The Simplex Algorithm

- So far, we have studied how to solve two-variable LP problems graphically.
- However, most real life problems have more than two variables!
- Therefore, we need to have another method to solve LPs with more than two variables.
- We are going to study **The Simplex Algorithm** which is quite useful in solving very large LP problems.
- Today, The Simplex Algorithm is used to solve LP problems in many industrial applications that involve thousands of variables and constraints.

- LP problems can have both equality and inequality constraints.
- LP problems can have nonnegative and urs (unrestricted in sign) variables.
- To use the Simplex Method, LP problems should be converted to ***Standard Form LP***.

Standard Form LP:

- Why? We know from 2-variables that extreme points are potential optimal solutions
- This will be true in higher dimensions as well
- We need an ALGEBRAIC way of characterizing extreme points. We can't draw the feasible region in higher dimensions
- Standard form LPs will provide an easy way to do this characterization

- **All constraints are equalities, with constant nonnegative right-hand sides (RHS),**
- **All variables are nonnegative.**
- **Any LP can be brought into standard form!**

Simplex Method:

- Start with an extreme point
- Move to a neighboring extreme point in the improving direction
- Stop if all neighbors are no better

- **Simple Greedy Logic**

  How to find a feasible extreme point?

  How to go to a neighbor?

Converting LP into Standard Form:

Ex: Consider the following LP problem:

$$\max \quad x_1 + 3x_2$$

s.t.

$$2x_1 + 3x_2 + x_3 \leq 5 \qquad (1)$$

$$4x_1 + x_2 + 2x_3 = -11 \qquad (2)$$

$$3x_1 + 4x_2 + 2x_3 \geq 8 \qquad (3)$$

$$x_1 \geq 0,\ x_2 \leq 0,\ x_3 \text{ urs (free)}$$

Converting LP into Standard Form:

a) Define a "slack" variable for each of the "≤" constraint to convert the inequality constraint into an equality constraint:

$$s_1 = 5 - 2x_1 - 3x_2 - x_3 \text{ , } s_1 \geq 0 \qquad (1)$$

So that, the first constraint becomes:

$$2x_1 + 3x_2 + x_3 + s_1 = 5 \qquad (1)$$

Converting LP into Standard Form:

b) Multiply the second constraint by -1 to get a nonnegative right hand side value, i.e. replace

$$4x_1 + x_2 + 2x_3 = -11 \qquad (2)$$

with:

$$-4x_1 - x_2 - 2x_3 = 11 \qquad (2)$$

Converting LP into Standard Form:

c) Define an “excess (surplus)” variable for each of the “≥” constraints to convert the inequality constraint into an equality constraint:

$$e_3 = 3x_1 + 4x_2 + 2x_3 - 8, \; e_3 \geq 0 \qquad (3)$$

so that, the third constraint becomes:

$$3x_1 + 4x_2 + 2x_3 - e_3 = 8 \qquad (3)$$

Converting LP into Standard Form:

d) Variable x_2 has a reverse sign restriction:

Replace x_2 with $-x_2'$ throughout.

If x_2 is nonpositive then $-x_2'$ will be nonnegative

Converting LP into Standard Form:

e) Variable x_3 is unrestricted in sign:

Replace x_3 with $x_3' - x_3''$ and force both x_3' and x_3''

to both be nonnegative

Converting LP into Standard Form:

max $x_1 + 3\,x_2$

s.t.

$$2x_1 + 3x_2 + x_3 \leq 5 \quad (1)$$

$$4x_1 + x_2 + 2x_3 = -11 \quad (2)$$

$$3x_1 + 4x_2 + 2x_3 \geq 8 \quad (3)$$

$x_1 \geq 0,\ x_2 \leq 0,\ x_3$ urs

max $z = x_1 - 3\,x_2'$

s.t.

$$2x_1 - 3\,x_2' + x_3' - x_3'' + \mathbf{s_1} = 5 \quad (1)$$

$$-4x_1 + x_2' - 2\,x_3' + 2\,x_3'' = 11 \quad (2)$$

$$3x_1 - 4\,x_2' + 2\,x_3' - 2\,x_3'' - \mathbf{e_3} = 8 \quad (3)$$

$x_1 \geq 0,\ x_2' \geq 0,\ x_3' \geq 0,\ x_3'' \geq 0,$
$s_1 \geq 0,\ e_3 \geq 0.$

To Convert an LP into Standard Form:

- each inequality constraint is converted into an equality constraint by adding or subtracting nonnegative slack/excess variables,
- an inequality (equality) can be multiplied by -1 to get nonnegative RHS,
- unrestricted variable can be represented as the difference of two new nonnegative variables.

  If x_i is urs, then let $x_i = x_i' - x_i''$ where $x_i', x_i'' \geq 0$.

  Replace every occurrence of x_i with $x_i' - x_i''$ and add sign restrictions $x_i', x_i'' \geq 0$.
- For sign restriction $x_k \leq 0$, let $x_k' = -x_k$ and replace every occurrence of x_k with $-x_k'$ and add the sign restriction $x_k' \geq 0$.

Standard Form LP:

Suppose that we converted an LP with *m* constraints into a standard form. Also assume that after the conversion, we have *n* variables as x_1, x_2, x_3,…, x_n.

Standard Form LP:

Suppose that we converted an LP with *m* constraints into a standard form. Also assume that after the conversion, we have *n* variables as $x_1, x_2, x_3, \ldots, x_n$.

$$\max\ (\min) \quad c_1 x_1 + c_2 x_2 + \cdots + c_n x_n$$

$$\text{s.t.}$$

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
\vdots \qquad\quad \vdots \qquad\qquad\quad \vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m
\end{aligned}
$$

$$x_i \geq 0 \quad (i = 1, 2, \ldots, n)$$

Standard Form LP:

If we define:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

and

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \qquad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

Standard Form LP:

If we define:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

and

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \qquad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

LP can be written as the system of equations :

$$\Rightarrow \quad Ax=b$$

Standard Form LP:

- For Ax=b to have a solution, rank (A|b)=rank (A).
- We also assume that all redundant constraints are removed, so rank(A)=m.

- i.e.

$$x_1 + 2x_2 + x_3 = 4 \quad (1)$$

$$x_2 - x_3 = 1 \quad (2)$$

$$2x_1 + 6x_2 = 10 \quad (3)$$

Standard Form LP:

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- We also assume that all redundant constraints are removed, so rank$(A)=m$.

  i.e.

$$x_1 + 2x_2 + x_3 = 4 \quad (1)$$

$$x_2 - x_3 = 1 \quad (2)$$

$$2x_1 + 6x_2 = 10 \quad (3)$$

Constraint (3) can be written as a linear combination of (1) and (2):

2 [(1)+(2)] = (3)

Remove the redundant constraint

Standard Form LP:

- Before proceeding any further with the discussion of the simplex algorithm, we should define the concept of a basic solution to a linear system.
- **Basic Solution:** A solution to Ax=b is called a basic solution if it is obtained by setting ***n-m*** variables equal to 0 and solving for the remaining ***m*** variables whose columns are linearly independent.

Standard Form LP:

- The ***n-m*** variables whose values are set to 0 are called **nonbasic variables**.
- The remaining ***m*** variables are called **basic variables**.

Standard Form LP:

For $Ax=b$, where A is a $m \times n$ matrix, $\text{rank}(A)=m$, $b \geq 0$,

- pick m linearly independent columns from A,
- rearrange A such that these chosen columns are the first m columns in A.

Standard Form LP:

For $Ax=b$, where A is a $m \times n$ matrix, $\text{rank}(A)=m$, $b \geq 0$,

- pick m linearly independent columns from A,
- rearrange A such that these chosen columns are the first m columns in A.

Since elementary column and row operations do not change the system of linear equations, matrix A can be brought into a form $A=[B_{m \times m} \; N_{m \times (n-m)}]$, where $B_{m \times m}$ is invertible. Such a B is called a basis matrix

Standard Form LP:

Let $x = \begin{pmatrix} x_B \\ x_N \end{pmatrix}$ be the corresponding partition in x.

$$Ax = b \equiv \begin{bmatrix} B & N \end{bmatrix} \begin{pmatrix} x_B \\ x_N \end{pmatrix} = b \Rightarrow Bx_B + Nx_N = b$$

Standard Form LP:

Let $x = \begin{pmatrix} x_B \\ x_N \end{pmatrix}$ be the corresponding partition in x.

$$(Ax = b) \equiv \left(\begin{bmatrix} B & N \end{bmatrix} \begin{pmatrix} x_B \\ x_N \end{pmatrix} = b \right) \Rightarrow Bx_B + Nx_N = b$$

If we set $x_N = 0$, then $x_B = B^{-1}b$ will be a unique solution.

For every such B choice, $\exists$ a unique solution $\begin{pmatrix} B^{-1}b \\ 0 \end{pmatrix}$.

Standard Form LP:

If a basic solution $x = \begin{pmatrix} B^{-1}b \\ 0 \end{pmatrix} \geq 0,$

then x is called a basic feasible solution (bfs).

Basic Solutions:

- Consider the system of equations:

$$
\begin{aligned}
x_1 + x_2 + x_3 &= 6 \\
x_2 + x_4 &= 3
\end{aligned}
$$

$$x_1, x_2, x_3, x_4 \geq 0$$

Basic Solutions:

- Consider the system of equations:

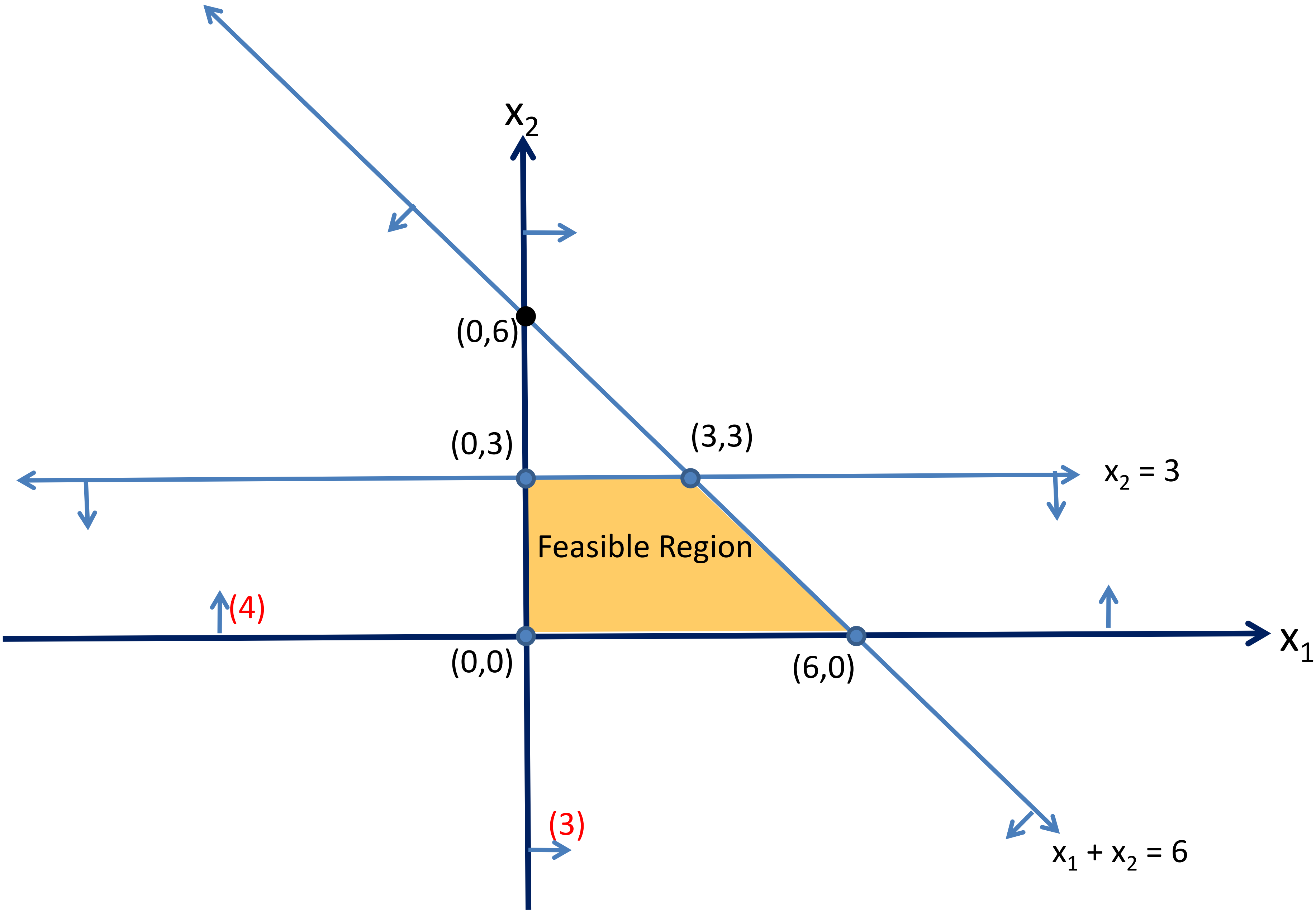
$$
\begin{aligned}
x_1 + x_2 &\leq 6 \\
x_2 &\leq 3 \\
x_1, x_2 &\geq 0
\end{aligned}
$$

(a)

- Converting into a standard form LP:

$$
\begin{aligned}
x_1 + x_2 + x_3 \quad\quad &= 6 \\
x_2 \quad\quad + x_4 &= 3 \\
x_1, x_2, x_3, x_4 &\geq 0
\end{aligned}
$$

(b)



x_2
x_1
(0,6)
(0,3)
(3,3)
$x_2 = 3$
Feasible Region
(4)
(0,0)
(6,0)
(3)
$x_1 + x_2 = 6$

Basic Solutions:

- Consider the system of equations:

$$x_1 + x_2 + x_3 = 6$$

$$x_2 + x_4 = 3$$

$$x_1, x_2, x_3, x_4 \geq 0$$

$$\Rightarrow$$

$$A = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \qquad b = \begin{pmatrix} 6 \\ 3 \end{pmatrix}$$

Basic Solutions:

$$A = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

1. Let us chose:

$$\underbrace{B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}}_{\text{The columns are linearly independent}} \quad x_B = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad x_N = \begin{pmatrix} x_3 \\ x_4 \end{pmatrix}$$

Basic Solutions:

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$$\text{Setting } x_N = 0 \text{ and solving for } x_B : \; Bx_B = b \Rightarrow x_B = B^{-1}b$$

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The columns are linearly independent

$$\text{Setting } x_N = 0 \text{ and solving for } x_B : \ Bx_B = b \Rightarrow x_B = B^{-1}b$$

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 6 \\ 3 \end{pmatrix} \Rightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^{-1} \begin{pmatrix} 6 \\ 3 \end{pmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{pmatrix} 6 \\ 3 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \end{pmatrix}$$

Basic Solutions:

$$\text{Since } x_B = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \end{pmatrix} \geq 0, \quad x = \begin{pmatrix} x_B \\ x_N \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \\ 0 \\ 0 \end{pmatrix} \text{ is a bfs.}$$

Basic Solutions:

Let us consider different ways of forming B:

2.

$$B = \underbrace{\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}}_{\text{The columns are Linearly dependent!}} \quad x_B = \begin{pmatrix} x_1 \\ x_3 \end{pmatrix} \quad x_N = \begin{pmatrix} x_2 \\ x_4 \end{pmatrix}$$

Basic Solutions:

Let us consider different ways of forming B:

2.

$$\underbrace{B = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}}_{\text{The columns are Linearly dependent!}} \quad x_B = \begin{pmatrix} x_1 \\ x_3 \end{pmatrix} \quad x_N = \begin{pmatrix} x_2 \\ x_4 \end{pmatrix}$$

Hence the choice of $x_B = \begin{pmatrix} x_1 \\ x_3 \end{pmatrix}$ cannot be a basic solution.

Basic Solutions:

3. Let us chose:

$$\underbrace{B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{\text{The columns are linearly independent}} \qquad x_B = \begin{pmatrix} x_1 \\ x_4 \end{pmatrix} \qquad x_N = \begin{pmatrix} x_2 \\ x_3 \end{pmatrix}$$

Basic Solutions:

3. Let us chose:

$$B = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{\substack{\text{The columns are} \\ \text{linearly independent}}} \qquad x_B = \begin{pmatrix} x_1 \\ x_4 \end{pmatrix} \qquad x_N = \begin{pmatrix} x_2 \\ x_3 \end{pmatrix}$$

Setting $x_N = 0$ and solving for x_B:

$$x_B = B^{-1}b = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{pmatrix} 6 \\ 3 \end{pmatrix} = \begin{pmatrix} 6 \\ 3 \end{pmatrix} \Rightarrow x = \begin{pmatrix} 6 \\ 0 \\ 0 \\ 3 \end{pmatrix} \text{is a bfs.}$$

Basic Solutions:

4. Let us chose:

$$B = \underbrace{\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}}_{\text{The columns are linearly independent}} \qquad x_B = \begin{pmatrix} x_2 \\ x_3 \end{pmatrix} \qquad x_N = \begin{pmatrix} x_1 \\ x_4 \end{pmatrix}$$

Basic Solutions:

4. Let us chose:

$$B = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \quad x_B = \begin{pmatrix} x_2 \\ x_3 \end{pmatrix} \quad x_N = \begin{pmatrix} x_1 \\ x_4 \end{pmatrix}$$

The columns are linearly independent

Setting $x_N = 0$ and solving for x_B:

$$x_B = B^{-1}b = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} \begin{pmatrix} 6 \\ 3 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \end{pmatrix} \Rightarrow x = \begin{pmatrix} 0 \\ 3 \\ 3 \\ 0 \end{pmatrix} \text{ is a bfs.}$$

Basic Solutions:

5. Let us chose:

$$\underbrace{B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}}_{\text{The columns are linearly independent}} \qquad x_B = \begin{pmatrix} x_2 \\ x_4 \end{pmatrix} \qquad x_N = \begin{pmatrix} x_1 \\ x_3 \end{pmatrix}$$

Basic Solutions:

5. Let us chose:

$$B = \underbrace{\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}}_{\text{The columns are linearly independent}} \quad x_B = \begin{pmatrix} x_2 \\ x_4 \end{pmatrix} \quad x_N = \begin{pmatrix} x_1 \\ x_3 \end{pmatrix}$$

Setting $x_N = 0$ and solving for x_B:

$$x_B = B^{-1}b = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{pmatrix} 6 \\ 3 \end{pmatrix} = \begin{pmatrix} 6 \\ -3 \end{pmatrix} \Rightarrow x = \begin{pmatrix} 0 \\ 6 \\ 0 \\ -3 \end{pmatrix},$$

Basic Solutions:

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$$B = \underbrace{\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}}_{\text{The columns are linearly independent}} \qquad x_B = \begin{pmatrix} x_2 \\ x_4 \end{pmatrix} \qquad x_N = \begin{pmatrix} x_1 \\ x_3 \end{pmatrix}$$

Setting $x_N = 0$ and solving for x_B:

$$x_B = B^{-1}b = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{pmatrix} 6 \\ 3 \end{pmatrix} = \begin{pmatrix} 6 \\ -3 \end{pmatrix} \Rightarrow x = \begin{pmatrix} 0 \\ 6 \\ 0 \\ -3 \end{pmatrix},$$

This basic solution is not feasible!

Basic Solutions:

6. Let us chose:

$$B = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{\text{The columns are linearly independent}} \qquad x_B = \begin{pmatrix} x_3 \\ x_4 \end{pmatrix} \qquad x_N = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Basic Solutions:

6. Let us chose:

$$B = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{\substack{\text{The columns are} \\ \text{linearly independent}}} \qquad x_B = \begin{pmatrix} x_3 \\ x_4 \end{pmatrix} \qquad x_N = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

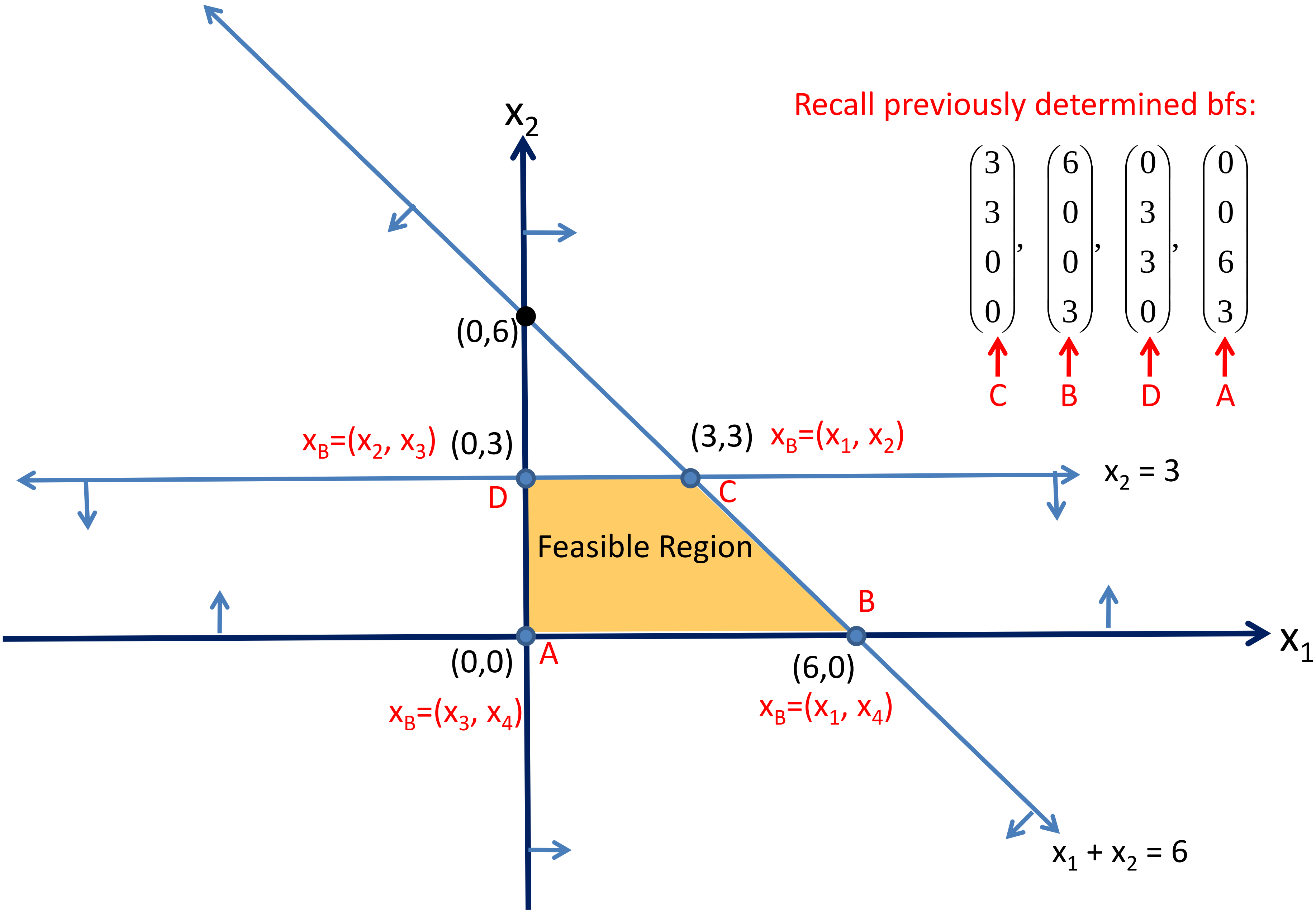
Setting $x_N = 0$ and solving for x_B :

$$x_B = B^{-1}b = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{pmatrix} 6 \\ 3 \end{pmatrix} = \begin{pmatrix} 6 \\ 3 \end{pmatrix} \Rightarrow x = \begin{pmatrix} 0 \\ 0 \\ 6 \\ 3 \end{pmatrix} \text{ is a bfs.}$$

Basic Feasible Solutions:

So, this system of equations has 4 basic feasible solutions:

$$\begin{pmatrix} 3 \\ 3 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 6 \\ 0 \\ 0 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \\ 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 6 \\ 3 \end{pmatrix}$$



Recall previously determined bfs:
(3, 3, 0, 0), (6, 0, 0, 3), (0, 3, 3, 0), (0, 0, 6, 3)
C B D A
x2
x1
(0,6)
(0,3)
(3,3)
(0,0)
(6,0)
xB=(x2, x3)
xB=(x1, x2)
xB=(x3, x4)
xB=(x1, x4)
D
C
B
A
Feasible Region
x2 = 3
x1 + x2 = 6

Basic Feasible Solutions:

$$\text{The number of basic feasible solutions} \leq \binom{n}{m}$$

Theorem:

A point in the feasible region of an LP is an extreme point if and only if it is a basic feasible solution to the LP.

Fundamental Theorem of LP (Revisited):

For an LP in standard form:

- If feasible set of an LP is non-empty, then there is at least one bfs.
- If an LP has an optimal solution, then there is a bfs which is optimal.

This Theorem Implies:

- Finding an optimal solution to an LP problem is equivalent to finding the best bfs!

$$\text{The number of bfs} \leq \binom{n}{m}$$

- Therefore, we should search for the best bfs.

- One way is to enumerate all bfs and choose the one that gives he best objective function value.

- One way is to enumerate all bfs and choose the one that gives he best objective function value.

- However, enumerating all bfs can be very expensive!

$$\text{For example, for } n = 20 \text{ and } m = 10, \binom{n}{m} \text{ is } 184756!$$

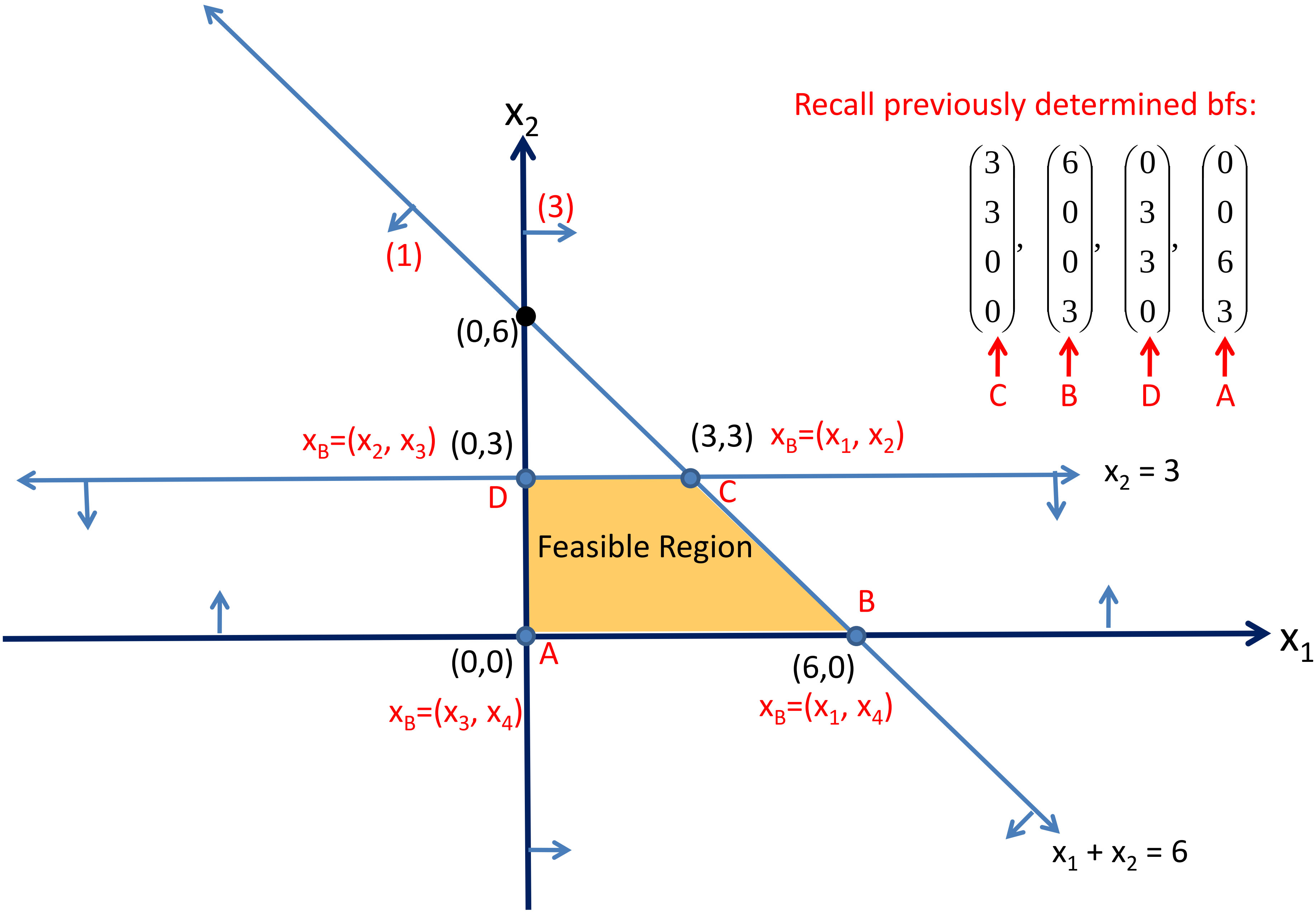
- Simplex Algorithm does this in a clever way. Usually it finds an optimal solution within *3m* enumeration.

Neighboring extreme points (bfs solutions):

- For any LP with *m* constraints, two bfs are said to be **adjacent** if their set of basic variables have *m-1* basic variables in common.

  (Intuitively, two bfs are adjacent if they both lie on the same edge of the boundary of the feasible region)

- Simplex Algorithm goes from an extreme point to an adjacent extreme point with a better objective value.



Recall previously determined bfs:
$\begin{pmatrix}3\\3\\0\\0\end{pmatrix}, \begin{pmatrix}6\\0\\0\\3\end{pmatrix}, \begin{pmatrix}0\\3\\3\\0\end{pmatrix}, \begin{pmatrix}0\\0\\6\\3\end{pmatrix}$
C B D A
x_2
(3)
(1)
(0,6)
$x_B=(x_2, x_3)$ (0,3)
(3,3) $x_B=(x_1, x_2)$
$x_2 = 3$
D
C
Feasible Region
B
x_1
(0,0) A
(6,0)
$x_B=(x_3, x_4)$
$x_B=(x_1, x_4)$
$x_1 + x_2 = 6$

General Description of the Simplex Algorithm:

1. Convert the LP problem into a standard form LP.
2. Obtain a bfs to the LP. This bfs is called the **initial bfs**. In general, the most recent bfs is called the **current bfs**. Therefore, at the beginning the initial bfs is the current bfs.

General Description of the Simplex Algorithm:

3. Determine if the current bfs is an optimal solution or not.
4. If the current bfs is not optimal, then find an adjacent bfs with a better objective function value (one nonbasic variable becomes basic and one basic variable becomes nonbasic).
5. Go to Step 3.

The Simplex Algorithm:

- To begin the simplex algorithm, convert the LP into a standard form,

- Convert the objective function $z=c_1x_1+c_2x_2+\ldots+c_nx_n$ to the row-0 format:

$$z-c_1x_1-c_2x_2-\ldots-c_nx_n=0$$

The Simplex Algorithm:

max $\quad z = x_1 + 3\,x_2$

s.t.

$$x_1 + x_2 \leq 6 \qquad (1)$$

$$-x_1 + 2x_2 \leq 8 \qquad (2)$$

$$x_1, x_2 \geq 0$$

max $\quad z$

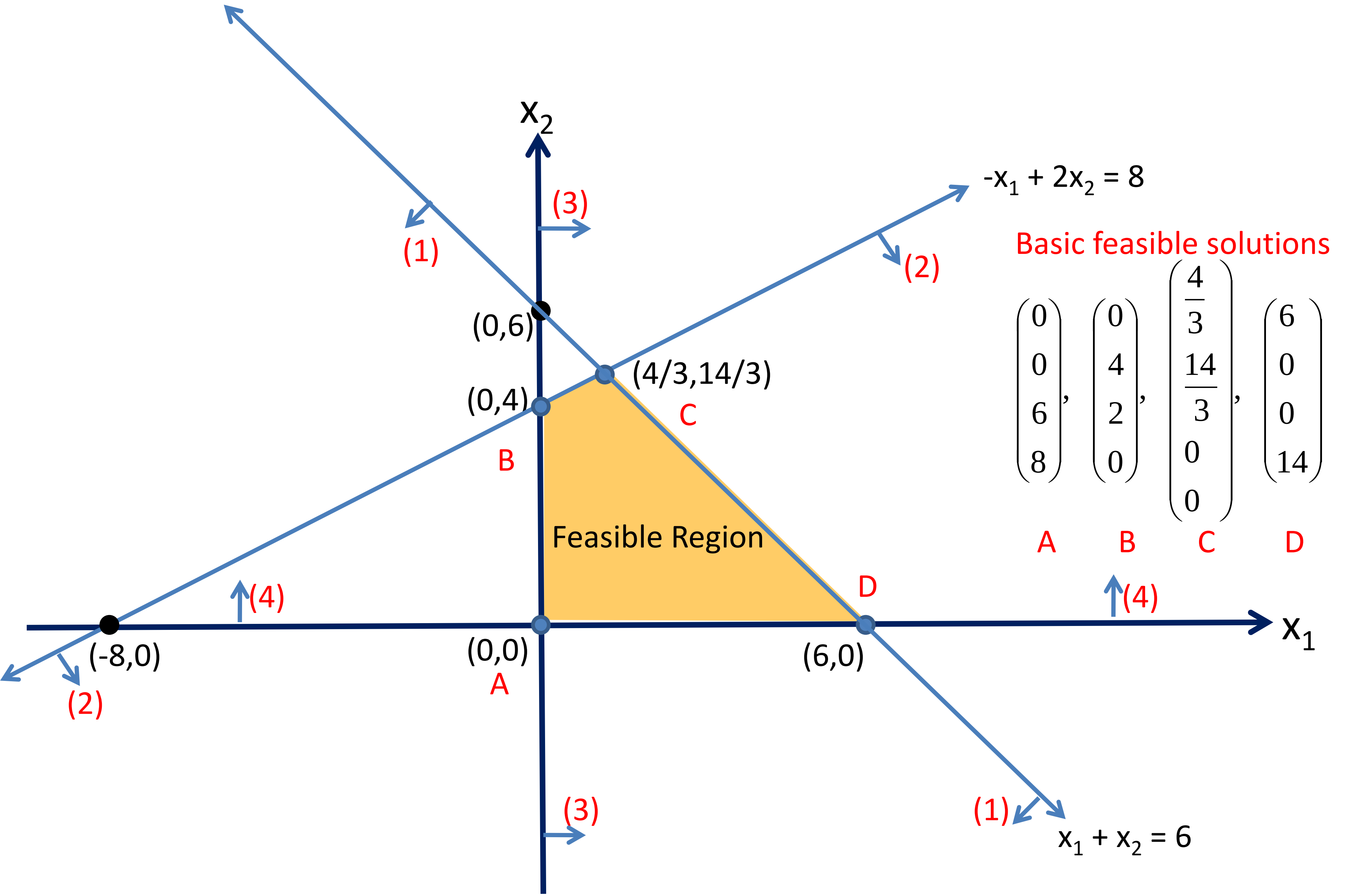
s.t.

$$z - x_1 - 3x_2 = 0 \qquad (0)$$

$$x_1 + x_2 + s_1 = 6 \qquad (1)$$

$$-x_1 + 2x_2 + s_2 = 8 \qquad (2)$$

$$x_1, x_2, s_1, s_2 \geq 0$$



x_2
x_1
$-x_1 + 2x_2 = 8$
$x_1 + x_2 = 6$
(1)
(2)
(3)
(4)
(0,6)
(0,4)
(4/3,14/3)
(-8,0)
(0,0)
(6,0)
A
B
C
D
Feasible Region
Basic feasible solutions
$\begin{pmatrix}0\\0\\6\\8\end{pmatrix}, \begin{pmatrix}0\\4\\2\\0\end{pmatrix}, \begin{pmatrix}\frac{4}{3}\\\frac{14}{3}\\0\\0\end{pmatrix}, \begin{pmatrix}6\\0\\0\\14\end{pmatrix}$
A B C D

The Simplex Algorithm

Canonical Form 0:

Row	**Basic Variable**											**RHS**
0		z	$-$	x_1	$-$	$3x_2$					$=$	0
1				x_1	$+$	x_2	$+$	s_1			$=$	6
2			$-$	x_1	$+$	$2x_2$			$+$	s_2	$=$	8

A system of linear equations in which each equation has a variable <u>with a coefficient 1</u> in that equation (and <u>a zero coefficient</u> in all other equations) is said to be in *canonical form*.

If the RHS of each constraint in a canonical form is nonnegative, a basic feasible solution can be obtained by inspection.

The Simplex Algorithm

Recall that the Simplex Algorithm begins with an initial bfs and attempts to find better ones. After obtaining a canonical form, we search for the **initial bfs**.

By inspection, if we set $x_1=x_2=0$, we can solve for the values of s_1 and s_2 by setting s_i equal to the RHS of row i.

BV=$\{s_1,s_2\}$ and NBV=$\{x_1,x_2\}$

The Simplex Algorithm

You may also verify the calculations for the initial basic feasible solution by:

$$B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad x_N = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad x_B = \begin{pmatrix} s_1 \\ s_2 \end{pmatrix}$$

The columns are linearly independent

$$\text{Setting } x_N = 0 \text{ and solving for } x_B : \; Bx_B = b \Rightarrow x_B = B^{-1}b$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{pmatrix} s_1 \\ s_2 \end{pmatrix} = \begin{pmatrix} 6 \\ 8 \end{pmatrix} \Rightarrow \begin{pmatrix} s_1 \\ s_2 \end{pmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}^{-1} \begin{pmatrix} 6 \\ 8 \end{pmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{pmatrix} 6 \\ 8 \end{pmatrix} = \begin{pmatrix} 6 \\ 8 \end{pmatrix}$$

The Simplex Algorithm

Notice that each basic variable is associated with the row of the canonical form in which the basic variable has a coefficient of 1.

To perform the simplex algorithm, we also need a basic variable **(not necessarily nonnegative!)** for row 0.

Observe that variable z appears in row 0 with a coefficient of 1, and z does not appear in any other row. Therefore, **we use z as basic variable for row 0**.

The Simplex Algorithm

Let us denote the initial canonical form as canonical form 0. With this convention, the basic and nonbasic variables for the canonical form 0 are BV={z, s_1, s_2} and NBV={x_1, x_2}.

For this basic feasible solution, $z=0$, $s_1=6$, $s_2=8$, $x_1=0$, $x_2=0$.

The Simplex Algorithm

In summary, the **canonical form**:

- LP has equality constraints and nonnegativity constraints ,
- There is one **basic** variable for each equality constraint,
- The column for the basic variable for constraint i has a 1 in constraint i and 0’s elsewhere,
- The remaining variables are called **nonbasic**.

The Simplex Algorithm

Canonical Form 0:

Row	Basic Variable											RHS
0		z	$-$	x_1	$-$	$3x_2$					$=$	0
1				x_1	$+$	x_2	$+$	s_1			$=$	6
2			$-$	x_1	$+$	$2x_2$			$+$	s_2	$=$	8

The Simplex Algorithm

Canonical Form 0:

Row	Basic Variable											RHS
0	$z=0$	z	$-$	x_1	$-$	$3x_2$					$=$	0
1	$s_1=6$			x_1	$+$	x_2	$+$	s_1			$=$	6
2	$s_2=8$		$-$	x_1	$+$	$2x_2$			$+$	s_2	$=$	8

Is this current basic feasible solution optimal?

To answer this question, we should determine whether there is any way that z can be increased by increasing some nonbasic variable from its current value of zero while holding all other nonbasic variables at their current values of zero ***(To reach an adjacent bfs)***.

The Simplex Algorithm

Canonical Form 0:

Row	Basic Variable											RHS
0	$z=0$	z	-	x_1	-	$3x_2$					=	0
1	$s_1=6$			x_1	+	x_2	+	s_1			=	6
2	$s_2=8$		-	x_1	+	$2x_2$			+	s_2	=	8

The Simplex Algorithm

Canonical Form 0:

Row	**Basic Variable**											**RHS**
0	$z=0$	z	-	x_1	-	$3x_2$					=	0
1	$s_1=6$			x_1	+	x_2	+	s_1			=	6
2	$s_2=8$		-	x_1	+	$2x_2$			+	s_2	=	8

Row 0 : $z - x_1 - 3x_2 = 0$

- if x_1 is increased by 1 unit, z increases by 1 unit
- if x_2 is increased by 1 unit, z increases by 3 units

So we choose x_2 as the **"entering variable".** If x_2 is to increase from its current value of zero, it has to become a **basic variable**.

The Simplex Algorithm

- For a max. problem, the entering variable has a negative coefficient in row 0. Usually we choose the variable with the most negative coefficient to be the entering variable (ties may be broken in an arbitrary fashion).

- x_2 will become basic. Therefore, one basic variable should become nonbasic. This will be the **“leaving variable”**.

The Simplex Algorithm

- Increasing x_2 may cause a basic variable to become negative. We look at how increasing x_2 (while holding x_1=0) changes the values of current set of basic variables:

$$\underbrace{x_1}_{=0} + x_2 + s_1 = 6 \qquad \Rightarrow \qquad x_2 + s_1 = 6$$

$$\underbrace{-x_1}_{=0} + 2x_2 + s_2 = 8 \qquad \qquad 2x_2 + s_2 = 8$$

The Simplex Algorithm

As $s_1 \geq 0$, $s_1 = 6 - x_2 \geq 0$ $\Rightarrow$ $x_2 \leq 6$

As $s_2 \geq 0$, $s_2 = 8 - 2x_2 \geq 0$ $\Rightarrow$ $x_2 \leq 4$

So, x_2 can be at most 4 (otherwise s_2 would become negative!)

The Simplex Algorithm

Observe that for any row in which the entering variable has a positive coefficient, the row's basic variable becomes negative if the entering variable exceeds:

$$\frac{\text{Right Hand Side of row}}{\text{Coefficient of entering variable in row}}$$

The Simplex Algorithm

Ratio Test: When entering a variable into the basis, for every *row i* in which the entering variable has a *positive coefficient*, we compute the ratio:

$$\frac{\text{RHS of row i}}{\text{Coefficient of entering variable in row i}},$$

and determine the smallest one.

The smallest ratio is the largest value of the entering variable that will keep all the current basic variables nonnegative.

The Simplex Algorithm

Canonical Form 0:

Row	**Basic Variable**											**RHS**
0	$z=0$	z	$-$	x_1	$-$	$3x_2$					$=$	0
1	$s_1=6$			x_1	$+$	x_2	$+$	s_1			$=$	6
2	$s_2=8$		$-$	x_1	$+$	$2x_2$			$+$	s_2	$=$	8

If $x_2=4$, then $s_2=0$ and $s_1=2$.

Therefore s_2 is the **leaving variable** and **becomes nonbasic**.

New basic variables = $\{s_1,x_2\}$; and new nonbasic variables = $\{x_1,s_2\}$
Hence, new $z = x_1+3x_2=12$.

The Simplex Algorithm

Canonical Form 0:

entering variable ↓ (column of x_2)

Row	**Basic Variable**												**RHS**
0	$z=0$	z	$-$	x_1	$-$	$3x_2$						$=$	0
1	$s_1=6$			x_1	$+$	x_2	$+$	s_1				$=$	6
2	$\boxed{s_2}=8$		$-$	x_1	$+$	$2x_2$			$+$	s_2		$=$	8

leaving variable

Always make the entering variable a basic variable in a row that wins the ratio test (ties may be broken arbitrarily).

The Simplex Algorithm

Canonical Form 0:

entering variable

Row	**Basic Variable**											**RHS**
0	z	z	-	x_1	-	$3x_2$					=	0
1	$s_1=6$			x_1	+	x_2	+	s_1			=	6
2	$\boxed{s_2}=8$		-	x_1	+	$2x_2$			+	s_2	=	8

leaving variable

Always make the entering variable a basic variable in a row that wins the ratio test (ties may be broken arbitrarily).

The Simplex Algorithm

Canonical Form 0:

Row	Basic Variable												RHS
0	z	z	-	x_1	-	$3x_2$						=	0
1	s_1			x_1	+	x_2	+	s_1				=	6
2	$\boxed{s_2}=8$		-	x_1	+	$2x_2$			+	s_2		=	8

entering variable

leaving variable

Always make the entering variable a basic variable in a row that wins the ratio test (ties may be broken arbitrarily).

The Simplex Algorithm

Row	Basic Variable											RHS
0	z	z	$-$	x_1	$-$	$3x_2$					$=$	0
1	s_1			x_1	$+$	x_2	$+$	s_1			$=$	6
2			$-$	x_1	$+$	$2x_2$			$+$	s_2	$=$	8

Always make the entering variable a basic variable in a row that wins the ratio test (ties may be broken arbitrarily).

The Simplex Algorithm

Row	Basic Variable											RHS
0	z	z	$-$	x_1	$-$	$3x_2$					$=$	0
1	s_1			x_1	$+$	x_2	$+$	s_1			$=$	6
2	$\boxed{x_2}$		$-$	x_1	$+$	$2x_2$			$+$	s_2	$=$	8

entering variable

Always make the entering variable a basic variable in a row that wins the ratio test (ties may be broken arbitrarily).

The Simplex Algorithm

Row	Basic Variable											RHS
0	z	z	-	x_1	-	$3x_2$					=	0
1	s_1			x_1	+	x_2	+	s_1			=	6
2	$\boxed{x_2}$		-	x_1	+	$2x_2$			+	s_2	=	8

entering variable

Always make the entering variable a basic variable in a row that wins the ratio test (ties may be broken arbitrarily).

To make x_2 a basic variable in row 2, we use elementary row operations to make x_2 has a coefficient of 1 in row 2 and coefficient of 0 in all other rows.

The Simplex Algorithm

Row	Basic Variable											RHS
0	z	z	$-$	x_1	$-$	$3x_2$					$=$	0
1	s_1			x_1	$+$	x_2	$+$	s_1			$=$	6
2	$\boxed{x_2}$		$-$	x_1	$+$	$2x_2$			$+$	s_2	$=$	8

entering variable

Always make the entering variable a basic variable in a row that wins the ratio test (ties may be broken arbitrarily).

To make x_1 a basic variable in row 2, we use elementary row operations to make x_1 has a coefficient of 1 in row 2 and coefficient of 0 in all other rows. This procedure is called **pivoting** on row 2.

The Simplex Algorithm

Row	Basic Variable											RHS
0	z	z	-	x_1	-	$3x_2$					=	0
1	s_1			x_1	+	x_2	+	s_1			=	6
2	$\boxed{x_2}$		-	x_1	+	$2x_2$			+	s_2	=	8

entering variable

Pivoting: Purpose is to rewrite the original problem in an equivalent form where columns corresponding to basic variables form an identity matrix. This allows us to determine the values of entering and leaving variables in the new solution.

The Simplex Algorithm

Row	**Basic Variable**											**RHS**
0	z	z	$-$	x_1	$-$	$3x_2$					$=$	0
1	s_1			x_1	$+$	x_2	$+$	s_1			$=$	6
2	x_2		$-$	x_1	$+$	$2x_2$			$+$	s_2	$=$	8

We may perform elementary row operations step by step, starting from the pivot row, one row at a time.

The Simplex Algorithm

Row	Basic Variable											RHS
0	z	z	-	x_1	-	$3x_2$					=	0
1	s_1			x_1	+	x_2	+	s_1			=	6
2	x_2		-	x_1	+	$2x_2$			+	s_2	=	8

To make x_2 has a coefficient of 1 in row 2:

- multiply row 2 by 0.5

The Simplex Algorithm

Row	Basic Variable											RHS
0	z	z	-	x_1	-	$3x_2$					=	0
1	s_1			x_1	+	x_2	+	s_1			=	6
2	x_2		-	$0.5\ x_1$	+	x_2			+	$0.5\ s_2$	=	4

To make x_2 has a coefficient of 1 in row 2:

- multiply row 2 by 0.5

The Simplex Algorithm

Row	Basic Variable											RHS
0	z	z	-	x_1	-	$3x_2$					=	0
1	s_1			x_1	+	x_2	+	s_1			=	6
2	x_2		-	$0.5\ x_1$	+	x_2			+	$0.5\ s_2$	=	4

To make x_2 has a coefficient of 0 in row 1:

- replace row 1 with row 1 – row 2

The Simplex Algorithm

Row	**Basic Variable**		**RHS**
0	z	$z - x_1 - 3x_2 =$	0
1	s_1	$1.5\,x_1 + s_1 - 0.5\,s_2 =$	2
2	x_2	$-\,0.5\,x_1 + x_2 + 0.5\,s_2 =$	4

To make x_2 has a coefficient of 0 in row 1:

- replace row 1 with row 1 – row 2

The Simplex Algorithm

Row	Basic Variable											RHS
0	z	z	-	x_1	-	$3x_2$					=	0
1	s_1			$1.5\ x_1$			+	s_1	-	$0.5\ s_2$	=	2
2	x_2		-	$0.5\ x_1$	+	x_2			+	$0.5\ s_2$	=	4

To make x_2 has a coefficient of 0 in row 0:

- replace row 0 with row 0 + 3 (row 2)

The Simplex Algorithm

Row	Basic Variable												RHS
0	z	z	-	$2.5\ x_1$					+	$1.5\ s_2$	=		12
1	s_1			$1.5\ x_1$			+	s_1	-	$0.5\ s_2$	=		2
2	x_2		-	$0.5\ x_1$	+	x_2			+	$0.5\ s_2$	=		4

To make x_2 has a coefficient of 0 in row 0:

- replace row 0 with row 0 + 3 (row 2)

The Simplex Algorithm

Canonical Form 1:

Row	**Basic Variable**											**RHS**
0	z	z	-	$2.5\ x_1$					+	$1.5\ s_2$	=	12
1	s_1			$1.5\ x_1$			+	s_1	-	$0.5\ s_2$	=	2
2	x_2		-	$0.5\ x_1$	+	x_2			+	$0.5\ s_2$	=	4

Is this bfs optimal?

No, increasing the nonbasic variable x_1 will increase z!

So *x_1 is the entering variable*. Also note that x_1 is the variable with the most negative coefficient in row 0.

The Simplex Algorithm

We perform the ratio test to find the leaving variable:

Since $s_2 = 0$, the system is:

$1.5\, x_1 + s_1 = 2$, and $s_1 \geq 0$ $\Rightarrow$ $s_1 = 2 - 1.5\, x_1 \geq 0$

$\Rightarrow$ $x_1 \leq 4/3$

$-0.5\, x_1 + x_2 = 4$, and $x_2 \geq 0$ $\Rightarrow$ $x_2 = 4 + 0.5\, x_1 \geq 0$

$\Rightarrow$ $x_1 \geq -8$

So, $x_1 = 4/3$ and s_1 becomes the leaving variable

The Simplex Algorithm

The new bfs is:

$$x_1 = 4/3,$$
$$x_2 = 14/3,$$
$$s_1 = 0,$$
$$s_2 = 0,$$
$$z = 46/3.$$

Now, keep the above results in mind and let us have a look at the pivot of the simplex algorithm:

The Simplex Algorithm

Canonical Form 1:

Row	**Basic Variable**			entering variable								**RHS**
0	z	z	$-$	$2.5\ x_1$					$+$	$1.5\ s_2$	$=$	12
1	$\boxed{s_1}$			$1.5\ x_1$			$+$	s_1	$-$	$0.5\ s_2$	$=$	2
2	x_2		$-$	$0.5\ x_1$	$+$	x_2			$+$	$0.5\ s_2$	$=$	4

leaving variable

The Simplex Algorithm

entering variable

Row	Basic Variable		RHS
0	z	$z - 2.5\,x_1 + 1.5\,s_2 =$	12
1	$\boxed{}$	$1.5\,x_1 + s_1 - 0.5\,s_2 =$	2
2	x_2	$-0.5\,x_1 + x_2 + 0.5\,s_2 =$	4

leaving variable

The Simplex Algorithm

Canonical Form 2:

Row	Basic Variable										RHS
0	z	z	$-$	$2.5\ x_1$					$+$	$1.5\ s_2$	$=$	12
1	x_1			$1.5\ x_1$			$+$	s_1	$-$	$0.5\ s_2$	$=$	2
2	x_2		$-$	$0.5\ x_1$	$+$	x_2			$+$	$0.5\ s_2$	$=$	4

To make x_1 a basic variable in row 1, we use elementary row operations to make x_1 has a coefficient of 1 in row 1 and coefficient of 0 in all other rows.

The Simplex Algorithm

Canonical Form 2:

Row	Basic Variable												RHS
0	z	z	-	$2.5\,x_1$					+	$1.5\,s_2$	=		12
1	x_1			$1.5\,x_1$			+	s_1	-	$0.5\,s_2$	=		2
2	x_2		-	$0.5\,x_1$	+	x_2			+	$0.5\,s_2$	=		4

To make x_1 has a coefficient of 1 in row 1:

- multiply row 1 by 2/3

The Simplex Algorithm

Canonical Form 2:

Row	**Basic Variable**											**RHS**
0	z	z	-	$2.5\ x_1$					+	$1.5\ s_2$	=	12
1	x_1			x_1			+	$2/3\ s_1$	-	$1/3\ s_2$	=	4/3
2	x_2		-	$0.5\ x_1$	+	x_2			+	$0.5\ s_2$	=	4

To make x_1 has a coefficient of 1 in row 1:

- multiply row 1 by 2/3

The Simplex Algorithm

Canonical Form 2:

Row	**Basic Variable**											**RHS**
0	z	z	$-$	$2.5\ x_1$					$+$	$1.5\ s_2$	$=$	12
1	x_1			x_1			$+$	$2/3\ s_1$	$-$	$1/3\ s_2$	$=$	$4/3$
2	x_2		$-$	$0.5\ x_1$	$+$	x_2			$+$	$0.5\ s_2$	$=$	4

The Simplex Algorithm

Canonical Form 2:

Row	Basic Variable		RHS
0	z	$z - 2.5x_1 + 1.5s_2 =$	12
1	x_1	$x_1 + 2/3\, s_1 - 1/3\, s_2 =$	4/3
2	x_2	$- 0.5x_1 + x_2 + 0.5s_2 =$	4

To make x_1 has a coefficient of 0 in row 2:

- multiply row 1 by 0.5 and add to the row 2

The Simplex Algorithm

Canonical Form 2:

Row	Basic Variable											RHS
0	z	z	$-$	$2.5\ x_1$					$+$	$1.5\ s_2$	$=$	12
1	x_1			x_1		$+$	$2/3\ s_1$		$-$	$1/3\ s_2$	$=$	$4/3$
2	x_2				x_2	$+$	$1/3\ s_1$		$+$	$1/3\ s_2$	$=$	$14/3$

To make x_1 has a coefficient of 0 in row 2:

- multiply row 1 by 0.5 and add to the row 2

The Simplex Algorithm

Canonical Form 2:

Row	**Basic Variable**											**RHS**
0	z	z	$-$	$2.5\ x_1$				$+$	$1.5\ s_2$	$=$		12
1	x_1			x_1		$+$	$2/3\ s_1$	$-$	$1/3\ s_2$	$=$		4/3
2	x_2				x_2	$+$	$1/3\ s_1$	$+$	$1/3\ s_2$	$=$		14/3

The Simplex Algorithm

Canonical Form 2:

Row	Basic Variable		RHS
0	z	$z - 2.5\,x_1 + 1.5\,s_2 =$	12
1	x_1	$x_1 + 2/3\,s_1 - 1/3\,s_2 =$	4/3
2	x_2	$x_2 + 1/3\,s_1 + 1/3\,s_2 =$	14/3

To make x_1 has a coefficient of 0 in row 0:

- multiply row 1 by 2.5 and add to the row 0

The Simplex Algorithm

Canonical Form 2:

Row	Basic Variable									RHS
0	z	z			$+$	$5/3\ s_1$	$+$	$2/3\ s_2$	$=$	$46/3$
1	x_1		x_1		$+$	$2/3\ s_1$	$-$	$1/3\ s_2$	$=$	$4/3$
2	x_2			x_2	$+$	$1/3\ s_1$	$+$	$1/3\ s_2$	$=$	$14/3$

To make x_1 has a coefficient of 0 in row 0:

- multiply row 1 by 2.5 and add to the row 0

The Simplex Algorithm

Canonical Form 2:

Row	Basic Variable									RHS
0	z	z			$+$	$5/3\ s_1$	$+$	$2/3\ s_2$	$=$	$46/3$
1	x_1		x_1		$+$	$2/3\ s_1$	$-$	$1/3\ s_2$	$=$	$4/3$
2	x_2			x_2	$+$	$1/3\ s_1$	$+$	$1/3\ s_2$	$=$	$14/3$

This result is the same as we had calculated before!

The Simplex Algorithm

Previously calculated new bfs was:

$x_1 = 4/3,$

$x_2 = 14/3,$

$s_1 = 0,$

$s_2 = 0,$

$z = 46/3.$

The Simplex Algorithm

Canonical Form 2:

Row	Basic Variable									
										RHS
0	z	z			$+$	$5/3\ s_1$	$+$	$2/3\ s_2$	$=$	$46/3$
1	x_1		x_1		$+$	$2/3\ s_1$	$-$	$1/3\ s_2$	$=$	$4/3$
2	x_2			x_2	$+$	$1/3\ s_1$	$+$	$1/3\ s_2$	$=$	$14/3$

Is this bfs optimal?

YES! Because increasing nonbasic variables s_1 and s_2 will decrease z (Also note that there is no variable in row 0 with a negative coefficient!)

Representing Simplex Tableaus

- Instead of writing each variable in every constraint, we can use a shorthand display called a ***simplex tableau***.

Representing Simplex Tableaus

Canonical Form 0:

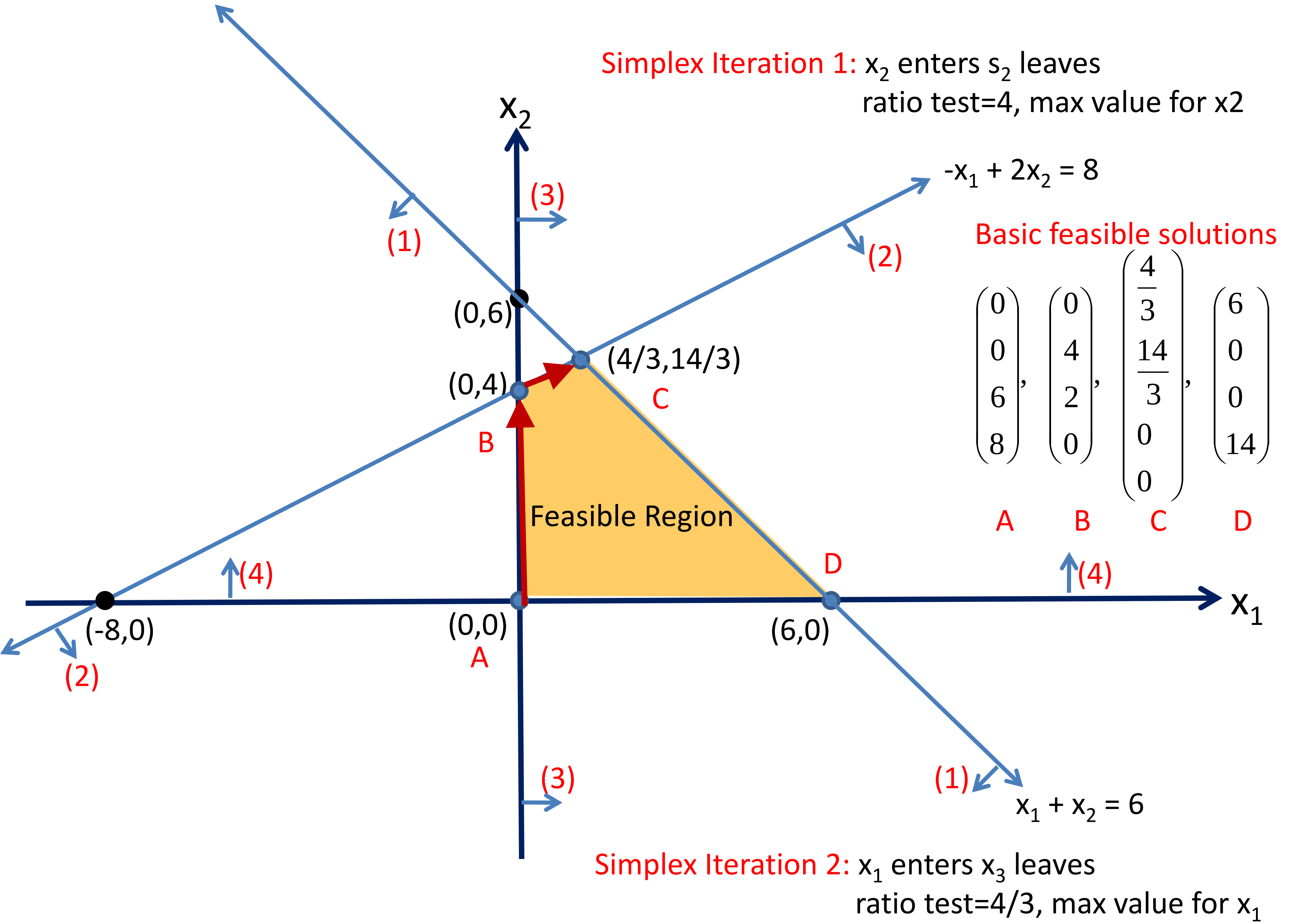
Row	**Basic Variable**											**RHS**
0	$z=0$	z	-	x_1	-	$3x_2$					=	0
1	$s_1=6$			x_1	+	x_2	+	s_1			=	6
2	$s_2=8$		-	x_1	+	$2x_2$			+	s_2	=	8

For example, canonical form 0 could be written as

Row	**Basic Variable**	**z**	x_1	x_2	s_1	s_2	**RHS**
0	$z=0$	1	-1	-3	0	0	0
1	$s_1=6$	0	1	1	1	0	6
2	$s_2=8$	0	-1	2	0	1	8

Representing Simplex Tableaus

- With this format, it is very easy to spot the basic variables. We just look for columns with a column of identity matrix underneath.


Simplex Iteration 1: x2 enters s2 leaves
ratio test=4, max value for x2
x2
-x1 + 2x2 = 8
(3)
(1)
(2)
Basic feasible solutions
(0,6)
(4/3,14/3)
(0,4)
C
B
Feasible Region
A
B
C
D
D
(4)
(4)
x1
(-8,0)
(0,0)
(6,0)
A
(2)
(3)
(1)
x1 + x2 = 6
Simplex Iteration 2: x1 enters x3 leaves
ratio test=4/3, max value for x1

Representing Simplex Tableaus

Simplex Tableau-0:

Basic Variable	**z**	x_1	x_2	s_1	s_2	**RHS**
z=0	1	-1	-3	0	0	0
s_1=6	0	1	1	1	0	6
s_2=8	0	-1	2	0	1	8

Note that z will always be a basic variable. Therefore, we won't be mentioning it unless it is necessary.
In addition, since row 0 corresponds to the objective function, it is indicated separately in the simplex tableau.

Also note that, since we are in canonical form, the basic variable of a row will be equal to the RHS of that row.

Representing Simplex Tableaus

Simplex Tableau-0:

Basic Variable	**z**	x_1	x_2	s_1	s_2	**RHS**
$z=0$	1	-1	-3	0	0	0
$s_1=6$	0	1	1	1	0	6
$s_2=8$	0	-1	2	0	1	8

Now let us summarize what we have done so far!

Representing Simplex Tableaus

Simplex Tableau-0:

Basic Variable	**z**	x_1	x_2	s_1	s_2	**RHS**
z	1	-1	-3	0	0	0
	0	1	1	1	0	6
	0	-1	2	0	1	8

For a max problem:

We start with an initial bfs in the canonical form above (if there are *m* slack variables, we use them as basic variables)

Representing Simplex Tableaus

Simplex Tableau-0:

Basic Variable	**z**	x_1	x_2	s_1	s_2	**RHS**
z	1	-1	-3	0	0	0
s_1	0	1	1	1	0	6
s_2	0	-1	2	0	1	8

For a max problem:

We start with an initial bfs in the canonical form above (if there are m slack variables, we use them as basic variables)

Representing Simplex Tableaus

Simplex Tableau-0:

entering variable ↓ (x_2)

Basic Variable	**z**	x_1	x_2	s_1	s_2	**RHS**
z	1	-1	-3	0	0	0
s_1	0	1	1	1	0	6
s_2	0	-1	2	0	1	8

Entering Variable: Choose a variable with the most negative coefficient in row 0.

Representing Simplex Tableaus

Simplex Tableau-0:

entering variable ↓ x_2

Basic Variable	**z**	x_1	x_2	s_1	s_2	**RHS**
z	1	-1	-3	0	0	0
s_1	0	1	1	1	0	6
s_2	0	-1	2	0	1	8

Representing Simplex Tableaus

Simplex Tableau-0: **entering variable** ↓ (x_2)

Basic Variable	**z**	x_1	x_2	s_1	s_2	**RHS**
z	1	-1	-3	0	0	0
s_1	0	1	1	1	0	6
s_2	0	-1	2	0	1	8

Leaving Variable: compute ratios:

$$\frac{\text{RHS of row } i}{\text{Coefficient of entering variable in row } i},$$

The smallest positive ratio wins and the basic variable of the winning row leaves the basis.

Representing Simplex Tableaus

Simplex Tableau-0:

entering variable

Basic Variable	z	x_1	x_2	s_1	s_2	RHS	Ratio Test
z	1	-1	-3	0	0	0	
s_1	0	1	1	1	0	6	6
$\boxed{s_2}$	0	-1	2	0	1	8	$\boxed{4}$

leaving variable

pivot term

Leaving Variable: compute ratios:

$$\frac{\text{RHS of row } i}{\text{Coefficient of entering variable in row } i},$$

The smallest positive ratio wins and the basic variable of the winning row leaves the basis.

Representing Simplex Tableaus

Simplex Tableau-0:

entering variable

Basic Variable	z	x_1	x_2	s_1	s_2	RHS	Ratio Test
z	1	-1	-3	0	0	0	
s_1	0	1	1	1	0	6	6
$\boxed{s_2}$	0	-1	2	0	1	8	$\boxed{4}$

leaving variable

pivot term

Leaving Variable: compute ratios:

$$\frac{\text{RHS of row } i}{\text{Coefficient of entering variable in row } i},$$

The smallest positive ratio wins and the basic variable of the winning row leaves the basis.

Representing Simplex Tableaus

Simplex Tableau-0:

Basic Variable	**z**	x_1	x_2	s_1	s_2	**RHS**	**Ratio Test**
z	1	-1	-3	0	0	0	
s_1	0	1	1	1	0	6	6
$\boxed{x_2}$	0	-1	2	0	1	8	4

pivot term

Leaving Variable: compute ratios:

$$\frac{\text{RHS of row } i}{\text{Coefficient of entering variable in row } i},$$

The smallest positive ratio wins and the basic variable of the winning row leaves the basis.

Representing Simplex Tableaus

Simplex Tableau-0:

Basic Variable	z	x_1	x_2	s_1	s_2	RHS
z	1	-1	-3	0	0	0
s_1	0	1	1	1	0	6
$\boxed{x_2}$	0	-1	2	0	1	8

pivot term

Leaving Variable: compute ratios:

$$\frac{\text{RHS of row } i}{\text{Coefficient of entering variable in row } i},$$

The smallest positive ratio wins and the basic variable of the winning row leaves the basis.

Representing Simplex Tableaus

Simplex Tableau-0:

Basic Variable	**z**	x_1	x_2	s_1	s_2	**RHS**
z	1	-1	-3	0	0	0
s_1	0	1	1	1	0	6
$\boxed{x_2}$	0	-1	2	0	1	8

pivot term

Representing Simplex Tableaus

Simplex Tableau-0:

Basic Variable	**z**	x_1	x_2	s_1	s_2	**RHS**
z	1	-1	-3	0	0	0
s_1	0	1	1	1	0	6
$\boxed{x_2}$	0	-1	2	0	1	8

pivot term

Pivot: Transform the tableau so that the new basic variable (entering variable) has 1 in the row of the leaving variable (pivot row) and 0 in other rows.

Representing Simplex Tableaus

Simplex Tableau-0:

Basic Variable	**z**	x_1	x_2	s_1	s_2	**RHS**
z	1	-1	-3	0	0	0
s_1	0	1	1	1	0	6
$\boxed{x_2}$						

pivot term

Pivot: Transform the tableau so that the new basic variable (entering variable) has 1 in the row of the leaving variable (pivot row) and 0 in other rows.

Representing Simplex Tableaus

Simplex Tableau-0:

Basic Variable	**z**	x_1	x_2	s_1	s_2	**RHS**
z	1	-1	-3	0	0	0
s_1	0	1	1	1	0	6
$\boxed{x_2}$	0	-1/2	1	0	1/2	4

pivot term

Pivot: Transform the tableau so that the new basic variable (entering variable) has 1 in the row of the leaving variable (pivot row) and 0 in other rows.

Representing Simplex Tableaus

Simplex Tableau-0:

Basic Variable	**z**	x_1	x_2	s_1	s_2	**RHS**
z	1	-1	-3	0	0	0
s_1						
$\boxed{x_2}$	0	-1/2	1	0	1/2	4

pivot term

Pivot: Transform the tableau so that the new basic variable (entering variable) has 1 in the row of the leaving variable (pivot row) and 0 in other rows.

Representing Simplex Tableaus

Simplex Tableau-0:

Basic Variable	**z**	x_1	x_2	s_1	s_2	**RHS**
z	1	-1	-3	0	0	0
s_1	0	3/2	0	1	-1/2	2
$\boxed{x_2}$	0	-1/2	1	0	1/2	4

pivot term

Pivot: Transform the tableau so that the new basic variable (entering variable) has 1 in the row of the leaving variable (pivot row) and 0 in other rows.

Representing Simplex Tableaus

Simplex Tableau-0:

Basic Variable	**z**	x_1	x_2	s_1	s_2	**RHS**
z						
s_1	0	3/2	0	1	-1/2	2
$\boxed{x_2}$	0	-1/2	1	0	1/2	4

pivot term

Pivot: Transform the tableau so that the new basic variable (entering variable) has 1 in the row of the leaving variable (pivot row) and 0 in other rows.

Representing Simplex Tableaus

Simplex Tableau-0:

Basic Variable	**z**	x_1	x_2	s_1	s_2	**RHS**
z	1	-5/2	0	0	3/2	12
s_1	0	3/2	0	1	-1/2	2
$\boxed{x_2}$	0	-1/2	1	0	1/2	4

pivot term

Pivot: Transform the tableau so that the new basic variable (entering variable) has 1 in the row of the leaving variable (pivot row) and 0 in other rows.

Representing Simplex Tableaus

Simplex Tableau-1:

enter ↓ (x_1)

Basic Variable	**z**	x_1	x_2	s_1	s_2	**RHS**
z	1	-5/2	0	0	3/2	12
s_1	0	3/2	0	1	-1/2	2
x_2	0	-1/2	1	0	1/2	4

Representing Simplex Tableaus

Simplex Tableau-1:

Basic Variable	z	x_1 (enter)	x_2	s_1	s_2	RHS	Ratio Test
z	1	-5/2	0	0	3/2	12	
leave $\boxed{s_1}$	0	3/2	0	1	-1/2	2	$\boxed{4/3}$
x_2	0	-1/2	1	0	1/2	4	no ratio

Representing Simplex Tableaus

Simplex Tableau-1:

enter ↓ x_1

Basic Variable	**z**	x_1	x_2	s_1	s_2	**RHS**	**Ratio Test**
z	1	-5/2	0	0	3/2	12	
leave ← $\boxed{}$	0	3/2	0	1	-1/2	2	$\boxed{4/3}$
x_2	0	-1/2	1	0	1/2	4	no ratio

Representing Simplex Tableaus

Simplex Tableau-1:

Basic Variable	**z**	x_1	x_2	s_1	s_2	**RHS**	**Ratio Test**
z	1	-5/2	0	0	3/2	12	
x_1	0	3/2	0	1	-1/2	2	4/3
x_2	0	-1/2	1	0	1/2	4	no ratio

Representing Simplex Tableaus

Simplex Tableau-1:

Basic Variable	**z**	x_1	x_2	s_1	s_2	**RHS**
z	1	-5/2	0	0	3/2	12
x_1	0	3/2	0	1	-1/2	2
x_2	0	-1/2	1	0	1/2	4

Representing Simplex Tableaus

Simplex Tableau-2:

Basic Variable	**z**	x_1	x_2	s_1	s_2	**RHS**
z	1	0	0	5/3	2/3	46/3
x_1	0	1	0	2/3	-1/3	4/3
x_2	0	0	1	1/3	1/3	14/3

Representing Simplex Tableaus

Simplex Tableau-2:

Basic Variable	**z**	x_1	x_2	s_1	s_2	**RHS**
z	1	0	0	5/3	2/3	46/3
x_1	0	1	0	2/3	-1/3	4/3
x_2	0	0	1	1/3	1/3	14/3

Can we iterate more?

No, because all row 0 coefficients are nonnegative. We stop here. The current bfs is optimal!

Summary of the Simplex Algorithm

(For a max problem)

1) Convert the LP into the standard form and then obtain the canonical form

2) Find an initial bfs (if there are *m* slack variables, use them as basic variables).

   If all nonbasic variables have nonnegative coefficients in row 0, then the current LP is optimal.

   If there are any variables with a negative coefficient, then we should decide the entering variable.

   Entering variable: choose a variable with the most negative coefficient in row 0 to enter the basis.

Summary of the Simplex Algorithm

3) For any row in which the entering variable has a *positive coefficient*, compute the ratios:

$$\frac{\text{RHS of row i}}{\text{Coefficient of the entering variable in row i}}$$

The smallest ratio wins (ties may be broken arbitrarily) and the basic variable of the winning row leaves the basis.

4) **Pivot:** Transform the tableau so that the new basic variable (entering variable) has coefficient of 1 in the row of the leaving variable (pivot row) and coefficient of 0 in all other rows. In the end, we get a tableau with a new canonical form.

Summary of the Simplex Algorithm

After the pivoting, note that:

$$\text{New pivot row} = \frac{1}{\begin{pmatrix}\text{coefficient of the entering} \\ \text{variable in the pivot row}\end{pmatrix}} \cdot (\text{old pivot row})$$

$$\text{new row i} = \text{old row i} - \begin{pmatrix}\text{coefficient of the entering} \\ \text{variable in the pivot row}\end{pmatrix} \cdot (\text{new pivot row})$$

Summary of the Simplex Algorithm

5) **Repeat steps 1,2,3, and 4** until all row 0 coefficients becomes nonnegative. If each nonbasic variable has a nonnegative coefficient in a canonical form's row 0 (remember that basic variables have coefficient of 0 in row 0 of a canonical form), then the canonical form is optimal. ***We stop here and the current bfs is optimal!***

More on Simplex Method

Simplex for min problems

Alternative optimal solutions

Unboundedness

Degeneracy

Big M method

Two phase method

Simplex for min Problems

Simplex for min Problems

Alternative 1: Use the algorithm for max problems

Remember,

min f$(x_1, x_2, \ldots, x_n)$ $\equiv$ - max - f$(x_1, x_2, \ldots, x_n)$

minimize $z = 2x_1 - 3x_2$ $\equiv$ -maximize $z = -2x_1 + 3x_2$

subject to

$$x_1 + x_2 \leq 4$$
$$x_1 - x_2 \leq 6$$
$$x_1, x_2 \geq 0$$

subject to

$$x_1 + x_2 \leq 4$$
$$x_1 - x_2 \leq 6$$
$$x_1, x_2 \geq 0$$

Don’t forget to negate the optimal value when you solve it as max problem!

Simplex for min Problems

Alternative 2: Direct way

In Row 0 format, choose the variable with the most positive coefficent as the entering variable.

Alternate Optimal Solutions

An Example*

The Dakota Furniture Company manufactures desks, tables, and chairs. The manufacture of each type of furniture requires lumber and two types of skilled labor: finishing and carpentry. The amount of each resource needed to make each type of furniture is given in Table 4.

Currently, 48 board feet of lumber, 20 finishing hours, and 8 carpentry hours are available. A desk sells for $60, a table for $30, and a chair for $20. Dakota believes that demand for desks and chairs is unlimited, but at most five tables can be sold. Because the available resources have already been purchased, Dakota wants to maximize total revenue.

Resource Requirements for Dakota Furniture

Resource	Desk	Table	Chair
Lumber (board ft)	8	6	1
Finishing hours	4	2	1.5
Carpentry hours	2	1.5	0.5

*from our textbook: "Operations Research: Applications and Algorithms" by Wayne Winston

Model

Resource Requirements for Dakota Furniture

Resource	Desk	Table	Chair
Lumber (board ft)	8	6	1
Finishing hours	4	2	1.5
Carpentry hours	2	1.5	0.5

Defining the decision variables as

$$x_1 = \text{number of desks produced}$$
$$x_2 = \text{number of tables produced}$$
$$x_3 = \text{number of chairs produced}$$

it is easy to see that Dakota should solve the following LP:

$$\begin{aligned}
\max z &= 60x_1 + 30x_2 + 20x_3 \\
\text{s.t.} \quad & 8x_1 + 6x_2 + x_3 \le 48 && \text{(Lumber constraint)} \\
& 4x_1 + 2x_2 + 1.5x_3 \le 20 && \text{(Finishing constraint)} \\
& 2x_1 + 1.5x_2 + 0.5x_3 \le 8 && \text{(Carpentry constraint)} \\
& x_2 \le 5 && \text{(Limitation on table demand)} \\
& x_1, x_2, x_3 \ge 0
\end{aligned}$$

Simplex Iterations

Canonical Form 0

Row		Basic Variable
0	$z - 60x_1 - 30x_2 - 20x_3 = 0$	$z = 0$
1	$8x_1 + 6x_2 + x_3 + s_1 = 48$	$s_1 = 48$
2	$4x_1 + 2x_2 + 1.5x_3 + s_2 = 20$	$s_2 = 20$
3	$2x_1 + 1.5x_2 + 0.5x_3 + s_3 = 8$	$s_3 = 8$
4	$x_2 + s_4 = 5$	$s_4 = 5$

Canonical Form 1

Row		Basic Variable
Row 0′	$z + 15x_2 - 5x_3 + 30s_3 = 240$	$z = 240$
Row 1′	$- x_3 + s_1 - 4s_3 = 16$	$s_1 = 16$
Row 2′	$- x_2 + 0.5x_3 + s_2 - 2s_3 = 4$	$s_2 = 4$
Row 3′	$x_1 + 0.75x_2 + 0.25x_3 + 0.5s_3 = 4$	$x_1 = 4$
Row 4′	$x_2 + s_4 = 5$	$s_4 = 5$

Canonical Form 2

Row		Basic Variable
0″	$z + 5x_2 + 10s_2 + 10s_3 = 280$	$z = 280$
1″	$- 2x_2 + s_1 + 2s_2 - 8s_3 = 24$	$s_1 = 24$
2″	$- 2x_2 + x_3 + 2s_2 - 4s_3 = 8$	$x_3 = 8$
3″	$x_1 + 1.25x_2 - 0.5s_2 + 1.5s_3 = 2$	$x_1 = 2$
4″	$x_2 + s_4 = 5$	$s_4 = 5$

Alternate Optimal Solutions

Now, reconsider the example with the modification that tables sell for $35 instead of $30.

TABLE 13

Initial Tableau for Dakota Furniture ($35/Table)

z	x_1	x_2	x_3	s_1	s_2	s_3	s_4	rhs	Basic Variable	Ratio
1	−60	$\boxed{-35}$	−20	0	0	0	0	0	$z = 0$	
0	8	6	1	1	0	0	0	48	$s_1 = 48$	$\frac{48}{8} = 6$
0	4	2	1.5	0	1	0	0	20	$s_2 = 20$	$\frac{20}{4} = 5$
0	2	1.5	0.5	0	0	1	0	8	$s_3 = 8$	$\frac{8}{2} = 4^*$
0	0	1	0	0	0	0	1	5	$s_4 = 5$	None

TABLE 14

First Tableau for Dakota Furniture ($35/Table)

z	x_1	x_2	x_3	s_1	s_2	s_3	s_4	rhs	Basic Variable	Ratio
1	0	10	−5	0	0	30	0	240	$z = 240$	
0	0	0	−1	1	0	−4	0	16	$s_1 = 16$	None
0	0	−1	0.5	0	1	−2	0	4	$s_2 = 4$	$\frac{4}{0.5} = 8^*$
0	1	0.75	0.25	0	0	0.5	0	4	$x_1 = 4$	$\frac{4}{0.25} = 16$
0	0	1	0	0	0	0	1	5	$s_4 = 5$	None

Alternate Optimal Solutions

TABLE 15

Second (and Optimal) Tableau for Dakota Furniture ($35/Table)

z	x_1	x_2	x_3	s_1	s_2	s_3	s_4	rhs	Basic Variable
1	0	0	0	0	10	10	0	280	$z = 280$
0	0	−2	0	1	2	−8	0	24	$s_1 = 24$
0	0	−2	1	0	2	−4	0	8	$x_3 = 8$
0	1	1.25	0	0	−0.5	1.5	0	2	$x_1 = 2$*
0	0	1	0	0	0	0	1	5	$s_4 = 5$

Recall that all basic variables must have a zero coefficient in row 0 (or else they wouldn't be basic variables). However, in our optimal tableau, there is a nonbasic variable, x_2, which also has a zero coefficient in row 0. Let us see what happens if we enter x_2 into the basis. The

TABLE 16

Another Optimal Tableau for Dakota Furniture ($35/Table)

z	x_1	x_2	x_3	s_1	s_2	s_3	s_4	rhs	Basic Variable
1	0	0	0	0	10	10	0	280	$z = 280$
0	1.6	0	0	1	1.2	−5.6	0	27.2	$s_1 = 27.2$
0	1.6	0	1	0	1.2	−1.6	0	11.2	$x_3 = 11.2$
0	0.8	1	0	0	−0.4	1.2	0	1.6	$x_2 = 1.6$
0	−0.8	0	0	0	0.4	−1.2	1	3.4	$s_4 = 3.4$

Remember,

change in objective value=|coefficient of entering variable| * ratio test result

Alternate Optimal Solutions

Note that their convex combinations are also optimal.

	x1	x2	x3	ObjFnVal
ObjCoeff	60	35	20	-
opt1	2.00	0.00	8.00	280
opt2	0.00	1.60	11.20	280

lambda	Convex Combinations			ObjFnVal
0.0	0.00	1.60	11.20	280
0.1	0.20	1.44	10.88	280
0.2	0.40	1.28	10.56	280
0.3	0.60	1.12	10.24	280
0.4	0.80	0.96	9.92	280
0.5	1.00	0.80	9.60	280
0.6	1.20	0.64	9.28	280
0.7	1.40	0.48	8.96	280
0.8	1.60	0.32	8.64	280
0.9	1.80	0.16	8.32	280
1.0	2.00	0.00	8.00	280

Alternate Optimal Solutions - Remark

- In Simplex algorithm, alternative solutions are detected when there are *0* valued coefficients for nonbasic variables in row-0 of the optimal tableau.
- If there is no nonbasic variable with a zero coefficient in row 0 of the optimal tableau, the LP has a unique optimal solution.
- Even if there is a nonbasic variable with a zero coefficient in row 0 of the optimal tableau, it is possible that the LP may not have alternative optimal solutions.

Alternate Optimal Solutions

Practice example:

maximize $z = 2x_1 + 4x_2$

subject to

$$x_1 + 2x_2 \leq 5$$

$$x_1 + x_2 \leq 4$$

$$x_1, x_2 \geq 0$$

Alternate Optimal Solutions

<u>Practice example:</u>

maximize $z = 2x_1 + 4x_2$

subject to

$$x_1 + 2x_2 \leq 5$$
$$x_1 + x_2 \leq 4$$
$$x_1, x_2 \geq 0$$

Set of alternate optimal solutions=

$$\left\{ \begin{pmatrix} x_1 \\ x_2 \\ s_1 \\ s_2 \end{pmatrix} : \begin{pmatrix} x_1 \\ x_2 \\ s_1 \\ s_2 \end{pmatrix} = \lambda \begin{pmatrix} 0 \\ \frac{5}{2} \\ 0 \\ \frac{3}{2} \end{pmatrix} + (1-\lambda) \begin{pmatrix} 3 \\ 1 \\ 0 \\ 0 \end{pmatrix} \; where \; \lambda \in [0,1] \right\}$$

Unboundedness

Unbounded LPs

For some LPs, there exist points in the feasible region for which *z* assumes arbitrarily large (in max problems) or arbitrarily small (in min problems) values. When this occurs, we say the LP is unbounded.

Consider the following LP:

$$\text{maximize } z = x_1 + 2x_2$$

subject to

$$\begin{aligned} x_1 - x_2 &\leq 10 \\ x_1 \quad\quad &\leq 40 \\ x_1, x_2 &\geq 0 \end{aligned}$$

Unbounded LPs

Practice Example:

In standard form:

maximize $z = x_1 + 2x_2$

subject to

$$
\begin{aligned}
x_1 - x_2 + s_1 \quad\quad &= 10 \\
x_1 \quad\quad\quad + s_2 &= 40 \\
x_1, x_2, s_1, s_2 &\geq 0
\end{aligned}
$$

Apply Simplex Method.

Consider $x_1 = 0$; $s_2 = 40$; $x_2 = a$; $s_1 = 10+a$.

The objective function value is then $2a$ for any $a \in \mathbb{R}^+$.

Unbounded LPs

- An unbounded LP occurs in a max (min) problem if there is a nonbasic variable with a negative (positive) coefficient in row 0 and there is no constraint that limits how large we can make this nonbasic variable.
- Specifically, an unbounded LP for a max (min) problem occurs when a variable with a negative (positive) coefficient in row 0 has a non positive coefficient in each constraint.

There is an entering variable but no leaving variable, since ratio test does not give a finite bound!

Degeneracy

Degeneracy

- An LP is a degenerate LP if in a basic feasible solution, one of the basic variables takes on a zero value. This bfs is called degenerate bfs.
- Degeneracy could cost simplex method extra iterations.
- When degeneracy occurs, obj fn value will not increase.
- A cycle in the simplex method is a sequence of K+1 iterations with corresponding bases B_0, …, B_K, B_0 and $K \geq 1$.
- If cycling occurs, then the algorithm will loop, or cycle, forever among a set of basic feasible solutions and never get to an optimal solution.

Example of Cycling

Example 1 *(Degenerate Pivoting)*
Pivot rules:

- *Choose entering variable with largest reduced cost.*
- *Choose leaving variable with smallest subscript.*

$$
\begin{array}{rrcrcrcrcl}
\textit{maximize} & 10x_1 & - & 57x_2 & - & 9x_3 & - & 24x_4 & & \\
\textit{subject to} & 0.5x_1 & - & 5.5x_2 & - & 2.5x_3 & + & 9x_4 & \leq & 0 \\
 & 0.5x_1 & - & 1.5x_2 & - & 0.5x_3 & + & x_4 & \leq & 0 \\
 & x_1 & & & & & & & \leq & 1 \\
 & x_1 & , & x_2 & , & x_3 & , & x_4 & \geq & 0.
\end{array}
$$

Introduce slacks in initial tableau. Initial basis: $\{5,6,7\}$.

$$
\begin{array}{crrrrrrrcl}
z & -10x_1 & +57x_2 & +9x_3 & +24x_4 & & & & = & 0 \\
 & \boxed{0.5x_1} & -5.5x_2 & -2.5x_3 & +9x_4 & +x_5 & & & = & 0 \\
 & 0.5x_1 & -1.5x_2 & -0.5x_3 & +x_4 & & +x_6 & & = & 0 \\
 & x_1 & & & & & & +x_7 & = & 1.
\end{array}
$$

$$
\begin{array}{crrrrrrrcl}
z & & -53x_2 & -41x_3 & +204x_4 & +20x_5 & & & = & 0 \\
 & x_1 & -11x_2 & -5x_3 & +18x_4 & +2x_5 & & & = & 0 \\
 & & \boxed{4x_2} & +2x_3 & -8x_4 & -x_5 & +x_6 & & = & 0 \\
 & & 11x_2 & +5x_3 & -18x_4 & -2x_5 & & +x_7 & = & 1.
\end{array}
$$

$$
\begin{array}{crrrrrrrcl}
z & & & -14.5x_3 & +98x_4 & +6.75x_5 & +13.25x_6 & & = & 0 \\
 & x_1 & & \boxed{+0.5x_3} & -4x_4 & -0.75x_5 & +2.75x_6 & & = & 0 \\
 & & x_2 & +0.5x_3 & -2x_4 & -0.25x_5 & +0.25x_6 & & = & 0 \\
 & & & -0.5x_3 & +4x_4 & +0.75x_5 & -2.75x_6 & +x_7 & = & 1.
\end{array}
$$

$$
\begin{array}{crrrrrrrcl}
z & +29x_1 & & & -18x_4 & -15x_5 & +93x_6 & & = & 0 \\
 & 2x_1 & & +x_3 & -8x_4 & -1.5x_5 & +5.5x_6 & & = & 0 \\
 & -x_1 & +x_2 & & \boxed{+2x_4} & +0.5x_5 & -2.5x_6 & & = & 0 \\
 & x_1 & & & & & & +x_7 & = & 1.
\end{array}
$$

$$
\begin{array}{crrrrrrrcl}
z & +20x_1 & +9x_2 & & & -10.5x_5 & +70.5x_6 & & = & 0 \\
 & -2x_1 & +4x_2 & +x_3 & & \boxed{+0.5x_5} & -4.5x_6 & & = & 0 \\
 & -0.5x_1 & +0.5x_2 & & +x_4 & +0.25x_5 & -1.25x_6 & & = & 0 \\
 & x_1 & & & & & & +x_7 & = & 1.
\end{array}
$$

$$
\begin{array}{crrrrrrrcl}
z & -22x_1 & +93x_2 & +21x_3 & & & -24x_6 & & = & 0 \\
 & -4x_1 & +8x_2 & +2x_3 & & +x_5 & -9x_6 & & = & 0 \\
 & +0.5x_1 & -1.5x_2 & -0.5x_3 & +x_4 & & \boxed{+x_6} & & = & 0 \\
 & x_1 & & & & & & +x_7 & = & 1.
\end{array}
$$

New basis, $\{5,6,7\}$, *is identical with the first basis, so now we* ***CYCLE!***

*Lecture 13 by Prof. David P. Williamson of Cornell Uni. ORIE Dept.

Degeneracy

- Consider the following example*:

$$\begin{aligned} \max z &= 5x_1 + 2x_2 \\ \text{s.t.} \quad x_1 + x_2 &\le 6 \\ x_1 - x_2 &\le 0 \\ x_1, x_2 &\ge 0 \end{aligned}$$

A Degenerate LP

z	x_1	x_2	s_1	s_2	rhs	Basic Variable	Ratio
1	−5	−2	0	0	0	$z = 0$	
0	1	1	1	0	6	$s_1 = 6$	6
0	1	−1	0	1	0	$s_2 = 0$	0*

First Tableau for (16)

z	x_1	x_2	s_1	s_2	rhs	Basic Variable	Ratio
1	0	−7	0	5	0	$z = 0$	
0	0	2	1	−1	6	$s_1 = 6$	$\frac{6}{2} = 3$*
0	1	−1	0	1	0	$x_1 = 0$	None

*from our textbook: "Operations Research: Applications and Algorithms" by Wayne Winston

Degeneracy

A Degenerate LP

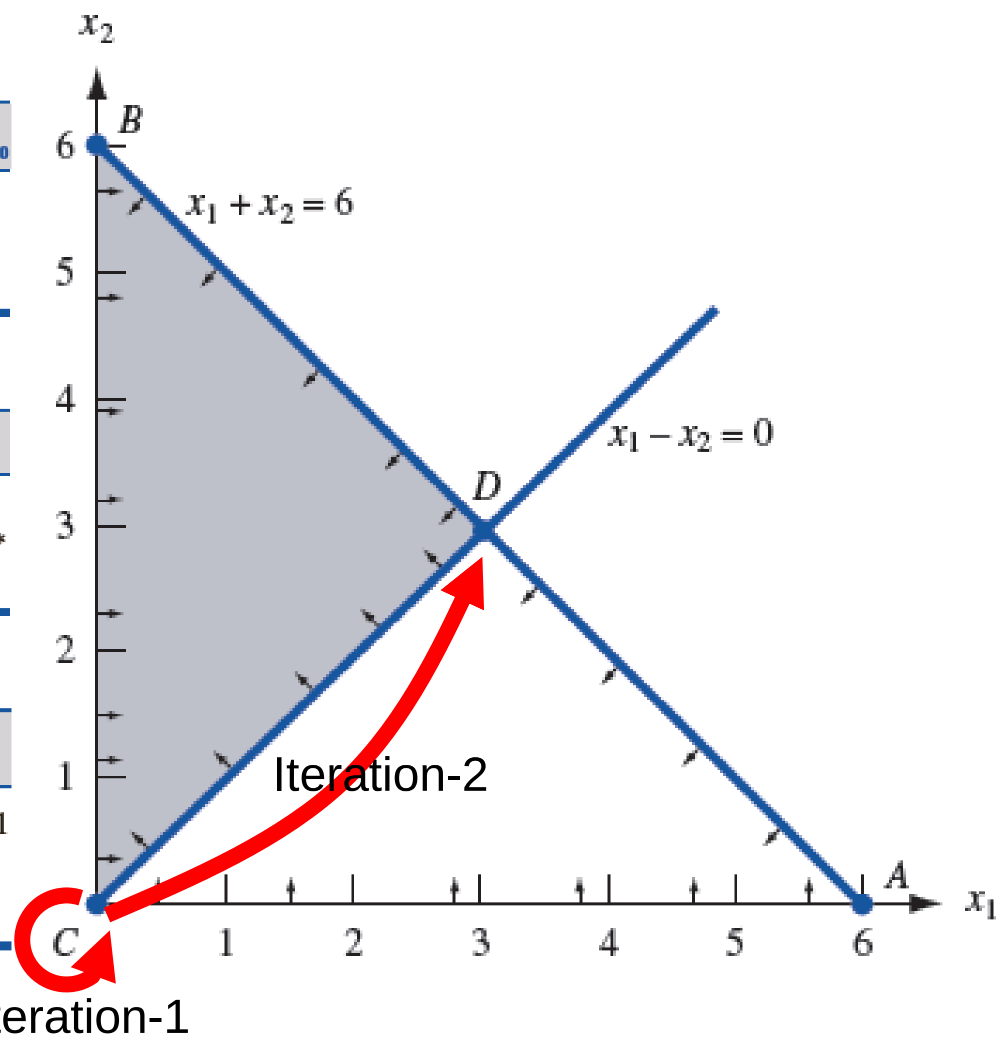
z	x_1	x_2	s_1	s_2	rhs	Basic Variable	Ratio
1	−5	−2	0	0	0	$z = 0$	
0	1	1	1	0	6	$s_1 = 6$	6
0	1	−1	0	1	0	$s_2 = 0$	0*

First Tableau for (16)

z	x_1	x_2	s_1	s_2	rhs	Basic Variable	Ratio
1	0	−7	0	5	0	$z = 0$	
0	0	2	1	−1	6	$s_1 = 6$	$\frac{6}{2} = 3$*
0	1	−1	0	1	0	$x_1 = 0$	None

Optimal Tableau for (16)

z	x_1	x_2	s_1	s_2	rhs	Basic Variable
1	0	0	3.5	1.5	21	$z = 21$
0	0	1	0.5	−0.5	3	$x_2 = 3$
0	1	0	0.5	0.5	3	$x_1 = 3$



Degeneracy

- In the simplex algorithm, degeneracy is detected when there is a tie for the minimum ratio test. In the following iteration, the solution is degenerate.

- <u>Example (for practice):</u>

$$\text{maximize } z = 3x_1 + 9x_2$$

subject to

$$x_1 + 4x_2 \le 8$$

$$x_1 + 2x_2 \le 4$$

$$x_1, x_2 \ge 0$$

Degeneracy – Bland's Rule

- When degeneracy occurs, obj fn value will not increase and algorithm cycles same basic feasible solutions. To prevent this:
- Bland showed that cycling can be avoided by applying the following rules (assume that the slack and excess variables are numbered x_{n+1}, x_{n+2} etc.)
- Choose an entering variable (in a max problem) the variable with a negative coefficient in row 0 that has the smallest index
- If there is a tie in the ratio test, then break the tie by choosing the winner of the ratio test so that the variable leaving the basis has the smallest index
- Using Bland's rule, the Simplex Algorithm terminates in finite time with optimal solution (i.e. no cycling)

Start Applying Bland's rule when a degenerate bfs is encountered

Big-M Method

Alternative 1 for finding and initial bfs.

Big M Method

- The simplex method algorithm requires a starting bfs.
- Previous problems have found starting bfs by using the slack variables as our basic variables.
  - If an LP has ≥ or = constraints, however, a starting bfs may not be readily apparent.
- In such a case, the Big M method may be used to solve the problem.

Big M Method

- Consider the following LP:

$$
\begin{aligned}
\text{minimize} \quad & z = 2x_1 + 3x_2 \\
\text{subject to} \quad & 0.5x_1 + 0.25x_2 \leq 4 \\
& x_1 + 3x_2 \geq 20 \\
& x_1 + x_2 = 10 \\
& x_1, x_2 \geq 0
\end{aligned}
$$

Big M Method

- Consider the following LP:

$$\begin{array}{llcl} \text{minimize} & z = 2x_1 + 3x_2 & & \text{- maximize} \quad z = -2x_1 - 3x_2 \\ \text{subject to} & 0.5x_1 + 0.25x_2 \leq 4 & \equiv & \text{subject to} \quad 0.5x_1 + 0.25x_2 \leq 4 \\ & x_1 + 3x_2 \geq 20 & & x_1 + 3x_2 \geq 20 \\ & x_1 + x_2 = 10 & & x_1 + x_2 = 10 \\ & x_1, x_2 \geq 0 & & x_1, x_2 \geq 0 \end{array}$$

Big M Method

- The LP in standard form has z and s_1 which could be used for BVs but row 2 would violate sign restrictions and row 3 no readily apparent basic variable.

$$
\begin{array}{lrcrcrcrcl}
\text{Row 0:} & z + & 2x_1 & + & 3x_2 & & & & = & 0 \\
\text{Row 1:} & & 0.5x_1 & + & 0.25x_2 & + & s_1 & & = & 4 \\
\text{Row 2:} & & x_1 & + & 3x_2 & & & - e_2 & = & 20 \\
\text{Row 3:} & & x_1 & + & x_2 & & & & = & 10
\end{array}
$$

- In order to use the simplex method, a bfs is needed.
  - To remedy the predicament, **artificial variables** are created.
  - The variables will be labeled according to the row in which they are used.

Big M Method

Row 0: $z + 2x_1 + 3x_2 = 0$

Row 1: $0.5x_1 + 0.25x_2 + s_1 = 4$

Row 2: $x_1 + 3x_2 - e_2 + a_2 = 20$

Row 3: $x_1 + x_2 + a_3 = 10$

- In the optimal solution, all artificial variables must be set equal to zero.
  - To accomplish this, in a min LP, a term Ma_i is added to the objective function for each artificial variable a_i.
  - For a max LP, the term $-Ma_i$ is added to the objective function for each a_i.
  - M represents some very large number.

Big M Method

- The modified LP in standard form then becomes:

$$
\begin{array}{llr}
\text{Row 0:} & z + 2x_1 + 3x_2 + Ma_2 + Ma_3 & = 0 \\
\text{Row 1:} & 0.5x_1 + 0.25x_2 + s_1 & = 4 \\
\text{Row 2:} & x_1 + 3x_2 - e_2 + a_2 & = 20 \\
\text{Row 3:} & x_1 + x_2 + a_3 & = 10
\end{array}
$$

- Modifying the objective function this way makes it extremely costly for an artificial variable to be positive. The optimal solution should force $a_2 = a_3 = 0$ (whenever possible!)

Big M Method

Row	Basic Variable	z	x_1	x_2	s_1	e_2	a_2	a_3	RHS
0	z	1	2	3	0	0	M	M	0
1	s_1	0	1/2	1/4	1	0	0	0	4
2	a_2	0	1	3	0	-1	1	0	20
3	a_3	0	1	1	0	0	0	1	10

Because basic variables a_2 and a_3 have nonzero Row 0 coefficients, do elementary row operations to zero them out: Add -M(Row2) and –M(Row 3) to Row 0 to achieve a proper Row 0 for simplex to start

Big M Method

Row	**Basic Variable**	**z**	x_1	x_2	s_1	e_2	a_2	a_3	**RHS**
~~0~~	~~z~~	~~1~~	~~2~~	~~3~~	~~0~~	~~0~~	~~M~~	~~M~~	~~0~~
0	z	1	2-2M	3-4M	0	M	0	0	-30M
1	s_1	0	1/2	1/4	1	0	0	0	4
2	a_2	0	1	3	0	-1	1	0	20
3	a_3	0	1	1	0	0	0	1	10

Big M Method

Row	Basic Variable	z	x_1	x_2	s_1	e_2	a_2	a_3	RHS	Min Ratio Test
0	z	1	2-2M	3-4M	0	M	0	0	-30M	
1	s_1	0	1/2	1/4	1	0	0	0	4	16
2	a_2	0	1	3	0	-1	1	0	20	20/3
3	a_3	0	1	1	0	0	0	1	10	10

Big M Method

Row	Basic Variable	z	x_1	x_2	s_1	e_2	a_2	a_3	RHS	Min Ratio Test
0	z	1	1-2M/3	0	0	1-M/3		0	-20-10M/3	
1	s_1	0	5/12	0	1	1/12		0	7/3	28/5
2	x_2	0	1/3	1	0	-1/3		0	20/3	20
3	a_3	0	2/3	0	0	1/3		1	10/3	5

Since a_2 has left the basis, we can forget about that column for good!

Big M Method

Row	Basic Variable	z	x_1	x_2	s_1	e_2	a_2	a_3	RHS
0	z	1	0	0	0	1/2			-25
1	s_1	0	0	0	1	-1/8			1/4
2	x_2	0	0	1	0	-1/2			5
3	x_1	0	1	0	0	1/2			5

Since a_3 has left the basis, we can also forget about that column for good!

Big M Method

Row	Basic Variable	z	x_1	x_2	s_1	e_2	a_2	a_3	RHS
0	z	1	0	0	0	1/2			-25
1	s_1	0	0	0	1	-1/8			1/4
2	x_2	0	0	1	0	-1/2			5
3	x_1	0	1	0	0	1/2			5

Final Tableau!

The optimal solution is $z=-25$, $x_1=x_2 = 5$, $s_1=1/4$, $e_2=0$.

Big M Method

- The optimal solution (for the original min problem) is $z=25$, $x_1=x_2 = 5$, $s_1=1/4$, $e_2=0$.
- *Remark*: once an artificial variable is NB, it can be dropped from the future tableaus since it will never become basic again.
- *Remark*: when choosing the entering variable, remember that M is a very large number. For example,
  - $4M-2 > 3M + 5000$,
  - $-6M-5 < -3M - 10000$.

Big M Method

- Another example LP:

$$\begin{aligned} \text{maximize} \quad & z = x_1 + x_2 \\ \text{subject to} \quad & x_1 - x_2 \geq 1 \\ & -x_1 + x_2 \geq 1 \\ & x_1, x_2 \geq 0 \end{aligned}$$

Big M Method

Row	Basic Variable	z	x_1	x_2	e_1	e_2	a_1	a_2	RHS
0	z	1	-1	-1	0	0	M	M	0
1	a_1	0	1	-1	-1	0	1	0	1
2	a_2	0	-1	1	0	-1	0	1	1

Because basic variables a_1 and a_2 have nonzero Row 0 coefficients, do elementary row operations to zero them out: Add -M(Row1) and –M(Row 2) to Row 0 to achieve a proper Row 0 for simplex to start

Big M Method

Row	**Basic Variable**	**z**	x_1	x_2	e_1	e_2	a_1	a_2	**RHS**
~~0~~	~~z~~	~~1~~	~~-1~~	~~-1~~	~~0~~	~~0~~	~~M~~	~~M~~	~~0~~
0	z	1	-1	-1	M	M	0	0	-2M
1	a_1	0	1	-1	-1	0	1	0	1
2	a_2	0	-1	1	0	-1	0	1	1

Because basic variables a_1 and a_2 have nonzero Row 0 coefficients, do elementary row operations to zero them out: Add -M(Row1) and –M(Row 2) to Row 0 to achieve a proper Row 0 for simplex to start

Big M Method

Row	Basic Variable	**z**	x_1	x_2	e_1	e_2	a_1	a_2	**RHS**
0	z	1	-1	-1	M	M	0	0	-2M
1	a_1	0	1	-1	-1	0	1	0	1
2	a_2	0	-1	1	0	-1	0	1	1

Big M Method

Row	Basic Variable	z	x_1	x_2	e_1	e_2	a_1	a_2	RHS
0	z	1	0	-2	M-1	M		0	-2M+1
1	x_1	0	1	-1	-1	0		0	1
2	a_2	0	0	0	-1	-1		1	2

The final tableau indicates that the solution is unbounded (no exiting variable) and one of the artificial variables is nonzero.

Thus, the original LP is infeasible.

Big M Method

1. Modify the constraints so that the rhs of each constraint is nonnegative. Identify each constraint that is now an = or ≥ constraint.
2. Convert each inequality constraint to standard form (add a slack variable for ≤ constraints, add an excess variable for ≥ constraints).
3. For each ≥ or = constraint, add artificial variables. Add sign restriction $a_i \geq 0$.
4. Let M denote a very large positive number. Add (for each artificial variable) Ma_i to min problem objective functions or $-Ma_i$ to max problem objective functions.
5. Since each artificial variable will be in the starting basis, all artificial variables must be eliminated from row 0 before beginning the simplex. Remembering M represents a very large number, solve the transformed problem by the simplex.

Big M Method

- If all artificial variables in the optimal solution equal zero, the solution is ?

Big M Method

- If all artificial variables in the optimal solution equal zero, the solution is optimal.
- If any artificial variables are positive in the optimal solution, the problem is ?

Big M Method

- If all artificial variables in the optimal solution equal zero, the solution is optimal.
- If any artificial variables are positive in the optimal solution, the problem is infeasible.
- When the LP (with the artificial variables) is solved, the final tableau may indicate that the LP is unbounded. If the final tableau indicates the LP is unbounded and all artificial variables in this tableau equal zero, then the original LP is ?

Big M Method

- If all artificial variables in the optimal solution equal zero, the solution is optimal.
- If any artificial variables are positive in the optimal solution, the problem is infeasible.
- When the LP (with the artificial variables) is solved, the final tableau may indicate that the LP is unbounded. If the final tableau indicates the LP is unbounded and all artificial variables in this tableau equal zero, then the original LP is unbounded. If the final tableau indicates that the LP is unbounded and at least one artificial variable is positive, then the original LP is ?

Big M Method

- If all artificial variables in the optimal solution equal zero, the solution is optimal.
- If any artificial variables are positive in the optimal solution, the problem is infeasible.
- When the LP (with the artificial variables) is solved, the final tableau may indicate that the LP is unbounded. If the final tableau indicates the LP is unbounded and all artificial variables in this tableau equal zero, then the original LP is unbounded. If the final tableau indicates that the LP is unbounded and at least one artificial variable is positive, then the original LP is infeasible.

Big M Method - Remark

For computer programs, it is difficult to determine how large M should be. Generally, M is chosen to be at least 100 times larger than the largest coefficient in the original objective function. The introduction of such large numbers into the problem can cause roundoff errors and other computational difficulties. For this reason, most computer codes solve LPs by using the two-phase simplex method.

Two-Phase Simplex

Alternative 2 for finding and initial bfs.

Two-Phase Simplex Method - Example

- Solve the same LP with the two-phase method

$$\begin{aligned} \text{minimize} \quad & z = 2x_1 + 3x_2 \\ \text{subject to} \quad & 0.5x_1 + 0.25x_2 \le 4 \\ & x_1 + 3x_2 \ge 20 \\ & x_1 + x_2 = 10 \\ & x_1, x_2 \ge 0 \end{aligned} \quad \equiv \quad \begin{aligned} -\ \text{maximize} \quad & z = -2x_1 - 3x_2 \\ \text{subject to} \quad & 0.5x_1 + 0.25x_2 \le 4 \\ & x_1 + 3x_2 \ge 20 \\ & x_1 + x_2 = 10 \\ & x_1, x_2 \ge 0 \end{aligned}$$

Two-Phase Simplex Method - Example

- Solve the same LP with the two-phase method

$$\begin{aligned} \text{maximize} \quad & z = -2x_1 - 3x_2 \\ \text{subject to} \quad & 0.5x_1 + 0.25x_2 \le 4 \\ & x_1 + 3x_2 \ge 20 \\ & x_1 + x_2 = 10 \\ & x_1, x_2 \ge 0 \end{aligned}$$

$$\begin{aligned} \text{Row 0:} \quad & z + 2x_1 + 3x_2 = 0 \\ \text{Row 1:} \quad & 0.5x_1 + 0.25x_2 + s_1 = 4 \\ \text{Row 2:} \quad & x_1 + 3x_2 - e_2 + a_2 = 20 \\ \text{Row 3:} \quad & x_1 + x_2 + a_3 = 10 \end{aligned}$$

Two-Phase Simplex Method - Example

Phase I: Change objective function and solve the following LP

$$
\begin{array}{llllll}
\text{Min} & w = a_2 + a_3 & & & & \\
\text{s.t.} & 0.5x_1 + 0.25x_2 + s_1 & & & & = 4 \\
& x_1 + 3x_2 & - e_2 + a_2 & & & = 20 \\
& x_1 + x_2 & & + a_3 & & = 10
\end{array}
$$

Two-Phase Simplex Method – Phase I

Row	Basic Variable	w	x_1	x_2	s_1	e_2	a_2	a_3	RHS
0	w	1	0	0	0	0	-1	-1	0
1	s_1	0	1/2	1/4	1	0	0	0	4
2	a_2	0	1	3	0	-1	1	0	20
3	a_3	0	1	1	0	0	0	1	10

Because basic variables a_2 and a_3 have nonzero Row 0 coefficients, do elementary row operations to zero them out: Add (Row2) and (Row 3) to Row 0 to achieve a proper Row 0 for simplex to start

Two-Phase Simplex Method – Phase I

Row	Basic Variable	w	x_1	x_2	s_1	e_2	a_2	a_3	RHS
~~0~~	~~w~~	~~1~~	~~0~~	~~0~~	~~0~~	~~0~~	~~-1~~	~~-1~~	~~0~~
0	w	1	2	4	0	-1	0	0	30
1	s_1	0	1/2	1/4	1	0	0	0	4
2	a_2	0	1	3	0	-1	1	0	20
3	a_3	0	1	1	0	0	0	1	10

Because basic variables a_2 and a_3 have nonzero Row 0 coefficients, do elementary row operations to zero them out: Add (Row2) and (Row 3) to Row 0 to achieve a proper Row 0 for simplex to start

Two-Phase Simplex Method – Phase I

Row	Basic Variable	w	x_1	x_2	s_1	e_2	a_2	a_3	RHS	Min Ratio Test
0	w	1	2	4	0	-1	0	0	30	
1	s_1	0	1/2	1/4	1	0	0	0	4	16
2	a_2	0	1	3	0	-1	1	0	20	20/3
3	a_3	0	1	1	0	0	0	1	10	10

Two-Phase Simplex Method – Phase I

Row	Basic Variable	w	x_1	x_2	s_1	e_2	~~a_2~~	a_3	RHS	Min Ratio Test
0	w	1	2/3	0	0	1/3		0	10/3	
1	s_1	0	5/12	0	1	1/12		0	7/3	28/5
2	x_2	0	1/3	1	0	-1/3		0	20/3	20
3	a_3	0	2/3	0	0	1/3		1	10/3	5

Since a_2 has left the basis, we can forget about that column for good!

Two-Phase Simplex Method – Phase I

Row	**Basic Variable**	**w**	x_1	x_2	s_1	e_2	a_2	a_3	**RHS**
0	w	1	0	0	0	0			0
1	s_1	0	0	0	1	-1/8			1/4
2	x_2	0	0	1	0	-1/2			5
3	x_1	0	1	0	0	1/2			5

Since a_3 has left the basis, we can also forget about that column for good! This is the end of Phase I. Since w=0, move to Phase II with this bfs.

Two-Phase Simplex Method – Phase II

$$\begin{aligned} \text{maximize} \quad & z = -2x_1 - 3x_2 \\ \text{subject to} \quad & 0.5x_1 + 0.25x_2 \le 4 \\ & x_1 + 3x_2 \ge 20 \\ & x_1 + x_2 = 10 \\ & x_1, x_2 \ge 0 \end{aligned}$$

Row 0: $z + 2x_1 + 3x_2 = 0$

Row 1: $0.5x_1 + 0.25x_2 + s_1 = 4$

Row 2: $x_1 + 3x_2 - e_2 + a_2 = 20$

Row 3: $x_1 + x_2 + a_3 = 10$

Two-Phase Simplex Method – Phase II

Row	Basic Variable	z	x_1	x_2	s_1	e_2	RHS
0	z	1	2	3	0	0	0
1	s_1	0	0	0	1	-1/8	1/4
2	x_2	0	0	1	0	-1/2	5
3	x_1	0	1	0	0	1/2	5

Bring in the original objective.
Zero out the nonzero coefficients of basic variables in Row 0.
Add -2(Row3) – 3(Row2) to Row 0

Two-Phase Simplex Method – Phase II

Row	**Basic Variable**	**z**	x_1	x_2	s_1	e_2	**RHS**
~~0~~	~~z~~	~~1~~	~~2~~	~~3~~	~~0~~	~~0~~	~~0~~
0	z	1	0	0	0	1/2	-25
1	s_1	0	0	0	1	-1/8	1/4
2	x_2	0	0	1	0	-1/2	5
3	x_1	0	1	0	0	1/2	5

Bring in the original objective.
Zero out the nonzero coefficients of basic variables in Row 0.
Add -2(Row3) – 3(Row2) to Row 0

Two-Phase Simplex Method – Phase II

Row	Basic Variable	z	x_1	x_2	s_1	e_2	RHS
0	z	1	0	0	0	1/2	-25
1	s_1	0	0	0	1	-1/8	1/4
2	x_2	0	0	1	0	-1/2	5
3	x_1	0	1	0	0	1/2	5

This is a max problem so the current tableau is optimal!
End of Phase II
The optimal solution is $z=-25$, $x_1=x_2 = 5$, $s_1=1/4$, $e_2=0$.

Two-Phase Simplex Method

- Solve the second LP with the two-phase method

maximize $z = x_1 + x_2$

subject to $x_1 - x_2 \geq 1$

$-x_1 + x_2 \geq 1$

$x_1, x_2 \geq 0$

First convert to standard form

maximize $z = x_1 + x_2$

subject to $x_1 - x_2 - e_1 = 1$

$-x_1 + x_2 - e_2 = 1$

$x_1, x_2, e_1, e_2 \geq 0$

Two-Phase Simplex Method

Phase I: Change objective function and solve the following LP

minimize $w = a_1 + a_2$

subject to $x_1 - x_2 - e_1 + a_1 = 1$

$-x_1 + x_2 - e_2 + a_2 = 1$

$x_1, x_2, e_1, e_2, a_1, a_2 \geq 0$

Phase I

Row	Basic Variable	z	x_1	x_2	e_1	e_2	a_1	a_2	RHS
0	w	1	0	0	0	0	-1	-1	0
1	a_1	0	1	-1	-1	0	1	0	1
2	a_2	0	-1	1	0	-1	0	1	1

Because basic variables a_1 and a_2 have nonzero Row 0 coefficients, do elementary row operations to zero them out: Add (Row1) and (Row 2) to Row 0 to achieve a proper Row 0 for simplex to start

Phase I

Row	**Basic Variable**	**z**	x_1	x_2	e_1	e_2	a_1	a_2	**RHS**
~~0~~	~~w~~	~~1~~	~~0~~	~~0~~	~~0~~	~~0~~	~~-1~~	~~-1~~	~~0~~
0	w	1	0	0	-1	-1	0	0	2
1	a_1	0	1	-1	-1	0	1	0	1
2	a_2	0	-1	1	0	-1	0	1	1

Because basic variables a_1 and a_2 have nonzero Row 0 coefficients, do elementary row operations to zero them out: Add (Row1) and (Row 2) to Row 0 to achieve a proper Row 0 for simplex to start

Phase I

Row	Basic Variable	z	x_1	x_2	e_1	e_2	a_1	a_2	RHS
0	w	1	0	0	-1	-1	0	0	2
1	a_1	0	1	-1	-1	0	1	0	1
2	a_2	0	-1	1	0	-1	0	1	1

Since this is a minimization problem, the most positive nonbasic variable should enter in Row 0 format. Since no such variable, this is the optimal tableau!

Since w>0 at the end of Phase I, we declare original problem as infeasible. This is the end of Two Phase Method, no need to move to Phase II.

Two-Phase Simplex Method - Summary

- When a basic feasible solution is not readily available, the two-phase simplex method may be used as an alternative to the Big M method.
- In this method, artificial variables are added to the same constraints, then a bfs to the original LP is found by solving Phase I LP.
- In Phase I LP, the objective function is to minimize the sum of all artificial variables.
- At completion, reintroduce the original LPs objective function and determine the optimal solution to the original LP.

Two-Phase Simplex Method – Phase I

- Replace the objective function with:

  min w = (sum of all artificial variables).

- The act of solving the Phase I LP will force the artificial variables to be zero.
- Since the artificial variables are in the starting basis, we should create zeros for each artificial variables in row 0 and then solve the minimization problem.
- Solving the Phase I LP will result in one of the following three cases:

Two-Phase Simplex Method – Phase I cont’

- CASE 1: The optimal value of w is greater than zero. In this case, the original LP has no feasible solution (which means at least one of the $a_i > 0$).
- CASE 2: The optimal value of w is equal to zero, and no artificial a_i’s are in the optimal Phase I basis. Then a basic feasible solution to the original problem is found. Continue to Phase II by bringing in the original objective function.
- CASE 3: The optimal value of w is zero and at least one artificial variable is in the optimal Phase I basis. Recall that we wanted a bfs of the original problem. But this means that we don’t want the basis to contain any artificial variables. Then we can perform an additional pivot and get rid of the artificial variable.

So that in the end, we will get w is zero and no artificial variables are in the optimal Phase I basis.

Two-Phase Simplex Method – Phase II

- Drop all columns in the optimal Phase I tableau that correspond to the artificial variables. And combine the original objective function with the constraints from the optimal Phase I tableau.
- Make sure that all basic variables have zero in row 0 by performing elementary row operations.
- Solve the problem starting with this tableau. The optimal solution to the Phase II LP is the optimal solution to the original LP.

Why does it work?

- Suppose the original LP is feasible. Then this feasible solution (with all a_i's being zero) is feasible in the Phase I LP with w=0. w=0 is the lowest value that w can get. Hence, it is optimal to Phase I. Therefore, if the original LP has a feasible solution then the optimal Phase I solution will have w = 0.
- If the original LP is infeasible then the only way to obtain a feasible solution to the Phase I LP is to let at least one artificial variable to be positive. In this situation, w > 0, hence optimal w will be greater than zero.

Two-Phase Simplex Method - Remarks

- As with the Big M method, the column for any artificial variable may be dropped from future tableaus as soon as the artificial variable leaves the basis.
- The Big M method and Phase I of the two-phase method make the same sequence of pivots in case the original problem is feasible. For the infeasible case, since Phase I can never be unbounded, they might differ.
- The two-phase method does not cause roundoff errors and other computational difficulties.

Practice Example

Solve the following LP with both the big-M and the two-phase method

$$
\begin{aligned}
\text{minimize} \quad & z = 3x_1 + 4x_2 \\
\text{subject to} \quad & 4x_1 - 3x_2 = 9 \\
& -2x_1 + 8x_2 \geq 2 \\
& x_1 - 2x_2 \leq 1 \\
& x_1, x_2 \geq 0
\end{aligned}
$$

Practice Example

The first step in both methods is to get the standard form by adding slack and surplus variables if necessary. The standard form is:

minimize $z = 3x_1 + 4x_2$

subject to $4x_1 - 3x_2 = 9$

$-2x_1 + 8x_2 - e_2 = 2$

$x_1 - 2x_2 + s_3 = 1$

$x_1, x_2, e_2, s_3 \geq 0$

Practice Example with Big-M Method

Add as many artificial variables as necessary to have a basic variable in each equation and penalize them appropriately in the objective function. Solve the following artificial model.

minimize $z = 3x_1 + 4x_2 + Ma_1 + Ma_2$

subject to

$$4x_1 - 3x_2 + a_1 = 9$$

$$-2x_1 + 8x_2 - e_2 + a_2 = 2$$

$$x_1 - 2x_2 + s_3 = 1$$

$$x_1, x_2, e_2, s_3, a_1, a_2 \geq 0$$

Big M Method in Tableau Format

Row	Basic Variable	z	x_1	x_2	e_2	s_3	a_1	a_2	RHS
0	z	1	-3	-4	0	0	-M	-M	0
1	a_1	0	4	-3	0	0	1	0	9
2	a_2	0	-2	8	-1	0	0	1	2
3	s_3	0	1	-2	0	1	0	0	1

Because basic variables a_1 and a_2 have nonzero Row 0 coefficients, do elementary row operations to zero them out: Add M(Row1) and M(Row 2) to Row 0 to achieve a proper Row 0 for simplex to start

Big M Method in Tableau Format

Row	**Basic Variable**	**z**	x_1	x_2	e_2	s_3	a_1	a_2	**RHS**
~~0~~	~~z~~	~~1~~	~~-3~~	~~-4~~	~~0~~	~~0~~	~~-M~~	~~-M~~	~~0~~
0	z	1	2M-3	5M-4	-M	0	0	0	11M
1	a_1	0	4	-3	0	0	1	0	9
2	a_2	0	-2	8	-1	0	0	1	2
3	s_3	0	1	-2	0	1	0	0	1

Big M Method in Tableau Format

Row	Basic Variable	z	x_1	x_2	e_2	s_3	a_1	a_2	RHS	Min Ratio Test
0	z	1	2M-3	5M-4	-M	0	0	0	11M	
1	a_1	0	4	-3	0	0	1	0	9	No ratio
2	a_2	0	-2	8	-1	0	0	1	2	2/8
3	s_3	0	1	-2	0	1	0	0	1	No ratio

Big M Method in Tableau Format

Row	Basic Variable	z	x_1	x_2	e_2	s_3	a_1	RHS
0	z	1	13M/4-4	0	-3M/8-1/2	0	0	1+39M/4
1	a_1	0	13/4	0	-3/8	0	1	39/4
2	x_2	0	-1/4	1	-1/8	0	0	1/4
3	s_3	0	1/2	0	-1/4	1	0	3/2

Since a_2 has left the basis, we can forget about that column for good!

Big M Method in Tableau Format

Row	Basic Variable	z	x_1	x_2	e_2	s_3	a_1	RHS	Min Ratio Test
0	z	1	13M/4-4	0	-3M/8-1/2	0	0	1+39M/4	
1	a_1	0	13/4	0	-3/8	0	0	39/4	39/13
2	x_2	0	-1/4	1	-1/8	0	0	1/4	No ratio
3	s_3	0	1/2	0	-1/4	1	0	3/2	3

There is a tie in the ratio test. We favor making artificial variables nonbasic so leaving variable is a_1

Big M Method in Tableau Format

Row	Basic Variable	z	x_1	x_2	e_2	s_3	RHS
0	z	1	0	0	-1/26-M/4	0	13
1	x_1	0	1	0	-3/26	0	3
2	x_2	0	0	1	-2/13	0	1
3	s_3	0	0	0	-1/16	1	0

No entering variable so this is the optimal tableau! $optimal\ solution = \begin{pmatrix} x_1 \\ x_2 \\ e_2 \\ s_3 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 0 \\ 0 \end{pmatrix}$

with optimal value = 13

Practice Example with Two Phase Method

Phase I Artificial Model

minimize $w = a_1 + a_2$

subject to $4x_1 - 3x_2 + a_1 = 9$

$-2x_1 + 8x_2 - e_2 + a_2 = 2$

$x_1 - 2x_2 + s_3 = 1$

$x_1, x_2, e_2, s_3, a_1, a_2 \geq 0$

Phase I in Tableau Format

Row	Basic Variable	w	x_1	x_2	e_2	s_3	a_1	a_2	RHS
0	w	1	0	0	0	0	-1	-1	0
1	a_1	0	4	-3	0	0	1	0	9
2	a_2	0	-2	8	-1	0	0	1	2
3	s_3	0	1	-2	0	1	0	0	1

Because basic variables a_1 and a_2 have nonzero Row 0 coefficients, do elementary row operations to zero them out: Add (Row1) and (Row 2) to Row 0 to achieve a proper Row 0 for simplex to start

Phase I in Tableau Format

Row	**Basic Variable**	**w**	x_1	x_2	e_2	s_3	a_1	a_2	**RHS**
~~0~~	~~w~~	~~1~~	~~0~~	~~0~~	~~0~~	~~0~~	~~-1~~	~~-1~~	~~0~~
0	w	1	2	5	-1	0	0	0	11
1	a_1	0	4	-3	0	0	1	0	9
2	a_2	0	-2	8	-1	0	0	1	2
3	s_3	0	1	-2	0	1	0	0	1

Phase I in Tableau Format

Row	Basic Variable	w	x_1	x_2	e_2	s_3	a_1	a_2	RHS	Min Ratio Test
0	w	1	2	5	-1	0	0	0	11	
1	a_1	0	4	-3	0	0	1	0	9	No ratio
2	a_2	0	-2	8	-1	0	0	1	2	2/8
3	s_3	0	1	-2	0	1	0	0	1	No ratio

Phase I in Tableau Format

Row	Basic Variable	**w**	x_1	x_2	e_2	s_3	a_1	**RHS**
0	w	1	13/4	0	-3/8	0	0	39/4
1	a_1	0	13/4	0	-3/8	0	1	39/4
2	x_2	0	-1/4	1	-1/8	0	0	1/4
3	s_3	0	1/2	0	-1/4	1	0	3/2

Since a_2 has left the basis, we can forget about that column for good!

Phase I in Tableau Format

Row	Basic Variable	w	x_1	x_2	e_2	s_3	a_1	RHS	Min Ratio Test
0	w	1	13/4	0	-3/8	0	0	39/4	
1	a_1	0	13/4	0	-3/8	0	1	39/4	39/13
2	x_2	0	-1/4	1	-1/8	0	0	1/4	No ratio
3	s_3	0	1/2	0	-1/4	1	0	3/2	3

There is a tie in the ratio test. We favor making artificial variables nonbasic so leaving variable is a_1

Phase I in Tableau Format

Row	Basic Variable	**w**	x_1	x_2	e_2	s_3	**RHS**
0	w	1	0	0	-1/4	0	13
1	x_1	0	1	0	-3/26	0	3
2	x_2	0	0	1	-2/13	0	1
3	s_3	0	0	0	-1/16	1	0

No entering variable so this is the optimal tableau for Phase I!
Bring in original objective
minimize $z = 3x_1 + 4x_2$
and move to Phase II to optimize starting with this bfs

$$bfs = \begin{pmatrix} x_1 \\ x_2 \\ e_2 \\ s_3 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

with value = 13

Phase II in Tableau Format

Row	Basic Variable	z	x_1	x_2	e_2	s_3	RHS
0	z	1	-3	-4	0	0	0
1	x_1	0	1	0	-3/26	0	3
2	x_2	0	0	1	-2/13	0	1
3	s_3	0	0	0	-1/16	1	0

Row 0 corresponds to minimize $z = 3x_1 + 4x_2$

Phase II in Tableau Format

Row	Basic Variable	z	x_1	x_2	e_2	s_3	RHS
~~0~~	~~z~~	~~1~~	~~-3~~	~~-4~~	~~0~~	~~0~~	~~0~~
0	z	1	0	0	-25/26	0	13
1	x_1	0	1	0	-3/26	0	3
2	x_2	0	0	1	-2/13	0	1
3	s_3	0	0	0	-1/16	1	0

Add 3(Row1) + 4(Row 2) to Row 0 to make it in proper format

Phase II in Tableau Format

Row	Basic Variable	z	x_1	x_2	e_2	s_3	RHS
0	z	1	0	0	-25/26	0	13
1	x_1	0	1	0	-3/26	0	3
2	x_2	0	0	1	-2/13	0	1
3	s_3	0	0	0	-1/16	1	0

There is no entering variable so this is the end of Phase II and it corresponds to an optimal bfs.

$$optimal\ solution = \begin{pmatrix} x_1 \\ x_2 \\ e_2 \\ s_3 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

with optimal value = 13