The Three Crises in Mathematics: Logicism, Intuitionism and Formalism

Crises in classical philosophy reveal doubts about mathematical and philosophical criteria for a satisfactory foundation for mathematics.

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The three schools, mentioned in the title, all tried to give a firm foundation to mathematics. The three crises are the failures of these schools to complete their tasks. This article looks at these crises "through modern eyes," using whatever mathematics is available today and not just the mathematics which was available to the pioneers who created these schools. Hence, this article does not approach the three crises in a strictly historical way. This article also does not discuss the large volume of current, technical mathematics which has arisen out of the techniques introduced by the three schools in question. One reason is that such a discussion would take a book and not a short article. Another one is that all this technical mathematics has very little to do with the philosophy of mathematics, and in this article I want to stress those aspects of logicism, intuitionism, and formalism which show clearly that these schools are founded in philosophy.

Logicism

This school was started in about 1884 by the German philosopher, logician and mathematician, Gottlob Frege (1848–1925). The school was rediscovered about eighteen years later by Bertrand Russell. Other early logicists were Peano and Russell's coauthor of Principia Mathematica, A. N. Whitehead. The purpose of logicism was to show that classical mathematics is part of logic. If the logicists had been able to carry out their program successfully, such questions as "Why is classical mathematics free of contradictions?" would have become "Why is logic free of contradictions?". This latter question is one on which philosophers have at least a thorough handle and one may say in general that the successful completion of the logicists' program would have given classical mathematics a firm foundation in terms of logic.

Clearly, in order to carry out this program of the logicists, one must first, somehow, define what "classical mathematics" is and what "logic" is. Otherwise, what are we supposed to show is part of what? It is precisely at these two definitions that we want to look through modern eyes, imagining that the pioneers of logicism had all of present-day mathematics available to them. We begin with classical mathematics.

In order to carry out their program, Russell and Whitehead created *Principia Mathematica* [10] which was published in 1910. (The first volume of this classic can be bought for $3.45! Thank heaven, only modern books and not the classics have become too expensive for the average reader.) *Principia*, as we will refer to *Principia Mathematica*, may be considered as a formal set theory. Although the formalization was not entirely complete, Russell and Whitehead thought that it was and planned to use it to show that mathematics can be reduced to logic. They showed that all classical mathematics, known in their time, can be derived from set theory and hence from the axioms of *Principia*. Consequently, what remained to be done, was to show that all the axioms of *Principia* belong to logic.

Of course, instead of *Principia*, one can use any other formal set theory just as well. Since today the formal set theory developed by Zermelo and Fraenkel (ZF) is so much better known than *Principia*, we shall from now on refer to ZF instead of *Principia*. ZF has only nine axioms and, although several of them are actually axiom schemas, we shall refer to all of them as "axioms." The formulation of the logicists' program now becomes: Show that all nine axioms of ZF belong to logic.

This formulation of logicism is based on the thesis that classical mathematics can be defined as the set of theorems which can be proved within ZF. This definition of classical mathematics is far from perfect, as is discussed in [12]. However, the above formulation of logicism is satisfactory for the purpose of showing that this school was not able to carry out its program. We now turn to the definition of logic.

In order to understand logicism, it is very important to see clearly what the logicists meant by "logic." The reason is that, whatever they meant, they certainly meant more than classical logic. Nowadays, one can define classical logic as consisting of all those theorems which can be proven in first order languages (discussed below in the section on formalism) without the use of nonlogical axioms. We are hence restricting ourselves to first order logic and use the deduction rules and logical axioms of that logic. An example of such a theorem is the law of the excluded middle which says that, if p is a proposition, then either p or its negation $\neg p$ is true; in other words, the proposition $p \vee \neg p$ is always true where $\vee$ is the usual symbol for the inclusive "or."

If this definition of classical logic had also been the logicists' definition of logic, it would be a folly to think for even one second that all of ZF can be reduced to logic. However, the logicists' definition was more extensive. They had a general concept as to when a proposition belongs to logic, that is, when a proposition should be called a "logical proposition." They said: *A logical proposition is a proposition which has complete generality and is true in virtue of its form rather than its content.* Here, the word "proposition" is used as synonymous with "theorem."

For example, the above law of the excluded middle "$p \vee \neg p$" is a logical proposition. Namely, this law does not hold because of any special content of the proposition p; it does not matter whether p is a proposition of mathematics or physics or what have you. On the contrary, this law holds with "complete generality," that is, for any proposition p whatsoever. Why then does it hold? The logicists answer: "Because of its form." Here they mean by form "syntactical form," the form of $p \vee \neg p$ being given by the two connectives of everyday speech, the inclusive "or" and the negation "not" (denoted by $\vee$ and $\neg$, respectively).

On the one hand, it is not difficult to argue that all theorems of classical logic, as defined above, are logical propositions in the sense of logicism. On the other hand, there is no *a priori* reason to believe that there could not be logical propositions which lie outside of classical logic. This is why we said that the logicists' definition of logic is more extensive than the definition of classical logic. And now the logicists' task becomes clearer: It consists in showing that all nine axioms of ZF are logical propositions in the sense of logicism.

The only way to assess the success or failure of logicism in carrying out this task is by going through all nine axioms of ZF and determining for each of them whether it falls under the logicists' concept of a logical proposition. This would take a separate article and would be of interest only to readers who are thoroughly familiar with ZF. Hence, instead, we simply state that at least two of these axioms, namely, the axiom of infinity and the axiom of choice, cannot possibly be considered as logical propositions. For example, the axiom of infinity says that there exist infinite sets. Why do we accept this axiom as being true? The reason is that everyone is familiar with so many infinite sets, say, the set of the natural numbers or the set of points in Euclidean 3-space. Hence, we accept this axiom on grounds of our everyday experience with sets, and this clearly shows that we accept it in virtue of its content and not in virtue of its syntactical form. In general, when an axiom claims the existence of objects with which we are familiar on grounds of our common everyday experience, it is pretty certain that this axiom is not a logical proposition in the sense of logicism.
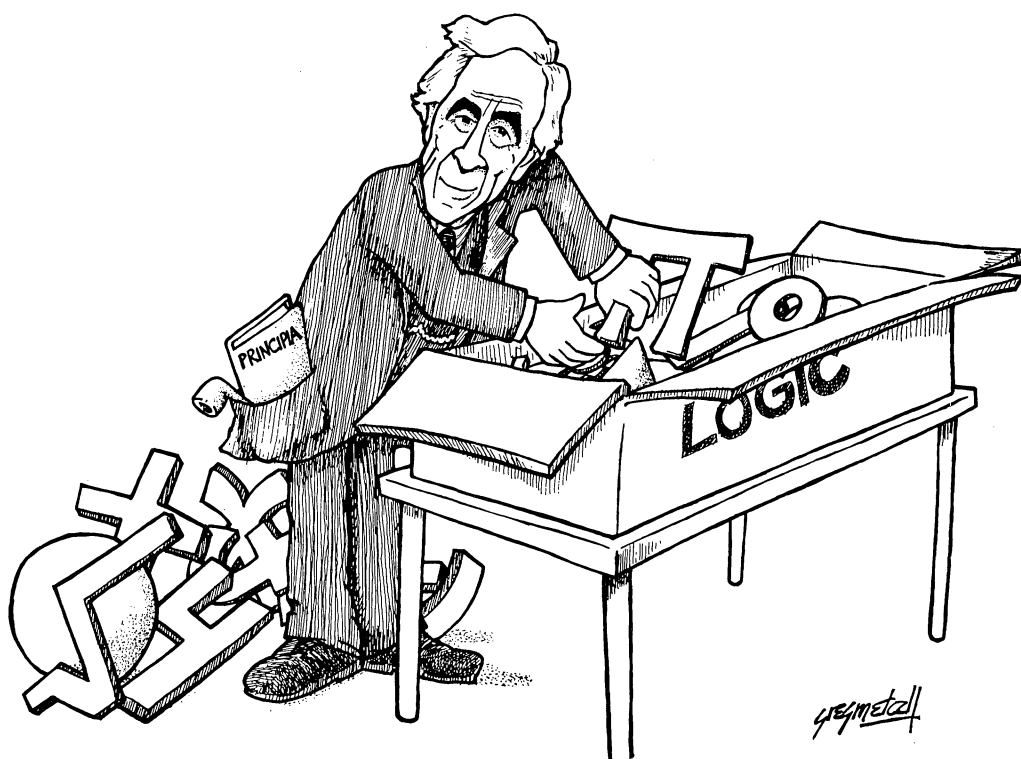
And here then is the first crisis in mathematics: Since at least two out of the nine axioms of ZF are not logical propositions in the sense of logicism, it is fair to say that this school failed by about 20% in its effort to give mathematics a firm foundation. However, logicism has been of the greatest importance for the development of modern mathematical logic. In fact, it was logicism which started mathematical logic in a serious way. The two quantifiers, the "for all" quantifier ∀ and the "there exists" quantifier ∃ were introduced into logic by Frege [5], and the influence of *Principia* on the development of mathematical logic is history.

It is important to realize that logicism is founded in philosophy. For example, when the logicists tell us what they mean by a logical proposition (above), they use philosophical and not mathematical language. They have to use philosophical language for that purpose since mathematics simply cannot handle definitions of so wide a scope.

The philosophy of logicism is sometimes said to be based on the philosophical school called "realism." In medieval philosophy "realism" stood for the Platonic doctrine that abstract entities have an existence independent of the human mind. Mathematics is, of course, full of abstract entities such as numbers, functions, sets, etc., and according to Plato all such entities exist outside our mind. The mind can discover them but does not create them. This doctrine has the advantage that one can accept such a concept as "set" without worrying about how the mind can construct a set. According to realism, sets are there for us to discover, not to be constructed, and the same holds for all other abstract entities. In short, realism allows us to accept many more abstract entities in mathematics than a philosophy which had limited us to accepting only those entities the human mind can construct. Russell was a realist and accepted the abstract entities which occur in classical mathematics without questioning whether our own minds can construct them. This is the fundamental difference between logicism and intuitionism, since in intuitionism abstract entities are admitted only if they are man made.

Excellent expositions of logicism can be found in Russell's writing, for example [9], [10] and [11].

Intuitionism

This school was begun about 1908 by the Dutch mathematician, L. E. J. Brouwer (1881–1966). The intuitionists went about the foundations of mathematics in a radically different way from the logicists. The logicists never thought that there was anything wrong with classical mathematics; they simply wanted to show that classical mathematics is part of logic. The intuitionists, on the contrary, felt that there was plenty wrong with classical mathematics.

By 1908, several paradoxes had arisen in Cantor's set theory. Here, the word "paradox" is used as synonymous with "contradiction." Georg Cantor created set theory, starting around 1870, and he did his work "naively," meaning nonaxiomatically. Consequently, he formed sets with such abandon that he himself, Russell and others found several paradoxes within his theory. The logicists considered these paradoxes as common errors, caused by erring mathematicians and not by a faulty mathematics. The intuitionists, on the other hand, considered these paradoxes as clear indications that classical mathematics itself is far from perfect. They felt that mathematics had to be rebuilt from the bottom on up.

The "bottom," that is, the beginning of mathematics for the intuitionists, is their explanation of what the natural numbers $1, 2, 3, \ldots$ are. (Observe that we do not include the number zero among the natural numbers.) According to intuitionistic philosophy, all human beings have a primordial intuition for the natural numbers within them. This means in the first place that we have an immediate certainty as to what is meant by the number 1 and, secondly, that the mental process which goes into the formation of the number 1 can be repeated. When we do repeat it, we obtain the concept of the number 2; when we repeat it again, the concept of the number 3; in this way, human beings can construct any *finite* initial segment $1, 2, \ldots, n$ for any natural number n. This mental construction of one natural number after the other would never have been possible if we did not have an awareness of time within us. "After" refers to time and Brouwer agrees with the philosopher Immanuel Kant (1724–1804) that human beings have an immediate awareness of time. Kant used the word "intuition" for "immediate awareness" and this is where the name "intuitionism" comes from. (See Chapter IV of [4] for more information about this intuitionistic concept of natural numbers.)

It is important to observe that the intuitionistic construction of natural numbers allows one to construct only arbitrarily long *finite* initial segments $1, 2, \ldots, n$. It does not allow us to construct that whole closed set of all the natural numbers which is so familiar from classical mathematics. It is equally important to observe that this construction is both "inductive" and "effective." It is inductive in the sense that, if one wants to construct, say, the number 3, one has to go through all the mental steps of first constructing the 1, then the 2, and finally the 3; one cannot just grab the number 3 out of the sky. It is effective in the sense that, once the construction of a natural number has been finished, that natural number has been constructed in its entirety. It stands before us as a completely finished mental construct, ready for our study of it. When someone says, "I have finished the mental construction of the number 3," it is like a bricklayer saying, "I have finished that wall," which he can say only after he has laid every stone in place.

We now turn to the intuitionistic definition of mathematics. According to intuitionistic philosophy, mathematics should be defined as a mental activity and not as a set of theorems (as was done above in the section on logicism). It is the activity which consists in carrying out, one after the other, those mental constructions which are inductive and effective in the sense in which the intuitionistic construction of the natural numbers is inductive and effective. Intuitionism maintains that human beings are able to recognize whether a given mental construction has these two properties. We shall refer to a mental construction which has these two properties as a **construct** and hence the intuitionistic definition of mathematics says: *Mathematics is the mental activity which consists in carrying out constructs one after the other.*

A major consequence of this definition is that all of intuitionistic mathematics is effective or "constructive" as one usually says. We shall use the adjective "constructive" as synonymous with "effective" from now on. Namely, every construct is constructive, and intuitionistic mathematics is nothing but carrying out constructs over and over. For instance, if a real number

r occurs in an intuitionistic proof or theorem, it never occurs there merely on grounds of an existence proof. It occurs there because it has been constructed from top to bottom. This implies for example that each decimal place in the decimal expansion of *r* can in principle be computed. In short, all intuitionistic proofs, theorems, definitions, etc., are entirely constructive.

Another major consequence of the intuitionistic definition of mathematics is that mathematics cannot be reduced to any other science such as, for instance, logic. This definition comprises too many mental processes for such a reduction. And here, then, we see a radical difference between logicism and intuitionism. In fact, the intuitionistic attitude toward logic is precisely the opposite from the logicists' attitude: According to the intuitionists, whatever valid logical processes there are, they are all constructs; hence, the valid part of classical logic is part of mathematics! Any law of classical logic which is not composed of constructs is for the intuitionist a meaningless combination of words. It was, of course, shocking that the classical law of the excluded middle turned out to be such a meaningless combination of words. This implies that this law cannot be used indiscriminately in intuitionistic mathematics; it can often be used, but not always.

Once the intuitionistic definition of mathematics has been understood and accepted, all there remains to be done is to do mathematics the intuitionistic way. Indeed, the intuitionists have developed intuitionistic arithmetic, algebra, analysis, set theory, etc. However, in each of these branches of mathematics, there occur classical theorems which are not composed of constructs and, hence, are meaningless combinations of words for the intuitionists. Consequently, one cannot say that the intuitionists have reconstructed all of classical mathematics. This does not bother the intuitionists since whatever parts of classical mathematics they cannot obtain are meaningless for them anyway. Intuitionism does not have as its purpose the justification of classical mathematics. Its purpose is to give a valid definition of mathematics and then to "wait and see" what mathematics comes out of it. Whatever classical mathematics cannot be done intuitionistically simply is not mathematics for the intuitionist. We observe here another fundamental difference between logicism and intuitionism: The logicists wanted to justify all of classical mathematics. (An excellent introduction to the actual techniques of intuitionism is [8].)

Let us now ask how successful the intuitionistic school has been in giving us a good foundation for mathematics, acceptable to the majority of mathematicians. Again, there is a sharp difference between the way this question has to be answered in the present case and in the case of logicism. Even hard-nosed logicists have to admit that their school so far has failed to give mathematics a firm foundation by about 20%. However, a hard-nosed intuitionist has every right in the world to claim that intuitionism has given mathematics an entirely satisfactory foundation. There is the meaningful definition of intuitionistic mathematics, discussed above; there is the intuitionistic philosophy which tells us why constructs can never give rise to contradictions and, hence, that intuitionistic mathematics is free of contradictions. In fact, not only this problem (of freedom from contradiction) but all other problems of a foundational nature as well receive perfectly satisfactory solutions in intuitionism.

Yet if one looks at intuitionism from the outside, namely, from the viewpoint of the classical mathematician, one has to say that intuitionism has failed to give mathematics an adequate foundation. In fact, the mathematical community has almost universally rejected intuitionism. Why has the mathematical community done this, in spite of the many very attractive features of intuitionism, some of which have just been mentioned?

One reason is that classical mathematicians flatly refuse to do away with the many beautiful theorems that are meaningless combinations of words for the intuitionists. An example is the Brouwer fixed point theorem of topology which the intuitionists reject because the fixed point cannot be constructed, but can only be shown to exist on grounds of an existence proof. This, by the way, is the same Brouwer who created intuitionism; he is equally famous for his work in (nonintuitionistic) topology.

A second reason comes from theorems which can be proven both classically and intuitionistically. It often happens that the classical proof of such a theorem is short, elegant, and devilishly

clever, but not constructive. The intuitionists will of course reject such a proof and replace it by their own constructive proof of the same theorem. However, this constructive proof frequently turns out to be about ten times as long as the classical proof and often seems, at least to the classical mathematician, to have lost all of its elegance. An example is the fundamental theorem of algebra which in classical mathematics is proved in about half a page, but takes about ten pages of proof in intuitionistic mathematics. Again, classical mathematicians refuse to believe that their clever proofs are meaningless whenever such proofs are not constructive.

Finally, there are the theorems which hold in intuitionism but are false in classical mathematics. An example is the intuitionistic theorem which says that every real-valued function which is defined for *all* real numbers is continuous. This theorem is not as strange as it sounds since it depends on the intuitionistic concept of a function: A real-valued function f is defined in intuitionism for all real numbers only if, for every real number r whose intuitionistic construction has been completed, the real number $f(r)$ can be constructed. Any obviously discontinuous function a classical mathematician may mention does not satisfy this constructive criterion. Even so, theorems such as this one seem so far out to classical mathematicians that they reject any mathematics which accepts them.

These three reasons for the rejection of intuitionism by classical mathematicians are neither rational nor scientific. Nor are they pragmatic reasons, based on a conviction that classical mathematics is better for applications to physics or other sciences than is intuitionism. They are all emotional reasons, grounded in a deep sense as to what mathematics is all about. (If one of the readers knows of a truly scientific rejection of intuitionism, the author would be grateful to hear about it.) We now have the second crisis in mathematics in front of us: It consists in the failure of the intuitionistic school to make intuitionism acceptable to at least the majority of mathematicians.

It is important to realize that, like logicism, intuitionism is rooted in philosophy. When, for instance, the intuitionists state their definition of mathematics, given earlier, they use strictly philosophical and not mathematical language. It would, in fact, be quite impossible for them to use mathematics for such a definition. The mental activity which is mathematics can be defined in philosophical terms but this definition must, by necessity, use some terms which do not belong to the activity it is trying to define.

Just as logicism is related to realism, intuitionism is related to the philosophy called "conceptualism." This is the philosophy which maintains that abstract entities exist only insofar as they are constructed by the human mind. This is very much the attitude of intuitionism which holds that the abstract entities which occur in mathematics, whether sequences or order-relations or what have you, are all mental constructions. This is precisely why one does not find in intuitionism the staggering collection of abstract entities which occur in classical mathematics and hence in logicism. The contrast between logicism and intuitionism is very similar to the contrast between realism and conceptualism.

A very good way to get into intuitionism is by studying [8], Chapter IV of [4], [2] and [13], in this order.

Formalism

This school was created in about 1910 by the German mathematician David Hilbert (1862–1943). True, one might say that there were already formalists in the nineteenth century since Frege argued against them in the second volume of his *Grundgesetze der Arithmetik* (see the book by Geach and Black under [5], pages 182–233); the first volume of the *Grundgesetze* appeared in 1893 and the second one in 1903. Nevertheless, the modern concept of formalism, which includes finitary reasoning, must be credited to Hilbert. Since modern books and courses in mathematical logic usually deal with formalism, this school is much better known today than either logicism or intuitionism. We will hence discuss only the highlights of formalism and begin by asking, "What is it that we formalize when we formalize something?"
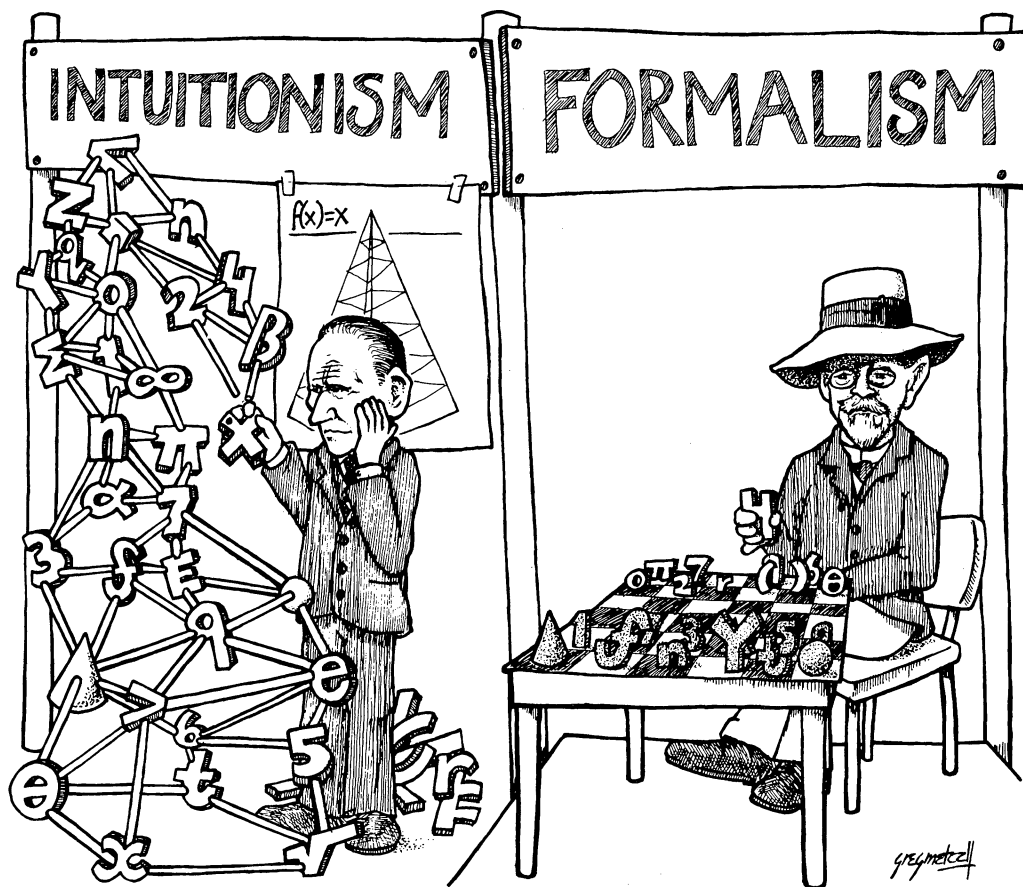
The answer is that we formalize some given *axiomatized* theory. One should guard against confusing axiomatization and formalization. Euclid axiomatized geometry in about 300 B.C., but formalization started only about 2200 years later with the logicists and formalists. Examples of axiomatized theories are Euclidean plane geometry with the usual Euclidean axioms, arithmetic with the Peano axioms, ZF with its nine axioms, etc. The next question is: "How do we formalize a given axiomatized theory?"

Suppose then that some axiomatized theory T is given. Restricting ourselves to first order logic, "to formalize T" means to choose an appropriate first order language for T. The vocabulary of a first order language consists of five items, four of which are always the same and are not dependent on the given theory T. These four items are the following: (1) A list of denumerably many variables—who can talk about mathematics without using variables? (2) Symbols for the connectives of everyday speech, say $\neg$ for "not," $\wedge$ for "and," $\vee$ for the inclusive "or," $\rightarrow$ for "if then," and $\leftrightarrow$ for "if and only if"—who can talk about anything at all without using connectives? (3) The equality sign $=$; again, no one can talk about mathematics without using this sign. (4) The two quantifiers, the "for all" quantifier $\forall$ and the "there exist" quantifier $\exists$; the first one is used to say such things as "*all* complex numbers have a square root," the second one to say things like "*there exist* irrational numbers." One can do without some of the above symbols, but there is no reason to go into that. Instead, we turn to the fifth item.

Since T is an axiomatized theory, it has so called "undefined terms." One has to choose an appropriate symbol for every undefined term of T and these symbols make up the fifth item. For

instance, among the undefined terms of plane Euclidean geometry, occur "point," "line," and "incidence," and for each one of them an appropriate symbol must be entered into the vocabulary of the first order language. Among the undefined terms of arithmetic occur "zero," "addition," and "multiplication," and the symbols one chooses for them are of course 0, +, and ×, respectively. The easiest theory of all to formalize is ZF since this theory has only one undefined term, namely, the membership relation. One chooses, of course, the usual symbol ∈ for that relation. These symbols, one for each undefined term of the axiomatized theory T, are often called the "parameters" of the first order language and hence the parameters make up the fifth item.

Since the parameters are the only symbols in the vocabulary of a first order language which depend on the given axiomatized theory T, one formalizes T simply by choosing these parameters. Once this choice has been made, the whole theory T has been completely formalized. One can now express in the resulting first order language L not only all axioms, definitions, and theorems of T, but more! One can also express in L all axioms of classical logic and, consequently, also all proofs one uses to prove theorems of T. In short, one can now proceed entirely within L, that is, entirely "formally."

But now a third question presents itself: "Why in the world would anyone want to formalize a given axiomatized theory?" After all, Euclid never saw a need to formalize his axiomatized geometry. It is important to ask this question, since even the great Peano had mistaken ideas about the real purpose of formalization. He published one of his most important discoveries in differential equations in a formalized language (very similar to a first order language) with the result that nobody read it until some charitable soul translated the article into common German.

Let us now try to answer the third question. If mathematicians do technical research in a certain branch of mathematics, say, plane Euclidean geometry, they are interested in discovering and proving the important theorems of the branch of mathematics. For that kind of technical work, formalization is usually not only no help but a definite hindrance. If, however, one asks such foundational questions as, for instance, "Why is this branch of mathematics free of contradictions?", then formalization is not just a help but an absolute necessity.

It was really Hilbert's stroke of genius to understand that formalization is the proper technique to tackle such foundational questions. What he taught us can be put roughly as follows. Suppose that T is an axiomatized theory which has been formalized in terms of the first order language L. This language has such a precise syntax that it itself can be studied as a *mathematical* object. One can ask for instance: "Can one possibly run into contradictions if one proceeds entirely formally within L, using only the axioms of T and those of classical logic, all of which have been expressed in L?" If one can prove mathematically that the answer to this question is "no," one has there a mathematical proof that the theory T is free of contradictions!

This is basically what the famous "Hilbert program" was all about. The idea was to formalize the various branches of mathematics and then to prove *mathematically* that each one of them is free of contradictions. In fact if, by means of this technique, the formalists could have just shown that ZF is free of contradictions, they would thereby already have shown that all of classical mathematics is free of contradictions, since classical mathematics can be done axiomatically in terms of the nine axioms of ZF. In short, the formalists tried to create a mathematical technique by means of which one could prove that mathematics is free of contradictions. This was the original purpose of formalism.

It is interesting to observe that both logicists and formalists formalized the various branches of mathematics, but for entirely different reasons. The logicists wanted to use such a formalization to show that the branch of mathematics in question belongs to logic; the formalists wanted to use it to prove mathematically that that branch is free of contradictions. Since both schools "formalized," they are sometimes confused.

Did the formalists complete their program successfully? No! In 1931, Kurt Gödel showed in [6] that formalization cannot be considered as a mathematical technique by means of which one

can prove that mathematics is free of contradictions. The theorem in that paper which rang the death bell for the Hilbert program concerns axiomatized theories which are free of contradictions and whose axioms are strong enough so that arithmetic can be done in terms of them. Examples of theories whose axioms are that strong are, of course, Peano arithmetic and ZF. Suppose now that T is such a theory and that T has been formalized by means of the first order language L. Then Gödel's theorem says, in nontechnical language, "No sentence of L which can be interpreted as asserting that T is free of contradictions can be proven formally within the language L." Although the interpretation of this theorem is somewhat controversial, most mathematicians have concluded from it that the Hilbert program cannot be carried out: Mathematics is not able to prove its own freedom of contradictions. Here, then, is the third crisis in mathematics.

Of course, the tremendous importance of the formalist school for present-day mathematics is well known. It was in this school that modern mathematical logic and its various offshoots, such as model theory, recursive function theory, etc., really came into bloom.

Formalism, as logicism and intuitionism, is founded in philosophy, but the philosophical roots of formalism are somewhat more hidden than those of the other two schools. One can find them, though, by reflecting a little on the Hilbert program.

Let again T be an axiomatized theory which has been formalized in terms of the first order language L. In carrying out Hilbert's program, one has to talk about the language L as one object, and while doing this, one is not talking within that safe language L itself. On the contrary, one is talking about L in ordinary, everyday language, be it English or French or what have you. While using our natural language and not the formal language L, there is of course every danger that contradictions, in fact, any kind of error, may slip in. Hilbert said that the way to avoid this danger is by making absolutely certain that, while one is talking in one's natural language about L, one uses only reasonings which are absolutely safe and beyond any kind of suspicion. He called such reasonings "finitary reasonings," but had, of course, to give a definition of them. The most explicit definition of finitary reasoning known to the author was given by the French formalist Herbrand ([7], the footnote on page 622). It says, if we replace "intuitionistic" by "finitary":

> By a finitary argument we understand an argument satisfying the following conditions: In it we never consider anything but a given finite number of objects and of functions; these functions are well defined, their definition allowing the computation of their values in a univocal way; we never state that an object exists without giving the means of constructing it; we never consider the totality of all the objects x of an infinite collection; and when we say that an argument (or a theorem) is true for all these x, we mean that, for each x taken by itself, it is possible to repeat the general argument in question, which should be considered to be merely the prototype of these particular arguments.

Observe that this definition uses philosophical and not mathematical language. Even so, no one can claim to understand the Hilbert program without an understanding of what finitary reasoning amounts to. The philosophical roots of formalism come out into the open when the formalists define what they mean by finitary reasoning.

We have already compared logicism with realism, and intuitionism with conceptualism. The philosophy which is closest to formalism is "nominalism." This is the philosophy which claims that abstract entities have no existence of any kind, neither outside the human mind as maintained by realism, nor as mental constructions within the human mind as maintained by conceptualism. For nominalism, abstract entities are mere vocal utterances or written lines, mere names. This is where the word "nominalism" comes from, since in Latin *nominalis* means "belonging to a name." Similarly, when formalists try to prove that a certain axiomatized theory T is free of contradictions, they do not study the abstract entities which occur in T but, instead, study that first order language L which was used to formalize T. That is, they study how one can form sentences in L by the proper use of the vocabulary of L; how certain of these sentences can be proven by the proper use of those special sentences of L which were singled out as

axioms; and, in particular, they try to show that no sentence of L can be proven and disproven at the same time, since they would thereby have established that the original theory T is free of contradictions. The important point is that this whole study of L is a strictly syntactical study, since no meanings or abstract entities are associated with the sentences of L. This language is investigated by considering the sentences of L as meaningless expressions which are manipulated according to explicit, syntactical rules, just as the pieces of a chess game are meaningless figures which are pushed around according to the rules of the game. For the strict formalist "to do mathematics" is "to manipulate the meaningless symbols of a first order language according to explicit, syntactical rules." Hence, the strict formalist does not work with abstract entities, such as infinite series or cardinals, but only with their meaningless names which are the appropriate expressions in a first order language. Both formalists and nominalists avoid the direct use of abstract entities, and this is why formalism should be compared with nominalism.

The fact that logicism, intuitionism, and formalism correspond to realism, conceptualism, and nominalism, respectively, was brought to light in Quine's article, "On What There Is" ([1], pages 183–196). Formalism can be learned from any modern book in mathematical logic, for instance [3].

Epilogue

Where do the three crises in mathematics leave us? They leave us without a firm foundation for mathematics. After Gödel's paper [6] appeared in 1931, mathematicians on the whole threw up their hands in frustration and turned away from the philosophy of mathematics. Nevertheless, the influence of the three schools discussed in this article has remained strong, since they have given us much new and beautiful mathematics. This mathematics concerns mainly set theory, intuitionism and its various constructivist modifications, and mathematical logic with its many offshoots. However, although this kind of mathematics is often referred to as "foundations of mathematics," one cannot claim to be advancing the philosophy of mathematics just because one is working in one of these areas. Modern mathematical logic, set theory, and intuitionism with its modifications are nowadays technical branches of mathematics, just as algebra or analysis, and unless we return directly to the philosophy of mathematics, we cannot expect to find a firm foundation for our science. It is evident that such a foundation is not necessary for technical mathematical research, but there are still those among us who yearn for it. The author believes that the key to the foundations of mathematics lies hidden somewhere among the philosophical roots of logicism, intuitionism, and formalism and this is why he has uncovered these roots, three times over.

Excellent literature on the foundations of mathematics is contained in [1] and [7].

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