The Unit Circle

The unit circle can be used to calculate the trigonometric functions sin(θ), cos(θ), tan(θ), sec(θ), csc(θ), and cot(θ). It utilizes (x,y) coordinates to label the points on the circle, where *x* represents cos(θ) of a given angle, *y* represents sin(θ), and $\frac{y}{x}$ represents tan(θ). Theta, or θ, represents the angle in degrees or radians. This handout will describe unit circle concepts, define degrees and radians, and explain the conversion process between degrees and radians. It will also demonstrate an additional way of solving unit circle problems called the triangle method.

What is the unit circle?

The unit circle has a radius of one. The intersection of the x and y-axes (0,0) is known as the origin. The angles on the unit circle can be in degrees or radians.

Degrees

Degrees, denoted by °, are a measurement of angle size that is determined by dividing a circle into 360 equal pieces.

Radians

Radians are unit-less but are always written with respect to π. They measure an angle in relation to a section of the unit circle's circumference.

The circle is divided into 360 degrees starting on the right side of the x–axis and moving counterclockwise until a full rotation has been completed. In radians, this would be 2π. The unit circle is shown on the next page.

Converting Between Degrees and Radians

In trigonometry, most calculations use radians. Therefore, it is important to know how to convert between degrees and radians using the following conversion factors.

Conversion Factors

$$\boldsymbol{Degrees \times \frac{\pi}{180^\circ} = Radians}$$

$$\boldsymbol{Radians \times \frac{180^\circ}{\pi} = Degrees}$$

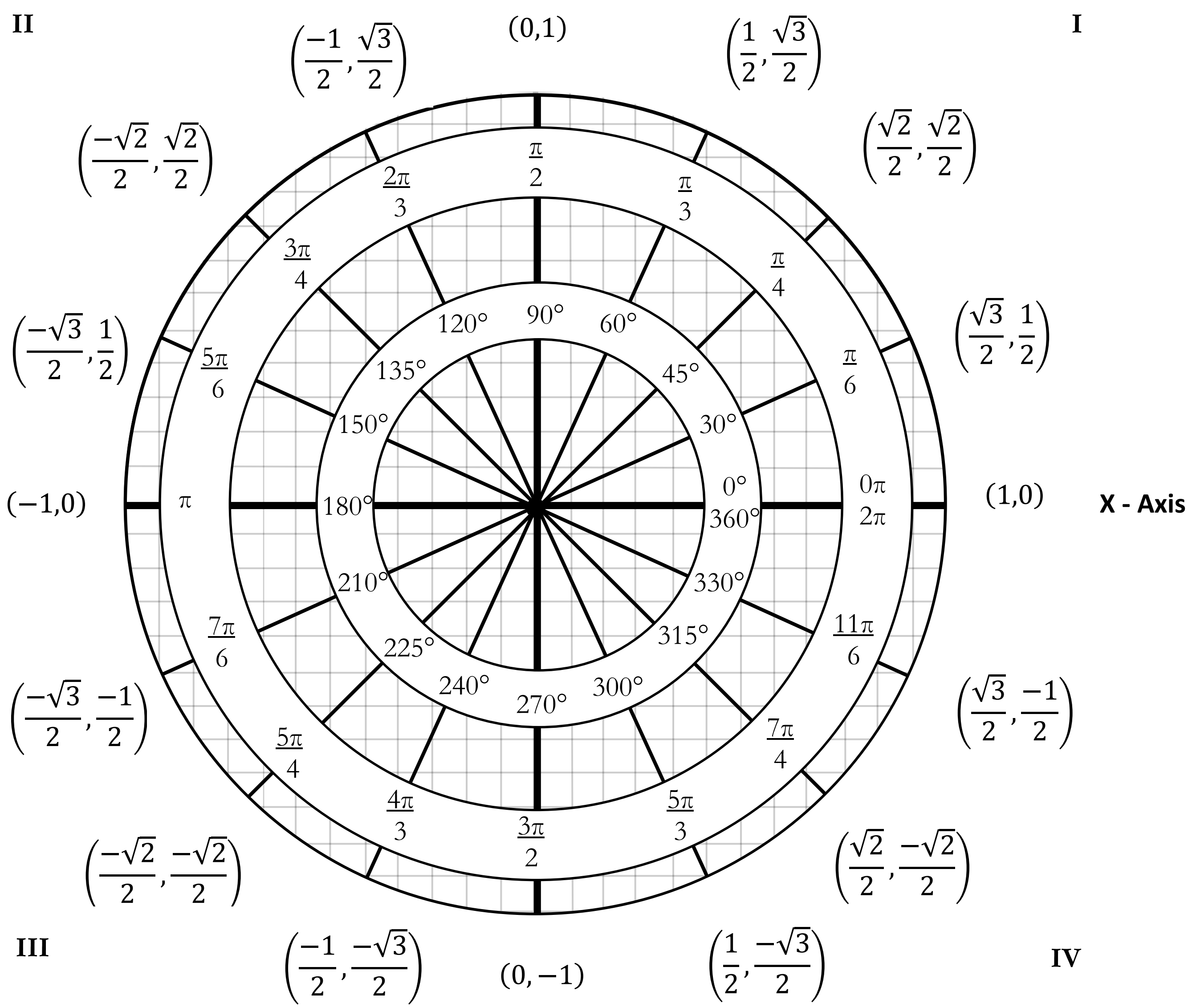
Example 1:

Convert 120° to radians.

Step 1: If starting with degrees, 180° should be on the bottom of the conversion factor so that the degrees cancel.

$$120^\circ \times \frac{\pi}{180^\circ} = \frac{120\cancel{^\circ}(\pi)}{1(180\cancel{^\circ})} = \boxed{\frac{2\pi}{3}}$$

The Standard Unit Circle



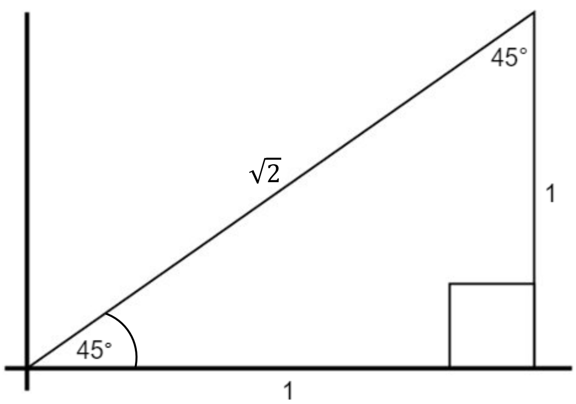
Key: $(\mathbf{Cos(\theta)}, \mathbf{Sin(\theta)})$

$$\boldsymbol{Tan(\theta)} = \frac{\mathbf{Sin(\theta)}}{\mathbf{Cos(\theta)}}$$

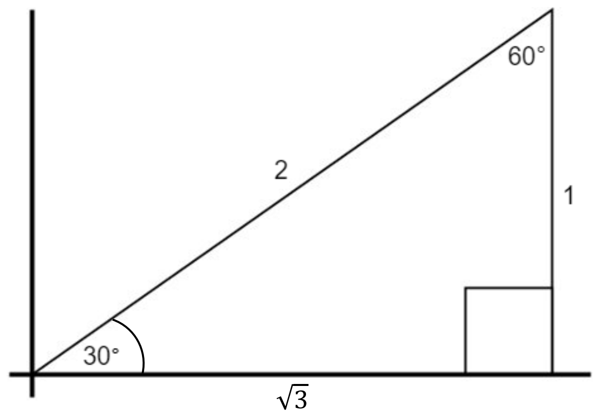
The Unit Circle by Triangles

Another method for solving trigonometric functions is the triangle method. To do this, the unit circle is broken up into more common triangles: the 45°−45°−90° and 30°−60°−90° triangles. Some examples of how these triangles can be drawn are below.

45°−45°−90° Triangle



30°−60°−90° Triangle

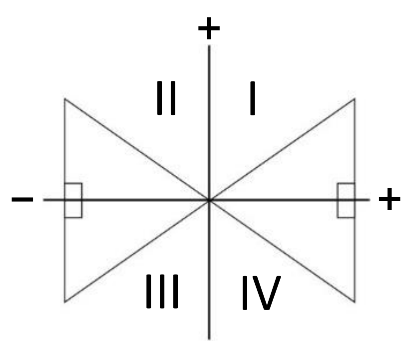


Sides:	1	1	$\sqrt{2}$
	$\updownarrow$	$\updownarrow$	$\updownarrow$
Angles:	45°	45°	90°

Sides:	1	$\sqrt{3}$	2
	$\updownarrow$	$\updownarrow$	$\updownarrow$
Angles:	30°	60°	90°

Triangle Method Steps

1. Choose a triangle.
    - If the angle inside the trigonometric function is divisible by 45, use the 45°−45°−90° triangle.
    - If the angle is divisible by 30 or 60, use the 30°−60°−90° triangle.
2. Draw the triangle in the correct quadrant, with the hypotenuse pointed towards the origin.
    - Add negative signs on the sides if necessary.
3. Analyze the triangle.
4. Rationalize and simplify.



Example 2:

Use the triangle method to solve:

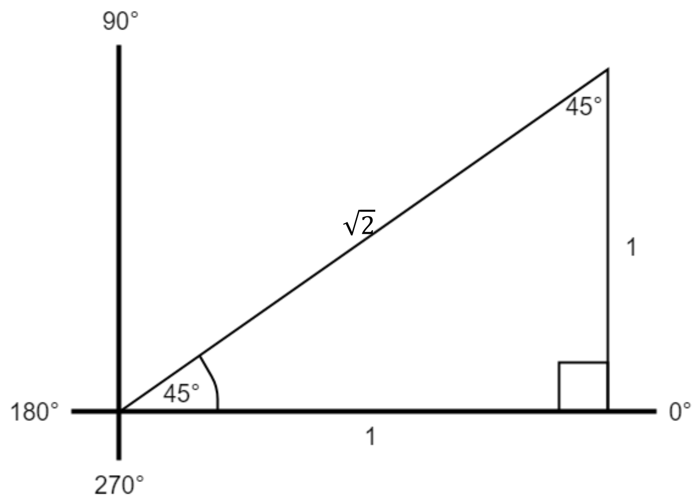
$Cos(45°)$

Step 1: Choose a triangle.

Because 45 is divisible by 45, use the 45°−45°−90° triangle.

Step 2: Draw the triangle in the correct quadrant.

This triangle will be in quadrant I because 45° is between 0° and 90°.



Step 3: Analyze the triangle.

Remember that cos(θ) represents $\frac{Adjacent}{Hypotenuse}$. Here, the adjacent side to θ (or 45°) is 1, and the hypotenuse is $\sqrt{2}$. This results in $cos(45°) = \frac{1}{\sqrt{2}}$.

Step 4: Rationalize the denominator.

The denominator is rationalized by removing the square roots. Do this by multiplying the numerator and denominator of the resulting fraction $\frac{1}{\sqrt{2}}$ by the radical in the denominator $\sqrt{2}$.

$$\text{Cos}(45°) = \frac{1}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$$

$$\boxed{\text{Cos}(45°) = \frac{\sqrt{2}}{2}}$$

Example 3:

Use the triangle method to solve:

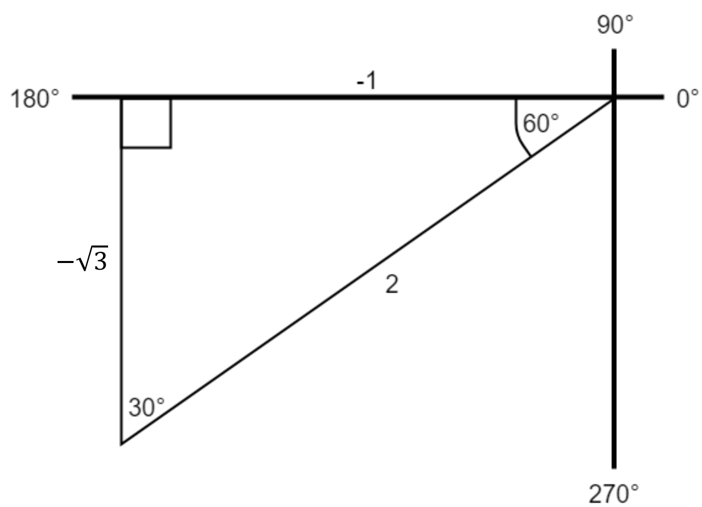
$Tan(240°)$

Step 1: Choose a triangle.

Because 240 is divisible by 30, use the 30°−60°−90° triangle.

Step 2: Draw the triangle in the correct quadrant.

This triangle will be in quadrant III because 240° is between 180° and 270°. Additionally, 60° will be the angle near the origin because 240° is 60° more than 180°.



Step 3: Analyze the triangle.

Note that tan(θ) represents $\frac{Opposite}{Adjacent}$. Here, the opposite side is $-\sqrt{3}$ while the adjacent side is -1. This results in $tan(240°) = \frac{-\sqrt{3}}{-1}$.

Step 4: Simplify.

The negatives cancel each other out to leave $\frac{\sqrt{3}}{1}$, which is $\sqrt{3}$.

$\boxed{Tan(240°) = \sqrt{3}}$

Practice Problems:

Find the exact value of the problems below using either the standard unit circle or the triangle method.

1.) Sin $\frac{4\pi}{3}$

2.) Cos $\frac{11\pi}{6}$

3.) Tan $\frac{\pi}{3}$

4.) Cos $\frac{-2\pi}{3}$ (Hint: Instead of rotating counterclockwise around the circle, go clockwise.)

5.) Sin $\frac{-\pi}{2}$

6.) Tan 2π

7.) Tan $\frac{\pi}{2}$

8.) Cos$\frac{9\pi}{4}$ (Hint: For angles larger than 360°, continue going around the circle.)

Answers:

1.) $\frac{-\sqrt{3}}{2}$

2.) $\frac{\sqrt{3}}{2}$

3.) $\sqrt{3}$

4.) $\frac{-1}{2}$

5.) -1

6.) 0

9.) Undefined ($\frac{1}{0}$ cannot occur/does not exist)

7.) $\frac{\sqrt{2}}{2}$