1, which you can factorise to get $(x+1)(x-1)$.

Let's look at an example:

E16

The positive numbers x and y satisfy the equations $x^4-y^4=2009$ and $x^2+y^2=49$. What is the value of y?

A 1 B 2 C 3 D 4 E more information is needed

Answer: We have $x^4-y^4=2009$ so $(x^2+y^2)(x^2-y^2)=2009$. Since $(x^2+y^2)=49$, $49(x^2-y^2)=2009$ i.e. $(x^2-y^2)=\frac{2009}{49}=41$. So now we have $(x^2+y^2)-(x^2-y^2)=2y^2=49-41=8$ so $y^2=4$ and thus $y=2$ ($y\neq-2$ because we are given that $y>0$)

[Notice again that it is possible to work out x $(=3\sqrt{5})$ *but this is not necessary to answer the question.]*

So the answer is **B**.

Practice questions

Q48

What is the value of $2006\times2008-2007\times2007$?

A -2007 B -1 C 0 D 1 E 4 026 042

Q49

How many prime numbers p are there such that $199p + 1$ is a perfect square?

A 0 B 1 C 2 D 4 E 8

Q50

What is the sum of the values of n for which both n and $\frac{n^2-9}{n-1}$ are integers?

A -8 B -4 C 0 D 4 E 8

Q51

Which of the following is equal to $\dfrac{1}{\sqrt{2005+\sqrt{2005^2-1}}}$?

A $\sqrt{1003}-\sqrt{1002}$ B $\sqrt{1005}-\sqrt{1004}$ C $\sqrt{1007}-\sqrt{1005}$
D $\sqrt{2005}-\sqrt{2003}$ E $\sqrt{2007}-\sqrt{2005}$

[For Q51, see also the section on Surds on page 25. This question comes with a health warning – it's really hard!]

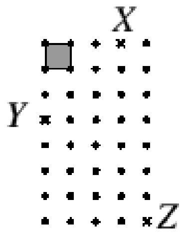
Watch out for the reverse of common theorems

For example, Pythagoras works in reverse: if you have a triangle ABC where sides $a^2 + b^2 = c^2$, then ABC is right-angled. All of the circle theorems work in reverse too. For example, if you have a quadrilateral with opposite angles that add up to 180°, then it is cyclic (i.e. you can draw a circle which goes through all four of its vertices).

Let's have a look at this example:

E17

The shaded square of the lattice shown has area 1. What is the area of the circle through the points X, Y and Z?



A $\frac{9\pi}{2}$ B 8π C $\frac{25\pi}{2}$ D 25π E 50π

Answer: We have $(\mathrm{XY})^2 = 3^2 + 3^2 = 18$, $(\mathrm{YZ})^2 = 4^2 + 4^2 = 32$ and $(\mathrm{XZ})^2 = 1^2 + 7^2 = 50$. This gives us $(\mathrm{XZ})^2 = (\mathrm{XY})^2 + (\mathrm{YZ})^2$, so, by Pythagoras' Theorem, $\mathrm{X\hat{Y}Z} = 90°$. Hence the line XZ will be a diameter of the circle which passes through X, Y and Z (by the right-angled triangle in a semicircle theorem). So the area of the circle will be equal to $\pi\left(\frac{\mathrm{XZ}}{2}\right)^2 = \pi\left(\frac{\sqrt{50}}{2}\right)^2 = \frac{50\pi}{4} = \frac{25\pi}{2}$

So the answer is **C**.

Approximation

There are often questions which will ask for an approximate answer to a question (often an exceedingly large or exceedingly small number). When doing these, look out for numbers which multiply together to give a number of convenient size. e.g. if you have 4×24, replace it with 100. It's worth working through an example:

E18

A newspaper headline read 'Welsh tortoise recaptured 1.8 miles from home after 8 months on the run'. Assuming the tortoise travelled in a straight line, roughly how many minutes did the tortoise take on average to 'run' one foot?
[1 mile = 5280 feet]

A 3 B 9 C 16 D 36 E 60

$8 \text{ mths} \approx 8 \times 30 \times 24 \times 60 \approx 10 \times 30 \times 20 \times 60 \text{ min.}$ $\quad 1.8 \text{ miles} = 1.8 \times 5280 = 18 \times 528 \approx 20 \times 500 \text{ ft.}$

So the tortoise covered one foot in approximately $\frac{10\times30\times20\times60}{20\times500} = \frac{1\cancel{0}\times3\cancel{0}\times\cancel{20}\times60}{\cancel{20}\times5\cancel{00}} = \frac{3\times60}{5} = 36$ minutes.

So the answer is **D**.

The skill of making decent approximations can turn out to be useful elsewhere – a slightly infamous Oxbridge interview question (which are often designed to see how a candidate thinks on his/her feet) went something along the lines of: "Estimate the percentage of the world's water which can be found in a cow".

[Have a go yourself if you're interested! A worked solution is provided in the answers booklet.]

Practice questions

Q52 A giant thresher shark weighing 1250 pounds, believed to be the heaviest ever caught, was landed by fisherman Roger Nowell off the Cornish coast in November 2007. The fish was sold by auction at Newlyn Fish Market for £255. Roughly, what was the cost per pound?

A 5p B 20p C 50p D £2 E £5

Q53 In 1954, a total of 6 527 mm of rain fell at Sprinkling Tarn and this set a UK record for annual rainfall. The tarn has a surface area of 23 450 m^2. Roughly how many million litres of water fell on Sprinkling Tarn in 1954?

A 15 B 150 C 1 500 D 15 000 E 150 000

TOPICS YOU'RE LIKELY TO ENCOUNTER

Factorials

n factorial, written $n!$, is the product of all of the positive whole numbers up to and including n. It is the number of different ways you can order any n items. You will quite possibly have met these already, but it's worth getting used to manipulating them (e.g. what's $8! \div 5!$? You could work the two out and then complete a massive division calculation. Bif if you notice that $8!$ is $5! \times 6 \times 7 \times 8$, then you can quickly see that $8! \div 5! = 6 \times 7 \times 8 = 336$).

Remember that $0!$ is 1. *[Why? Well, it is clear that $n! = (n+1)! \div (n+1)$. This gives $0! = 1! \div 1 = 1$.]*

A relatively common question involves working out the number of zeroes on the end of a factorial number. Say we're working with some number equal to $\mathrm{x}!$, which we shall call X. The number of zeroes at the end of X is the same as the number of times X can be divided by 10 (think about it for a second). This is the same as the minimum of the number of times that 2 divides X and the number of times 5 divides X (seeing as $10 = 2 \times 5$). It should be relatively obvious that if you multiply the numbers from 1 to x together, the prime factorisation of the result is going to have more 2s in it than 5s, so we just need to know the number of times 5 appears in the prime factorisation of X in order to know how many zeroes there are at the end of X.

Counting the number of times 5 divides X is relatively straightforward. Each time we pass a multiple of five when counting from 1 to x, we add one to the total. When we get to a multiple of 25 $(= 5^2)$ we need to add an extra one to the total, because 25 has two factors of five in it ($25 \times 4 = 100$, which has **two** zeroes on the end). Similarly, for every time we get to a multiple of 125 $(= 5^3)$, we add another extra one (so in total we add three) to the total, because there are 3 fives in the prime factorisation of 125 (and so on for higher powers of five e.g. $625=5^4$).

This description is rather cumbersome, so it may be easier to see how this works in an example:

How many 0s are there in 150!?

From 1 to 150 inclusive, there are thirty multiples of 5, six multiples of 25 and one multiple of 125. So, in total, we know that 5 appears $(30 + 6 + 1)$ times in the prime factorisation of $100!$. Each of these fives will yield a 0 on the end of $150!$, so there are 37 zeroes on the end of $150!$.

*[Notice you *could* sit there and work out 150! (= 57133839564458545904789328652610540031895535786011264 182548375833179829124845398393126574488675311145377107878746854204162666250198684504466355949 1959220665749425920957357789293253572904449624724054167907221184454371222696755200000000000000 00000000000000000000000) and count the zeroes, but then you'd run out of time to do any of the other questions.]*

Practice questions

Q54

Which of the following gives the exact number of seconds in the last six complete weeks of 2007?

A $9!$ B $10!$ C $11!$ D $12!$ E $13!$

{Note that $n! = 1 \times 2 \times 3 \times \ldots \times n$.}

Q55

The factorial of n, written $n!$, is defined by $n! = 1 \times 2 \times 3 \times \ldots \times (n-2) \times (n-1) \times n$. For how many positive integer values of k less than 50 is it impossible to find a value of n such that $n!$ ends in exactly k zeros?

A 0 B 5 C 8 D 9 E 10

Modulo arithmetic

Modulo arithmetic is the fancy name for arithmetic done with remainders. Working "modulo x" means working with remainders when dividing by x. So if you're working modulo x, you reduce all numbers down to a number between 0

and $x - 1$, which is their remainder when divided by x. (e.g. when working modulo seven, you only work with the numbers 0 to 6, because $6 + 1$ " $=$ " 0, i.e. a remainder of 6 plus a remainder of 1 "equals" no remainder). For example, $9 \equiv 0 \pmod 3$ and $9 \equiv 2 \pmod 7$. You can continue with arithmetic as before, so $3 \times 5 \pmod 7$ is the remainder when (3×5) is divided by 7, so it is 1. The powerful thing about modulo arithmetic is that if you have any number equivalent to $3 \pmod 7$, such as 52, and you multiply it by any number equivalent to $5 \pmod 7$, like 89, then the remainder remains the same. So without working out 89×52, you know that $(89 \times 52) \div 7$ has a remainder of 1. It even works with negative numbers: $3 \times -2 \pmod 7 \equiv 3 \times 5 \pmod 7 \equiv 1 \pmod 7$.

[The little ≡ symbol is what you use to describe equivalence in modulo arithmetic]

Practice Question

Q56 A square number is divided by 6. Which of the following could not be the remainder?

A 0 B 1 C 2 D 3 E 4

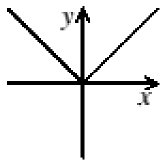
The modulus function ($|x|$)

[The modulus function and modulo arithmetic are unrelated! Don't confuse the two…]

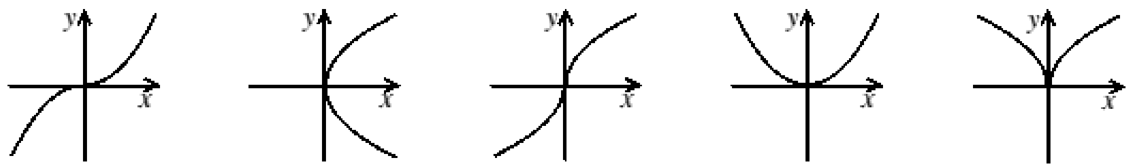
The modulus of x, written $|x|$, means the value or magnitude of x, ignoring the sign. So $|x| = |-x|$ for all values of x. When tackling questions involving modulus, you are best to break it down into different cases (e.g. when x is positive and when x is negative) and consider each of them separately.

Practice questions

Q57 The graph of $y = |x|$ is shown alongside.
Which of the following could be a sketch of the graph of $y = x\,|x|$?



A B C D E



Q58 What is the area of the polygon formed by all points (x, y) in the plane satisfying the inequality $\big||x| - 2\big| + \big||y| - 2\big| \leqslant 4$?

A 24 B 32 C 64 D 96 E 112

[Hint: draw a picture!]

Algebraic manipulation

You shouldn't have any difficulties with the algebraic manipulation you will meet during the SMC: it will include basic things like multiplying out brackets, factorising, using algebraic fractions and rearranging to make something else the subject of an equation. For the most successful candidates all of these processes should be pretty much automatic. One or two of the later questions may involve some work with quadratics. It is very useful to remember that a square number can never be negative, but remember that a number has **two** square roots (positive & negative).

There is often a particularly tricky algebraic manipulation question towards the very end of the paper (quite often question 25 involves some more complicated algebra). If you're careful and methodical with your manipulations then you should be able to work through them successfully, but be aware that these questions (and consequently the last three given here – 62 to 64) are pretty challenging.

Q59

Cheryl finds a bag of coins. There are 50 coins inside and the value of the contents is £1.81. Given that it contains only two-pence and five-pence coins, how many more five-pence coins are there inside the bag than two-pence coins?

A 4 B 6 C 8 D 10 E 12

Q60

Last year Rachel took part in a swimathon. Every day for 9 weeks she swam the same number of lengths, either in a 25m indoor pool or a 20m outdoor pool. Later she discovered that she had swum the same total distance in each pool. On how many days did Rachel swim in the indoor pool?

A 45 B 42 C 35 D 32 E 28

Q61

How many pairs of real numbers (x, y) satisfy the equation $(x + y)^2 = (x + 3)(y - 3)$?

A 0 B 1 C 2 D 4 E infinitely many

Q62

The line with equation $y = x$ is an axis of symmetry of the curve with equation $y = \dfrac{px + q}{rx + s}$, where p, q, r, s are all non-zero. Which of the following is necessarily true?

A $p+q=0$ B $r+s=0$ C $p+r=0$ D $p+s=0$ E $q+r=0$

Q63

Four positive integers a, b, c and d are such that

$$abcd + abc + bcd + cda + dab + ab + bc + cd + da + ac + bd + a + b + c + d = 2009.$$

What is the value of $a + b + c + d$?

A 73 B 75 C 77 D 79 E 81

Q64

X is a positive integer in which each digit is 1; that is, X is of the form 11111… .
Given that every digit of the integer $pX^2 + qX + r$ (where p, q and r are fixed integer coefficients and $p > 0$) is also 1, irrespective of the number of digits X, which of the following is a possible value of q?

A -2 B -1 C 0 D 1 E 2

[Health warning: Q64 is a seriously hard question!]

Simultaneous equations

A question involving simultaneous equations very well may be in the SMC. You should have little difficulty solving a system of two simultaneous equations.

Speed

Speeds often turn up in some guise or other, perhaps forming a pair of simultaneous equations. Speeds can sometimes be counterintuitive so have your wits about you.

Practice questions

Q65

Travelling at an average speed of 100 km/hr, a train took 3 hours to travel to Birmingham. Unfortunately the train then waited just outside the station, which reduced the average speed for the whole journey to 90 km/hr. For how many minutes was the train waiting?

A 1 B 5 C 10 D 15 E 20

Q66

Hamish and his friend Ben live in villages which are 51 miles apart. During the summer holidays, they agreed to cycle towards each other along the same main road. Starting at noon, Hamish cycled at x mph. Starting at 2 pm, Ben cycled at y mph. They met at 4 pm. If they had both started at noon, they would have met at 2.50 pm. What is the value of y?

A 7.5 B 8 C 10.5 D 12 E 12.75

Surds

You don't have a calculator in the SMC, but you shouldn't need one. All of the answers are expressed in exact form, including using pi and surds (numbers involving square roots or roots of higher order). There are a couple of useful rules to know when working with surds:

$$\sqrt{a \times b} = \sqrt{a} \times \sqrt{b}$$

$$\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$$

If you remember that $\sqrt{a} = a^{\frac{1}{2}}$ then you can see that these rules are equivalent to the rules of indices.

Using these rules, you can see that two surds which look very different can in fact be the same. For example:

$$\sqrt{8} = \sqrt{2 \times 2 \times 2} = \sqrt{2 \times 2} \times \sqrt{2} = \sqrt{2^2} \times \sqrt{2} = 2\sqrt{2}$$

$$\sqrt{\frac{1}{2}} = \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

and (more tricky):

$$\sqrt{3 + 2\sqrt{2}} = \sqrt{1 + 2\sqrt{2} + 2} = \sqrt{\left(1 + \sqrt{2}\right)^2} = 1 + \sqrt{2}$$

Knowing this, if you arrive at an answer to a problem which doesn't look like one of the five options, you shouldn't just give up on it! Check whether you can see a way to rearrange it to look like one of them first.

Practice questions

Q67

What is the value of $\sqrt{2^4 + \sqrt{3^4}}$?

A 4 B $\sqrt{20}$ C 5 D 7 E $\sqrt{97}$

Q68

For what value of x is $\sqrt{2} + \sqrt{2} + \sqrt{2} + \sqrt{2} = 2^x$ true?

A $\frac{1}{2}$ B $1\frac{1}{2}$ C $2\frac{1}{2}$ D $3\frac{1}{2}$ E $4\frac{1}{2}$

Manipulating percentages

You have been playing with percentages for years, so I won't insult your intelligence by telling you how they work. Suffice it to say that you should know what you're doing with them.

In this current day and age, VAT is now 20%. This makes some calculations even easier than before! A quick tip in case you are asked to work with 17.5%:

$17.5 = 10 + 5 + 2.5$, so if you work out ten percent, then work out half of that and half again, you will have worked out 10%, 5% and 2.5%. Adding these three numbers together will give you 17.5%.

Practice questions

Q69

The promotion 'AMAZING! 20% OFF ALL OUR BEDFRAMES' appears on the cover of the 2006 brochure of a well-known furniture company. If 20% were to be taken off the length of a bedframe originally 2.10 m long, what would be the resulting length of the bedframe?

A 2.00 m B 1.90 m C 1.89 m D 1.78 m E 1.68 m

Q70

Matt black paint absorbs 97% of light, the remainder being reflected. Scientists have developed a new superblack coating, "10 times blacker" than matt black paint, meaning that it reflects $\frac{1}{10}$ of the light reflected by matt black paint. What percentage of light does the new coating absorb?

A 9.7 B 90.3 C 99.7 D 99.9 E 970

Q71

200 T-shirts have been bought for a Fun Run at a cost of £400 plus VAT at $17\frac{1}{2}$%.
The cost of entry for the run is £5 per person. What is the minimum number of entries needed in order to cover the total cost of the T-shirts?

A 40 B 47 C 80 D 84 E 94

Q72

In a sale, a shopkeeper reduced the advertised selling price of a dress by 20%. This resulted in a profit of 4% over the cost price of the dress. What percentage profit would the shopkeeper have made if the dress had been sold at the original selling price?

A 16% B 20% C 24% D 25% E 30%

OTHER THINGS IT'S WORTH KNOWING

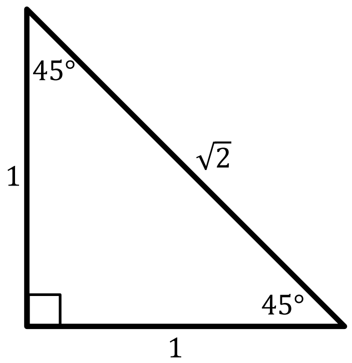
A couple of probability things

- There are 52 cards in a deck of cards, 54 if you include the jokers.
- If you're working with a die (or several dice), $6^2 = 36$ and $6^3 = 216$ are useful numbers.
- Remember that coin tosses and dice throws and are independent events which will not be affected by any previous coin tosses or die throws.
- When randomly picking any number (n, say) of things from a larger set (of, say, m things), the odds that they come out in a given order is always $\mathrm{n}!$, regardless of the values of either n or m or any relationship between the two.

Sin, Cos and Tan

A couple of useful triangles:

The 1,1,$\sqrt{2}$ triangle

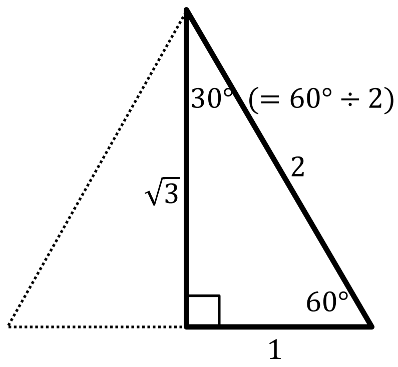


Thinking about a right angle triangle with two 45° angle. If the two shorter sides are of length 1, then the hypotenuse has length √2 by Pythagoras. Basic trig quickly leads us to:

- $\sin 45° = \cos 45° = \frac{1}{\sqrt{2}}$
- $\tan 45° = 1$

The 1,2,$\sqrt{3}$ triangle

If we cut an equilateral triangle in half, we get a right-angled triangle with angles of 60° and 30°:



If we started with an equilateral triangle with side length 2 (notice this is a value deliberately chosen to make the arithmetic easy!), we now have a triangle with sides of 1, 2 and √3 by Pythagoras. As before, we can now establish some useful sin and cos values:

- $\sin 30° = \cos 60° = \frac{1}{2}$
- $\sin 60° = \cos 30° = \frac{2}{\sqrt{3}}$
- $\tan 30° = \frac{1}{\sqrt{3}}$; $\tan 60° = \sqrt{3}$

Potentially useful Sin and Cos identities

These are unlikely to be essential to solve an SMC question, but the more weapons you have in your arsenal the better. There are a couple of identities you might find useful:

- $\sin(-\theta) = -\sin\theta$; $\quad \cos(-\theta) = \cos\theta$; $\quad \tan(-\theta) = -\tan\theta$
- If θ and ϕ add up to 90°, then $\sin\theta = \cos\phi$ and $\cos\theta = \sin\phi$ (think of a right-angled triangle)

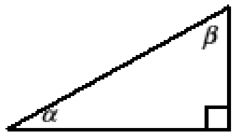
Practice questions

Q73 Which of the following has the greatest value?

A $\cos 50°$ B $\sin 50°$ C $\tan 50°$ D $\dfrac{1}{\sin 50°}$ E $\dfrac{1}{\cos 50°}$

Q74

If $\alpha < \beta$, how many different values are there among the following expressions?

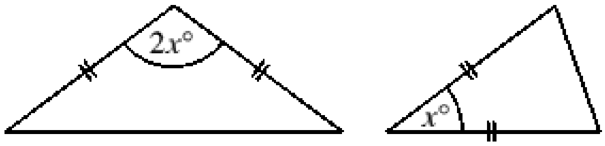


$\sin\alpha\,\sin\beta$ $\quad$ $\sin\alpha\,\cos\beta$ $\quad$ $\cos\alpha\,\sin\beta$ $\quad$ $\cos\alpha\,\cos\beta$

A 1 $\quad$ B 2 $\quad$ C 3 $\quad$ D 4 $\quad$ E It depends on the value of α

Q75

The two triangles have equal areas and the four marked lengths are equal.
What is the value of x?



A 30 $\quad$ B 45 $\quad$ C 60 $\quad$ D 75 $\quad$ E more information needed

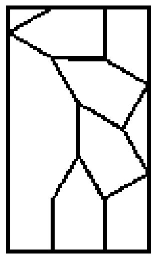
Q76

It takes two weeks to clean the 3312 panes of glass in the 6000m^2 glass roof of the British Museum, a task performed once every two years. Assuming that all the panes are equilateral triangles of the same size, roughly how long is the side of each pane?

A 50 cm $\quad$ B 1 m $\quad$ C 2 m $\quad$ D 3 m $\quad$ E 4 m

Q77

A pentagon is made by attaching an equilateral triangle to a square with the same edge length. Four such pentagons are placed inside a rectangle, as shown.
What is the ratio of the length of the rectangle to its width?



A $\sqrt{3}:1$ $\quad$ B $2:1$ $\quad$ C $\sqrt{2}:1$ $\quad$ D $3:2$ $\quad$ E $4:\sqrt{3}$

Arithmetic progressions

An arithmetic progression is one where the terms go up by the same amount each time. These are all examples:

- 1, 2, 3, 4, 5, … (starting at 1 and going up by 1 every time)
- 279, 281, 283, 285, 287 … (starting at 279 and going up by 2 every time)
- $6 - \pi$, $10 + \pi$, $14 + 3\pi$, $18 + 5\pi$ … (starting at $6 - \pi$ and going up by $4 + 2\pi$ every time)

Finding the *n*th term of an arithmetic progression

You can generate the n^{th} term of an arithmetic progression by adding $\mathrm{n} - 1$ times the difference to the first value. For example, the 20$^{\text{th}}$ term of the second sequence above is $279 + 19 \times 2 = 317$

Finding the sum of the first *n* terms of an arithmetic progression

You can also work out the sum of the first n terms of an arithmetic progression without working out the first n terms. If you call the first term a, and the n^{th} term l, then the sum of the first n terms is given by:

$$\text{Sum} = \frac{1}{2}n(a + l)$$

This isn't difficult to prove (just think about it!) so it is left as a challenge.

Practice Question

Q78

Given that $S = (x + 20) + (x + 21) + (x + 22) + \ldots + (x + 100)$, where x is a positive integer, what is the smallest value of x such that S is a perfect square?

A 1 $\quad$ B 2 $\quad$ C 4 $\quad$ D 8 $\quad$ E 64

Distances and volumes

Some useful conversions to be familiar with:

- There are 1760 yards or roughly 1600 metres in a mile (1 mile × ⅝ ≈ 1km)
- 1 pint = 568mm (you can remember an approximation to this with the little two-line poem: "a litre of water's a pint and three quarters").
- 1 gallon = 8 pints ≈ 4.5 litres.
- 1 yard = 36 inches ≈ 90cm.
- 1 pound ≈ 454g; 5kg ≈ 11lb

These may turn out to be useful in questions where some approximation is involved *[see page 20]*.

Practice Question

Q79

The 80 spokes of the giant wheel *The London Eye* are made from 4 miles of cable. Roughly what is the circumference of the wheel in metres?

A 50 B 100 C 500 D 750 E 900

Formulae for surface area and volume

You will probably be given most of the formulae that you need, but it's worth being familiar with these, including the formulae for spheres and cones:

Sphere: of radius r

Surface Area $= 4\pi r^2$; Volume $= \frac{4\pi r^3}{3}$

Cone: with base radius r, height h and slant height (distance from circumference of base to point) $l\ \left(= \sqrt{h^2 + r^2}\right)$

Surface Area: $= \pi r^2 + \pi r l$; Volume $= \frac{\pi r^2 h}{3}$

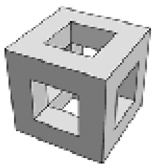
Be prepared to chop a shape up into smaller bits to work out its surface area or volume. Make sure you're methodical!

Practice questions

Q80

A $4 \times 4 \times 4$ cube has three $2 \times 2 \times 4$ holes drilled symmetrically all the way through, as shown.

What is the surface area of the resulting solid?



A 192 B 144 C 136 D 120 E 96

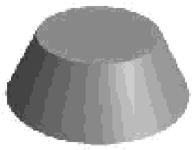
Q81

The base of a pyramid has n edges. What is the difference between the number of edges the pyramid has and the number of faces the pyramid has?

A $n - 2$ B $n - 1$ C n D $n + 1$ E $n + 2$

Q82

A frustum is the solid obtained by slicing a right-circular cone perpendicular to its axis and removing the small cone above the slice. This leaves a shape with two circular faces and a curved surface. The original cone has base radius 6 cm and height 8 cm, and the curved surface area of the frustum is equal to the area of the two circles. What is the height of the frustum?



A 3 cm B 4 cm C 5 cm D 6 cm E 7 cm

UKMT

UK SENIOR MATHEMATICAL CHALLENGE

Thursday 4 November 2010

Organised by the **United Kingdom Mathematics Trust**

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The Actuarial Profession

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RULES AND GUIDELINES (to be read before starting)

1. Do not open the question paper until the invigilator tells you to do so.
2. **Use B or HB pencil only**. Mark *at most one* of the options A, B, C, D, E on the Answer Sheet for each question. Do not mark more than one option.
3. Time allowed: **90 minutes.**
   No answers or personal details may be entered on the Answer Sheet after the 90 minutes are over.
4. The use of rough paper is allowed.
   Calculators, measuring instruments and squared paper are forbidden.
5. Candidates must be full-time students at secondary school or FE college, and must be in Year 13 or below (England & Wales); S6 or below (Scotland); Year 14 or below (Northern Ireland).
6. There are twenty-five questions. Each question is followed by five options marked A, B, C, D, E. Only one of these is correct. Enter the letter A-E corresponding to the correct answer in the corresponding box on the Answer Sheet.
7. **Scoring rules**: all candidates start out with 25 marks;
   0 marks are awarded for each question left unanswered;
   4 marks are awarded for each correct answer;
   1 mark is deducted for each incorrect answer.
8. **Guessing**: Remember that there is a penalty for wrong answers. Note also that later questions are deliberately intended to be harder than earlier questions. You are thus advised to concentrate first on solving as many as possible of the first 15-20 questions. Only then should you try later questions.

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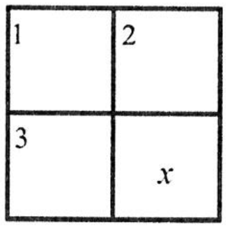
1. What is the digit x in this cross-number?



Across
1. A cube
3. A cube

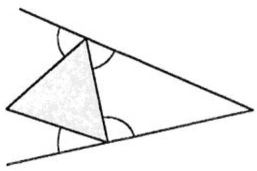
Down
1. One less than a cube

A 2 B 3 C 4 D 5 E 6

2. What is the smallest possible value of $20p + 10q + r$ when p, q and r are *different* positive integers?

A 31 B 43 C 53 D 63 E 2010

3. The diagram shows an equilateral triangle touching two straight lines.
What is the sum of the four marked angles?



A 120° B 180° C 240° D 300° E 360°

4. The year 2010 is one in which the sum of its digits is a factor of the year itself. How many more years will it be before this is next the case?

A 3 B 6 C 9 D 12 E 15

5. A notice on Morecambe promenade reads: 'It would take 20 million years to fill Morecambe Bay from a bath tap.' Assuming that the flow from the bath tap is 6 litres a minute, what does the notice imply is the approximate capacity of Morecambe Bay in litres?

A 6×10^{10} B 6×10^{11} C 6×10^{12} D 6×10^{13} E 6×10^{14}

6. Dean runs up a mountain road at 8 km per hour. It takes him one hour to get to the top. He runs down the same road at 12 km per hour. How many minutes does it take him to run down the mountain?

A 30 B 40 C 45 D 50 E 90

7. There are 120 different arrangements of the five letters in the word ANGLE. If all 120 are listed in alphabetical order starting with AEGLN and finishing with NLGEA, which position in the list does ANGLE occupy?

A 18th B 20th C 22nd D 24th E 26th

8. Which of the following is equivalent to $(x + y + z)(x - y - z)$?

A $x^2 - y^2 - z^2$ B $x^2 - y^2 + z^2$ C $x^2 - xy - xz - z^2$ D $x^2 - (y + z)^2$ E $x^2 - (y - z)^2$

9. The symbol $\Diamond$ is defined by $x \Diamond y = x^y - y^x$. What is the value of $(2 \Diamond 3) \Diamond 4$?

A -3 B $-\frac{3}{4}$ C 0 D $\frac{3}{4}$ E 3

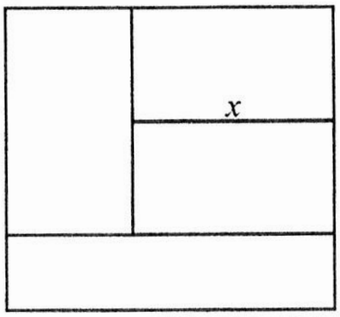
10. A square is cut into 37 squares of which 36 have area 1 cm^2. What is the length of the side of the *original* square?

A 6 cm B 7 cm C 8 cm D 9 cm E 10 cm

11. What is the median of the following numbers?

A $9\sqrt{2}$ B $3\sqrt{19}$ C $4\sqrt{11}$ D $5\sqrt{7}$ E $6\sqrt{5}$

12. The diagram, which is not to scale, shows a square with side length 1, divided into four rectangles whose areas are equal. What is the length labelled x?

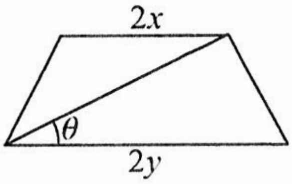


A $\frac{2}{3}$ B $\frac{17}{24}$ C $\frac{4}{5}$ D $\frac{49}{60}$ E $\frac{5}{6}$

13. How many two-digit numbers have remainder 1 when divided by 3 and remainder 2 when divided by 4?

A 8 B 7 C 6 D 5 E 4

14. The parallel sides of a trapezium have lengths $2x$ and $2y$ respectively. The diagonals are equal in length, and a diagonal makes an angle θ with the parallel sides, as shown. What is the length of each diagonal?



A $x + y$ B $\dfrac{x + y}{\sin\theta}$ C $(x + y)\cos\theta$ D $(x + y)\tan\theta$ E $\dfrac{x + y}{\cos\theta}$

15. What is the smallest prime number that is equal to the sum of two prime numbers and is also equal to the sum of three different prime numbers?

A 7 B 11 C 13 D 17 E 19

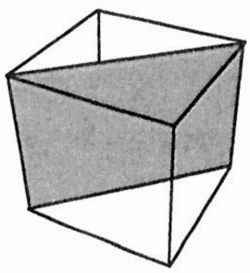
16. *PQRS* is a quadrilateral inscribed in a circle of which *PR* is a diameter. The lengths of *PQ*, *QR* and *RS* are 60, 25 and 52 respectively. What is the length of *SP*?

A $21\frac{2}{3}$ B $28\frac{11}{13}$ C 33 D 36 E 39

17. One of the following is equal to $\sqrt{9^{16x^2}}$ for all values of x. Which one?

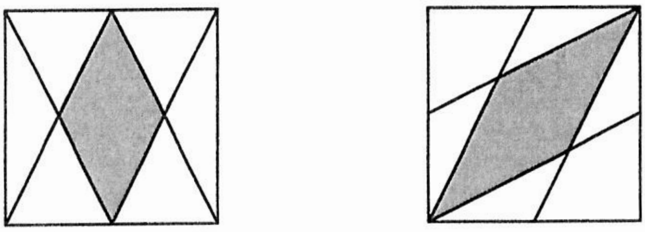
A 3^{4x} B 3^{4x^2} C 3^{8x^2} D 9^{4x} E 9^{8x^2}

18. A solid cube of side 2 cm is cut into two triangular prisms by a plane passing through four vertices, as shown. What is the total surface area of these two prisms?



A $8(3 + \sqrt{2})$ B $2(8 + \sqrt{2})$ C $8(3 + 2\sqrt{2})$
D $16(3 + \sqrt{2})$ E $8\sqrt{2}$

19. The diagrams show two different shaded rhombuses, each inside a square with sides of length 6.



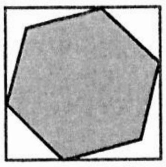
Each rhombus is formed by joining vertices of the square to midpoints of the sides of the square. What is the difference between the shaded areas?

A 4 B 3 C 2 D 1 E 0

20. There are 10 girls in a mixed class. If two pupils from the class are selected at random to represent the class on the School Council, then the probability that both are girls is 0.15. How many boys are in the class?

A 10 B 12 C 15 D 18 E 20

21. The diagram shows a regular hexagon, with sides of length 1, inside a square. Two vertices of the hexagon lie on a diagonal of the square and the other four lie on the edges.



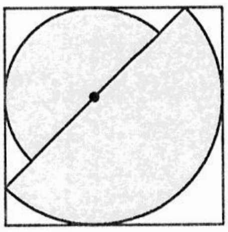
What is the area of the square?

A $2 + \sqrt{3}$ B 4 C $3 + \sqrt{2}$ D $1 + \dfrac{3\sqrt{3}}{2}$ E $\dfrac{7}{2}$

22. If $x^2 - px - q = 0$, where p and q are positive integers, which of the following could not equal x^3?

A $4x + 3$ B $8x + 5$ C $8x + 7$ D $10x + 3$ E $26x + 5$

23. The diagram shows two different semicircles inside a square with sides of length 2. The common centre of the semicircles lies on a diagonal of the square.
What is the total shaded area?



A π B $6\pi(3 - 2\sqrt{2})$ C $\pi\sqrt{2}$ D $3\pi(2 - \sqrt{2})$ E $8\pi(2\sqrt{2} - 3)$

24. Three spheres of radius 1 are placed on a horizontal table and inside a vertical hollow cylinder of height 2 units which is just large enough to surround them. What fraction of the internal volume of the cylinder is occupied by the spheres?

A $\dfrac{2}{7 + 4\sqrt{3}}$ B $\dfrac{2}{2 + \sqrt{3}}$ C $\dfrac{1}{3}$ D $\dfrac{3}{2 + \sqrt{3}}$ E $\dfrac{6}{7 + 4\sqrt{3}}$

25. All the digits of a number are different, the first digit is not zero, and the sum of the digits is 36. There are $N \times 7!$ such numbers. What is the value of N?

A 72 B 97 C 104 D 107 E 128

SPACE FOR NOTES