The Smith Chart

One-to-one mapping between Γ and z_L

$$\boxed{\Gamma = \frac{z_L - 1}{z_L + 1}} \qquad \boxed{z_L = \frac{1+\Gamma}{1-\Gamma}}$$

More generally, anywhere along the transmission line

$$\boxed{z(d) \equiv \frac{Z(d)}{Z_0} = \frac{1+\Gamma_d}{1-\Gamma_d}}$$

The math looks simple. But Γ and z_L are generally both complex.
Actual calculation is tedious. A graphical tool desired (esp. before the computer age).

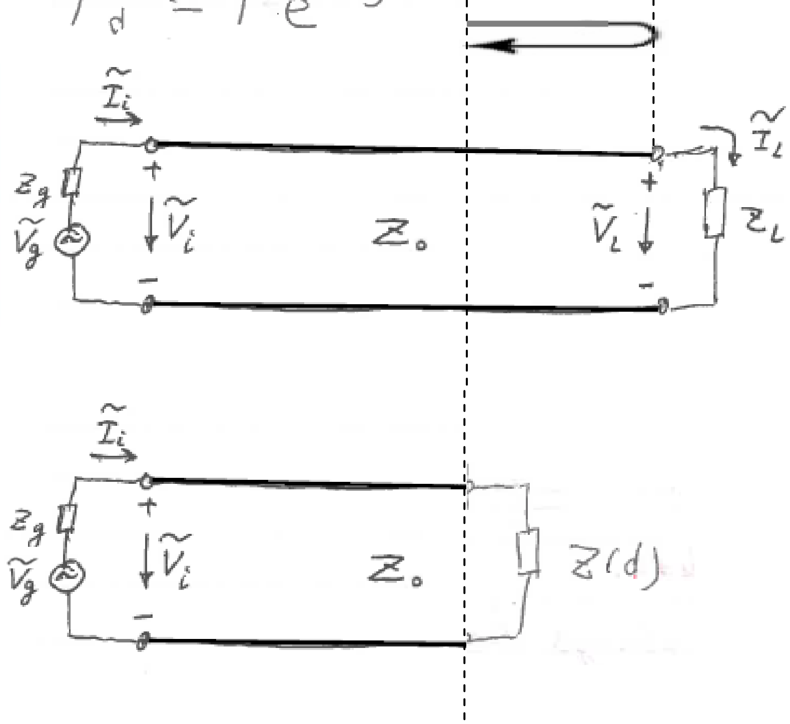
Same math for load and point d. So same tool.

Let's now develop the tool.

Sign!

d

$$\Gamma_d = \Gamma e^{-2j\beta d}$$



$$\Gamma = |\Gamma| e^{j\theta_r} \doteq |\Gamma| \angle \theta_r = \Gamma_r + j\Gamma_i$$

$$z_L = r_L + j x_L$$ (normalized)

Insert $\begin{cases} \Gamma = \Gamma_r + j\Gamma_i \\ z_L = r_L + jx_L \end{cases}$ into $\boxed{z_L = \dfrac{1+\Gamma}{1-\Gamma}}$, then use

Re(left side) = Re(right side) $\Rightarrow$ $r_L = \dfrac{1 - \Gamma_r^2 - \Gamma_i^2}{(1-\Gamma_r)^2 + \Gamma_i^2}$

$$\Rightarrow \boxed{\left(\Gamma_r - \frac{r_L}{1+r_L}\right)^2 + \Gamma_i^2 = \left(\frac{1}{1+r_L}\right)^2}$$

This is the equation for the locus of all $\Gamma = \Gamma_r + j\Gamma_i$ that correspond to a given r_L.

Example: A circle is the locus of all points that are at a given distance from a given point, which is called the center; the given distance is the radius.

Insert $\begin{cases} \Gamma = \Gamma_r + j\Gamma_i \\ z_L = r_L + j x_L \end{cases}$ into $\boxed{z_L = \dfrac{1+\Gamma}{1-\Gamma}}$, then use

Re(left side) = Re(right side) $\Rightarrow$ $r_L = \dfrac{1-\Gamma_r^2-\Gamma_i^2}{(1-\Gamma_r)^2+\Gamma_i^2}$

$$\Rightarrow \boxed{\left(\Gamma_r - \frac{r_L}{1+r_L}\right)^2 + \Gamma_i^2 = \left(\frac{1}{1+r_L}\right)^2}$$

This is the equation for the locus of all $\Gamma = \Gamma_r + j\Gamma_i$ that correspond to a given r_L.

The equation represents a circle centered at

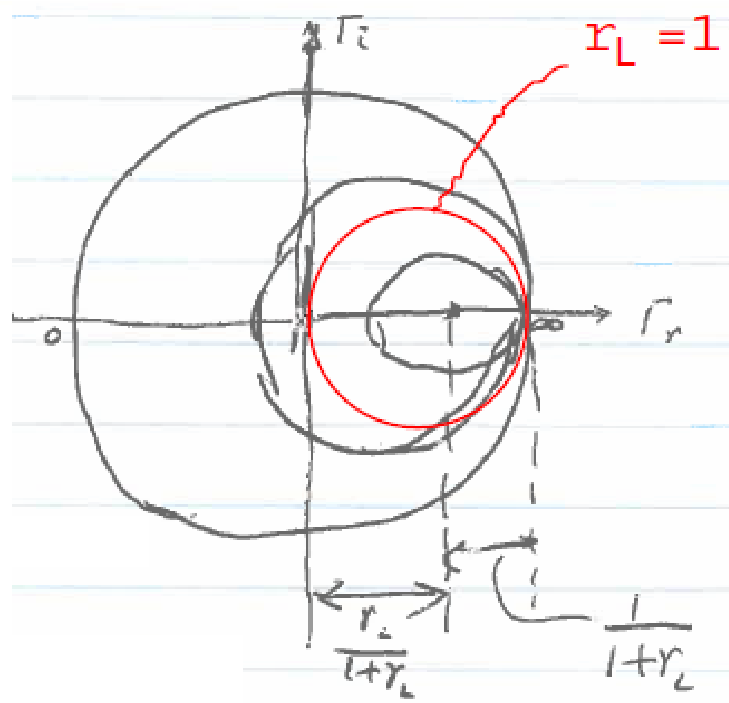
$$\left(\frac{r_L}{1+r_L}, 0\right)$$

and with radius

$$\frac{1}{1+r_L}$$

Notice that $\dfrac{r_L}{1+r_L} + \dfrac{1}{1+r_L} = 1$

So every such circle goes through the point (1, 0).

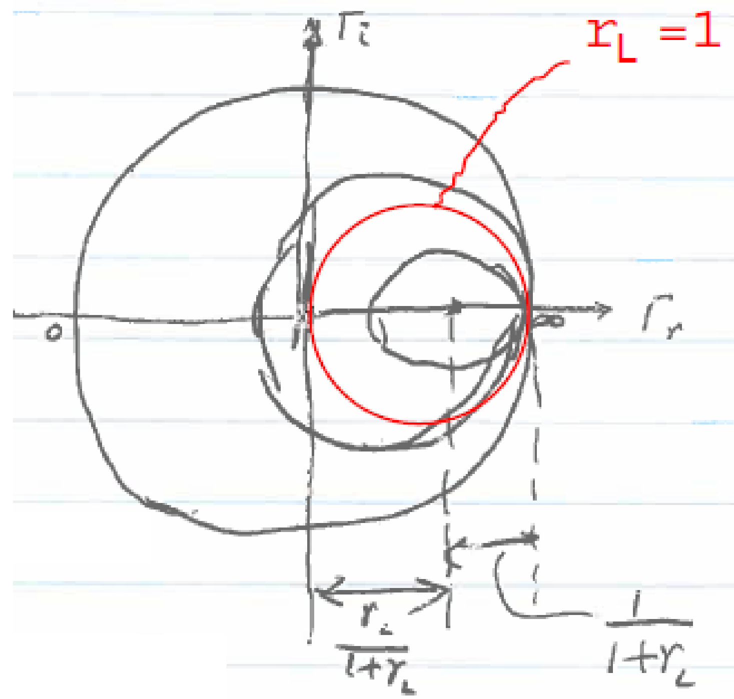


Each of these circles represent all $\Gamma = \Gamma_r + j\Gamma_i$ that correspond to a given r_L.

Every such circle goes through the point (1, 0).

Each circle centered at $\left(\frac{r_L}{1+r_L}, 0 \right)$

with radius $\frac{1}{1+r_L}$



$r_L = 1$ $\Rightarrow$ center at (1/2, 0), radius = 1/2.

$r_L = 0$ $\Rightarrow$ center at (0, 0), radius = 1.

$|\Gamma| = 1$

What kind of impedances correspond to this circle?

$r_L = \infty$ $\Rightarrow$ center at (1, 0), radius = 0.

Insert $\begin{cases} \Gamma = \Gamma_r + j\Gamma_i \\ z_L = r_L + jx_L \end{cases}$ into $\boxed{z_L = \dfrac{1+\Gamma}{1-\Gamma}}$, then use

Im(left side) = Im(right side) $\Rightarrow$ $x_L = \dfrac{2\Gamma_i}{(1-\Gamma_r)^2 + \Gamma_i^2}$

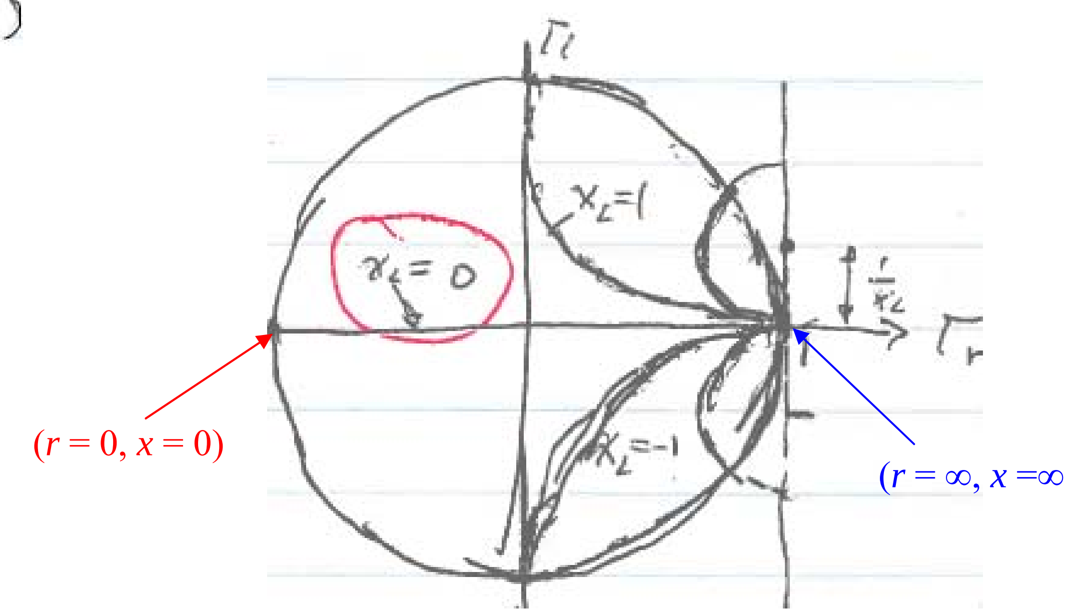
$$\Rightarrow \boxed{(\Gamma_r - 1)^2 + \left(\Gamma_i - \frac{1}{x_L}\right) = \left(\frac{1}{x_L}\right)^2}$$

This is the equation for the locus of all $\Gamma = \Gamma_r + j\Gamma_i$ that correspond to a given x_L.

Each circle centered at $\left(1, \dfrac{1}{x_L}\right)$

with radius $\dfrac{1}{|x_L|}$

Look at special cases $x_L = 0$, $x_L = \pm 1$, $x_L = \pm\infty$



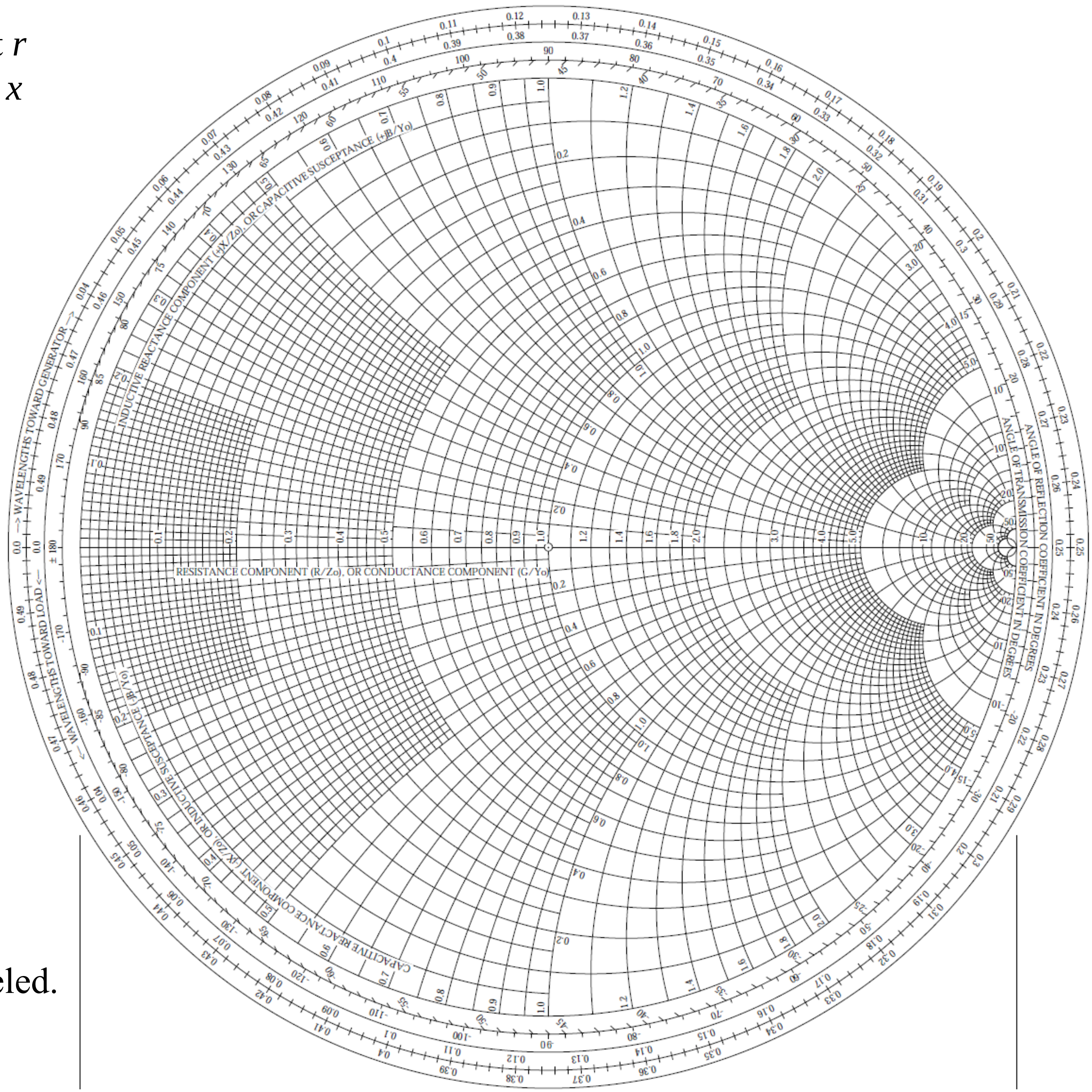
Plot circles of constant *r* and circles of constant *x* on the same chart, and you have the Smith Chart.

Given any $z = r + jx$, you can locate the intersection between circle *r* and circle *x*.

The intersection represents Γ.

Connect the intersection to origin. Measure length to get $|\Gamma|$ (with reference $|\Gamma| = 1$ for the big circle). Measure angle to get phase.

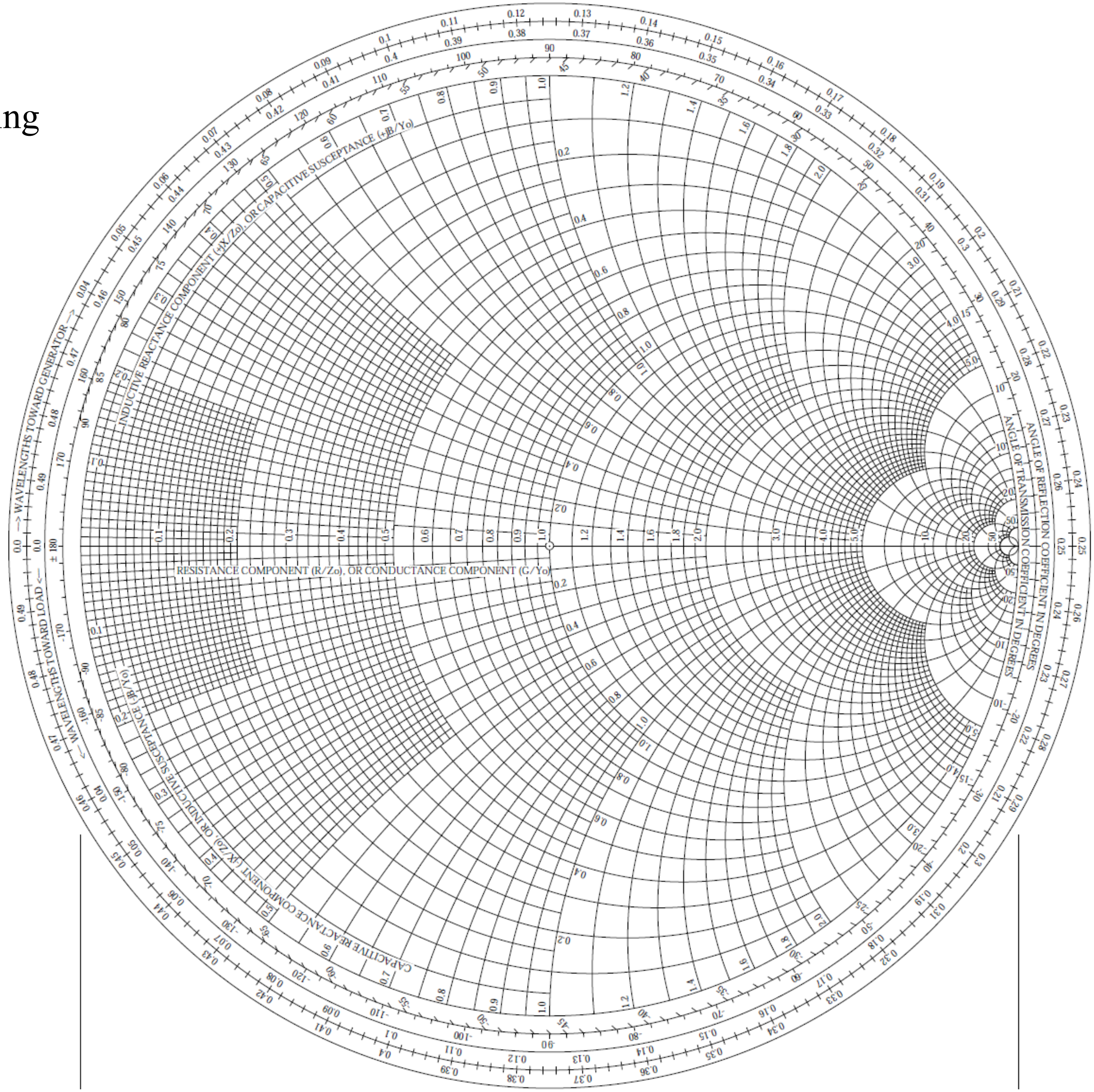
Signs for *x* are not labeled.



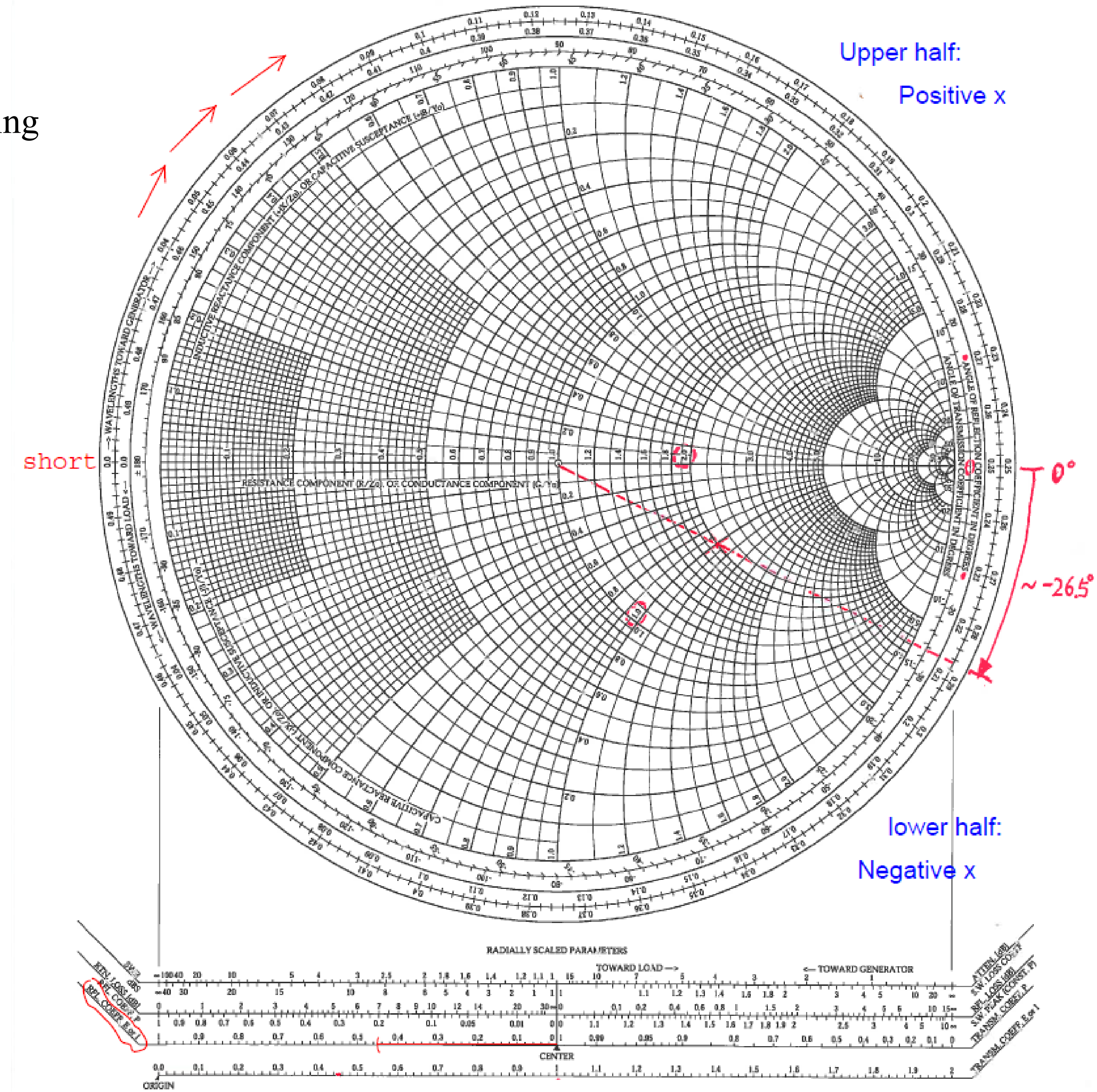
Let's have an exercise:

Find the Γ corresponding to a load $z_L = 2 - j$.

BTW, if $Z_0 = 50\ \Omega$, what is Z_L?



Find the Γ corresponding to a load $z_L = 2 - j$.



Answer: $\boxed{|\Gamma| \sim 0.45 \quad \theta_r \sim -27^\circ}$

How: (not the only method; for your reference)

- Find the intersection of the $r_L = 2$ circle & the $x_L = -1$ circle
- Draw a straight line from origin to this intersection point, extend it beyond the $|\Gamma| = 1$ circle (a.k.a. the unit circle)
- Measure the distance between origin and the intersection point with regard to the radius of the unit circle, using the "RFL COEFF" bar at bottom – you get $|\Gamma|$
- Read the outer tick "ANGLE OF REFLECTION COEFFICIENT IN DEGREES" – you get θ_r

Notes:

- The $|\Gamma| = 1$ circle is the circle next to the inner scale
- The outer ticks of this inner scale stand for θ_r

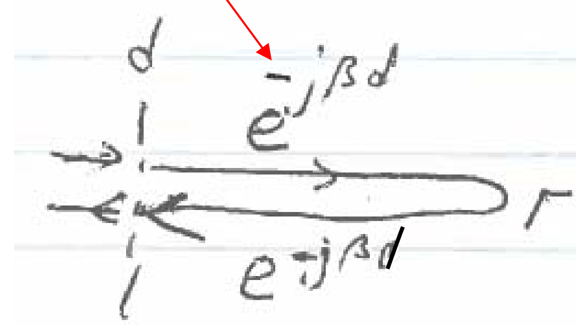
What are the two sets of ticks on the outer scale?

Recall that $z(d) = \dfrac{1+\Gamma_d}{1-\Gamma_d}$ at distance d from the load,

just as $z_L = \dfrac{1+\Gamma}{1-\Gamma}$ at the load.

$$\boxed{\Gamma_d = \Gamma e^{-2j\beta d}}$$

Sign!



Γ_d is just phase-shifted by $-2\beta d$ from Γ.
Therefore, $|\Gamma_d| = |\Gamma|$. Just rotate from Γ by $2\beta d$ clockwise.

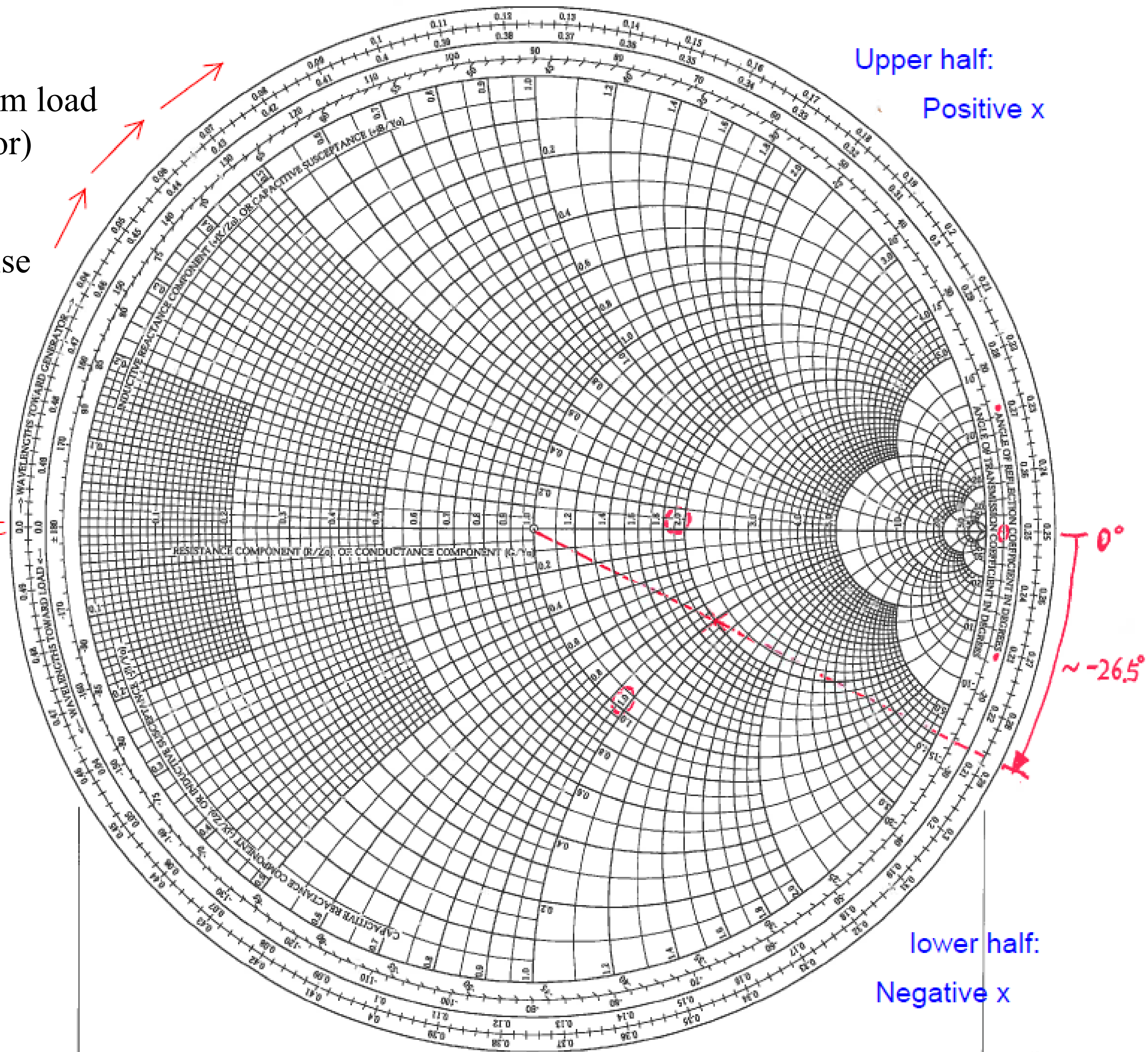
$$\boxed{\text{Moving away from load (towards generator)} \iff \text{Rotating clockwise}}$$

Moving away from load
(towards generator)

⇕

Rotating clockwise

short



Same for $d + \Delta d$ from d:

$\Gamma_{d+\Delta d}$ is just phase-shifted by $-2\beta\Delta d$ from Γ_d .
Therefore, $|\Gamma_{d+\Delta d}| = |\Gamma_d|$. Just rotate from Γ_d by $2\beta\Delta d$ clockwise.

If $d = \frac{\lambda}{2}$, $2\beta d = 2\pi = 360°$: one round!

Recall $\beta = \frac{2\pi}{\lambda}$

$$\boxed{\Gamma_{\frac{\lambda}{2}} = \Gamma}$$

Similarly, $\boxed{\Gamma_{d+\frac{\lambda}{2}} = \Gamma_d}$

The outer set of ticks of the outer scale is distance d in λ's towards the generator, i.e., away from the load.

The assumed load is a short. Therefore $d = 0.25\lambda$ for an open, corresponding to $\Gamma = +1$.

But an actual load can be anything. So take this scale to be relative.

Let's have another exercise.

Get the chart you used for the last example, where we found, for $z_L = 2 - j$,

$|\Gamma| \sim 0.45 \quad \theta_r \sim -27°$

Now, find $z(d)$ for $d = 0.1\lambda$.

(See chart on next slide)

Extend the line we drew to cross the outer scale.
Read the outer ticks to get the offset distance "*d*".

"*d*" = 0.287λ .

$0.287 + 0.1 = 0.387$

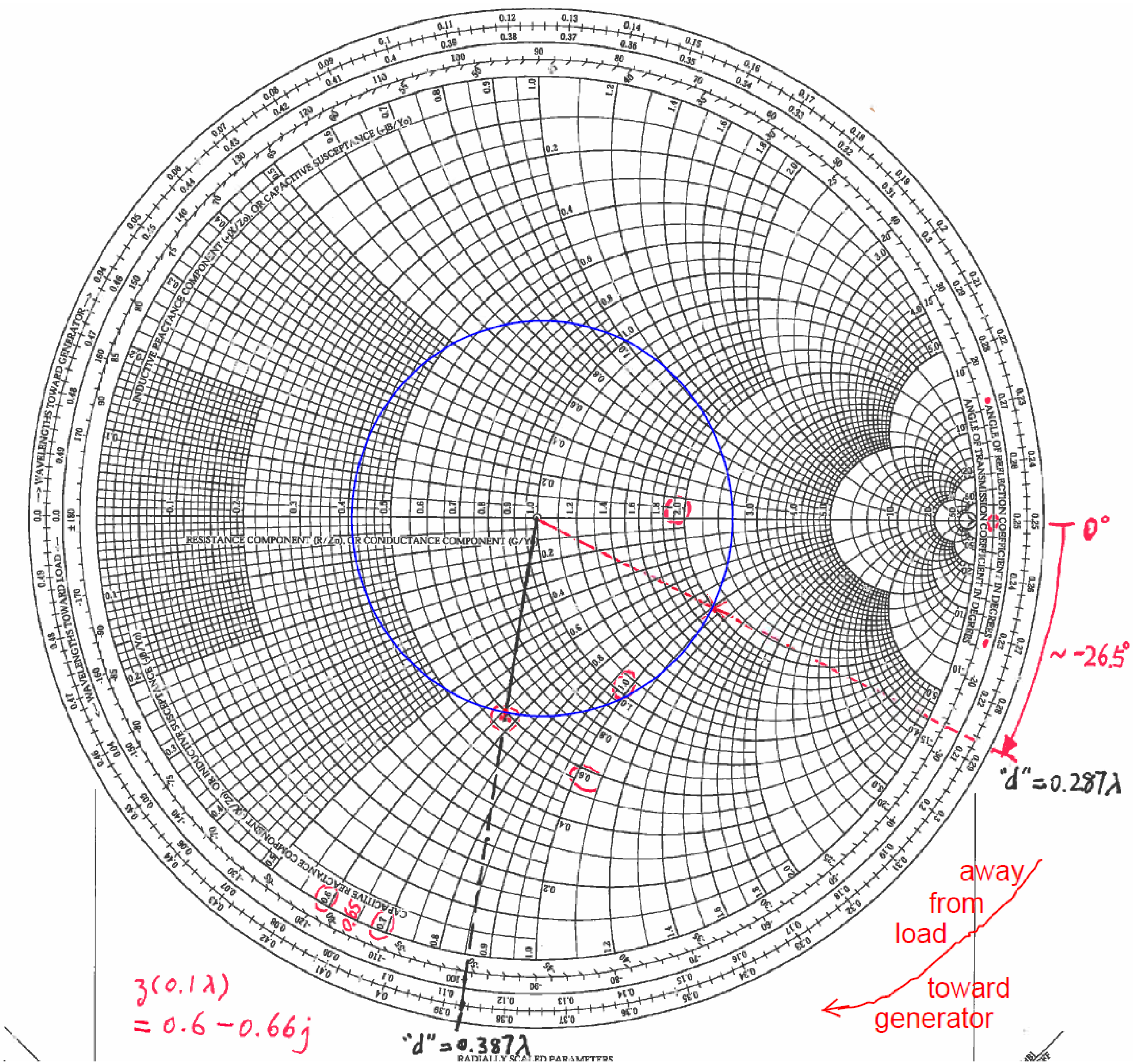
Go clockwise to "*d*" = 0.387λ .
Connect this point on the outermost scale to the origin.

Recall that $|\Gamma_d| = |\Gamma|$. So you get the Γ_d for $d = 0.1\lambda$.

Now read $z(d)$ from the chart.

The scale is relative.

Offset because the chart assumes short.



$z(0.1\lambda) = 0.6 - 0.66j$

Answer: $z(d) = z(0.1\lambda) = 0.6 - j0.66$.

Sign!

(Question: What is $Z(d)$ if $Z_0 = 50\ \Omega$?)

BTW, the inner tick labels of the outer scale are distance in wavelengths towards the load, i.e., away from the generator (source).

You may use either set of labels. Just remember that going away from the load on the transmission line is rotating clockwise on the chart. (Why?)
You just need to be consistent, and do the addition/subtraction right.

The smith chart was developed in 1939, when doing tedious algebra with complex numbers was a big deal.
Now, with computers, why do we still teach this tool here?

Recall that the SWR $S = \dfrac{|\tilde{V}|_{max}}{|\tilde{V}|_{min}} = \dfrac{1+|\Gamma|}{1-|\Gamma|}$

So, a constant $|\Gamma|$ means a constant S. The blue circle we drew is called the constant-SWR circle.

It's just a constant-$|\Gamma|$ circle.

Hyphen , not minus/negative

The SWR is a property of the transmission line/load system.
When we say "rotating" on the chart, we mean moving on the constant-SWR circle.

Going away from the load on the transmission line is moving clockwise on the constant-SWR circle.

Let's have one more hands-on exercise (using the chart we used).

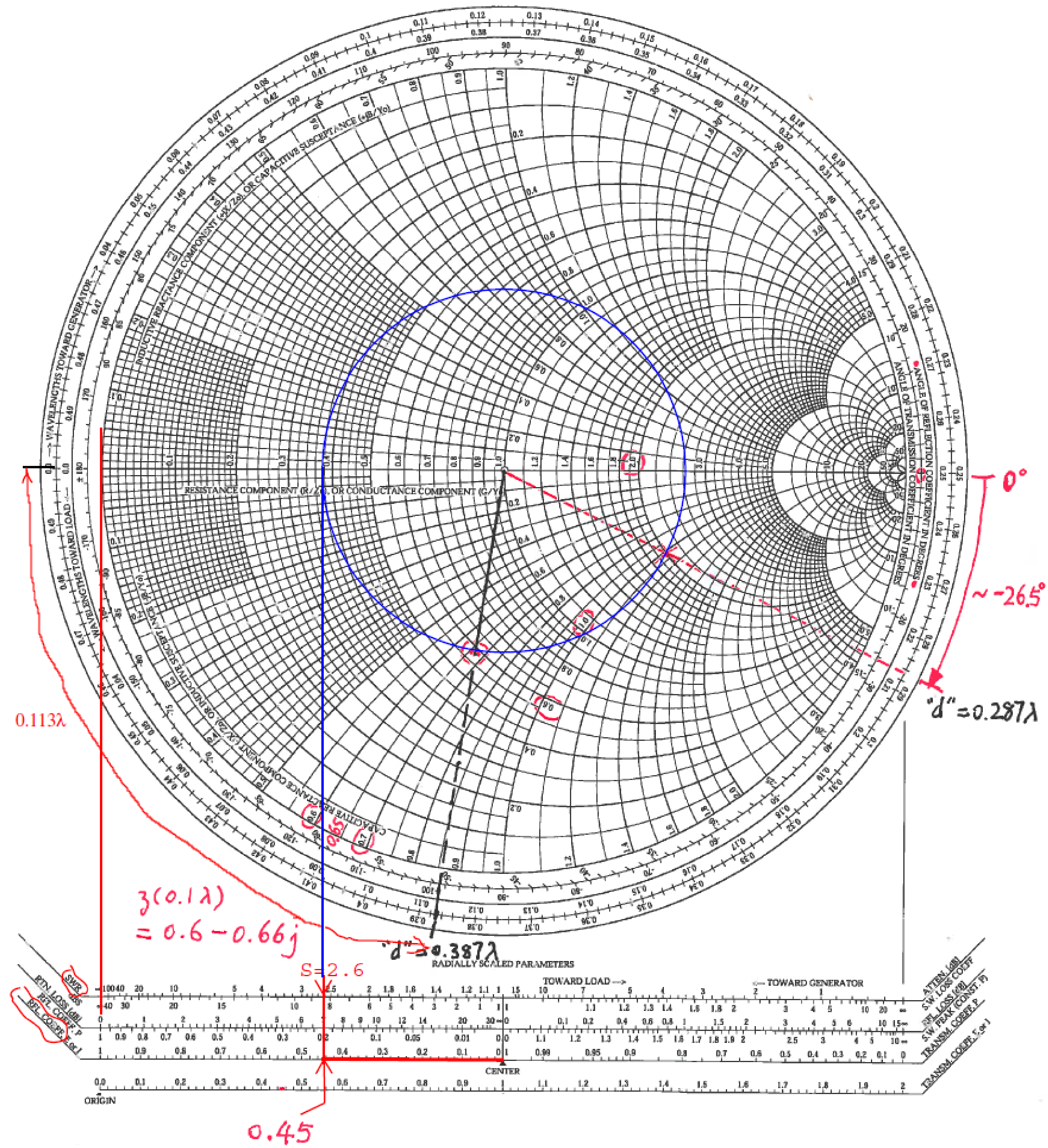
We had a load $z_L = 2 - j$, and found $|\Gamma| \sim 0.45 \quad \theta_r \sim -27^\circ$

Then we move a tenth of a wavelength and found found $z(0.1\lambda) = 0.6 - j0.66$.

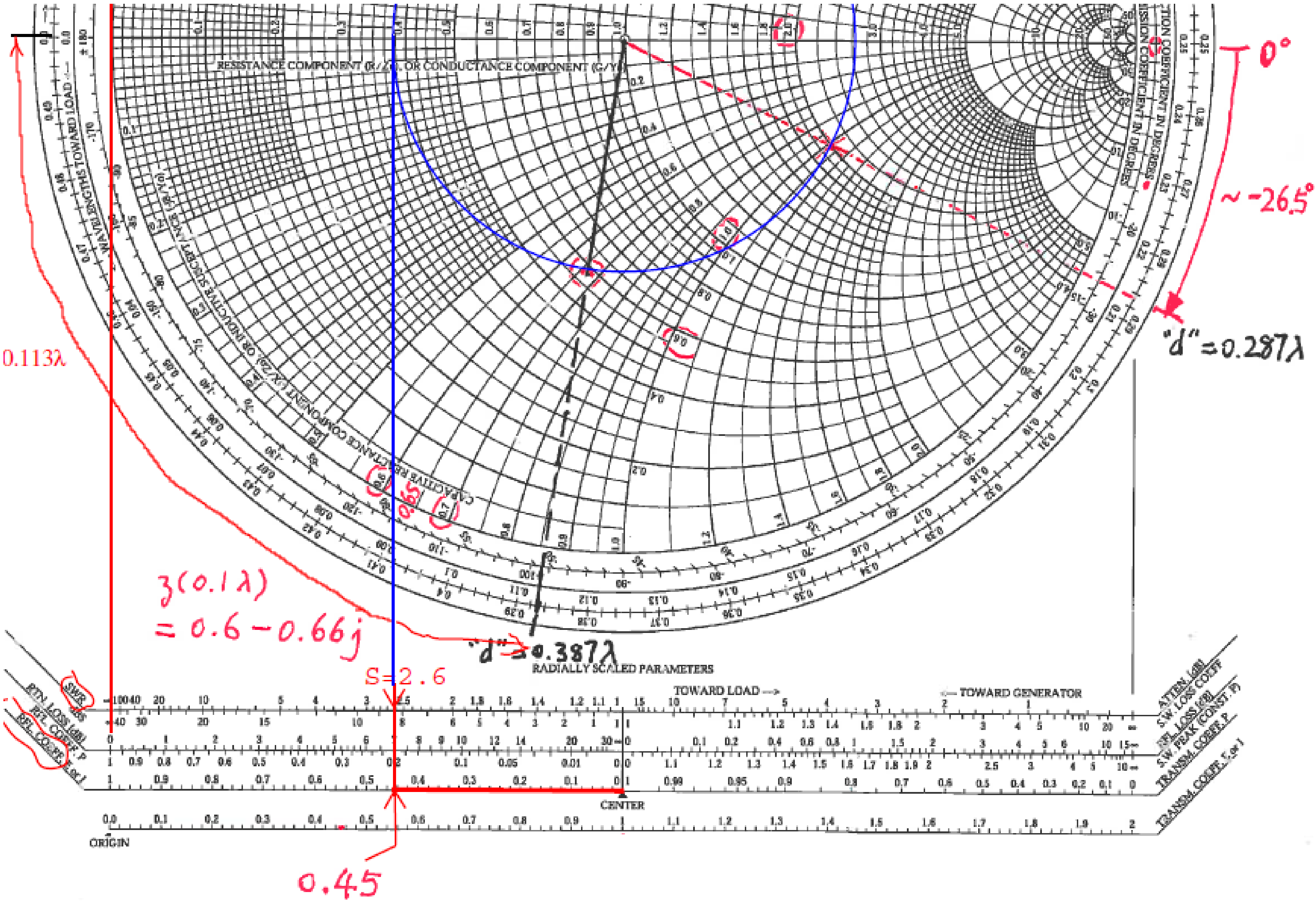
We moved on a constant-$|\Gamma|$, or constant-SWR circle. You can easily calculate S by

$$S = \frac{|\tilde{V}|_{max}}{|\tilde{V}|_{min}} = \frac{1+|\Gamma|}{1-|\Gamma|}$$

You may also graphically read from the chart.



0°
~ -26.5°
"d" = 0.287λ
0.113λ
z(0.1λ)
= 0.6 - 0.66j
"d" = 0.387λ
S = 2.6
0.45
RESISTANCE COMPONENT (R/Zo), OR CONDUCTANCE COMPONENT (G/Yo)
WAVELENGTHS TOWARD GENERATOR
WAVELENGTHS TOWARD LOAD
ANGLE OF REFLECTION COEFFICIENT IN DEGREES
ANGLE OF TRANSMISSION COEFFICIENT IN DEGREES
RADIALLY SCALED PARAMETERS
TOWARD LOAD
TOWARD GENERATOR
CENTER
ORIGIN


0°
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"d" = 0.287λ
0.113λ
3(0.1λ)
= 0.6 − 0.66j
"d" = 0.387λ
S = 2.6
0.45
RESISTANCE COMPONENT (R/Z0), OR CONDUCTANCE COMPONENT (G/Y0)
WAVELENGTHS TOWARD LOAD
CAPACITIVE REACTANCE COMPONENT (-jX/Z0), OR INDUCTIVE SUSCEPTANCE
ANGLE OF TRANSMISSION COEFFICIENT IN DEGREES
RADIALLY SCALED PARAMETERS
TOWARD LOAD
TOWARD GENERATOR
SWR
RTN. LOSS [dB]
RFL. COEFF. P
RFL. COEFF. E or I
ATTEN. [dB]
S.W. LOSS COEFF
RFL. LOSS [dB]
S.W. PEAK (CONST. P)
TRANSM. COEFF. P
TRANSM. COEFF. E or I
CENTER
ORIGIN

We now move further, and hit the horizontal axis.
Recall that on this axis, $x(d) = 0$, and we have a minimum, i.e. $2\beta d - \theta_r = \pi$

Find $d = d_{\min}$

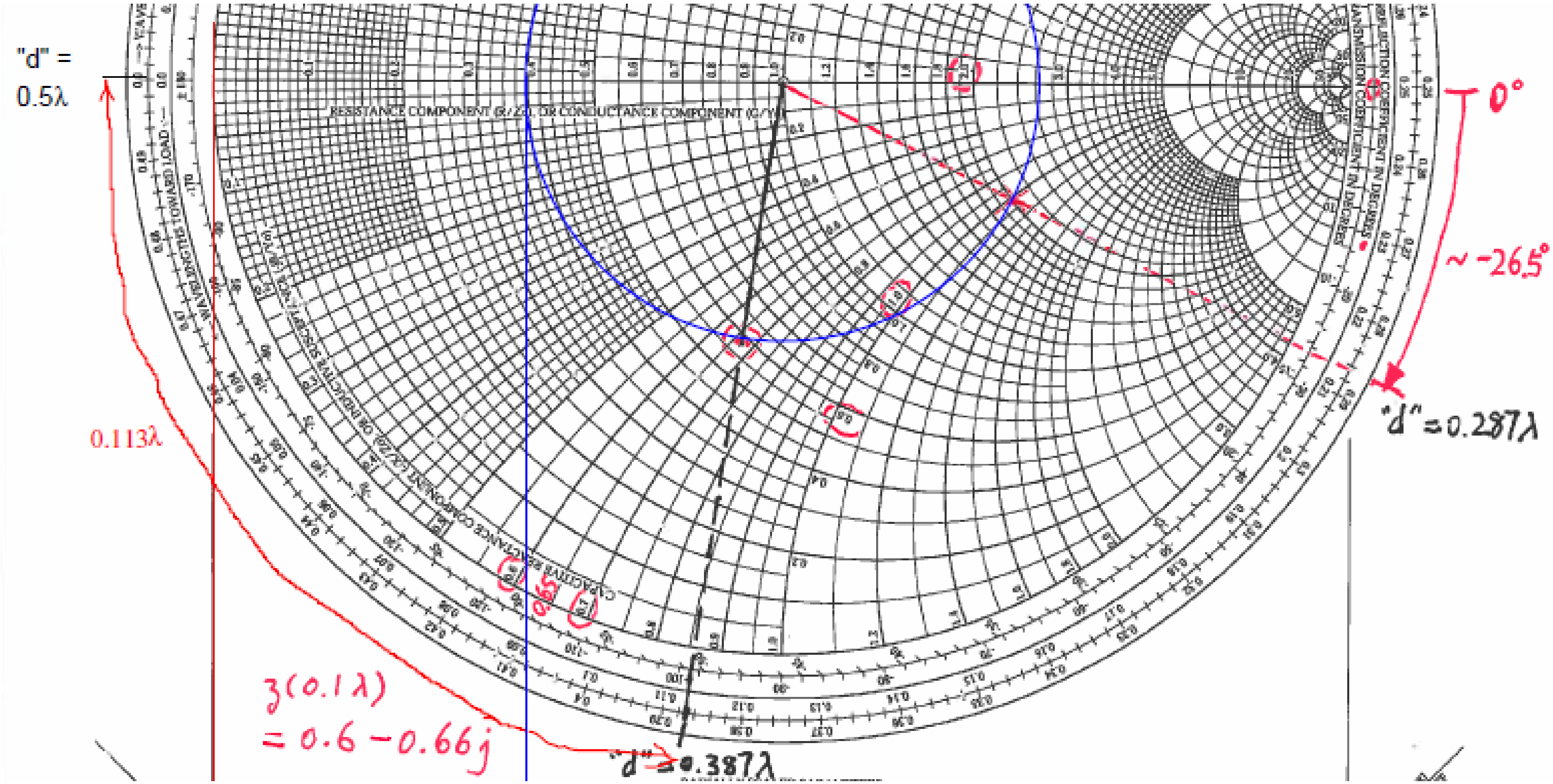
Recall the slotted line.

Now, moving further clockwise on the constant-SWR circle (i.e., away from the load or towards the generator), we will hit the horizontal axis again.

How much further have we moved from $d_{\min}$ to reach this point?

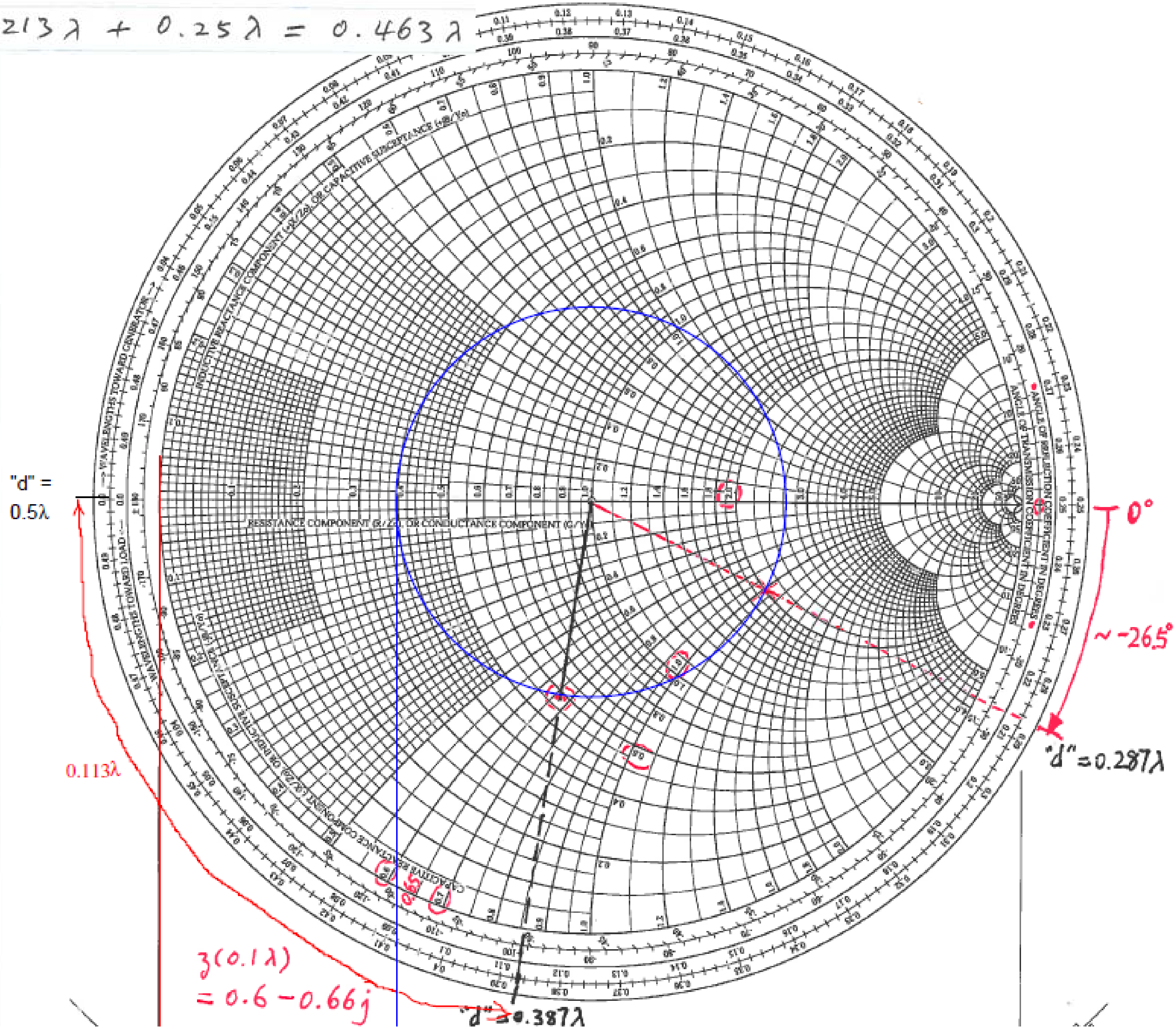
This is a maximum. Also purely resistive. $2\beta d - \theta_r = 2\pi$

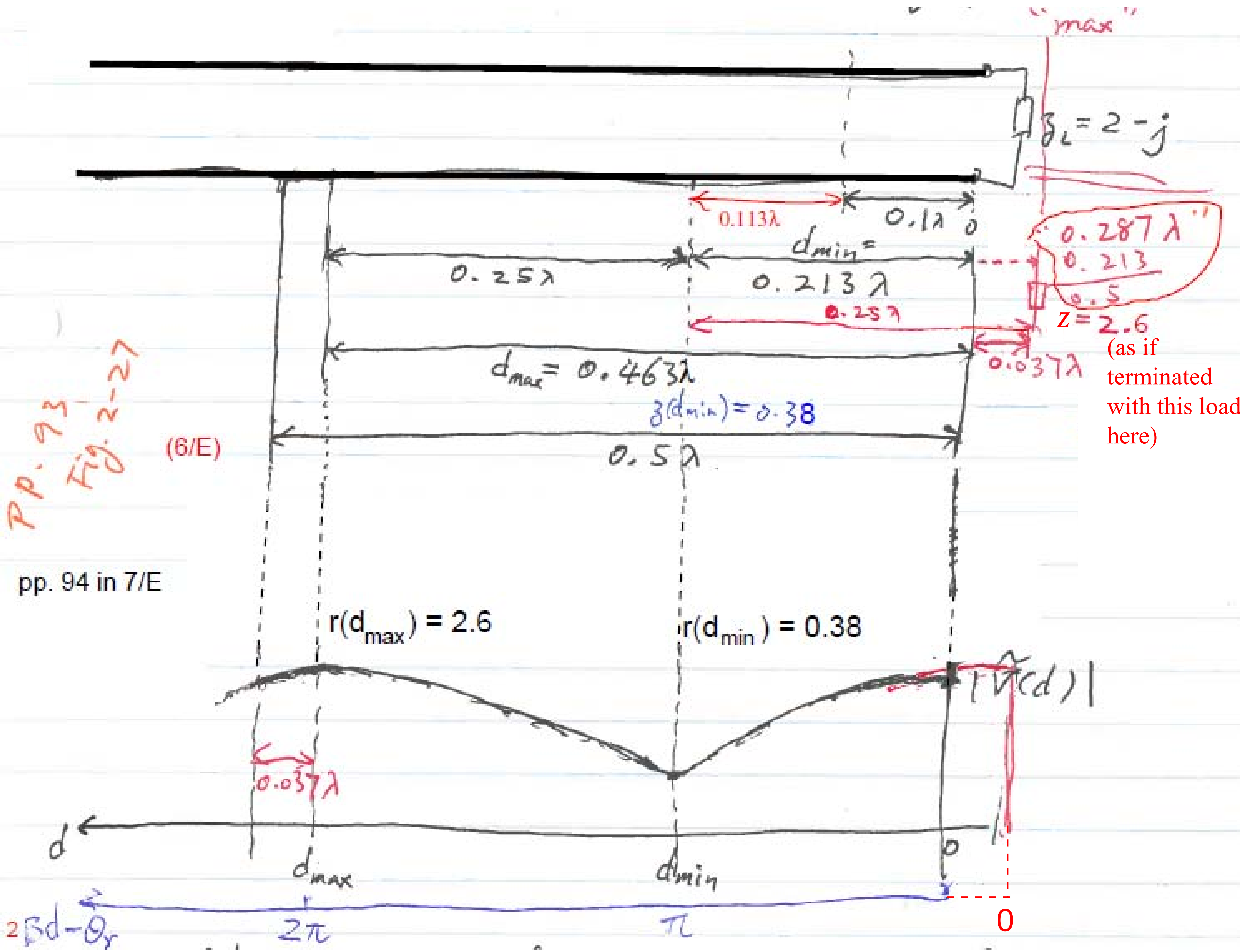
Find this $d = d_{\max}$



$$d_{min} = 0.5\lambda - 0.287\lambda = 0.213\lambda$$

$$d_{max} = 0.213\lambda + 0.25\lambda = 0.463\lambda$$




"max"
$z_L = 2 - j$
0.113λ
0.1λ
0
"0.287λ"
0.213
0.5
$d_{min} = 0.213\lambda$
0.25λ
0.25λ
z = 2.6
(as if terminated with this load here)
0.037λ
$d_{max} = 0.463\lambda$
$z(d_{min}) = 0.38$
0.5λ
pp. 93 Fig. 2-27
(6/E)
pp. 94 in 7/E
$r(d_{max}) = 2.6$
$r(d_{min}) = 0.38$
$|\tilde{V}(d)|$
0.037λ
d
d_{max}
d_{min}
0
$2\beta d - \theta_r$
2π
π
0

You can move along the constant-SWR circle round and round.

One round is $2\pi = 360°$, or $\lambda/2$.

$|\tilde{V}(d)|$ is periodic.

Half a round is $\pi = 180°$, or $\lambda/4$.

The maximum is a quarter wavelength away from the adjacent minimum.

At a minimum or a maximum, $x(d_{min}) = 0$ or $x(d_{max}) = 0$

Purely resistive.

If you have a purely resistive $z(d)$, you must have either a maximum or minimum,

At a minimum, the most destructive, $\Gamma < 0$

At a maximum, the most constructive, $\Gamma > 0$

Finish Homework 4 Problems 1-3

Smith Chart for $z \leftrightarrow \Gamma$ Mapping: A Recap

- Each point within the $|\Gamma| = 1$ circle represents a Γ, which uniquely corresponds to a $z = r + jx$

  (inside)
- One family of circles, each the locus of all Γ corresponding to a constant r
- One family of circles, each the locus of all Γ corresponding to a constant x
- Moving along the transmission line away from the load (i.e. towards the generator) is equivalent to moving along a constant-SWR circle (i.e. constant-$|\Gamma|$ circle) clockwise
- The chart we use assumes a short at the load, corresponding to $\Gamma = -1 = 1\angle 180°$, but that doesn't matter. Take the scales as relative.