# The sutras of Vedic Mathematics in Geometry

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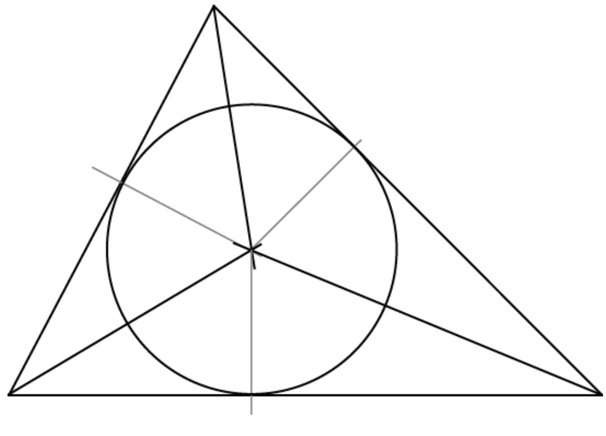
### Abstract

During the last fifty years or so geometry in education has diminished both in quantity and in quality and yet the pedagogic use and applications remain as important now as they were then. A new and holistic set of principles provides a simple approach to the fundamentals as well as to the vast array of applications. Since the sutras of Vedic Maths apply to all areas of mathematics it seems reasonable to suppose that they also apply to geometry. One of the hallmarks of the sutras is that they include the human experience. One example is *Vilokanam, By Observation*. Accepting both the mathematical and psychological nature of the sutras leads to a new orientation within geometry. This paper looks at eleven sutras which provide general principles. Each sutra expresses a simple concept that has a broad range of applications.

### Introduction

In its widest sense geometry deals with the relationship of form within space. Both form and space are innate properties of the known universe and the human psyche. Geometry informs and enables us to understand many aspects of the world around us. It develops our spatial awareness and is integral to mathematics. So much of the world in which we live depends upon this developed awareness, such as in design, architecture, physics, chemistry, crystallography, moelcular modelling, engineering, art, and even in sport such as cricket, that it renders the subject a vital part of modern education. Not only is geometry functional in providing engineers, and the like, with the concepts and faculties required for their work but also it contains an aesthetic appeal because it reveals beauty, harmony, symmetry and balance. Furthermore, it is a useful pedagogical tool because it gives visual representation to mathematical relationships some of which are far easier to understand in spacial terms than in digital format.

In education geometry has become analytical in the sense that it is invariably related to algebra. For instance, lines and circles are frequently associated with equations. There is nothing wrong with algebraic geometry but there are many geometric relationships that are more easily understood visually. For example, the incentre and circumcentre of triangles, in terms of the concurrence of angle bisectors and perpendicular bisectors, are far more easily understood visually than algebraically.



The angle bisectors all meet at the same point that is the incentre. All points along the bisectors are equidistant from their adjacent sides. Hence the perpendiculars from the incentre to the sides of the triangle are all the same length. Therefore a circle can be drawn that touches all three sides. This can be proved algebraically but the “visual proof” is far easier.

In the past geometry has been a pillar of western education. For nearly a thousand years the geometry of Euclid was taught at schools and universities throughout Europe. It was regarded as a powerful tool primarily because it provided a visual means to learn how to think logically, with rigour and clarity, and how to establish the validity of propositions through deductive reasoning. The discovery and invention of other geometries during the $19^{th}$ C and a less formal approach in education loosened the grip of Euclid in education. The decline of geometry in UK schools proliforated to the extent that today formal proofs are absent from all mathematics curricular for 11 – 18 year-olds. Geometry exists as a small fragment of primary education with the result that little progress is accomplished at secondary level. One of the beautiful aspects of geometry is its practicality and in an obvious way. The practicality and usefulness of teaching some aspects of algebra, such as the difference of two squares, is not so readily seen. It is not uncommon for disgruntled teenagers, who do not have a thirst for mathematical knowledge, to ask, Why am I having to learn this? In pure geometry, such questions are sometimes easier to answer because problems frequently admit to simple visual relationships. Diagrams can be drawn and so it appeals greatly to those who are inclined towards kinesthetic or visual learning. Yet, virtually no child enters a secondary school being able to draw a circle with a pair of compasses!

One of the great assets of geometry is that the spatial relationships involved are visual and, in the past, this has assisted education in deductive reasoning. Unfortunately, Euclidean propositions and proofs are very formal and this is not entirely suited to modern appraoches to education.

Euclid laid out a system of propositions based on three fundamental types of principles, axioms, postulates and definitions. His axioms are not specific to geometry. They are quite general statements of self-evident truths such as, if equals be added to equals the totals are equal. The five postulates were entirely practical, such as, a line can be drawn between two points, and intended to be specifically related to geometry and which had to be accepted in order to develop the propositions. The definitions were highly important as they sought to clarify anything that might not be clear. His aim was to demonstrate the validity of various propositions based only on the principles he set out. Euclid’s Elements is a highly systematic exposition but, for the modern reader, may seem rather pedantic.

A more modern approach to geometry, one that takes in all available methods of proof, can be much more user-friendly than pure Euclid. The sutras of Vedic Mathematics can be applied to all types of geometry, practical and theoretical and offer a new perspective. They do not go against established norms but emphasise the underlying structure of mathematical thought.

**The sutras of Vedic Mathematics in Geometry**

Inasmuch as the sutras express thought processes and mathematical principles, many of them are applicable to geometrical problems. Of course, a sutra such as “All form nine and the last from 10” seems only to apply to number but others, such as Particular and General, have a closer relationship to geometry.

By using the key idea within a sutra it is possible to formulate geometric principles or processes. Wherever possible I have tried to use a single idea or word to encapsulate how the sutras apply in geometry.

**1. Invariance**

This is based on the enigmatic sutra Sishyate Seshasanja, which Sri Tirtha translates as *The remainder remains constant*. A more literal translation can be *The remainder is what remains* and, at first glance, appears to be tautological. In mathematics, however, it largely relates to invarianceFor example, in geometric transformations an enlargement with scale factor 1 is invariant. What remains following a process is the same as before the process. Another example will be referred to later when sectioning off squares from a golden rectangle; the remaining rectangle's proportions are invariant.

**2. Balance**

Balance expresses equality and symmetry. When two things are equal they are in balance. This is an important principle not only in geometry but in all aspects of life. This concept is expressed in the sutra, Sunyam Samyasamuccaye, *When the total is the same it is zero*. The idea is that when two objects are the same then the difference is nothing. Seemingly tautological, this idea frequently occurs in mathematics whenever equality is seen for example with two sides of an equation. A simple example can be seen with congruence, that is, when two figures have the same size and same shape. Congruence is frequently used in geometrical proofs.

**3. Proportion**

The sutra involved here is Anurupyena, Proportionately. The Sanskrit word for form or shape is rupa and the literal meaning of the sutra is "by the same form". The classical meaning of proportion is equality of ratios. For example, two rectangles are similar when their sides are in the same ratio and so there is a proportion.

**4. Self-similarity**

This is part of the Vyashti Samashti sutra, Particular and general. It gives expression to the perception of a particular shape or form reflected in the form or shape of the whole. There are many examples of this in geometry, particularly in chaos theory.

**5. Transposition**

Transformations and transpositions are common throughout geometry. The sutra is Paravartya Yojayet meaning Transpose and Apply or Transpose and Adjust (Sri Tirtha uses both translations). It is a very far-reaching process. In fact, in his book, this is the most commonly used sutra. This sutra expresses such processes as applying a formula to a new situation, reversing processes, geometrical and algebraic transformations, and even when transposing a word problem into mathematical language.

**6. Deficiency**

The single-word sutra, *Yavadunam*, is applicable wherever a problem is solved by using or referring to a defficiency from some whole.

**7. Observation**

*Vilokanam* translates as *By mere observation* and comes into play wherever the answer to a problem appears just by inspection. The fact that this is included in the list of sutras indicates that the mental process of mathematical working is just as important as what is conventionally taken as mathematical working.

**8. Progression**

The *Ekadhikena Purvena* sutra, *By one more than the one before*, relates to sequencing and progressions.

The next set of principles are based on the couplet sutras, *Lopanasthapanabhyam – By elimination and retention, Puranapuranabhyam – By completion and non-completion* and *Sankalana Vyavakalanabhyam- By addition and subtraction*.

The "bhyam" ending is dual case and so indicates both. On first reading it appears that both processes are involved. For example, when something is eliminated it usually implies that something else is retained. Certainly, in the physical sense when something is added it must be subtracted from somewhere, for example, if money is added to an overdraft it must come from somewhere and so there is a subtraction from the place where it comes. Again, if I give £9 to a friend my pocket will be £9 shorter. The question then arises, if I set a problem for a child and create a sum such as 56 plus 9, where does the 9 come from? Is there a subtraction? Clearly there is no end to the potential of creating problems with plus 9 and so we may assume that the source of the 9 is unlimited. 9 can be subtracted, as it were, from an infinite source.

In practice we may focus on one or other part of these couplets. For example, we may solve a problem *By Addition*, and not refer to the subtraction. This implies that the dual case ending in the sutras can be taken as either meaning "and" or "or".

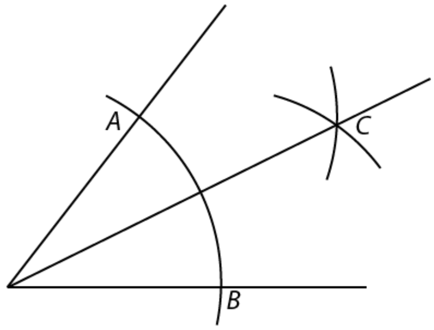
**9. Completion/non-completion**

**10. Addition/subtraction**

**11. Elimination/Retention**

The following eight examples are used to provide a brief illustratration of how these principles appear.

**1. Bisecting an angle**

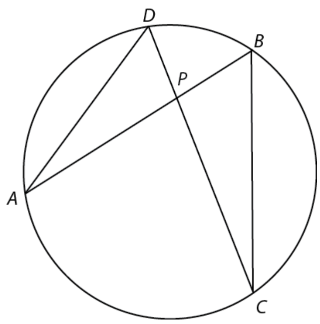



This uses the principle of **Balance**. An arc is drawn, cutting the two lines at *A* and *B*. With *A* and *B* as centres, two equal arcs are drawn intersecting at *C*. The angle bisector is the locus of points equidistant from the two lines.

The same principle is involved with bisecting a line into two equal parts.

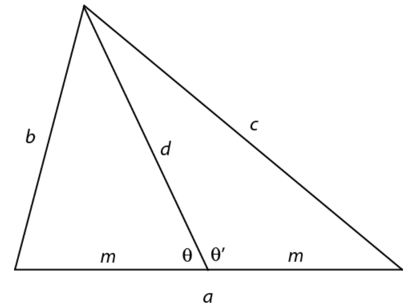
**2. Intersecting chord theorem**

This theorem uses the fact that angles in the same segment are equal.




| **Proof** | **Principles** |
| --- | --- |
| Angles in the same segment are equal and so $D\hat{A}P = B\hat{C}P$ and $A\hat{D}P = C\hat{B}P$. | **Balance** |
| It follows that the angles of triangles DAP and BCP are the same and so the triangles are similar. | **Proportion** |
| Since they are similar, corresponding sides are in ratio with, | |
| $\frac{AP}{CP} = \frac{DP}{BP}$ | |
| From this it follows that $AP \times BP = CP \times DP$. | **Transposition** |
| This is the intersecting chord theorem. | |

### 3. Apollonius' theorem



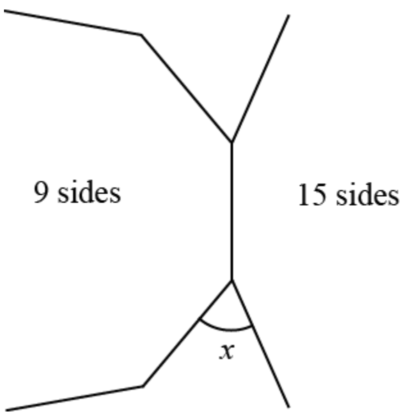

The triangle has sides *a*, *b* and *c* and a median is drawn that divides the side, *a*, into two equal parts, *m*.

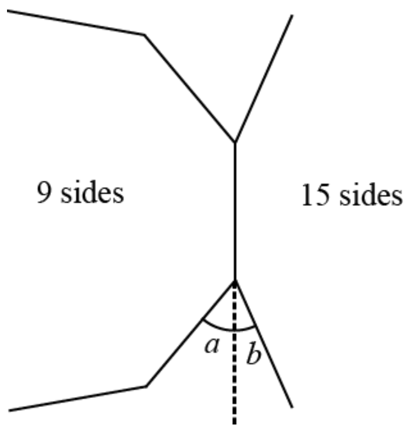
Apollonius' theorem states that $b^2 + c^2 = 2\left(m^2 + d^2\right)$

| **Proof** | **Principles** |
|---|---|
| Using the Cosine rule, | |
| $b^2 = m^2 + d^2 - 2md\cos\theta$ | **Transposition** |
| and $c^2 = m^2 + d^2 - 2md\cos\theta'$ | |
| But $\cos\theta' = -\cos\theta$ | **Transposition** |
| And so $c^2 = m^2 + d^2 + 2md\cos\theta$ | |
| Adding the two equations, | **Addition** |
| $b^2 + c^2 = 2\left(m^2 + d^2\right)$ | **Elimination and retention** |

### 4. Polygon problem

A common polygon problem found in school exams involves two adjacent regular polygons with different number of sides. The problem is to find the missng angle, x.

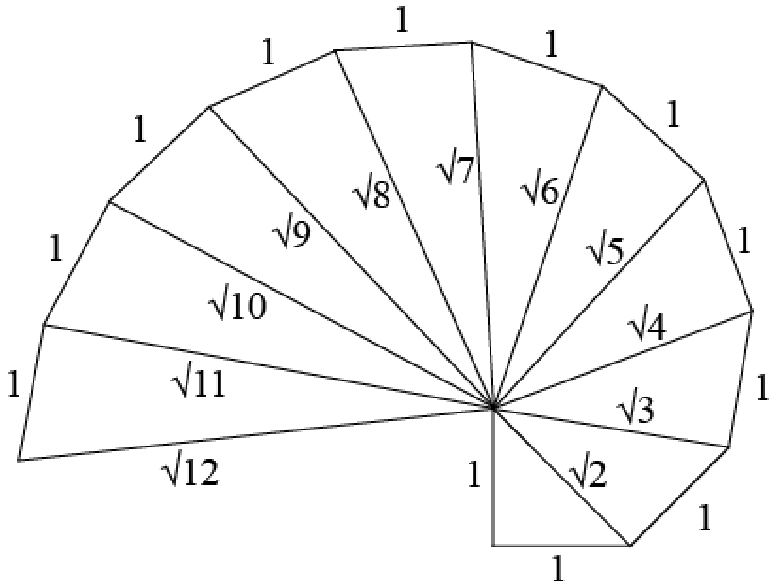


In the figures the polygons are incomplete (**Non-completion**) but it is easy to imagine them as being whole (Completion). It is possible to calculate the interior angles of each polygon and then subtract these from 360˚ to find *x*. But an easier method is found by extending **(Transposition**) the adjacent side to form two exterior angles, *a* and *b*. The figure is now set up (**Completion)** for using the exterior angles. By dividing 360˚ by the number of sides in each case the exterior angles are found to be 40˚ and 24˚. *x* is therefore 64˚.

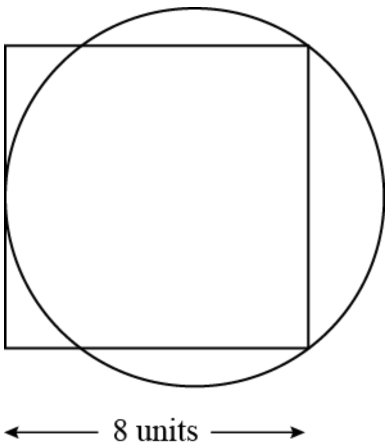
**5. Drawing square roots**

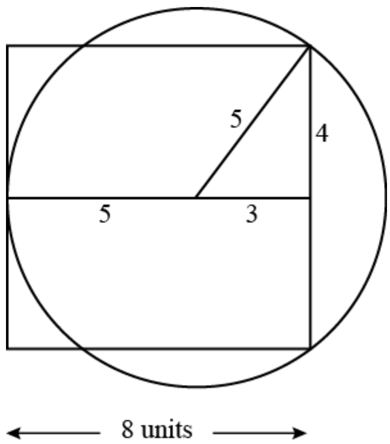
In the diagram each right-angled triangle has a unit side perpendicular to its base. The principle of **Progression**, demonstrated here is a formulation of the sutra, *By one more than the one before*.




**6. The problem of greater perimeter**

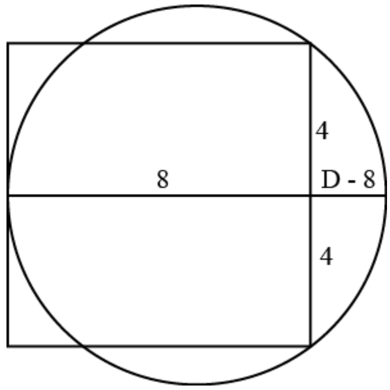
A square has side 8 units. A circle is drawn through two corners of the square so that the opposite side of the square is a tangent. Which has the greater perimeter?



The clue to an easy solution is in the side of 8 units. **By Observation** 8 can be split into 3 and 5 and 4 is half the side length. This menas that a 3, 4, 5 triangle can be drawn revealing the radius of the circle to be 5 units. The circumference is therefore $10\pi$ which is about 31.4 units whilst the square has perimeter of 32 units.

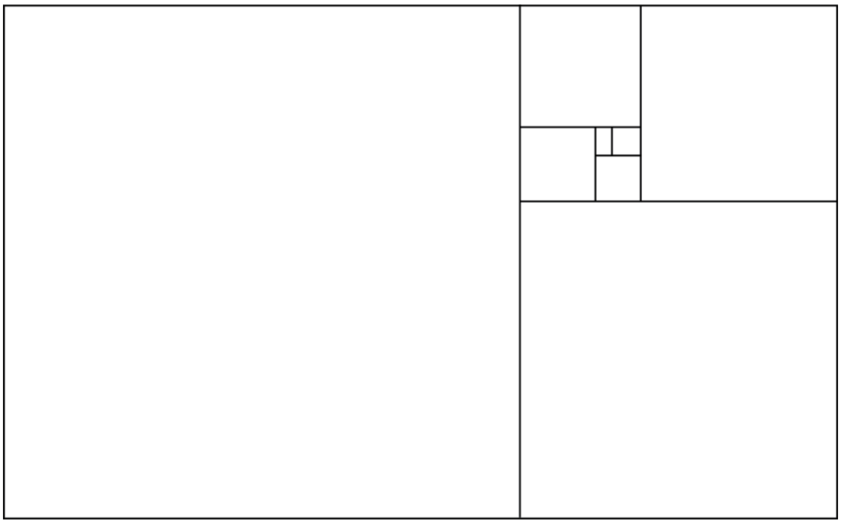
Another simple solution comes from extending the centre line of the square to form a diameter, by the principle of **Completion**.




Using the intersecting chord theorem, $8(D - 8) = 4^2$, from which $D = 10$. Again, this gives the circumference as 31.4 units.

A point to note is that in both cases previsouly known mathematics have been used. In the first instance the Pythagorean triple and in the second, the intersecting chord theorem. This is an example of a very common procedure and comes under the principle of **Transposition**. For example, the intersecting chord theorem is formulaic and is applied to the situation in hand. A transposition occurs whenever a formula is used or applied in mathematics.

**7. Golden Rectangle**

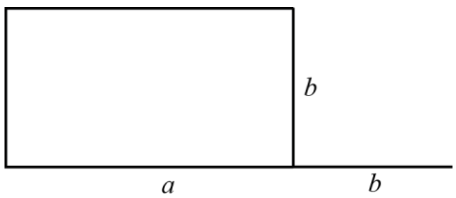


This rectangle has sides in the ratio $\phi : 1$, where $\phi = \frac{\sqrt{5}+1}{2} = 1.61803398...$

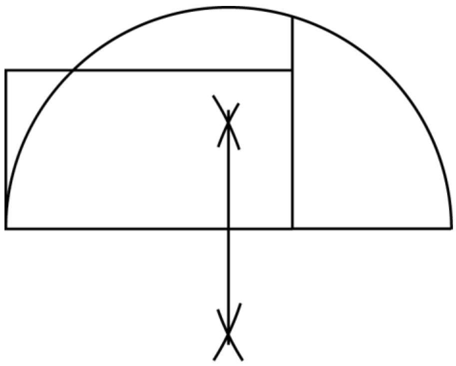
By cutting off a square the remaining rectangle has the same proportions. This can be repeated with successive squares being cut off, each time leaving a golden rectangle as the remainder. Three principles are involved, **Proportion**, **Invariance** and **Self-Similarity**. The proportionality of the rectangle is unique since no other rectangle has this property. Invariance ocurrs in the sense that the remainder remains constant (in shape). Self-similarity is where the shapes of the part is reflected in the shape of the whole and vice versa.

## 8. Construct a square equal in area to a given rectangle

Invariably, constructions in geometry require a number of steps each one of which can involve one or more of the principles.

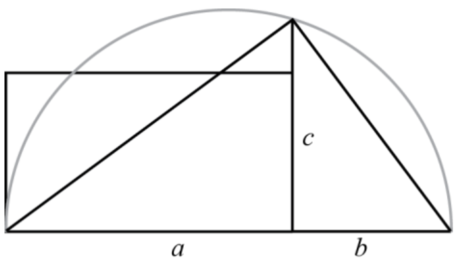



The rectangle has sides *a* and *b*. The side *b* is rotated to the horizontal to form a line of length *a* + *b*. (**Transposition**)



The midpoint of this line is found by contructing the perpendicular bisector. (**Balance**)

A semicircle with radius 0.5(a + b) is then drawn and the side b is extended upwards to meet the circle. This line has length *c*.

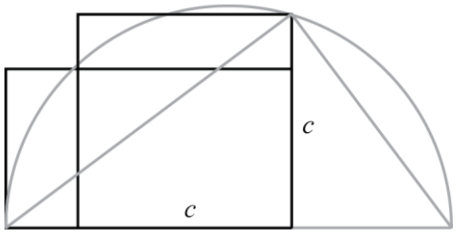



By drawing lines from the ends of the diameter three similar triangles are formed (**Completion**).

Then $\frac{a}{c} = \frac{c}{b}$ (**Proportion**)

This leads to $ab = c^2$ (**Transposition**)

In fact $c$ is the geometric mean of $a$ and $b$.




A square can now be drawn equal in area to the rectangle whose area is $ab$. (**Completion**)

**Concluding Remarks**

It should be noted that, in none of the cases above, is the mathematics unconventional. All of these are well-known solutions and problems often have alternative solutions. But it is the way they are looked at that is different. The Vedic Maths sutras provide a simple set of principles that can be applied singly or in combination in a very wide variety of ways. In some of the above problems there is a multi-step approach, such as with Apolloniius' theorem, intersecting chord theorem and creating a square equal to a rectangle, and in these it is possible to see the sutras at work within individual steps of solving the problem. So in one sense the sutras provide alternative names for steps or working that are entirely conventional. Additionally, there are cases, such as bisecting an angle, where the sutra simply express an idea or aspect of knowledge that lies behind the steps of working.

Uniquely, this approach reveals that behind mathematical thought processes and ideas there are very few principles, each one of which has huge potential in many apparantly diverse areas of mathematics. Not only do the sutras apply to arithmetic and algebra but also in pure geometry. The very fact that so few aphorisms find expression within a multitude of problems has a unifying effect.

Above all the sutras provide us with simplicity and this can be a great asset in teaching and learning. As the famous Renaissance artist and scientist Leonado Da Vinci said, "Simplicity is the ultimate sophistication".

In 2001 the Royal Society and the Joint Mathematical Council produced the report, Teaching and learning geometry 11 – 19. The report layed out key principles for the learning of geometry, together with sixteen recommendations for change. The curriculum authority within the department for Education has implemented none of them.

Also look at

Jones, K. (2000), Critical Issues in the Design of the Geometry Curriculum. In: Bill Barton (Ed), Readings in Mathematics Education. Auckland, New Zealand: University of Auckland. pp 75-90.