The simplicity conjecture

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IAS

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Section 1

Introduction

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Question (Fathi, 1980)

Is the group $Homeo_c(D^2, \omega)$ simple?

Today's theorem

Theorem ("Simplicity conjecture"; CG., Humiliere, Seyfadinni)

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Corollary

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S^2 the only closed manifold for which simplicity of $Homeo_0(M, \omega)$ not known.

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- Symplectic case: kernel of flux simple when manifold closed; if not closed, there's a Calabi homomorphism, kernel of Calabi simple (Banyaga)
- Volume preserving homeomorphisms: there is a "mass flow" homomorphism; kernel is simple for $n \geq 3$ (Fathi). $n = 2$ case mysterious before our work.

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- (Whittaker, '63): any iso. $Homeo_0(M) \longrightarrow Homeo_0(N)$ induced by a homeomorphism $M \longrightarrow N$.
- (Filipkiewicz, '82): an iso. $Diff_0^r(M) \longrightarrow Diff_0^s(N)$ implies $r = s$, M, N C^r-diffeomorphic (requires M, N compact)

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Our work shows this fragmentation property does not hold.

Section 2

Idea of the proof

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- Fact: $Cal(\varphi)$ doesn't depend on choice of H!

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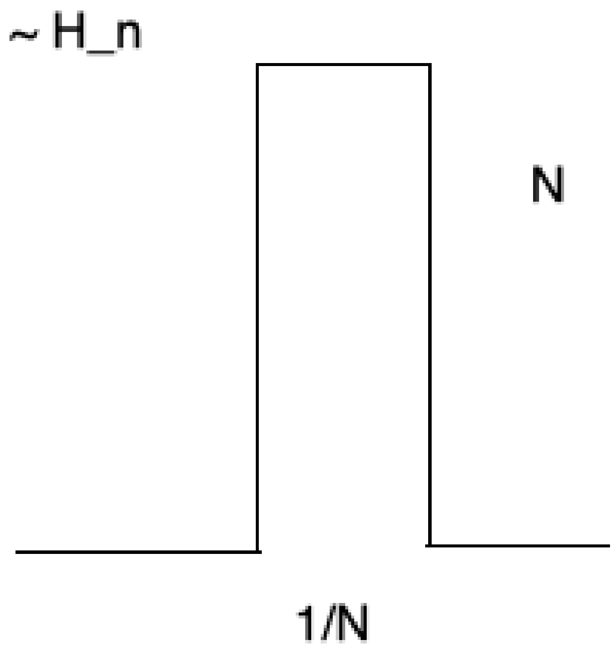
eg: Consider H_n, supported on disc around origin of area $1/n$, where $H_n \approx n$. $Cal(\varphi^1_{H_n}) \approx 1$, C^0 converges to the identity.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- Prove "enough" of Hutchings' conjecture:

$$lim_{d \longrightarrow \infty} \frac{c_d(\varphi)}{d} = Cal(\varphi)$$

on $Diffeo_c$. (Inspired by "Volume Conjecture" for ECH.)

Section 3

Outline of the argument

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Say $\varphi \in FHomeo_c(D^2, \omega)$ — "finite Hofer energy homeomorphisms" — if there exists

$$\varphi^1_{H_i} \longrightarrow_{C^0} \varphi, \quad ||H_i||_{1,\infty} \leq M,$$

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$$\varphi^1_{H_i} \longrightarrow_{C^0} \varphi, \quad ||H_i||_{1,\infty} \leq M,$$

for M independent of i. Here, $||H_i||_{1,\infty}$ is the **Hofer norm**

$$||H_i||_{1,\infty} = \int_0^1 max(H_i) - min(H_i) dt.$$

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Define a **monotone twist** φ_f to be

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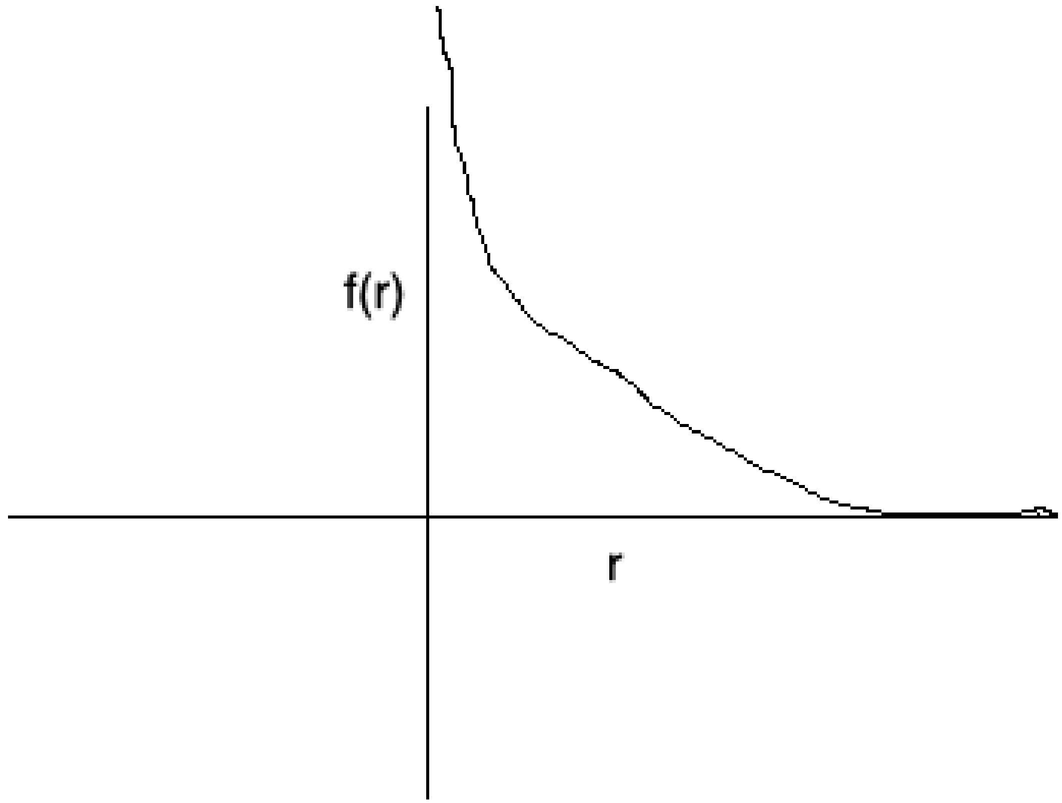
Call φ_f an **infinite twist** if

$$\int_0^1 \int_r^1 sf(s)ds\ r\ dr = \infty.$$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is that for monotone twists $\varphi \in Diffeo_c$,

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So, morally, infinite twists "should" have infinite Calabi invariant.

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- (A) For any $\varphi \in FHomeo_c$, there exists a constant M with

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- (B) For any infinite twist φ_f,

$$lim_{d \longrightarrow \infty} \frac{c_d(\varphi)}{d} = +\infty.$$

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Then, (A) follows easily from C^0 continuity and the fact that the $id = \varphi^1_K$ for $K = 0$.

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$$H \leq K \implies c_d(\varphi^1_H) \leq c_d(\varphi^1_K),$$

We then approximate φ_f with smooth φ_{f_i} such that:

$$f_i \leq f_j$$

hence

$$\frac{c_d(\varphi_f)}{d} \geq \frac{c_d(\varphi_{f_i})}{d}.$$

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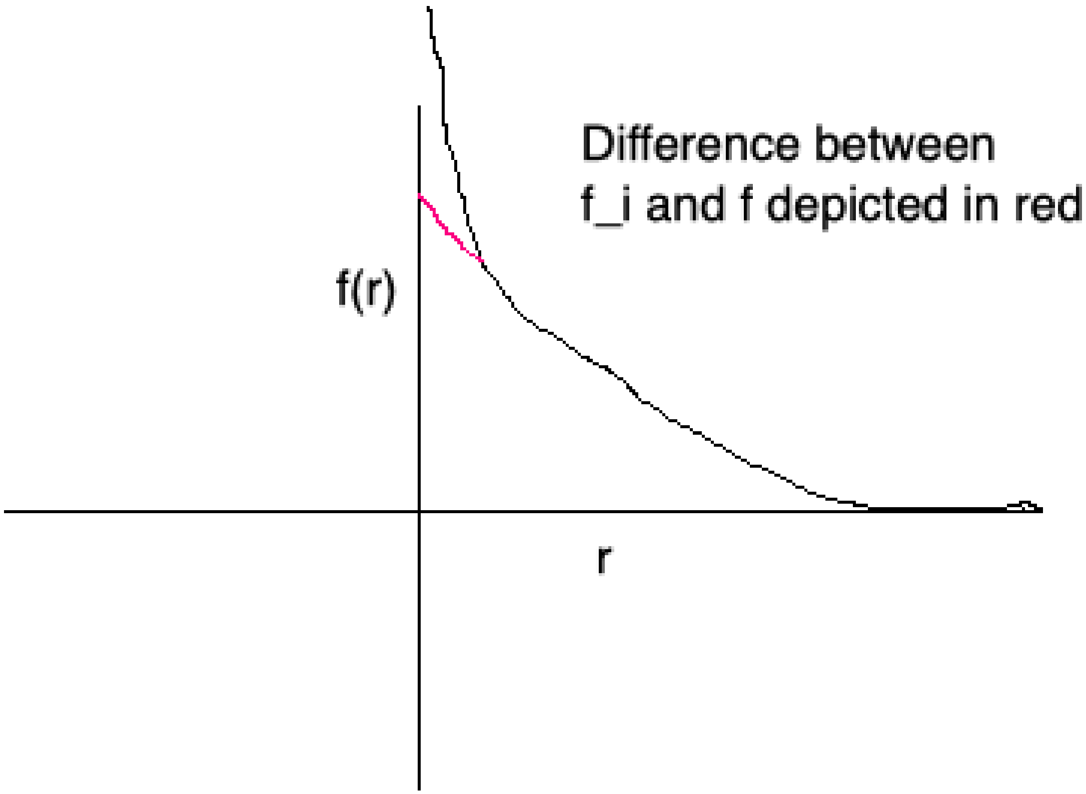

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$$lim_{d\longrightarrow\infty}\frac{c_d(\varphi_f)}{d} \geq lim_{d\longrightarrow\infty}\frac{c_d(\varphi_{f_i})}{d} = Cal(\varphi_{f_i}) \longrightarrow \infty.$$

We prove Hutchings' conjecture by direct computation in the monotone twist case.

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To recap, to prove $Homeo_c(D^2, \omega)$ is not simple, we have to:

- Define PFH spectral invariants
- Establish C^0 continuity, Hofer continuity, monotonicity for these invariants
- Prove Hutchings' conjecture for monotone twists
- Put it all together, as explained above.

Section 4

PFH spectral invariants — impressionistic sketch

We define PFH spectral invariants by embedding D^2 as the northern hemisphere of S^2, and then using the periodic Floer homology of S^2.

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and a canonical two-form ω_φ induced by ω.

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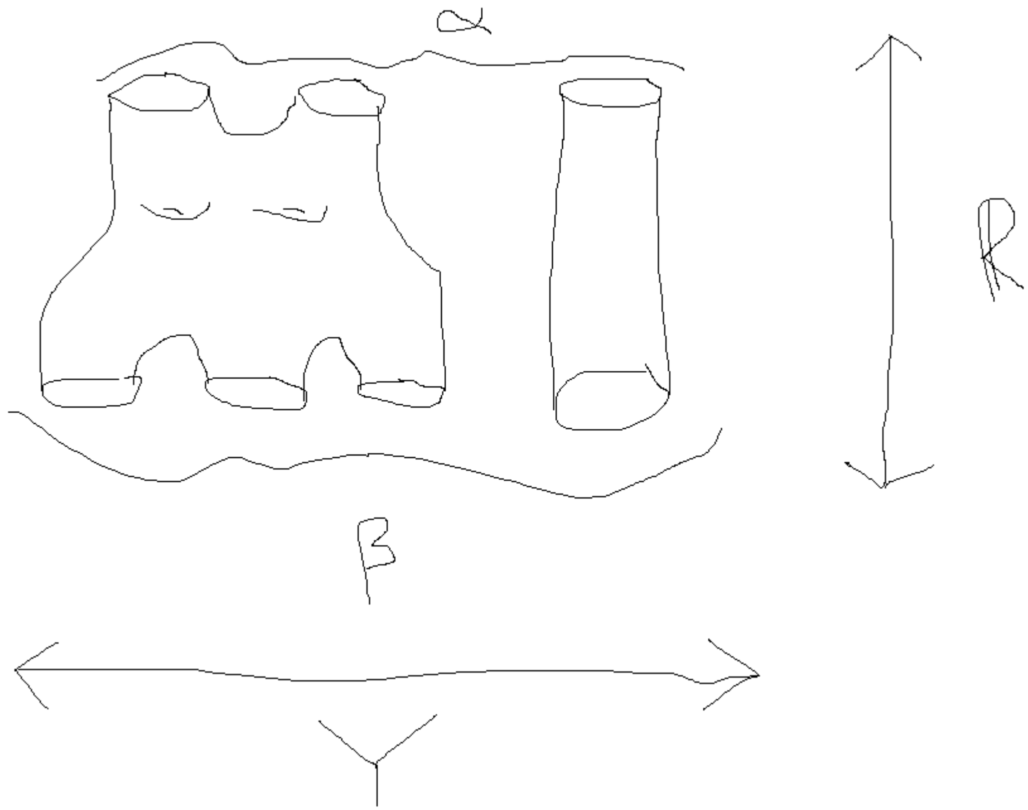
- Differential ∂ counts $I = 1$ J-holomorphic curves in $\mathbb{R} \times Y_\varphi$, for generic J, where I is the "ECH index"
- ECH index beyond scope of talk; basic idea: $I = 1$ forces curves to be mostly embedded,

The PFH differential:



More about PFH

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There's a splitting

$$PFH(\varphi) = \oplus_d PFH(\varphi, d),$$

where $PFH(\varphi, d)$ homology of subcomplex generated by degree d orbit sets.

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- ∂ counts $I = 1$ curves C from (α, Z) to (β, Z'):
    - this means: C a curve from α to β, with $Z = [C] + [Z']$.

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Section 5

Remarks on the rest of the proof

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    - — get a combinatorial model, involving lattice paths, lattice regions, inspired by work of Hutchings-Sullivan