The Status of Mathematics in India and Arabia during the "Dark Ages" of Europe*

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POLITICAL and economic world historians have found it convenient to divide the time from the fall of Rome to the discovery of America into two periods, and to designate the first of these by the term "Dark Ages." One work accounts for this name by "the inrush into Europe of the barbarians and the almost total eclipse of the light of classical culture." The period covers, roughly, the time from 500 to 1000 A.D. Part of these "barbarians" came down from the north and the rest attacked from the south, the latter bound together politically and religiously by the great, although probably totally illiterate, leader Mohammed into a vast dominion that at one time or another covered all of eastern Asia, northern Africa, Spain, France in part, and the European islands of the Mediterranean Sea. It was during this period that Europe was dark, learning at low ebb, and the development of mathematics almost negligible. The world as a whole was not dark, and as applied to general history the expression "Dark Ages" is a gross misnomer. Throughout the entire period there was considerable intellectual (including mathematical) activity among the Hindus and, beginning about 750, there developed many centers of Muslim civilization which rose to the very peak of mathematical productivity.

A number of facts combine to account for our heretofore slight emphasis upon oriental mathematical science. (1) We have been so enamored by the story of the Golden Age of Greece and the Modern Period that the Orient failed until recently to divert our attention. (2) There is such an overwhelming mass of mediaeval material left us and so much of it is worthless, due largely to the overabundance of study devoted to Scholasticism, that the search for gems among the rubbish has seemed hardly worth the effort. (3) Then, too, the difficulty of accessibility involved is enormous. Before many Greek and Hindu works could be assimilated in the main scientific current in the West, they had to be translated first into Syriac, then Arabic, then Latin and finally into our own language. Thus the completeness and the accuracy of these transmissions can only be determined by painstaking investigations of historians of science. (4) Furthermore, but few persons have ever been qualified to advance our knowledge of Muslim mathematics, due to the rare qualifications required. Not only does one need to know well mathematics and astronomy, but also Sanskrit, Syriac, Arabic and Persian, and, in addition, one needs to have a thorough training in paleography and a keen historical sense. This rare combination, together with a lifetime of ceaseless work, is the price that must be paid to increase our understanding of oriental mathematics.

Modern mathematics is easily transmitted; the process is simple. Articles appearing in any scientific journal are announced in other principal ones, and hence it is easily possible for a worker in any field to be fully informed on what is being done throughout the world regardless of the language used in the original publication. So simple is it that the modern scientist who lacks historical training can hardly comprehend the difficulties involved in the handing down of knowl-

* In the preparation of this article the author has drawn heavily from Dr. George Sarton's *Introduction to the History of Science,* volume I, a work of great value to the student of the history of mathematics.

edge from the early ages to the present.

For our knowledge of Greek mathematics we owe an unpayably large debt to two men who devoted years to the production of numerous accurate translations: J. L. Heiberg of Copenhagen, and T. L. Heath, a great English scholar in both mathematics and Greek. Unfortunately we have neither a Heiberg nor a Heath to enlighten us in Hindu and Muslim mathematics. In our knowledge of the former we are perhaps least fortunate of all. Cantor's three chapters are quite satisfying on the works of Brahmagupta and Bhāskara, being based upon Colebrooke's *Algebra with Arithmetic and Mensuration, from the Sanscrit of Brahmegupta and Bhascara*, but later historians place much dependence upon the interpretations of G. R. Kaye who was for many years a resident of India as a high government official, and whose work on Indian mathematics (1915) is shown to be erroneous in many respects by competent scholars of India today, such as Saradakanta Ganguli. One of the ablest scholars in the field of Muslim mathematics was Carl Schoy (1877–1925), who, between the years 1911 and 1925, contributed many valuable papers and books containing critical translations of Muslim mathematics and astronomy.

Recent scholars such as Schoy have shown us that, just as the greatest achievements of antiquity were due to Greek genius, so the greatest achievements of the Middle Ages were due to Hindu and Muslim genius. Furthermore, just as, for many centuries of antiquity, Greek was the dominant progressive language of the learned, so Arabic was the progressive scientific language of mankind during the period of the Middle Ages. We have learned further that the fall of ancient science and the dampening of the scientific spirit in Europe was far less due to the overrunning of southern Europe by the barbarians than it was to the passive indifference of the Romans themselves, and to the theological domination of a little later time. As soon as the Arabs were apprised of the Greek and Hindu sources of mathematical knowledge they were fired with a contagious and effective enthusiasm that led to numerous remarkable investigations in mathematics prosecuted from a number of cultural centers throughout the Muslim world, and that did not abate until the close of the 12th century when they had made a permanent impression on mathematics as a whole.

What I wish to do is to give as comprehensive a survey of Hindu and Muslim mathematics as space permits, pointing out principal achievements of leaders in the field, and indicating work still to be done to make the history complete.

Our attention is first called to Hindu mathematics in one of the five Hindu scientific works on astronomy called *Siddhāntas*, which were theoretical as opposed to karaṇas which were practical. Its date is very uncertain, but is placed in the first half of the fifth century. The *Sūrya-Siddhānta*, the only one we have in full, is composed of fourteen chapters of epic stanzas (ślokas) which show decided knowledge of Greek astronomy but also much Hindu originality, especially the consistent use throughout of sines (jyā) instead of chords, and the first mention of versed sines (utramadjyā).

Even more important is the *Pauliśa Siddhānta* which we have only indirectly through the commentator Varāhamihira (*c.* 505). It contains the foundation of trigonometry and a table of sines and versed sines of angles between 0° and 90° by intervals of 225′ (kramajyā). The sine and the arc of 225′ were taken to be equal, and sines of multiples of 225′ were obtained by a rule equivalent in our symbolism to

$$\sin (n+1)\, x = 2 \sin nx - \sin (n-1)x - \frac{\sin nx}{\sin x}; \sin x = x = 225'.$$

The next important Hindu advance is due to Āryabhaṭa (The Elder) who wrote in 499 a work, *Āryabhaṭīyam*, of four parts, the second of which—the *Gaṇitapāda*—was a mathematical treatise of 32 stanzas in verse, containing essentially the continued fraction process of solution of indeterminate equations of the first degree; an amazingly accurate value of π, namely 3 177/1250; the solution of the quadratic equation implied in the problem of finding n of an arithmetic series when a, d, and s are known; and the summing of an arithmetic progression after the pth term. These are expressed now by means of the formulas:

$$(1) \qquad n=\frac{d-2a\pm\sqrt{(2a-d)8ds}}{2d}$$

$$(2) \qquad s=n\left[a+\left(\frac{n-1}{2}+p\right)d\right].$$

To him also were due other startling truths less mathematical in character, among which was the theory that the apparent rotation of the heavens is due to the rotation of the earth about its axis.

Varāhamihira, astronomer-poet and contemporary of Āryabhaṭa, contributed the equivalent of trigonometric facts and formulas as follows:

$$\sin 30°=1/2; \qquad \cos 60°=\sqrt{1-1/4};$$

$$\sin\frac{x}{2}=\sqrt{\frac{1-\cos x}{2}}$$

$$\sin^2 x+\text{versin}^2\ x=4\sin^2\frac{x}{2};$$

$$\sin^2 x=\frac{\sin^2 2x}{4}+\frac{[1-\sin(90°-2x)]^2}{4}.$$

Then we have a leap of about a century when Brahmagupta (*c.* 628), one of India's greatest scientists, and the leading scientist of his time of all races, made his study of determinate and indeterminate equations of first and second degree, cyclic quadrilaterals and combinatorial analysis. He solved the quadratic for positive roots completely, and the Pellian equation $nx^2+1=y^2$ in part (it was finished by Bhāskara, 1150). If a, b, c, d, x, and y are the sides and diagonals of a cyclic quadrilateral, s the half-perimeter, and K the area, his results can be expressed by the equations:

$$K=\sqrt{(s-a)\ (s-b)\ (s-c)\ (s-d)};$$

$$x^2=\frac{(ab+cd)\ (ac+bd)}{(ad+bc)};$$

$$y^2=\frac{(ad+bc)\ (ac+bd)}{(ab+cd)}.$$

Another proposition, "Brahmagupta's trapezium," states that if $a^2+b^2=c^2$ and $x^2+y^2=z^2$, then az, cy, bz, cx form a cyclic quadrilateral whose diagonals are at right angles. His work in combinations and permutations is much like that now offered in a first course, but is not quite complete. Brahmagupta used three values of π: for rough work, 3; for "neat" work, $\sqrt{10}$; and for close accuracy, the finer value given by Āryabhaṭa.

By the close of the period of Brahmagupta Hindu mathematics had developed to such an extent that its influence reached out far toward both the east and the west. Two Chinese authors are of special value as witnesses of the influence of Hindu mathematics in China. (1) The first is Ch'ü-t'an Hsi-ta, a Hindu-Chinese astrologer of the first half of the eighth century, whose work gives a detailed account of a number of ancient systems of chronology, the most important of which, from our point of view, is the Hindu system, and in the explanation of which is implied the Hindu decimal notation and rules. Thus we think this must have been the very latest date of the introduction of the Hindu numerals and hence other Hindu mathematics into China. Much more probably they were introduced earlier, about the second half of the sixth century, for the catalog of the Sui dynasty (589–618) lists many books devoted to Hindu mathematics and astronomy. Then, too, it was about this time that Buddhism entered China from India. (2) The second author, I-hsing (683–727), or Chang Sui (the first being his religious name) was an astronomer of note who undertook, by order of the emperor, an investigation of chronological and arithmetical systems of India, but he failed to finish the work on account of a premature death.

The westward reach of Hindu mathematics is equally certain. The Syrian philosopher and scientist, Severus Sēbōkht (fl. 660), who studied also Greek philosophy and astronomy, was the first to mention the Hindu numerals outside of India, and expressed his full appreciation of Hindu learning in these words: "I will omit all discussion of the science of the Hindus, a people not the same as the Syrians; their subtle discoveries in this science of astronomy, discoveries that are more ingenious than those of the Greeks and the Babylonians; their valuable methods of calculation; and their computing that surpasses description. I wish only to say that this computation is done by means of nine signs. If those who believe, because they speak Greek, that they have reached the limits of science should know these things they would be convinced that there are also others who know something." (Quoted from Smith: *History of Mathematics*, v. I, p. 167.)

But few Hindu mathematicians after the day of Brahmagupta stand out prominently. One of uncertain date but probably of the 9th century is Mahāvīra, author of the *Gaṇita-Sāra-Sangraha*, which deals with arithmetic, including geometric progressions, the relation between the sides of a rational sided right triangle ($2mn$, m^2+n^2, m^2-n^2), and the solution of several types of equations involving the unknown and its square root. Two hundred years later, about 1030, another *Gaṇita-Sāra* (compendium of calculation) was produced by Śrīdhara, but was quite elementary. However, he wrote a work (now lost) on quadratic equations in which, according to the eminent Bhāskara of the 12th century, was found our present formula for the quadratic solution.

Now let us set our clock back at the second half of the 8th century. Here we find that practically all the work done in mathematics was done by Arabs. This is the beginning of a long period extending to the close of the 11th century, during the whole of which there was an overwhelming superiority of Muslim culture. Stimulated from the east by the Hindus and from the west by the eastward transmission of Greek mathematics, the Arabs began a remarkable and altogether too little emphasized flourish of activity. At first there were mainly students and translators of the five Hindu siddhāntas and ohter works. Chief of these were al-Fazārī, the elder, and his son. Then with the 9th century began a series of very important steps forward, especially in the field of trigonometry and the construction of astronomical tables; but there was also an imposing group of geometers, arithmeticians, algebraists and translators of Greek works.

The cause of science was greatly enhanced by the caliph al-Ma'mūn (813–833) who, although religiously exceedingly intolerant, was one of the world's greatest patrons of science. He collected all the Greek manuscripts he could, even sending a special mission into Armenia for that purpose, then ordered the translation of these into Arabic. He built two observatories, had made (probably by al-Khwārizmī) a large map of the world which was a much improved revision of Ptolemy's, organized at Bagdad a scientific academy, and stocked a library which was the finest since the Alexandrian (3d century B.C.). He then invited many of the world's greatest scientists to his court. Among them was al-Khwārizmī (d.*c.* 850) who wrote very important works on arithmetic and algebra and widely used astronomic and trigonometric tables of sines and tangents. He revised Ptolemy's geography, syncretized Hindu and Greek knowledge and is recognized by authorities as having influenced mathematical thought more than any other mediaeval writer.

A very notable astronomer of that period was al-Farghānī (fl. 860) who wrote the first comprehensive treatise in Arabic on astronomy, which was in wide use until the close of the 15th century, and was translated into Latin and Hebrew thus influencing European astronomy greatly before Regiomontanus.

Among the notes and extracts left us from Ḥabash al-Ḥāsib (*c.* 770–*c.* 870) we find record of the determination of time by the altitude of the sun and complete trigonometric tables of tangents, cotangents and cosecants which are at present preserved in the Staatsbibliothek in Berlin. In connection with the problem of finding the sun's altitude, the cotangent function arose. If $x=$the sun's altitude, $h=$the height of a stick, and $l=$the length of its shadow,

$$h=l\left(\frac{\cos\ x}{\sin\ x}\right),$$

and al-Ḥāsib constructed a table of values of h for $x=1,\ 2,\ 3,\ \ldots$ degrees, from which either x or h could be read if the other were known. According to Schoy, the Berlin MS. also contains the equivalent of

$$\sin x=\frac{\tan y \cos z}{\sin z},$$

where y is the declination and z the obliquity of the ecliptic.

One of the world's best general scientists and the greatest philosopher of the Arabs was al-Kindī (fl. 813–842), prolific author of between 250 and 275 works on astronomy, geography, mathematics and physics. He understood thoroughly the Greek mathematical works and influenced widely the early European scientists among whom were Girolamo Cardano, of cubic equation fame, and Roger Bacon.

The Banū Mūsā (Sons of Moses, also known as the Three Brothers) were wealthy scientists and patrons of science of this period. Through their efforts many Greek MSS. were collected, studied, translated and thus preserved for us. They were among the earliest to use the gardener's method of the construction of the ellipse (use of a string and two pins) and discovered a trammel based upon the conchoid for trisecting angles. Various problems of mechanics and geometry interested them most.

Perhaps chief among the many translators is to be mentioned al-Ḥajjāj (*c.* 825) who first (under Hārūn al-Rashīd) translated Euclid's *Elements* into Arabic and improved it later (under al-Ma'mūn) in a second translation. He also was an early translator of Ptolemy's *Syntaxis*, which he called *al-mijisṭi* (the greatest) from which term the name *Almagest* was derived. He was preceded, however, in this work by the Jewish Arab al-Ṭabarī of the same period. By the middle of the 9th century, then, these men and others of their day had made accessible to the Arabs the most important works of the Greeks and the earlier Hindus, had extended the sum total of astronomy and trigonometry, and had given tremendous impetus to independent investigation, which bore its fruit in the century to follow.

There was, however, much translating still to be done. Al-Māhānī (fl. 860) wrote commentaries on at least the first, fifth and tenth books of the *Elements*, and on Archimedes' *Sphere and Cylinder*, and studied also considerably the *Spherics* of Menelaos which led him to an equation of the form $x^3+a^2b=ax^2$, with which he wrestled long enough to cause his successors in the field to refer to it as "Al-Māhānī's equation." He never solved it. Al-Ḥimṣī (d.*c.* 883) translated the first four books of the *Conics* of Apollonius. Al-Nairīzī (d.*c.* 922) wrote (both commentaries lost) most authoritatively on the *Quadripartitum* and *Almagest* of Ptolemy as well as on the *Elements*. Thābit ibn Qurra (*c.* 826–901) founded within his own family a school of translators and enlisted outside scholars to aid him in producing translations of nearly all the Greek mathematical classics even including the commentary of Eutocius. Without naming others, suffice it to say that by the beginning of the 10th century there was probably not a single important work of the Golden Age of Greece that had not been translated and mastered by the Arabs.

As to the results of independent investigators of this period, the following must be mentioned. The best and most complete study among the Arabs on the spherical astrolabe was produced by Al-Nairīzī. The school of Thābit ibn Qurra wrote about 50 works of independent research, and 150 (about) translations. Many of these works are still extant. The most valuable ones are those on theory of numbers and the study of parabolas and paraboloids. A sample of the former on amicable numbers shows the fine reasoning employed: $2^n pq$ and $2^n r$ are amicable numbers if p, q, and r are prime to each other and if $p=(3)(2^n)-1$, $q=(3)(2^{n-1})-1$, and $r=(9)(2^{2n-1})-1$. The latter contains ingenious developments of Archimedes' Method.

In many respects the outstanding scholar of the century was al-Battānī (d. 929), Muslim's greatest astronomer, in whose principal work on astronomy are found numerous important facts. He gave the inclination of the ecliptic correct to 6″; calculated the precession at 54.5″ a year; did not believe in the trepidation of the equinoxes, which Copernicus still believed many years later. In trigonometry, to which his fifth chapter is devoted, he gave the equivalent of our formulas:

$$\sin x=\frac{1}{\csc x}; \qquad \cos x=\frac{1}{\sec x};$$

$$\tan x=\frac{\sin x}{\cos x}; \qquad \cot x=\frac{\cos x}{\sin x};$$

$$\csc x=\sqrt{1+\cot^2 x}; \quad \sec x=\sqrt{1+\tan^2 x};$$

$$\sin x=\frac{\tan x}{\sec x};$$

$\cos a=\cos b \cos c+\sin b \sin c \cos A$; and the sine law (doubtful). This work was translated into Latin by Plato of Tivoli in the 12th century and into Spanish the 13th, and exerted a tremendous influence for 500 years.

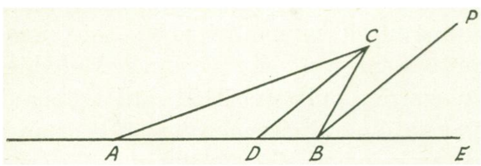
Considering now the first half of the 10th century, we note again that almost all the original work was done by Arabs in Arabic, but with the marked difference that there is a decided decline in activity. The development of mathematics may be compared with rainfall, which, after uncertain periods of drouth or average intensity, may burst forth in torrents. Some sort of unknown law of rhythm seems to hold rather than a law of uniform advance. The outstanding men of the period are Muslim. Ibrāhīm ibn Sinān (d. 946) wrote numerous commentaries on astronomy and geometry, but has received high recognition only since 1918 when Schoy's translation of his *Quadrature of the Parabola* revealed the fact that his method was superior to and simpler than that of Archimedes. Abū Kāmil (*c.* 925) was an able algebraist, improved essentially the algebra of al-Khwārizmī by some generalizations, development of algebraic multiplication and division, and operations with radicals. He also gave an algebraic treatment of regular inscribed polygons.

Toward the close of the century a decided renewal of creative work is to be observed. Far more names of famous mathematicians could be mentioned than space permits. But Abū-l-Fatḥ (fl. 982), al-Khāzin (fl. 950), al-Kūhī (fl. 988), al-Sijzī (fl. 1000), al-Ṣūfī (fl. 975), al-Khujandī (fl. 990) and Abū-l-Wafā' (fl. 990) cannot be omitted.

In Florence there is an untranslated commentary on the first five books of Apollonius' *Conics*, by Abū-l-Fatḥ, who also wrote a translation of the first seven books of the *Conics*. His work on books V–VII is considered highly important because the original Greek is no longer extant. Al-Khāzin solved the cubic equation of al-Māhānī (mentioned before in this paper). Al-Kūhī is chiefly known for his work on the solution of higher degree equations by means of the intersections of two conics. His problem on trisecting an angle I quote here (from French) from Woepcke's *L'Algèbre d'Omar Alkhayyami* (1851), page 118:

Let the given angle be CBE. Take on EB produced points A and D, and on the other side a point C so that (1) $AD=DC$, (2) $AB:BC::BC:BD$. Draw BP parallel to DC. Then the angle $CBP=1/3$ angle CBE.

The geometry of the figure shows clearly that the angle is trisected, but the construction involves the solution of a cubic equation.



Of the works of al-Sijzī, fourteen are now preserved in Cairo, Leyden, Paris, and the British Museum. All deal with conic sections, trisection problems and the resulting cubic equations. These works rank him as an outstanding Arabic geometer.

The greatest mathematician of the century, however, was Abū-l-Wafā', who wrote 15 or 20 works (mostly extant) on geometry, geometrical solutions of special fourth degree equations, and trigonometry, to which he added a number of formulas and the line values of the functions.

The 11th century continues in the first half to show remarkable activity, with an imposing array of first order mathematicians, the principal ones being Al-Bīrūnī (973–1048), Ibn Yūnus (d. 1009), al-Karkhī (d. 1025), Ibn Sīnā (980–1037), al-Ḥusain (?) and al-Nasawī (*c.* 1025). Of paramount importance is the work of Al-Bīrūnī. Two of his writings are of great mathematical significance. We have known the first, *A Summary of Mathematics*, for some time, but the other, *Al-Qānūn al-Masūdī*, has recently been given us in German, and proves its author to be of considerably more importance in the history of trigonometry than we have suspected.

Noteworthy among the contributions of Ibn Yūnus is the introduction of the prosthaphaeretical (sum and difference) formulas of trigonometry which were so useful before the time of logarithms. Al-Karkhī, whose work, *al-Fakhrī*, Woepcke has given us in French, was an algebraist of the first rank. He gave a splendid treatment of the solution of Diophantine equations, equations of quadratic form, operations with radicals, and summation of integro-geometric series. In connection with series he gives such results (not symbolically, of course) as

$$\sum_1^n i^2 = \left(\sum_1^n i\right)\left(\frac{2n+1}{3}\right)$$

and $$\sum_1^n i^3 = \left(\sum_1^n i\right)^2.$$

Ibn Sīnā was principally a philosopher and hence emphasized that phase of mathematics; al-Ḥusain wrote one of the few Arabic treatises on the construction of right triangles with rational sides; and al-Nasawī, an able arithmetician, explained extraction of square and cube roots by a method very similar to our own, and furthermore anticipated decimal fractions in the manner indicated in the equations

$$\sqrt{17} = \frac{1}{100}\sqrt{170000} = \frac{412}{100},$$

but he changed them to sexagesimals for the final form of his answer.

There is a distinct decline toward the close of the 11th century with a notable decrease in the number of mathematicians of the first rank. Of these I mention but one, Omar Khayyan, who, although living in a period of decline, was one of the greatest of the Middle Ages. His chief distinction results from his admirable work on *Algebra*, in which he classified equations by the number and degree of terms, treated 13 types of cubic equations, and referred to the general expansion of the binomial with positive integral coefficients which he treated in another work now unknown. His other mathematical writings dealt with the assumptions of Euclid and a very accurate reform of the calendar.

Thus closes the marvelous periods of scientific activity of the Hindus and Muslims in the field of mathematics. The Hindu period began shortly after the time of Proclus (d. 485), last straggler of the great Greek period. After the Hindu peak comes the tremendous Muslim flourish. Suter in his work, *Die Mathematiker und Astronomen der Araber und ihre Werke*, lists the works of 528 Arabic scholars who were active in mathematics from 750 to 1600. Certainly 400 of these came within the period of the "Dark Ages." We have learned much about them, but there is still much to be done. Tropfke tells us that "ueberreiche Schaetze" still lie untranslated in the large libraries of Europe.

During this same period in Europe, but few names deserve even feeble mention as writers on mathematics. There was no creative work. Boetius (d. 524) wrote texts on the quadrivium (arithmetic, music, geometry and astronomy), which, though comparatively poor, were so widely used in schools that they had tremendous influence. Anthemios (d. 534) is interesting for his history of conic sections and his use of the focus and directrix in the construction of the parabola. Eutocius (b. 480?) is important for his commentaries on the works of Archimedes and Apollonius. Bede (673–735) deserves mention because our information on finger reckoning is almost entirely dependent on his work. Alcuin or Albinus (735–804) wrote a work on puzzle problems which furnished material for textbook writers for ten centuries. Gerbert (Pope Sylvester II) (999) was great and influential as compared with other European writers of his day and doubtless did much to popularize the Hindu-Arabic numerals. But these men were all small as compared with the Muslim giants.

Let me repeat what I said at the beginning: Europe was dark, but India and the Muslim world were not. To quote Dr. George Sarton, "the 'Dark Ages' were never so dark as our ignorance of them."

DOES the harmony which human intelligence thinks it discovers in Nature exist apart from such intelligence? Assuredly no. A reality completely independent of the spirit that conceives it, sees it, or feels it, is an impossibility. A world so external as that, even if it existed, would be forever inaccessible to us. What we call *objective reality* is, strictly speaking, that which is common to several thinking beings and might be common to all; this common part, we shall see, can only be the harmony expressed by mathematical laws.—POINCARÉ. *The Value of Science.*

THERE IS no subject in the entire high school curriculum which by its very nature lends itself more admirably to a realization of some educational objectives as does the teaching of mathematics. I know of no other subject, save possibly foreign language, where a close day by day application of the student is so absolutely essential to success. When can the boy or girl be more impressed with the importance of doing his job day by day, and meeting his obligations that lead to his own success than in a class of mathematics? He is constantly confronted with the necessity of being alert, critical and observant. He learns to develop habits of inspection and inquiry concerning printed and spoken statements, habits worth cultivating by everyone today. He learns to take nothing for granted except certain hypotheses about which there can be no argument. He waits for all the evidence to come into the picture before drawing conclusions. He finally accepts no opinion, theory or notion that is not backed by facts. Empirical reasoning is soon detected and labeled as such.

Geometry is a powerful training in logic. When Lincoln had a difficult case to try in court he resorted to Euclid as the most helpful aid to jurisprudence. The practical value of geometry was much larger to Lincoln than its mere application to the arts and industries. From an article by S. W. Lavengood, Principal, Pershing School, Tulsa, Oklahoma in the Georgia Educational Journal, Feb. 1936, p. 27 on "Contributions of the Teaching of Mathematics."