The Two-Body Problem

In the previous lecture, we discussed a variety of conclusions we could make about the motion of an arbitrary collection of particles, subject only to a few restrictions. Today, we will consider a much simpler, very well-known problem in physics - an isolated system of two particles which interact through a central potential. This model is often referred to simply as the **two-body problem**. In the case of only two particles, our equations of motion reduce simply to

$$m_1 \ddot{\mathbf{r}}_1 = \mathbf{F}_{21} \; ; \; m_2 \ddot{\mathbf{r}}_2 = \mathbf{F}_{12} \tag{1}$$

A famous example of such a system is of course given by Newton's Law of Gravitation, where the two particles interact through a potential energy given by

$$U_{12}\left(|\mathbf{r}_1 - \mathbf{r}_2|\right) = U_{21}\left(|\mathbf{r}_2 - \mathbf{r}_1|\right) = G\frac{m_1 m_2}{|\mathbf{r}_1 - \mathbf{r}_2|^2}, \tag{2}$$

where G is Newton's constant,

$$G = 6.673 \times 10^{-11} \text{ N m}^2/\text{kg}^2. \tag{3}$$

How can we go about finding the most general solution to this set of equations?

As with any physics problem, the first thing we should do is make maximal use of the symmetries or conservation laws of our problem. First, because the two particles interact via a central potential, these two forces should obey Newton's third law, as we discussed in the previous lecture. We know that as a result, the total momentum of our system will be conserved, and so we should consider the center of mass,

$$\mathbf{R} = \frac{m_1 \mathbf{r}_1 + m_2 \mathbf{r}_2}{m_1 + m_2} = \frac{m_1 \mathbf{r}_1 + m_2 \mathbf{r}_2}{M} \tag{4}$$

the time derivative of which is given by the center of mass velocity,

$$\mathbf{v}_{CM} = \frac{m_1 \mathbf{v}_1 + m_2 \mathbf{v}_2}{M}. \tag{5}$$

Now, in the previous lecture, we found that the acceleration of the center of mass depended on the net external force,

$$\mathbf{F}_{\text{ext}} = M \mathbf{a}_{CM}. \tag{6}$$

Since our system is isolated, the center of mass acceleration must be zero, and hence the center of mass velocity must be a constant,

$$\mathbf{v}_{CM}^{(0)} = \frac{m_1 \mathbf{v}_1^{(0)} + m_2 \mathbf{v}_2^{(0)}}{M}. \tag{7}$$

Thus, the center of mass motion is given by

$$\mathbf{R}\left(t\right) = \mathbf{v}_{CM}^{(0)} t. \tag{8}$$

Since we already know the motion of the center of mass on general grounds, we can make use of this information to simplify our problem. To see how, let's define the vector of relative distance

$$\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2. \tag{9}$$

A simple algebraic rearrangement then yields

$$\mathbf{r}_1 = \mathbf{R} + \frac{m_2}{m_1 + m_2}\mathbf{r} \; ; \; \mathbf{r}_2 = \mathbf{R} - \frac{m_1}{m_1 + m_2}\mathbf{r}. \tag{10}$$

From this expression, and the fact that we already know $\mathbf{R}$, our problem simply reduces to finding $\mathbf{r}$.

To find the equation of motion satisfied by $\mathbf{r}$, we return to our original equations of motion and multiply the first by m_2 and the second by m_1, in order to find

$$m_1 m_2 \ddot{\mathbf{r}}_1 = m_2 \mathbf{F}_{21} \; ; \; m_1 m_2 \ddot{\mathbf{r}}_2 = m_1 \mathbf{F}_{12}. \tag{11}$$

If we then subtract the second equation from the first, we have

$$m_1 m_2 \left(\ddot{\mathbf{r}}_1 - \ddot{\mathbf{r}}_2 \right) = m_2 \mathbf{F}_{21} - m_1 \mathbf{F}_{12} \; \Rightarrow \; \frac{m_1 m_2}{(m_2 + m_1)}\ddot{\mathbf{r}} = \mathbf{F}_{21}, \tag{12}$$

where we have made use of Newton's third law in the second equation. Because the forces are derived from a central potential which only depends on the distance between the two particles, we have

$$\mathbf{F}_{21} = -\frac{\partial}{\partial \mathbf{r}_1} U_{12} \left(|\mathbf{r}_1 - \mathbf{r}_2| \right) = \nabla_1 U_{12} \left(|\mathbf{r}_1 - \mathbf{r}_2| \right) \tag{13}$$

Now, since the potential only depends on $\mathbf{r}$, and **not** the center of mass $\mathbf{R}$, we can use the chain rule to write for the x-component of the derivative, for example,

$$\frac{\partial}{\partial r_{1x}} U_{12} \left(|\mathbf{r}| \right) = \frac{\partial r_x}{\partial r_{1x}}\frac{\partial}{\partial r_x} U_{12} \left(|\mathbf{r}| \right) + \frac{\partial R_x}{\partial r_{1x}}\frac{\partial}{\partial R_x} U_{12} \left(|\mathbf{r}| \right) = \frac{\partial}{\partial r_x} U_{12} \left(|\mathbf{r}| \right), \tag{14}$$

and so on for the other coordinates. Thus, I find that I can write

$$m_* \ddot{\mathbf{r}} = -\frac{\partial}{\partial \mathbf{r}} U \left(|\mathbf{r}| \right) \equiv \mathbf{F} \left(\mathbf{r} \right), \tag{15}$$

where

$$m_* = \frac{m_1 m_2}{(m_2 + m_1)} \tag{16}$$

is the **reduced mass** of the system. Thus, our problem has effectively been reduced to a one-particle system - mathematically, it is no different than a single particle with position vector $\mathbf{r}$ and mass m_*, subject to an external force $\mathbf{F}$. Therefore, conservation of momentum has dramatically simplified our system.

Conservation of Angular Momentum

Since our two particles interact with each other through a central potential, we know that the total angular momentum of the system is conserved. However, since we have reduced our problem to a one-particle system, it makes more sense to reformulate this statement in terms of the angular momentum of this fictitious particle,

$$\mathbf{L} = m_* \mathbf{r} \times \mathbf{v}, \tag{17}$$

where

$$\mathbf{v} = \dot{\mathbf{r}}. \tag{18}$$

Now, a short exercise in the chain rule shows us that

$$\mathbf{F}(\mathbf{r}) = -\frac{\partial}{\partial \mathbf{r}} U(|\mathbf{r}|) = -\frac{\mathbf{r}}{r} \frac{dU(r)}{dr} \; ; \; r = |\mathbf{r}|. \tag{19}$$

Therefore, the torque on the particle due to $\mathbf{F}$ is

$$\tau = \mathbf{r} \times \mathbf{F} \propto \mathbf{r} \times \mathbf{r} = 0. \tag{20}$$

That is, the torque vanishes because the force is parallel to the displacement vector. Thus, in the absence of any torque, the angular momentum of the particle must be constant,

$$\frac{d}{dt} \mathbf{L} = 0. \tag{21}$$

This fact is a general result for the motion of a particle in an external central potential.

For our one-particle system, conservation of angular momentum allows us to make a further simplification. For any three vectors, we can form the **scalar triple product**,

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a}) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b}). \tag{22}$$

The fact that all three of these expressions are equal is left as an exercise on your homework. If we use this identity, we can see that

$$\mathbf{r} \cdot \mathbf{L} = m_* \mathbf{r} \cdot (\mathbf{r} \times \mathbf{v}) = m_* \mathbf{v} \cdot (\mathbf{r} \times \mathbf{r}) = 0. \tag{23}$$

Because this inner product is zero, it must be the case that $\mathbf{r}$ is always *perpendicular* to the angular momentum $\mathbf{L}$,

$$\mathbf{r} \perp \mathbf{L} \tag{24}$$

However, because the angular momentum is constant, *there must be a fixed vector in space which the position vector* $\mathbf{r}$ *is always perpendicular to.* Given that the position vector is always perpendicular to a certain orientation in space, it must be the case that the position vector *always lies in a plane.*

As a result of this fact, not only has our problem been reduced to a one-particle system, it has also been effectively reduced to two dimensions. Because

our problem is described by a radial force in two dimensions, at this point it is most convenient to switch over to polar coordinates,

$$r_x = r \cos \theta \; ; \; r_y = r \sin \theta. \tag{25}$$

We have chosen the convention that the plane which the particle travels in is the x-y plane, and that the angular momentum is oriented along the z-axis. In this set of coordinates, we can write

$$\frac{d\theta}{dt} = l/m_* r^2 \; ; \; l \equiv |\mathbf{L}| \, , \tag{26}$$

which you'll show on your homework. This expression for the time derivative of the angular coordinate makes another fact clear - the sign of $d\theta/dt$ is always positive, so that the particle always rotates around the center of our coordinate system in the same direction.

Conservation of Energy

There is one last conservation law we of course have at our disposal, which is the conservation of energy. Since our particle's motion is described in terms of a potential energy function, we know that the quantity

$$E = \frac{1}{2} m_* v^2 + U(r) \tag{27}$$

should be conserved. Now, as we discussion in section this week, we remember that in polar coordinates, we can write

$$v^2 = \dot{x}^2 + \dot{y}^2 = \dot{r}^2 + r^2 \dot{\theta}^2. \tag{28}$$

Therefore, our energy conservation constraint becomes, in polar coordinates,

$$E = \frac{1}{2} m_* \dot{r}^2 + \frac{1}{2} m_* r^2 \dot{\theta}^2 + U(r) \, . \tag{29}$$

However, we can eliminate $d\theta/dt$ from this expression, using the result we found above. Substituting in this result, we find

$$E = \frac{1}{2} m_* \dot{r}^2 + \frac{1}{2} m_* r^2 \left(\frac{l}{m_* r^2} \right)^2 + U(r) \, , \tag{30}$$

or,

$$E = \frac{1}{2} m_* \dot{r}^2 + \frac{1}{2} \frac{l^2}{m_* r^2} + U(r) = \frac{1}{2} m_* \dot{r}^2 + U_{\text{eff}}(r) \, , \tag{31}$$

where we have defined the **effective potential**

$$U_{\text{eff}}(r) = \frac{1}{2} \frac{l^2}{m_* r^2} + U(r) \, . \tag{32}$$

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Notice that aside from depending on the (constant) value l, the effective potential is only a function of r - there is no longer any θ dependence in the energy conservation equation. The expression above is mathematically identical to a single particle in one dimension, with a coordinate r, whose energy is the sum of its "kinetic energy"

$$K = \frac{1}{2}m_*\dot{r}^2, \tag{33}$$

and also its potential energy, described by the effective potential. For this reason, we have now effectively reduced our problem to a *one-dimensional system*. The time evolution of our system is now determined by the two equations

$$E = \frac{1}{2}m_*\dot{r}^2 + U_{\text{eff}}(r) \; ; \; \frac{d\theta}{dt} = l/m_*r^2. \tag{34}$$

The first equation can be solved using the methods we have already been developing in the course up until this point. Once we have done so, and determined the functional form of $r(t)$, we can find the angular coordinate by simply integrating

$$\theta(t) = \theta_0 + \frac{l}{m}\int_0^t \frac{dt'}{r^2(t')}. \tag{35}$$

The combination of linear momentum, angular momentum, and energy conservation in our system has led to a dramatic simplification - a system of two particles in three-dimensional space has been reduced to a problem of finding the motion of **one** particle travelling in **one** dimension. Notice that while two particles moving in three dimensions involves 12 pieces of information (the position and velocity of both particles), our current simplified problem only involves two - the position and velocity of a one-dimensional particle. Thus, we have **ten** fewer pieces of information to find. This dramatic simplification, along with the incredible ubiquity of the central potential, makes the two-body problem one of the most well-known and most important problems in all of physics.

Notice that in addition to being a nice mathematical trick, the reduction of our problem to one particle moving in an external central potential is often **physically motivated** as well. In many situations, especially in celestial mechanics, it is often the case that one of the two bodies is much, much more massive than the other. If we assume this body is m_2, then the reduced mass simply becomes

$$m_* = \frac{m_1 m_2}{(m_2 + m_1)} \approx m_1, \tag{36}$$

and the center of mass becomes

$$\mathbf{R} = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{m_1 + m_2} \approx \mathbf{r}_2 \tag{37}$$

Therefore, the difference vector

$$\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2 \approx \mathbf{r}_1 - \mathbf{R}, \tag{38}$$

simply defines the position of the smaller mass with respect to the position of the larger mass, which is the center of mass itself. Thus, to a good approximation, we can imagine that the lager mass is stationary (since the center of mass moves with a constant velocity, which we can take to be zero), and that the smaller mass is subject to an external potential due to this larger mass. So in the following discussion when we consider the motion of such a system, you may find this intuitive picture to be helpful. This idea is shown in Figure 1.

Figure 1: In the case that one of the two bodies is much more massive than the other, the location of the larger body effectively becomes the location of the center of mass, which we can take to be stationary. In this case, the smaller mass is considered to be moving in an external potential due to the larger mass.

Given that a recurring theme of this course has been the subject of approximation methods, you may be wondering why we are so concerned with one of the few exactly solvable cases which appear in physics. However, remember that whenever we perform a perturbative calculation, we also need an exactly solvable case to perturb away from. Many problems in astronomy (and physics in general) can be well-modelled by starting with the solution to some sort of two-body problem, and then perturbing away from that. This makes the two-body problem all the more important to understand.

Physical Interpretation of the Reduced System

Before moving on to solve for the motion of our system, we can say a little bit more about the physical interpretation of the reduced problem we have set up. Our simplified system is described by a position vector $\mathbf{r}$, which is constrained to lie in a plane while under the influence of the external potential $U(\mathbf{r})$. To understand the meaning of $\mathbf{r}$ somewhat better, let's write the position vectors of the two bodies, as measured with respect to the center of mass,

$$\mathbf{r}'_1 = \mathbf{r}_1 - \mathbf{R} = \frac{m_2}{m_1 + m_2}\mathbf{r} \; ; \; \mathbf{r}'_2 = \mathbf{r}_2 - \mathbf{R} = -\frac{m_1}{m_1 + m_2}\mathbf{r}. \tag{39}$$

Make sure to notice that the vectors $\mathbf{r}'_1$ and $\mathbf{r}'_2$ are the position vectors of m_1 and m_2, in a coordinate system where $\mathbf{R}$ is situated at the origin. From the above expressions, we can see that in such a coordinate system, the vector $\mathbf{r}$ points from the origin, towards the location of the first body. Likewise, the vector $-\mathbf{r}$ points from the origin, towards the location of the second body. Be careful to notice, however, that $\mathbf{r}$ does **not** have the same length as the position vectors - it only gives the correction directions. The factors of m_2/M and m_1/M scale the length of $\mathbf{r}$ in order to give the correct position vectors $\mathbf{r}'_1$ and $\mathbf{r}'_2$. These ideas are illustrated in Figure 2. In particular, when the mass of the second body is much larger than the first,

$$m_2 \gg m_1 \;\Rightarrow\; \mathbf{r}'_1 \approx \mathbf{r} \; ; \; \mathbf{r}'_2 \approx 0. \tag{40}$$

In addition to the physical interpretation of the difference vector, we can also ask about the quantity we defined previously

$$\mathbf{L} = m_* \mathbf{r} \times \mathbf{v}, \tag{41}$$

which we referred to as the angular momentum of the fictitious one-particle system with mass m_*. How does this relate to the actual angular momentum of the full two-particle system? To answer this question, let's write the expression for the total angular momentum in the center-of-mass coordinate system,

$$\mathbf{L} = m_1 \mathbf{r}'_1 \times \dot{\mathbf{r}}'_1 + m_2 \mathbf{r}'_2 \times \dot{\mathbf{r}}'_2. \tag{42}$$

Using the expressions for these position vectors in terms of the difference vector, we find

$$\mathbf{L} = \frac{m_1 m_2}{M^2}\left(m_2 \mathbf{r} \times \dot{\mathbf{r}} + m_1 \mathbf{r} \times \dot{\mathbf{r}}\right) = \frac{m_1 m_2}{M}\mathbf{r} \times \dot{\mathbf{r}}, \tag{43}$$

or

$$\mathbf{L} = m_* \mathbf{r} \times \dot{\mathbf{r}}. \tag{44}$$

Thus, the angular momentum we previously defined is nothing other than the full two-body angular momentum, evaluated in the center-of-mass frame (remember that the value of the angular momentum depends on our choice of coordinate system, although the fact that it is constant is always true in any particular frame).
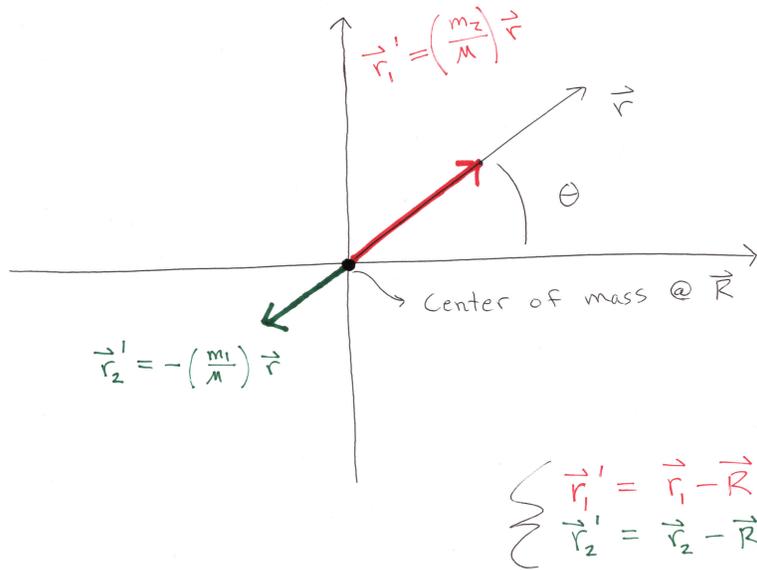
Figure 2: The two position vectors in the two-body problem, with respect to the center of mass coordinate system. Notice that the difference vector points from the origin towards the location of the first body, although it doesn't have the same length as the first body's position vector.

Lastly, we may ask how the energy of the fictitious one-particle system,

$$E = \frac{1}{2} m_* \dot{\mathbf{r}}^2 + U(\mathbf{r}), \tag{45}$$

compares with the energy of the full two-body system. To answer this question, let's write the total energy of the two-body system in terms of the center of mass quantities,

$$E_{\mathrm{CM}} = \frac{1}{2} m_1 \left(\dot{\mathbf{r}}_1' \right)^2 + \frac{1}{2} m_2 \left(\dot{\mathbf{r}}_2' \right)^2 + U(\mathbf{r}), \tag{46}$$

which we know can be written as

$$E_{\mathrm{CM}} = \frac{1}{2} m_1 \left(\frac{m_2}{M} \dot{\mathbf{r}} \right)^2 + \frac{1}{2} m_2 \left(\frac{m_1}{M} \dot{\mathbf{r}} \right)^2 + U(\mathbf{r}). \tag{47}$$

With some simple algebra, we can see that E_{CM} is precisely the quantity E that we had previously defined.

Thus, we see that all of the quantities we have introduced to describe the fictitious one-particles system with mass m_* have a natural interpretation in terms of the two-body quantities, as expressed in the center-of-mass coordinate system.

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Radial Oscillations

Now that our problem has been reduced to a one particle system moving in an external potential in one dimension, we can apply all of the same tools we used in the first half of the course. First, in the case that the effective potential admits a local minimum, we know from our previous considerations that we will find **oscillatory** behaviour for the radial coordinate r in the vicinity of this potential minimum. An example of this is shown in Figure 3 for the potential

$$U\left(r\right) = -\frac{1}{r}, \tag{48}$$

which as we will discuss shortly is of particular importance in physics. Notice that because the effective potential is explicitly dependent on the value of the angular momentum, its shape, and also the location of its minimum, will change with different values of the angular momentum. Smaller values of angular momentum will result in effective potentials with much deeper potential minima, while larger values of the angular momentum will result in very shallow effective potentials. In some sense, we can think of the increased angular momentum as creating a "centrifugal force" which is flinging the particle outwards, although keep in mind that this fictitious force is an artefact of reducing the problem to a one-dimensional system through the use of the effective potential.
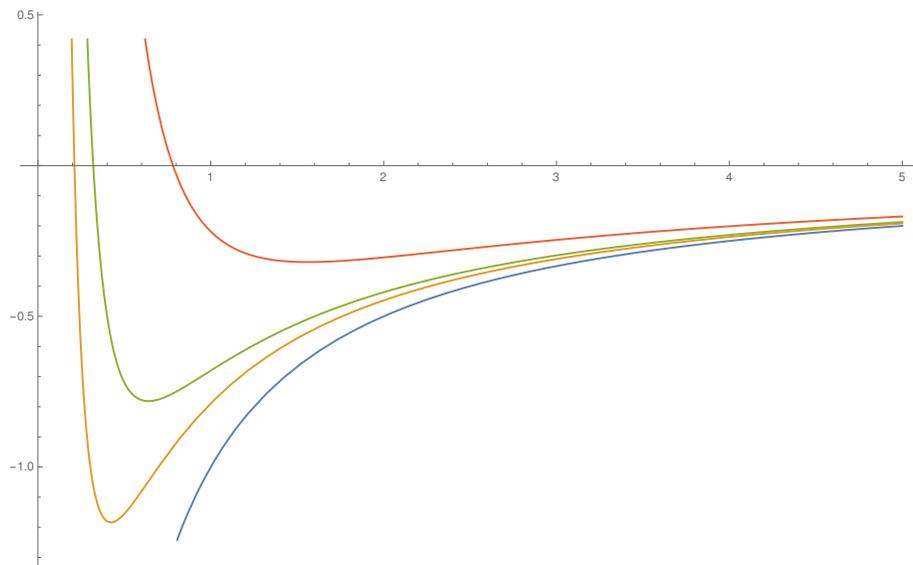

Figure 3: A collection of different effective potentials, for $m = 1$, and $U\left(r\right) = -1/r$. The various values of the angular momentum are $l = 0$ (blue curve), $l = 0.65$ (orange curve), $l = 0.8$ (green curve), and $l = 1.25$ (red curve).

Notice also that because the angular momentum part of the effective potential is always positive

$$U_l\left(r\right) = \frac{1}{2}\frac{l^2}{m_* r^2}, \tag{49}$$

the behaviour of the effective potential at small values of r will always increase to large positive values, since the inverse square behaviour of this term causes it to dominate at small values of r. For this reason, we often say that a non-zero angular momentum creates a "centrifugal barrier" which prohibits the particle from reaching the origin at $r = 0$. Only when $l = 0$ can the particle ever reach this point (of course, in a realistic system describing, for example, a satellite orbiting the Earth, it is certainly still possible for the satellite to crash into the Earth even with a non-zero angular momentum, since of course the surface of the Earth exists at a non-zero radius away from the center of mass of the system).

When the effective potential does **not** admit a minimum, or when the particle possesses a suitably large enough energy that it can escape the potential minimum, then it is possible for the particle to escape to infinity and never return. We will discuss this possibility in a moment when we discuss orbital shapes.

Assuming that the particle is indeed trapped within a potential minimum, as the particle moves, its radial coordinate will oscillate between some minimum and maximum values. At the point of closest approach to the origin, when the radial coordinate is at a minimum, we say that the particle is at its **perigee**. Similarly, when the particle is at its maximum distance from the origin, we say that it is at its **apogee**. If we want to be quantitative about this behaviour, we can say that the time it takes to travel from perigee to apogee is given by

$$T_{p\to a} = \sqrt{\frac{m_*}{2}}\int_{r_p}^{r_a}\frac{dr'}{\sqrt{E - U_{\text{eff}}\left(r'\right)}}, \tag{50}$$

with the positive sign chosen, since we are moving from a smaller radius to a larger one. In the case that the particle is sitting precisely at the minimum of the effective potential,

$$U'_{\text{eff}}\left(r_*\right) = 0, \tag{51}$$

the radial coordinate will be constant for all time. This of course corresponds to a **circular orbit**. Thus, we see that so long as the effective potential admits a minimum, there will exist a circular orbit.

Thanks to our heavy use of conservation laws, finding the radial motion of our particle has essentially been reduced to the previous two-page recap of our previous discussions in the course. Of course, understanding how the interplay between the angular momentum and bare potential determines the effective potential is an interesting question, and is precisely the sort of question you will explore more on the homework. But for now, let's move on to the angular motion.

Orbits

Returning to the angular motion of our particle, the equation we derived previously allows us to write

$$\frac{d\theta}{dt} = l/m_* r^2 \ \Rightarrow \ \theta(t) = \theta_0 + \frac{l}{m_*} \int_0^t \frac{dt'}{r^2(t')}. \tag{52}$$

After solving for the motion of the radial coordinate, we can always perform the integral in this equation, thus finding the angular motion of the particle. However, we can actually say something interesting about the motion of the particle without even solving for the motion as a function of time. Again using energy conservation, we can write write

$$dt = \pm \sqrt{\frac{m_*}{2}} \frac{dr}{\sqrt{E - U_{\text{eff}}(r)}}. \tag{53}$$

In the previous section we use this fact to find the travel time between any two radial points. However, we can also use this expression for dt to perform a change of variables in the integral for the angle. Making this substitution, we find

$$\theta(r) = \theta_0 \pm \frac{l}{\sqrt{2m_*}} \int_{r_0}^r \frac{dr'}{r'^2 \sqrt{E - U_{\text{eff}}(r')}}. \tag{54}$$

This integral equation we have found now relates the radius to the angle, and thus allows us to find the shape, or **orbit** of the particle - how the radius changes as the angle precesses. In many situations, determining the orbit is much more important than determining the actual time evolution of the radius and angle coordinates themselves.

In particular, for a particle bound in a potential minimum, this expression allows us to find the change in angle as the particle travels from perigee to apogee, given by

$$\theta_{p \to a} = \frac{l}{\sqrt{2m_*}} \int_{r_p}^{r_a} \frac{dr'}{r'^2 \sqrt{E - U_{\text{eff}}(r')}}. \tag{55}$$

This will in fact be equal to the angle traced out as the particle travels back from apogee to perigee,

$$\theta_{a \to p} = \frac{-l}{\sqrt{2m_*}} \int_{r_a}^{r_p} \frac{dr'}{r'^2 \sqrt{E - U_{\text{eff}}(r')}} = \frac{+l}{\sqrt{2m_*}} \int_{r_p}^{r_a} \frac{dr'}{r'^2 \sqrt{E - U_{\text{eff}}(r')}} = \theta_{p \to a}. \tag{56}$$

Again, this equality comes from choosing the opposite sign out in front of the integral, and then removing the sign by swapping back the order of integration.

In general, this angular difference could be some arbitrary number, which depends on the energy, the detailed shape of the potential, and the particle's mass and angular momentum. Thus, it could very well be an **irrational** multiple of π. In this case, as the particle travels from perigee to apogee, and back to

perigee, its angular coordinate will advance by some irrational multiple of π, and then repeat the process over again. Each iteration of this process will result in a new perigee (radial point of closest approach) at a different angle than the starting angle, and since the angular difference is not a rational multiple of π, there will never be another perigee at the original starting angle. A demonstration of this type of behaviour is shown in Figure 4, for the choice of potential

$$U\left(r\right) = -\frac{e^{-r}}{r}, \tag{57}$$

This type of potential is known as a **Yukawa** potential, and appears in many areas of physics (you'll explore some of its properties in the homework). Each color shows the motion of the particle during one radial cycle as it moves from perigee to apogee, and back to perigee again. Notice that after the particle completes its first radial cycle, it has advanced by an angle which is **more** than 2π. Thus, the point it returns to in two-dimensional space after completing one radial cycle is **not** the same as its starting point. This type of behaviour is known as orbital **precession**. For comparison, Figure 5 shows the radial coordinate as a function of time, while Figure 6 shows the angular coordinate as a function of time. A plot of the effective potential is shown in Figure 7.

The Kepler Problem

However, there is a very important result which occurs for two special choices of potential. The first choice is

$$U\left(r\right) = -\frac{\gamma}{r} \; ; \; \gamma > 0, \tag{58}$$

while the second choice is

$$U\left(r\right) = \gamma r^2 \; ; \; \gamma > 0, \tag{59}$$

where in both cases, γ is some positive constant. For the first choice, it turns out that we always have

$$\theta_{p \to a} = \pi, \tag{60}$$

while for the second choice, we always have

$$\theta_{p \to a} = \pi/2. \tag{61}$$

In these two special cases, we say that the orbit of the particle is always **closed**. In the first case, as the particle travels from perigee to apogee, and then back to perigee, a full 2π radians have been traced out by the angular coordinate. As such, once the particle returns to its initial starting radius, it also returns to its initial starting angle. Therefore, after one radial cycle, the particle returns to its original starting point in two-dimensional space. The radial oscillations then continue, and the particle continues to trace out one closed orbit, with
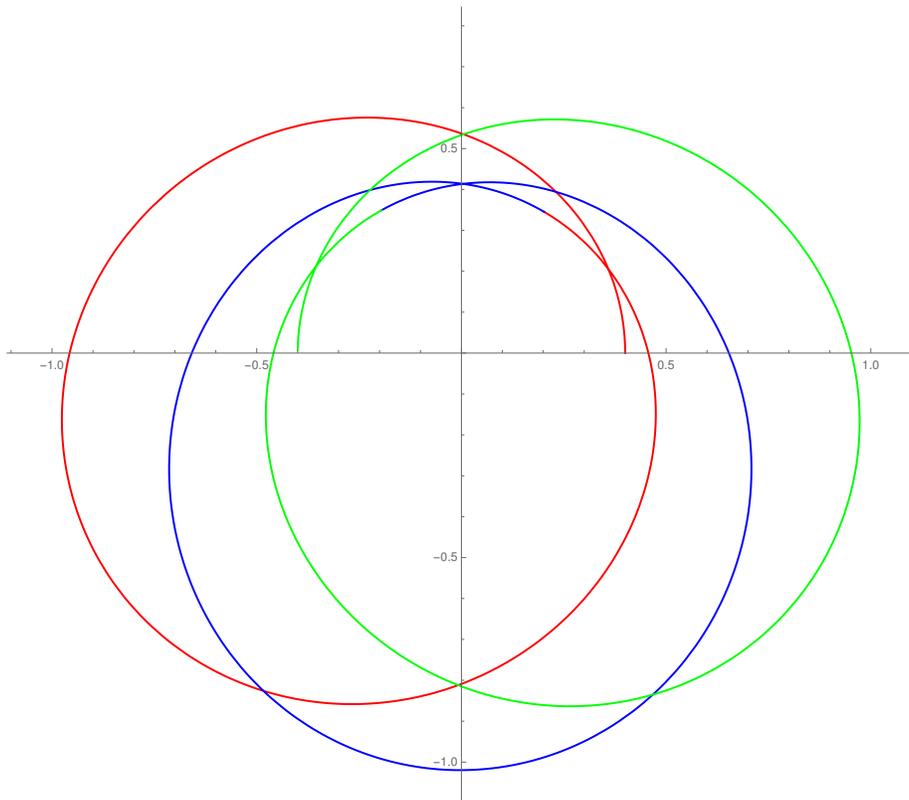
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Figure 4: The motion of a particle subject to a Yukawa potential, with $m_* = 1$ and $l^2 = 0.5$. The initial starting radius is chosen to be $r = 0.4$. The red curve shows the first radial cycle, while the blue and green curves show the second and third cycles, respectively. Notice that the points of closest approach (perigees) occur when the curve changes color.

absolutely no precession of its perigee. In the second case, after the particle has experienced one radial cycle, its angular coordinate will have advanced by π radians. After a second radial cycle, the angular coordinate will have traced out 2π radians, and the particle will have returned to its initial starting point. An example of this type of behaviour is shown in Figure 8. Notice that this result does **not** depend on the values of m_*, γ, or E.

This result regarding closed orbits is especially important, because many physical systems (including gravitational systems) are described by the $-\gamma/r$ potential. The two-body problem in the context of such a potential is often known as the **Kepler problem**, named after Johannes Kepler, whose work on the subject of astronomy helped lay the groundwork for Newtonian mechanics. For this reason, we want to spend some time understanding the orbits in such a potential in more detail. To do so, however, it will be most convenient to make
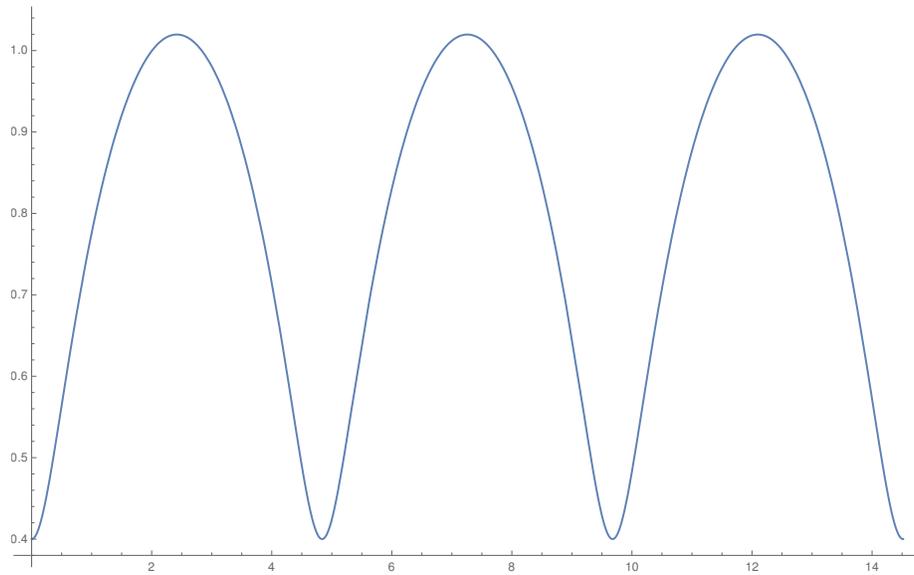
Figure 5: The radial coordinate as a function of time for the particle subject to the Yukawa potential.
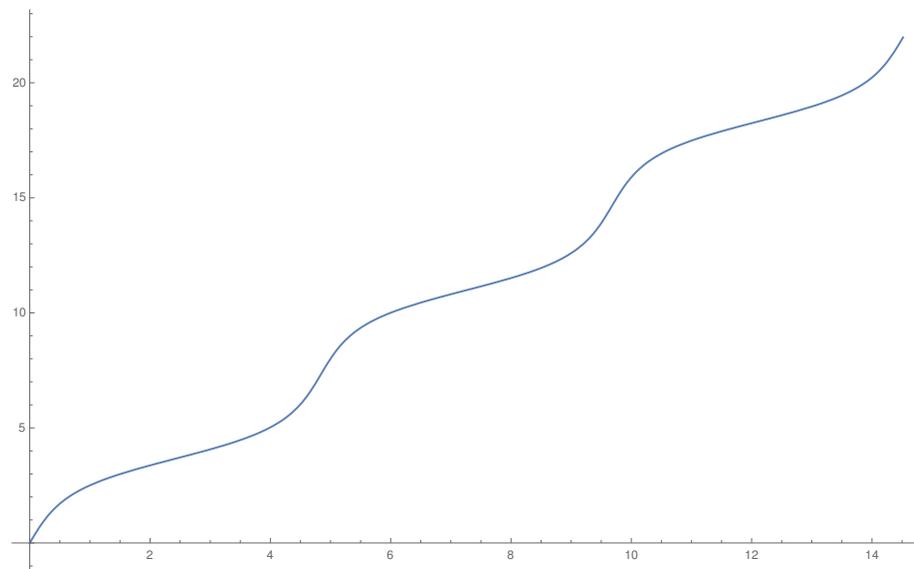

Figure 6: The angular coordinate as a function of time for the particle subject to the Yukawa potential.
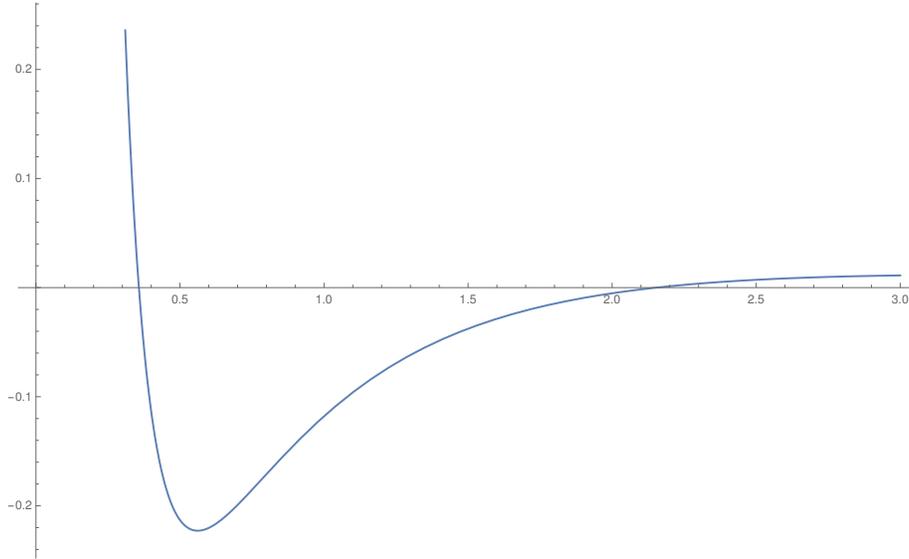
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Figure 7: The effective potential resulting from the bare Yukawa potential, with $m_* = 1$ and $l^2 = 0.5$.

a slight change of variables. For this potential, it is a straight-forward exercise to verify that there is a minimum of the effective potential at the radial value

$$r_* = l^2/\gamma m_*, \tag{62}$$

at which point the value of the effective potential is given by

$$U_{\text{eff}}(r_*) = -m_*\gamma^2/2l^2. \tag{63}$$

For a given angular momentum l and mass m_*, these two quantities set the characteristic length scale and energy scale of the problem. So, it is natural to introduce the new, dimensionless variables

$$\rho = r/r_* \ ; \ \varepsilon = E/|U_{\text{eff}}(r_*)|. \tag{64}$$

If we then make a change of variables in the integral describing the angular coordinate, it is a simple exercise in algebra to show that

$$\theta(\rho) = \theta_0 \pm \int_{\rho_0}^{\rho} \frac{d\rho'/(\rho')^2}{\sqrt{\varepsilon - 1/(\rho')^2 + 2/\rho'}}. \tag{65}$$

It turns out that performing this integral is not too difficult, through the use of an integral table, or Mathematica. Choosing to first consider motion in which the radial coordinate is increasing, the result that we find is

$$\theta(\rho) = \arccos\left[\frac{(1/\rho) - 1}{\sqrt{1 + \varepsilon}}\right] - \arccos\left[\frac{(1/\rho_0) - 1}{\sqrt{1 + \varepsilon}}\right] + \theta_0. \tag{66}$$

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Without loss of generality, we can always choose θ_0 to be whatever we wish, since this just amounts to rotating our choice of coordinates. Thus, if we assume we have chosen θ_0 in such a way as to cancel the second term, we find

$$\theta\left(\rho\right) = \arccos\left[\frac{(1/\rho) - 1}{\sqrt{1+\varepsilon}}\right] \;\; \Rightarrow \;\; \rho\left(\theta\right) = \frac{1}{1 + e\cos\theta}, \tag{67}$$

where we have defined the **eccentricity** e as the quantity

$$e = \sqrt{1+\varepsilon}. \tag{68}$$

Similar considerations show that this expression also holds as the radius is decreasing. We now have an expression which determines the (scaled) radius as a function of angle, which determines the shape of our orbit. It turns out that the above equation is in fact the defining equation of the **conic sections** - the family of geometric shapes which include the ellipse, the parabola, and the hyperbola. Which shape the orbit takes in particular will depend on the value of the eccentricity. Let's consider each of the various cases.

First, notice that as $r \to \infty$, the effective potential goes to zero. Thus, if a particle has an energy $E < 0$, its energy is less than the potential energy at infinity, and is thus incapable of escaping to infinity. In this case, we say that the orbit of the particle is **bounded** - its radial coordinate will oscillate between perigee and apogee forever. In the case that the energy of our particle is negative,

$$E < 0 \;\; \Rightarrow \;\; \varepsilon < 0 \;\; \Rightarrow \;\; e = \sqrt{1+\varepsilon} < 1. \tag{69}$$

Conic sections with an eccentricity less than one are in fact ellipses, with one focus at the origin. This is demonstrated in Figure 8. Also, notice that as promised, the radius is in fact periodic in θ with a period of 2π - the orbit is **closed**. An important special case of this result occurs when the particle is sitting precisely at the minimum of the effective potential, so that

$$\varepsilon = U_{\text{eff}}\left(r_*\right)/\left|U_{\text{eff}}\left(r_*\right)\right| = -1 \;\; \Rightarrow \;\; e = 0. \tag{70}$$

In this case, we simply have the circle defined by

$$\rho = 1 \;\; \Rightarrow \;\; r = r_*. \tag{71}$$

While it is not immediately obvious, our equation for the elliptical orbit can in fact be rewritten in the form

$$\frac{(x+d)^2}{a^2} + \frac{y^2}{b^2} = 1, \tag{72}$$

where

$$a = \frac{l^2}{\gamma m_*}\frac{1}{1-e^2} \;\; ; \;\; b = \frac{l^2}{\gamma m_*}\frac{1}{\sqrt{1-e^2}} \;\; ; \;\; d = ae. \tag{73}$$

This is shown in Figure 9. The parameters a and b are known as the semi-major and semi-minor axes, respectively. The appearance of the term d indicates that the center of the ellipse is **not** at the origin, but instead it is a distance d away.

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In particular, since Newton's Law of Gravitation involves precisely this type of potential energy function that we have been considering, these results have of course played a very important role in the history of physics and astronomy. While Johannes Kepler had proposed that the motions of planets were described by ellipses based on observational data, it was not until Newton's work on gravitation that this result could be derived from a simple assumption regarding the attraction between two bodies. Although our solar system is of course not a single two-body system, because the Sun is so much more massive than any other planet, to a good approximation, we can assume that the Sun remains sitting stationary at the origin, while all of the planets, comets, asteroids, and other members of the solar system orbit around it. Each planet's motion is an ellipse, with one focus at the location of the Sun. Corrections to the motions of the planets which stem from their interactions with each other can then be introduced as a perturbation on top of this two-body solution.

If instead the particle in question has a positive energy, then it is possible for it to escape to infinity - in this case, we say that the particle's orbit is **unbounded**. In this situation,

$$E > 0 \;\Rightarrow\; \varepsilon > 0 \;\Rightarrow\; e = \sqrt{1 + \varepsilon} > 1. \tag{74}$$

Conic sections with an eccentricity greater than one are known as **hyperbolas**. An example of this is shown in Figure 10. Notice that there is in fact a **maximum** angle that can be obtained by a particle subject to a hyperbolic orbit, given by

$$\rho\left(\theta\right) = \frac{1}{1 + e\cos\theta} = \infty \;\Rightarrow\; \theta_{\max} = \arccos\left[-\frac{1}{e}\right], \tag{75}$$

which comes from simply setting the denominator on the left equal to zero. In the special limiting case that the energy is exactly zero, we have

$$E = 0 \;\Rightarrow\; \varepsilon = 0 \;\Rightarrow\; e = \sqrt{1 + \varepsilon} = 1, \tag{76}$$

which defines a **parabola**, an example of which is shown in Figure 11. In some sense, the parabola is the marginal case between an ellipse and a hyperbola. Hyperbolic orbits in general are incredibly important in astronomy, since they describe **gravitational slingshots**. When planning missions that will send spacecraft on interplanetary missions, astronomers can use the gravity of other planets to influence the motion of the spacecraft, without necessarily sending the spacecraft into orbit.

While it is a relatively straight-forward (though tedious) task to explicitly work out the time dependence of the radial and angular coordinates, the resulting expressions are not as neat as the ones we have found for the orbital motion, and so we will not bother to state them here. However, you will explore some basic aspects of the time evolution (for example, how the orbital period related to the semi-major axis) on the homework.
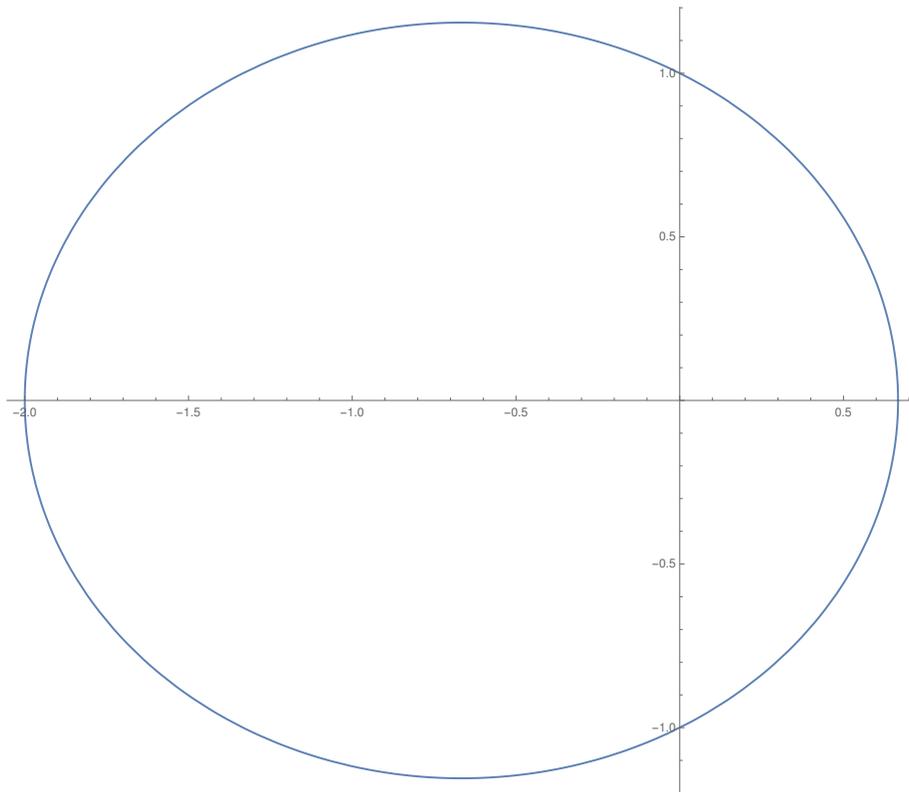
Figure 8: An ellipse with eccentricity $e = 0.5$. This type of shape describes the orbit of a particle with an energy less than zero. Due to a special property of the $1/r$ potential, this orbit will always be **closed**, regardless of the detailed physical parameters of the problem.
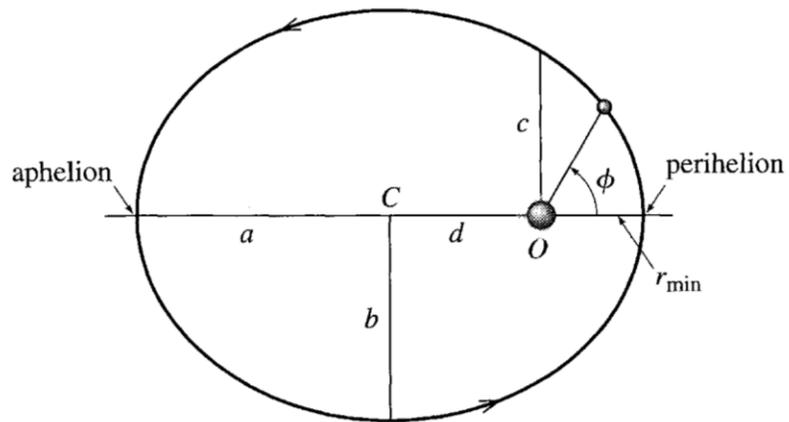
Figure 9: Figure 8.10 from Taylor. Notice the use of the alternative terminology perihelion and aphelion, instead of perigee and apogee. Also, Taylor makes use of the Greek letter ϕ, as opposed to θ, in order to describe the angular motion.
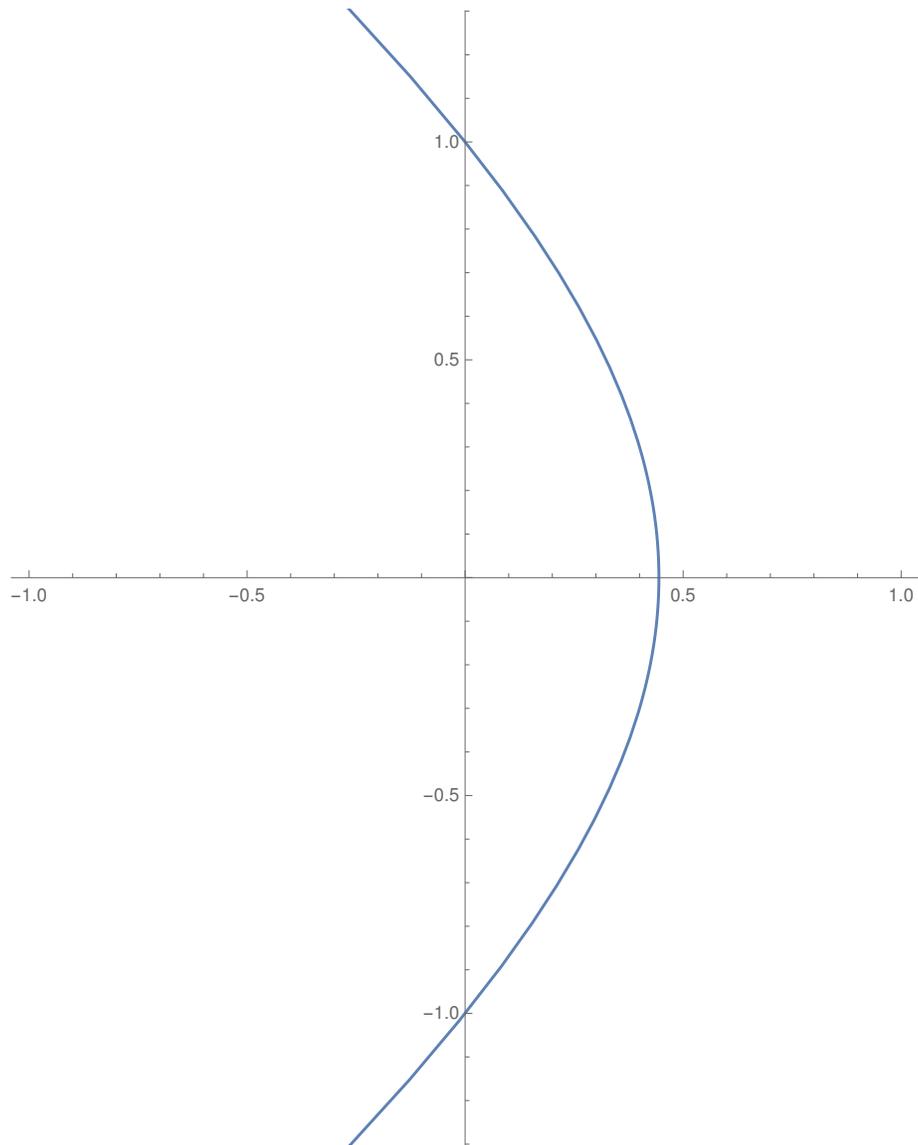
Figure 10: A hyperbola with eccentricity $e = 1.25$, which is an example of the motion of a particle with positive energy. This type of orbit describes a particle which can escape to infinity, and never return.
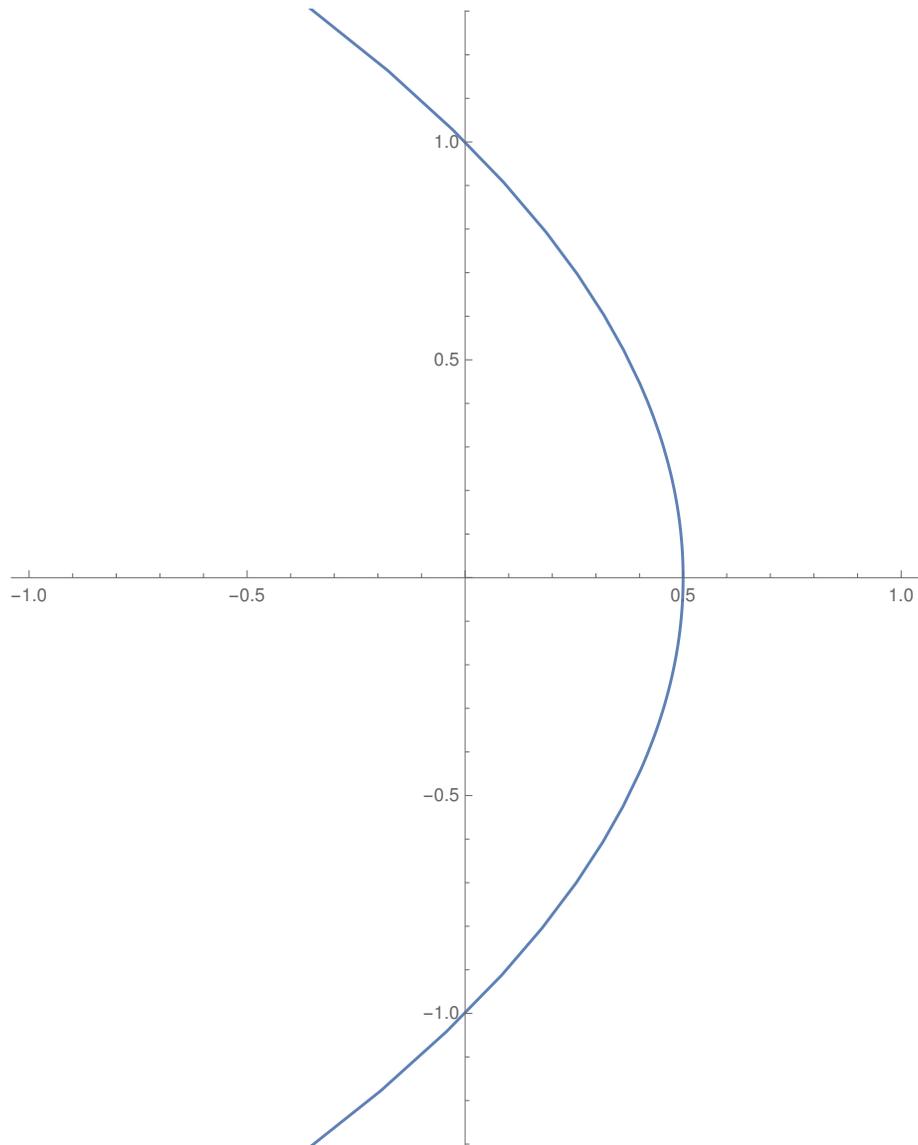
Figure 11: A parabola with eccentricity $e = 1$, which is in fact the only eccentricity a parabola can have, by definition. This type of orbit describes the motion of a particle with just barely enough energy to escape to infinity.