2

The vibration of structures with one degree of freedom

All real structures consist of an infinite number of elastically connected mass elements and therefore have an infinite number of degrees of freedom; hence an infinite number of coordinates are needed to describe their motion. This leads to elaborate equations of motion and lengthy analyses. However, the motion of a structure is often such that only a few coordinates are necessary to describe its motion. This is because the displacements of the other coordinates are restrained or not excited, being so small that they can be neglected. Now, the analysis of a structure with a few degrees of freedom is generally easier to carry out than the analysis of a structure with many degrees of freedom, and therefore only a simple mathematical model of a structure is desirable from an analysis viewpoint. Although the amount of information that a simple model can yield is limited, if it is sufficient then the simple model is adequate for the analysis. Often a compromise has to be reached, between a comprehensive and elaborate multi-degree of freedom model of a structure which is difficult and costly to analyse but yields much detailed and accurate information, and a simple few degrees of freedom model that is easy and cheap to analyse but yields less information. However, adequate information about the vibration of a structure can often be gained by analysing a simple model, at least in the first instance.

The vibration of some structures can be analysed by considering them as a one degree or single degree of freedom system; that is, a system where only one coordinate is necessary to describe the motion. Other motions may occur, but they are assumed to be negligible compared with the coordinate considered.

A system with one degree of freedom is the simplest case to analyse because only one coordinate is necessary to describe the motion of the system completely. Some real systems can be modelled in this way, either because the excitation of the system is such that the vibration can be described by one coordinate, although the system could vibrate in other directions if so excited, or the system really is simple as, for example, a clock

pendulum. It should also be noted that a one, or single degree of freedom model of a complicated system can often be constructed where the analysis of a particular mode of vibration is to be carried out. To be able to analyse one degree of freedom systems is therefore essential in the analysis of structural vibrations. Examples of structures and motions which can be analysed by a single degree of freedom model are the swaying of a tall rigid building resting on an elastic soil, and the transverse vibration of a bridge. Before considering these examples in more detail, it is necessary to review the analysis of vibration of single degree of freedom dynamic systems. For a more comprehensive study see *Engineering Vibration Analysis with Application to Control Systems* by C. F. Beards (Edward Arnold, 1995). It should be noted that many of the techniques developed in single degree of freedom analysis are applicable to more complicated systems.

2.1   FREE UNDAMPED VIBRATION

2.1.1 Translation vibration

In the system shown in Fig. 2.1 a body of mass m is free to move along a fixed horizontal surface. A spring of constant stiffness k which is fixed at one end is attached at the other end to the body. Displacing the body to the right (say) from the equilibrium position causes a spring force to the left (a restoring force). Upon release this force gives the body an acceleration to the left. When the body reaches its equilibrium position the spring force is zero, but the body has a velocity which carries it further to the left although it is retarded by the spring force which now acts to the right. When the body is arrested by the spring the spring force is to the right so that the body moves to the right, past its equilibrium position, and hence reaches its initial displaced position. In practice this position will not quite be reached because damping in the system will have dissipated some of the vibrational energy. However, if the damping is small its effect can be neglected.

If the body is displaced a distance x_0 to the right and released, the free-body diagrams (FBDs) for a general displacement x are as shown in Fig. 2.2(a) and (b).

The effective force is always in the direction of positive x. If the body is being retarded $\ddot{x}$ will be calculated to be negative. The mass of the body is assumed constant: this is usually so but not always, as, for example, in the case of a rocket burning fuel. The spring stiffness k is assumed constant: this is usually so within limits (see section 2.1.3). It is assumed that the mass of the spring is negligible compared with the mass of the body; cases where this is not so are considered in section 2.1.4.1.
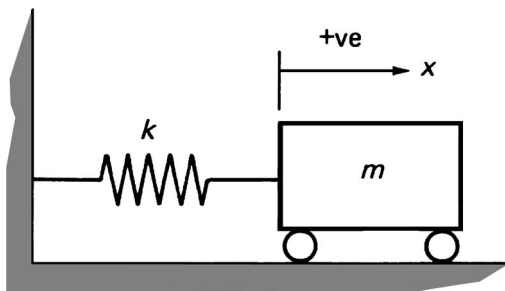

Fig. 2.1.   Single degree of freedom model – translation vibration.
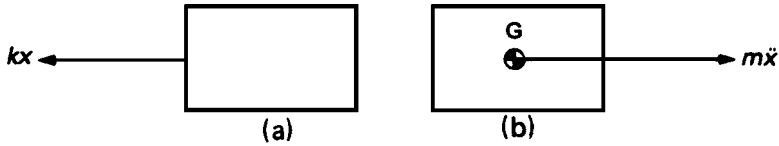
Fig. 2.2.   (a) Applied force; (b) effective force.

From the free-body diagrams the equation of motion for the system is

$$m\ddot{x} = -kx \quad \text{or} \quad \ddot{x} + (k/m)x = 0. \tag{2.1}$$

This will be recognized as the equation for simple harmonic motion. The solution is

$$x = A \cos \omega t + B \sin \omega t, \tag{2.2}$$

where A and B are constants which can be found by considering the initial conditions, and ω is the circular frequency of the motion. Substituting (2.2) into (2.1) we get

$$-\omega^2 (A \cos \omega t + B \sin \omega t) + (k/m) (A \cos \omega t + B \sin \omega t) = 0.$$

Since $(A \cos \omega t + B \sin \omega t) \neq 0$ (otherwise no motion),

$$\omega = \sqrt{(k/m)} \text{ rad/s},$$

and

$$x = A \cos \sqrt{(k/m)}t + B \sin \sqrt{(k/m)}t.$$

Now

$$x = x_0 \text{ at } t = 0,$$

thus

$$x_0 = A \cos 0 + B \sin 0, \quad \text{and therefore } x_0 = A,$$

and

$$\dot{x} = 0 \text{ at } t = 0,$$

thus

$$0 = -A\sqrt{(k/m)} \sin 0 + B\sqrt{(k/m)} \cos 0, \quad \text{and therefore } B = 0;$$

that is,

$$x = x_0 \cos \sqrt{(k/m)}t. \tag{2.3}$$

The system parameters control ω and the type of motion but not the amplitude x_0, which is found from the initial conditions. The mass of the body is important, but its weight is not, so that for a given system, ω is independent of the local gravitational field.

The frequency of vibration, f, is given by

$$f = \frac{\omega}{2\pi}, \quad \text{or} \quad f = \frac{1}{2\pi}\sqrt{\left(\frac{k}{m}\right)} \text{Hz}. \tag{2.4}$$
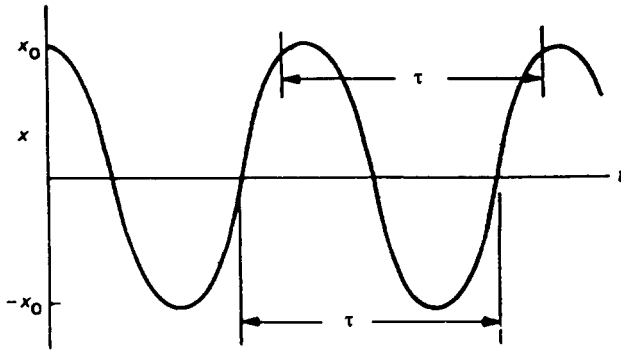
The motion is as shown in Fig. 2.3.

Fig. 2.3.   Simple harmonic motion.

The period of the oscillation, τ, is the time taken for one complete cycle so that

$$\tau = \frac{1}{f} = 2\pi\sqrt{(m/k)} \text{ seconds.} \tag{2.5}$$

The analysis of the vibration of a body supported to vibrate only in the vertical or y direction can be carried out in a similar way to that above.

It is found that for a given system the frequency of vibration is the same whether the body vibrates in a horizontal or vertical direction.

Sometimes more than one spring acts in a vibrating system. The spring, which is considered to be an elastic element of constant stiffness, can take many forms in practice; for example, it may be a wire coil, rubber block, beam or air bag. Combined spring units can be replaced in the analysis by a single spring of equivalent stiffness as follows.

2.1.1.1 Springs connected in series

The three-spring system of Fig. 2.4(a) can be replaced by the equivalent spring of Fig. 2.4(b).

Fig. 2.4.   Spring systems.

If the deflection at the free end, δ, experienced by applying the force F is to be the same in both cases,

$$\delta = F/k_e = F/k_1 + F/k_2 + F/k_3,$$

that is,

$$1/k_e = \sum_1^3 1/k_i.$$

In general, the reciprocal of the equivalent stiffness of springs connected in series is obtained by summing the reciprocal of the stiffness of each spring.

2.1.1.2 Springs connected in parallel

The three-spring system of Fig. 2.5(a) can be replaced by the equivalent spring of Fig. 2.5(b).
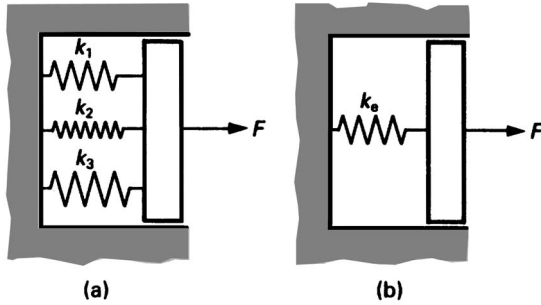

(a)   (b)

Fig. 2.5.   Spring systems.

Since the defection δ must be the same in both cases, the sum of the forces exerted by the springs in parallel must equal the force exerted by the equivalent spring. Thus

$$F = k_1\delta + k_2\delta + k_3\delta = k_e\delta,$$

that is,

$$k_e = \sum_{i=1}^{3} k_i.$$

In general, the equivalent stiffness of springs connected in parallel is obtained by summing the stiffness of each spring.

2.1.2 Torsional vibration

Fig. 2.6 shows the model used to study torsional vibration.

A body with mass moment of inertia I about the axis of rotation is fastened to a bar of torsional stiffness k_T. If the body is rotated through an angle θ_0 and released, torsional vibration of the body results. The mass moment of inertia of the shaft about the axis of rotation is usually negligible compared with I.

For a general displacement θ, the FBDs are as given in Fig. 2.7(a) and (b). Hence the equation of motion is

$$I\ddot{\theta} = -k_T\theta$$

or

$$\ddot{\theta} + \left(\frac{k_T}{I}\right)\theta = 0.$$

This is of a similar form to equation (2.1); that is, the motion is simple harmonic with frequency $(1/2\pi)\,\sqrt{(k_T/I)}$ Hz.
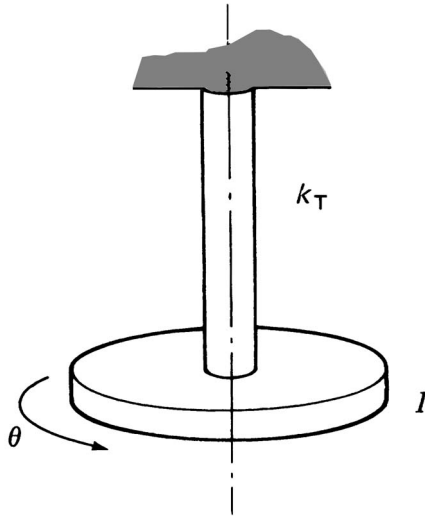
Fig. 2.6.   Single degree of freedom model – torsional vibration.
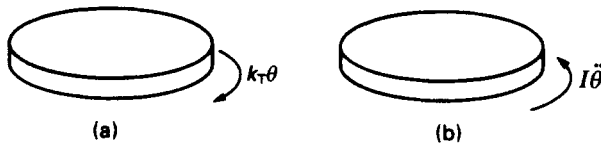
(a)   (b)

Fig. 2.7.   (a) Applied torque; (b) effective torque.

The torsional stiffness of the shaft, k_T, is equal to the applied torque divided by the angle of twist.
Hence

$$k_T = \frac{GJ}{l}, \text{ for a circular section shaft,}$$

where G = modulus of rigidity for shaft material,
$\quad\quad J$ = second moment of area about the axis of rotation, and
$\quad\quad l$ = length of shaft.

Hence

$$f = \frac{\omega}{2\pi} = \frac{1}{2\pi} \sqrt{(GJ/Il)} \text{ Hz,}$$

and

$$\theta = \theta_0 \cos \sqrt{(GJ/Il)}t,$$

when $\theta = \theta_0$ and $\dot{\theta} = 0$ at $t = 0$.

If the shaft does not have a constant diameter, it can be replaced analytically by an equivalent shaft of different length but with the same stiffness and a constant diameter.

For example, a circular section shaft comprising a length l_1 of diameter d_1 and a length l_2 of diameter d_2 can be replaced by a length l_1 of diameter d_1 and a length l of diameter d_1 where, for the same stiffness,

$$(GJ/l)_{\text{length } l_2 \text{ diameter } d_2} = (GJ/l)_{\text{length } l \text{ diameter } d_1}$$

that is, for the same shaft material, $d_2^4/l_2 = d_1^4/l$.

Therefore the equivalent length l_e of the shaft of constant diameter d_1 is given by

$$l_e = l_1 + (d_1/d_2)^4 l_2.$$

It should be noted that the analysis techniques for translational and torsional vibration are very similar, as are the equations of motion.

2.1.3 Non-linear spring elements

Any spring elements have a force–deflection relationship that is linear only over a limited range of deflection. Fig. 2.8 shows a typical characteristic.
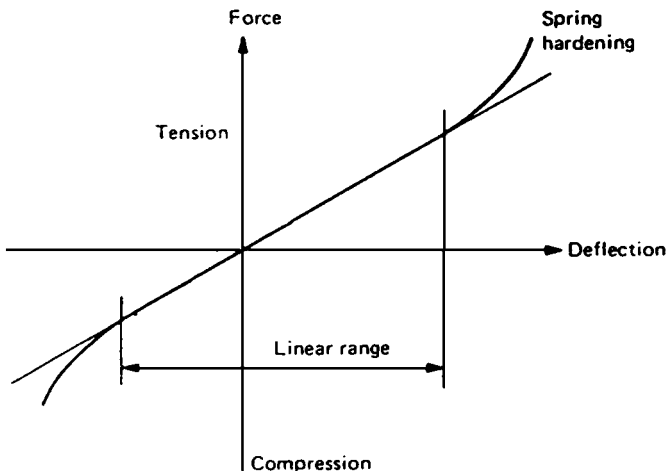

Fig. 2.8.   Non-linear spring characteristic.

The non-linearities in this characteristic may be caused by physical effects such as the contacting of coils in a compressed coil spring, or by excessively straining the spring material so that yielding occurs. In some systems the spring elements do not act at the same time, as shown in Fig. 2.9 (a), or the spring is designed to be non-linear as shown in Fig. 2.9 (b) and (c).

Analysis of the motion of the system shown in Fig. 2.9 (a) requires analysing the motion until the half-clearance a is taken up, and then using the displacement and velocity at this point as initial conditions for the ensuing motion when the extra springs are operating. Similar analysis is necessary when the body leaves the influence of the extra springs.
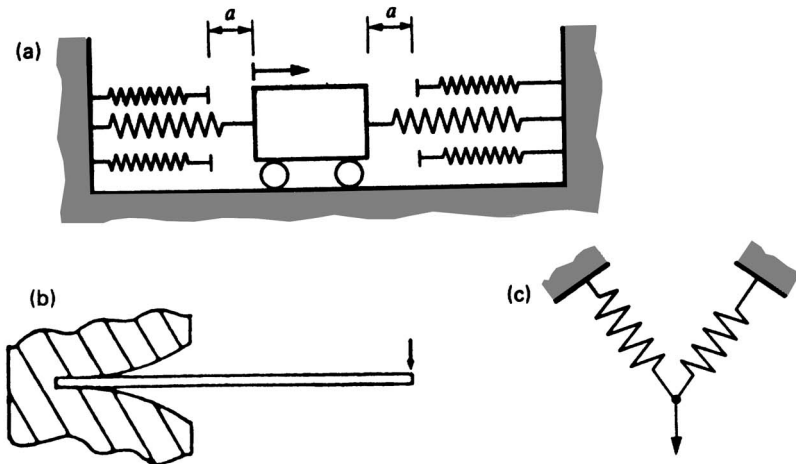
Fig. 2.9.   Non-linear spring systems.

2.1.4 Energy methods for analysis

For undamped free vibration the total energy in the vibrating system is constant throughout the cycle. Therefore the maximum potential energy V_{max} is equal to the maximum kinetic energy T_{max} although these maxima occur at different times during the cycle of vibration. Furthermore, since the total energy is constant,

$$T + V = \text{constant},$$

and thus

$$\frac{d}{dt}(T + V) = 0.$$

Applying this method to the case, already considered, of a body of mass m fastened to a spring of stiffness k, when the body is displaced a distance x from its equilibrium position,

strain energy (SE) in spring $= \frac{1}{2} kx^2$.
kinetic energy (KE) of body $= \frac{1}{2} m\dot{x}^2$.

Hence

$$V = \tfrac{1}{2}kx^2,$$

and

$$T = \tfrac{1}{2}m\dot{x}^2.$$

Thus

$$\frac{d}{dt}(\tfrac{1}{2}m\dot{x}^2 + \tfrac{1}{2}kx^2) = 0,$$

that is

$$m\ddot{x}\dot{x} + kx\dot{x} = 0,$$

or

$$\ddot{x} + \left(\frac{k}{m}\right)x = 0, \text{ as before in equation (2.1).}$$

This is a very useful method for certain types of problem in which it is difficult to apply Newton's laws of motion.

Alternatively, assuming SHM, if $x = x_0 \cos \omega t$,

the maximum SE, $V_{max} = \frac{1}{2}kx_0^2$,

and

the maximum KE, $T_{max} = \frac{1}{2}m(x_0\omega)^2$.

Thus, since $T_{max} = V_{max}$,

$$\frac{1}{2}kx_0^2 = \frac{1}{2}mx_0^2\omega^2,$$

or $\omega = \sqrt{(k/m)}$ rad/s.

Energy methods can also be used in the analysis of the vibration of continuous systems such as beams. It has been shown by Rayleigh that the lowest natural frequency of such systems can be fairly accurately found by assuming any reasonable deflection curve for the vibrating shape of the beam: this is necessary for the evaluation of the kinetic and potential energies. In this way the continuous system is modelled as a single degree of freedom system, because once one coordinate of beam vibration is known, the complete beam shape during vibration is revealed. Naturally the assumed deflection curve must be compatible with the end conditions of the system, and since any deviation from the true mode shape puts additional constraints on the system, the frequency determined by Rayleigh's method is never less than the exact frequency. Generally, however, the difference is only a few per cent. The frequency of vibration is found by considering the conservation of energy in the system; the natural frequency is determined by dividing the expression for potential energy in the system by the expression for kinetic energy.

2.1.4.1 The vibration of systems with heavy springs

The mass of the spring element can have a considerable effect on the frequency of vibration of those structures in which heavy springs are used.

Consider the translational system shown in Fig. 2.10, where a rigid body of mass M is connected to a fixed frame by a spring of mass m, length l, and stiffness k. The body moves in the x direction only. If the dynamic deflected shape of the spring is assumed to be the same as the static shape, the velocity of the spring element is $\dot{y} = (y/l)\dot{x}$, and the mass of the element is $(m/l)dy$.

Thus

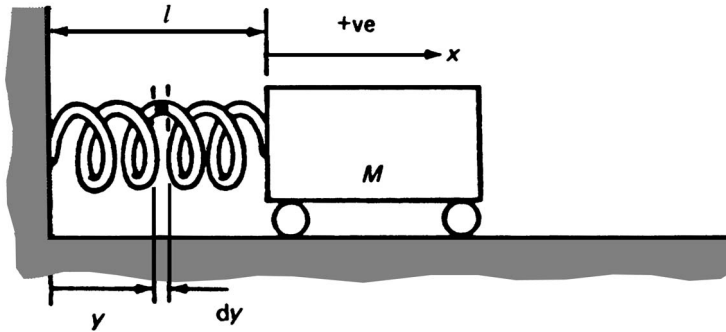$$T = \frac{1}{2}M\dot{x}^2 + \int_0^l \frac{1}{2}\left(\frac{m}{l}\right)\left[\frac{y}{l}\dot{x}\right]^2 dy$$

Fig. 2.10.   Single degree of freedom system with heavy spring.

$$= \frac{1}{2}\left(M + \frac{m}{3}\right)\dot{x}^2$$

and

$$V = \tfrac{1}{2}kx^2.$$

Assuming simple harmonic motion and putting $T_{max} = V_{max}$ gives the frequency of free vibration as

$$f = \frac{1}{2\pi}\sqrt{\left(\frac{k}{M + (m/3)}\right)} \text{ Hz},$$

that is, if the system is to be modelled with a massless spring, one third of the actual spring mass must be added to the mass of the body in the frequency calculation.

Alternatively, $\dfrac{\mathrm{d}}{\mathrm{d}t}(T + V) = 0$ can be used for finding the frequency of oscillation.

2.1.4.2 Transverse vibration of beams

For the beam shown in Fig. 2.11, if m is the mass unit length and y is the amplitude of the assumed deflection curve, then

$$T_{max} = \tfrac{1}{2}\int \dot{y}^2_{max} \, \mathrm{d}m = \tfrac{1}{2}\omega^2 \int y^2 \, \mathrm{d}m,$$

where ω is the natural circular frequency of the beam.

The strain energy of the beam is the work done on the beam which is stored as elastic energy. If the bending moment is M and the slope of the elastic curve is θ,
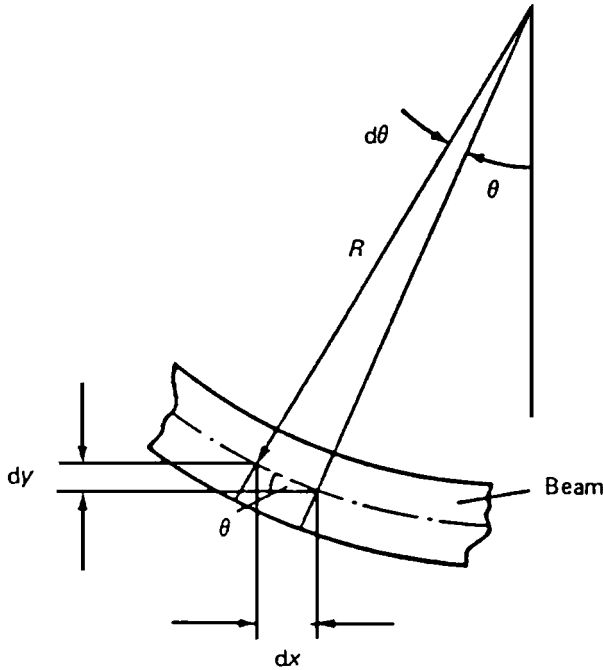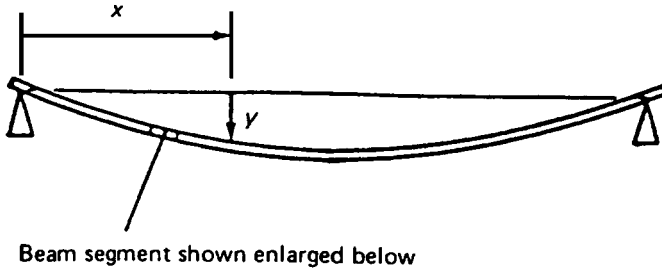
$$V = \tfrac{1}{2}\int M \, \mathrm{d}\theta.$$

Fig. 2.11.   Beam deflection.

Usually the deflection of beams is small so that the following relationships can be assumed to hold:

$$\theta = \frac{dy}{dx} \quad \text{and} \quad Rd\theta = dx;$$

thus

$$\frac{1}{R} = \frac{d\theta}{dx} = \frac{d^2y}{dx^2}.$$

From beam theory, $M/I = E/R$, where R is the radius of curvature and EI is the flexural rigidity. Thus

$$V = \tfrac{1}{2} \int \frac{M}{R} \mathrm{d}x = \tfrac{1}{2} \int EI \left(\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} \right)^2 \mathrm{d}x.$$

Since

$$T_{\max} = V_{\max};$$

$$\omega^2 = \frac{\int EI \left(\dfrac{\mathrm{d}^2 y}{\mathrm{d}x^2} \right)^2 \mathrm{d}x}{\int y^2 \, \mathrm{d}m}$$

This expression gives the lowest natural frequency of transverse vibration of a beam. It can be seen that to analyse the transverse vibration of a particular beam by this method requires y to be known as a function of x. For this the static deflected shape or a part sinusoid can be assumed, provided the shape is compatible with the beam boundary conditions.

2.1.5 The stability of vibrating structures

If a structure is to vibrate about an equilibrium position, it must be stable about that position; that is, if the structure is disturbed when in an equilibrium position, the elastic forces must be such that the structure vibrates about the equilibrium position. Thus the expression for ω^2 must be positive if a real value of the frequency of vibration about the equilibrium position is to exist, and hence the potential energy of a stable structure must also be positive.

The principle of minimum potential energy can be used to test the stability of structures that are conservative. Thus a structure will be stable at an equilibrium position if the potential energy of the structure is a minimum at that position. This requires that

$$\frac{\mathrm{d}V}{\mathrm{d}q} = 0 \quad \text{and} \quad \frac{\mathrm{d}^2 V}{\mathrm{d}q^2} > 0$$

where q is an independent or generalized coordinate. Hence the necessary conditions for vibration to take place are found, and the position about which the vibration occurs is determined.

Example 1

A link AB in a mechanism is a rigid bar of uniform section 0.3 m long. It has a mass of 10 kg, and a concentrated mass of 7 kg is attached at B. The link is hinged at A and is supported in a horizontal position by a spring attached at the mid-point of the bar. The stiffness of the spring is 2 kN/m. Find the frequency of small free oscillations of the system. The system is as follows.
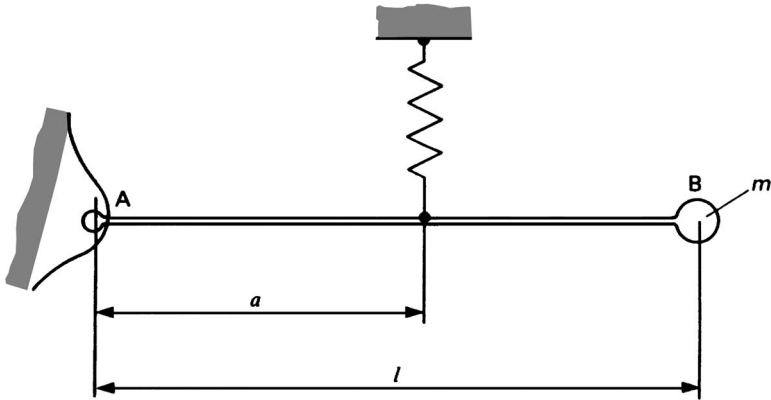
For rotation about A the equation of motion is

$$I_A\ddot{\theta} = -ka^2\theta$$

that is,

$$\ddot{\theta} + (ka^2/I_A)\theta = 0.$$

This is SHM with frequency

$$\frac{1}{2\pi}\sqrt{(ka^2/I_A)} \text{ Hz.}$$

In this case

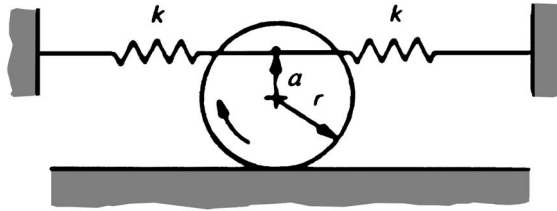$$a = 0.15 \text{ m, } l = 0.3 \text{ m, } k = 2000 \text{ N/m,}$$

and

$$I_A = 7(0.3)^2 + \tfrac{1}{3} \times 10 \, (0.3)^2 = 0.93 \text{ kg m}^2.$$

Hence

$$f = \frac{1}{2\pi}\sqrt{\left(\frac{2000(0.15)^2}{0.93}\right)} = 1.1 \text{ Hz.}$$

Example 2

A uniform cylinder of mass m is rotated through a small angle θ_0 from the equilibrium position and released. Determine the equation of motion and hence obtain the frequency of free vibration. The cylinder rolls without slipping.

If the axis of the cylinder moves a distance x and turns through an angle θ so that $x = r\theta$

$$KE = \tfrac{1}{2} m\dot{x}^2 + \tfrac{1}{2}I\dot{\theta}^2, \text{ where } I = \tfrac{1}{2} mr^2.$$

Hence

$$KE = \tfrac{3}{4}mr^2\dot{\theta}^2.$$
$$SE = 2 \times \tfrac{1}{2} \times k[(r + a)\theta]^2 = k(r + a)^2\theta^2.$$

Now, energy is conserved, so $(\tfrac{3}{4}mr^2\dot{\theta}^2 + k(r + a)^2\theta^2)$ is constant; that is,

$$\frac{d}{dt}(\tfrac{3}{4} mr^2\dot{\theta}^2 + k(r + a)^2\theta^2) = 0$$

or

$$\tfrac{3}{4}mr^2 2\dot{\theta}\ddot{\theta} + k(r + a)^2 2\theta\dot{\theta} = 0.$$

Thus the equation of the motion is

$$\ddot{\theta} + \frac{k(r + a)^2\theta}{(\tfrac{3}{4})mr^2} = 0.$$

Hence the frequency of free vibration is

$$\frac{1}{2\pi}\sqrt{\left[\frac{4k(r + a)^2}{3mr^2}\right]} \text{ Hz.}$$

Example 3

A uniform wheel of radius R can roll without slipping on an inclined plane. Concentric with the wheel, and fixed to it, is a drum of radius r around which is wrapped one end of a string. The other end of the string is fastened to an anchored spring, of stiffness k, as shown. Both spring and string are parallel to the plane. The total mass of the wheel/drum assembly is m and its moment of inertia about the axis through the centre of the wheel O is I. If the wheel is displaced a small distance from its equilibrium position and released, derive the equation describing the ensuing motion and hence calculate the frequency of the oscillations. Damping is negligible.
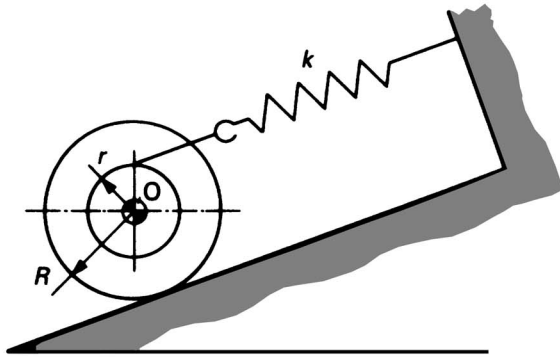
If the wheel is given an anti-clockwise rotation θ from the equilibrium position, the spring extension is $(R + r)\,\theta$ so that the restoring spring force is $k(R + r)\theta$.

The FBDs are:

Applied   **Effective**
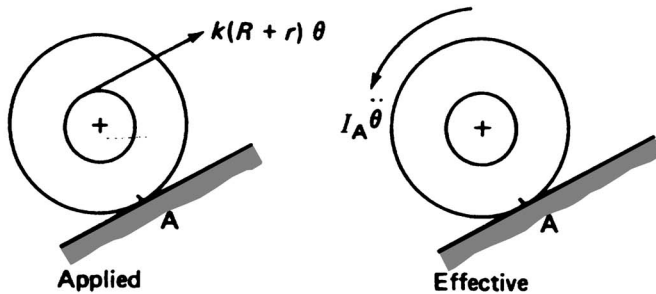
The rotation is instantaneously about the contact point A so that taking moments about A gives the equation of motion as

$$I_A\ddot{\theta} = -k(R + r)^2\theta.$$

(The moment due to the weight cancels with the moment due to the initial spring tension.)

Now $I_A = I + mR^2$, so

$$\ddot{\theta} + \left(\frac{k(R + r)^2}{I + mR^2}\right)\theta = 0,$$

and the frequency of oscillation is

$$\frac{1}{2\pi}\sqrt{\left(\frac{k(R + r)^2}{I + mR^2}\right)} \text{ Hz.}$$

An alternative method for obtaining the frequency of oscillation is to consider the energy in the system.

Now

SE, $V = \frac{1}{2}k(R + r)^2\theta^2$,

and

KE, $T = \frac{1}{2}I_A\dot{\theta}^2$,

(weight and initial spring tension effects cancel) so

$$T + V = \frac{1}{2}I_A\dot{\theta}^2 + \frac{1}{2}k(R + r)^2\theta^2,$$

and

$$\frac{d}{dt}(T + V) = \frac{1}{2}I_A 2\dot{\theta}\ddot{\theta} + \frac{1}{2}k(R + r)^2 2\theta\dot{\theta} = 0.$$

Hence

$$I_A\ddot{\theta} + k(R + r)^2\theta = 0,$$

which is the equation of motion.

Or, we can put $V_{max} = T_{max}$, and if $\theta = \theta_0 \sin \omega t$ is assumed,

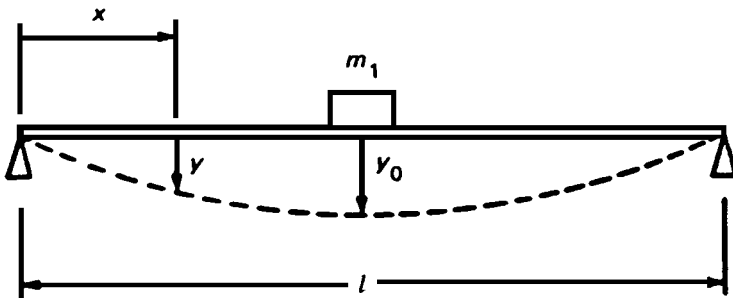$$\frac{1}{2}k(R + r)^2\theta_0^2 = \frac{1}{2}I_A\omega^2\theta_0^2,$$

so that

$$\omega = \sqrt{\left(\frac{k(R + r)^2}{I_A}\right)} \text{ rad/s},$$

where

$$I_A = I + mr^2 \quad \text{and} \quad f = (\omega/2\pi) \text{ Hz}.$$

Example 4

A simply supported beam of length l and mass m_2 carries a body of mass m_1 at its midpoint. Find the lowest natural frequency of transverse vibration.

The boundary conditions are $y = 0$ and $d^2y/dx^2 = 0$ at $x = 0$ and $x = l$. These conditions are satisfied by assuming that the shape of the vibrating beam can be represented by a half sine wave. A polynomial expression can be derived for the deflected shape, but the sinusoid is usually easier to manipulate.

$y = y_0 \sin(\pi x/l)$ is a convenient expression for the beam shape, which agrees with the boundary conditions. Now

$$\dot{y} = \dot{y}_0 \sin\left(\frac{\pi x}{l}\right) \quad \text{and} \quad \frac{d^2 y}{dx^2} = -y_0 \left(\frac{\pi}{l}\right)^2 \sin\left(\frac{\pi x}{l}\right).$$

Hence

$$\int_0^l EI\left(\frac{d^2 y}{dx^2}\right)^2 dx = \int_0^l EI y_0^2 \left(\frac{\pi}{l}\right)^4 \sin^2\left(\frac{\pi x}{l}\right) dx$$

$$= EI y_0^2 \left(\frac{\pi}{l}\right)^4 \frac{l}{2},$$

and

$$\int y^2 \, dm = \int_0^l y_0^2 \sin^2\left(\frac{\pi x}{l}\right) \frac{m_2}{l} dx + y_0^2 m_1$$

$$= y_0^2 \left(m_1 + \frac{m_2}{2}\right). \quad \text{Thus}$$

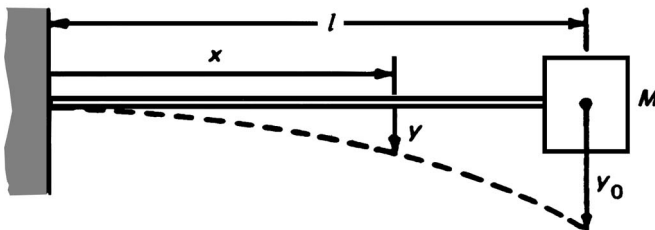$$\omega^2 = \frac{EI(\pi/l)^4 \, l/2}{(m_1 + m_2/2)}$$

If $m_2 = 0$,

$$\omega^2 = \frac{EI}{2} \frac{\pi^4}{l^3 m_1} = 48.7 \frac{EI}{m_1 l^3}.$$

The exact solution is $48 \, EI/m_1 l^3$, so the Rayleigh method solution is 1.4% high.

Example 5

Find the lowest natural frequency of transverse vibration of a cantilever of mass m, which has rigid body of mass M attached at its free end.

The static deflection curve is $y = (y_0/2l^3)(3lx^2 - x^3)$. Alternatively $y = y_0(1 - \cos \pi x/2l)$ could be assumed. Hence

$$\int_0^l EI\left(\frac{d^2y}{dx^2}\right)^2 dx = EI\int_0^l \left(\frac{y_0}{2l^3}\right)^2 (6l - 6x)^2 dx = \frac{3EI}{l^3}y_0^2,$$

and

$$\int y^2 dm = \int_0^l y^2 \frac{m}{l} dx + y_0^2 M$$

$$= \int_0^l \frac{y_0^2}{4l^6} \frac{m}{l}(3lx^2 - x^3)^2 dx + y_0^2 M$$

$$= y_0^2\left\{M + \frac{33}{140}m\right\}.$$

Thus

$$\omega^2 = \left\{\frac{3EI}{\left(M + \frac{33}{140}m\right)l^3}\right\} (rad/s)^2.$$

Example 6

Part of an industrial plant incorporates a horizontal length of uniform pipe, which is rigidly embedded at one end and is effectively free at the other. Considering the pipe as a cantilever, derive an expression for the frequency of the first mode of transverse vibration using Rayleigh's method.

Calculate this frequency, given the following data for the pipe:

Modulus of elasticity	200 GN/m^2
Second moment of area about bending axis	0.02 m^4
Mass	6×10^4 kg
Length	30 m
Outside diameter	1 m
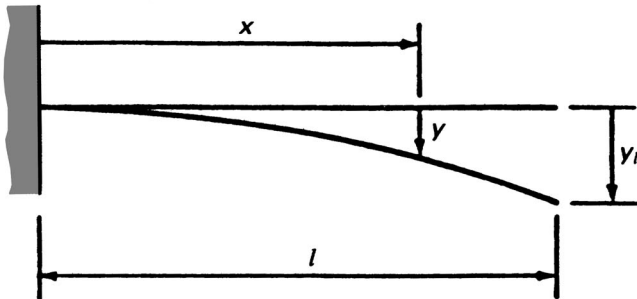
For a cantilever, assume

$$y = y_i \left(1 - \cos\frac{\pi x}{2l}\right).$$

This is compatible with zero deflection and slope when $x = 0$, and zero shear force and bending moment when $x = l$. Thus

$$\frac{d^2 y}{dx^2} = y_i \left(\frac{\pi}{2l}\right)^2 \cos\frac{\pi x}{2l}.$$

Now

$$\int_0^l EI\left(\frac{d^2 y}{dx^2}\right)^2 dx = EI\int_0^l y_i^2 \left(\frac{\pi}{2l}\right)^4 \cos^2\frac{\pi x}{2l} dx$$

$$= EIy_i^2 \left(\frac{\pi}{2l}\right)^4 \frac{l}{2}$$

and

$$\int_0^l y^2 \, dm = \int_0^l y_i^2 \left(1 - \cos\frac{\pi x}{2l}\right)^2 \frac{m}{l} dx$$

$$= y_i^2 m\left(\frac{3}{2} - \frac{4}{\pi}\right).$$

Hence, assuming the structure to be conservative, that is, the total energy remains constant throughout the vibration cycle,

$$\omega^2 = \frac{EIy_i^2 \left(\dfrac{\pi}{2l}\right)^4 \dfrac{l}{2}}{y_i^2 m\left(\dfrac{3}{2} - \dfrac{4}{\pi}\right)}$$

$$= \frac{EI}{ml^3} 13.4.$$

Thus

$$\omega = 3.66 \sqrt{\left(\frac{EI}{ml^3}\right)} \text{ rad/s} \quad \text{and} \quad f = \frac{3.66}{2\pi} \sqrt{\left(\frac{EI}{ml^3}\right)} \text{ Hz.}$$

In this case

$$\frac{EI}{ml^3} = \frac{200 \times 10^9 \times 0.02}{6 \times 10^4 \times 30^3} \bigg/ s^2.$$

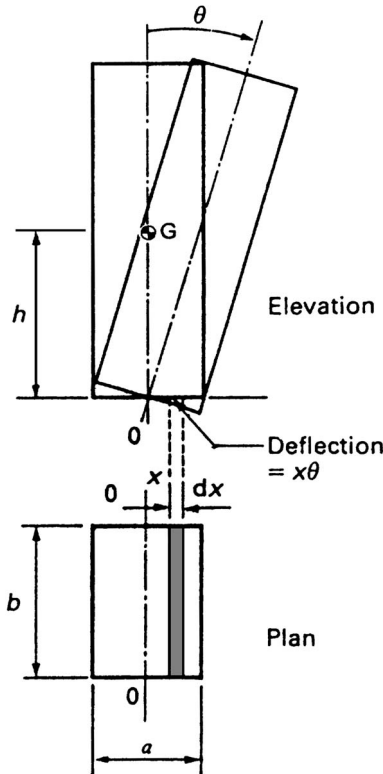Hence

$$\omega = 5.75 \text{ rad/s} \quad \text{and} \quad f = 0.92 \text{ Hz.}$$

Example 7

A uniform building of height $2h$ and mass m has a rectangular base $a \times b$ which rests on an elasic soil. The stiffness of the soil, k, is expressed as the force per unit area required to produce unit deflection.

Find the lowest frequency of free low-amplitude swaying oscillation of the building.

The lowest frequency of oscillation about the axis O–O through the base of the building is when the oscillation occurs about the shortest side, of length a.

I_0 is the mass moment of inertia of the building about axis O–O.

The FBDs are:
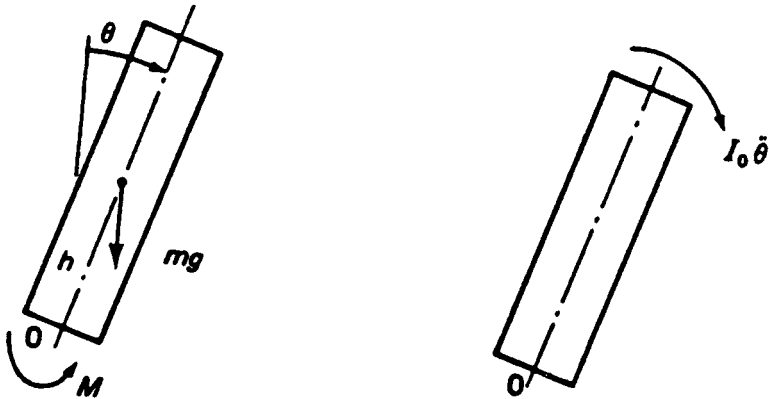
and the equation of motion for small θ is given by

$$I_0\ddot{\theta} = mgh\theta - M,$$

where M is the restoring moment from the elastic soil.

For the soil, k = force/(area × deflection), so considering an element of the base as shown, the force on element = $kb\,dx \times x\theta$, and the moment of this force about axis O–O = $kb\,dx \times x\theta x$. Thus the total restoring moment M, assuming the soil acts similarly in tension and compression, is

$$M = 2\int_0^{a/2} kbx^2\theta dx$$

$$= 2kb\theta\frac{(a/2)^3}{3} = \frac{ka^3b}{12}\theta.$$

Thus the equation of motion becomes

$$I_0\ddot{\theta} + \left(\frac{ka^3b}{12} - mgh\right)\theta = 0.$$

Motion is therefore simple harmonic, with frequency

$$f = \frac{1}{2\pi}\sqrt{\left(\frac{ka^3b/12 - mgh}{I_0}\right)} \text{ Hz.}$$

An alternative solution can be obtained by considering the energy in the system. In this case,

$$T = \tfrac{1}{2}I_0\dot{\theta}^2,$$

and

$$V = \tfrac{1}{2}.2\int_0^{a/2} kbdx \times x\theta \times x\theta - \frac{mgh\theta^2}{2},$$

where the loss in potential energy of the building weight is given by $mgh (1 - \cos \theta) \simeq mgh\theta^2/2$, since $\cos \theta \simeq 1 - \theta^2/2$ for small values of θ. Thus

$$V = \left(\frac{ka^3b}{24} - \frac{mgh}{2} \right) \theta^2.$$

Assuming simple harmonic motion, and putting $T_{max} = V_{max}$, gives

$$\omega^2 = \left(\frac{ka^3b/12 - mgh}{I_0} \right)$$

as before.

Note that for stable oscillation, $\omega > 0$, so that

$$\left(\frac{ka^3b}{12} - mgh \right) > 0,$$

that is, $ka^3b > 12mgh$.

This expression gives the minimum value of k, the soil stiffness, for stable oscillation of a particular building to occur. If k is less that $12\ mgh/a^3b$ the building will fall over when disturbed.

2.2   FREE DAMPED VIBRATION

All real structures dissipate energy when they vibrate. The energy dissipated is often very small, so that an undamped analysis is sometimes realistic; but when the damping is significant its effect must be included in the analysis, particularly when the amplitude of vibration is required. Energy is dissipated by frictional effects, for example that occurring at the connection between elements, internal friction in deformed members, and windage. It is often difficult to model damping exactly because many mechanisms may be operating in a structure. However, each type of damping can be analysed, and since in many dynamic systems one form of damping predominates, a reasonably accurate analysis is usually possible.

The most common types of damping are viscous, dry friction and hysteretic. Hysteretic damping arises in structural elements due to hysteresis losses in the material.

The type and amount of damping in a structure has a large effect on the dynamic response levels.

2.2.1 Vibration with viscous damping

Viscous damping is a common form of damping which is found in many engineering systems such as instruments and shock absorbers. The viscous damping force is proportional to the first power of the velocity across the damper, and it always opposes the motion, so that the damping force is a linear continuous function of the velocity. Because the analysis of viscous damping leads to the simplest mathematical treatment, analysts sometimes approximate more complex types of damping to the viscous type.

Consider the single degree of freedom model with viscous damping shown in Fig. 2.12.
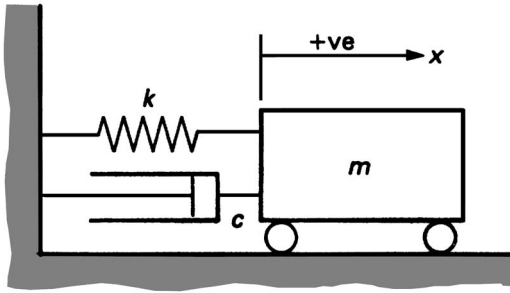
Fig. 2.12.   Single degree of freedom model with viscous damping.

The only unfamiliar element in the system is the viscous damper with coefficient c. This coefficient is such that the damping force required to move the body with a velocity $\dot{x}$ is $c\dot{x}$.

For motion of the body in the direction shown, the free body diagrams are as in Fig. 2.13(a) and (b).

Fig. 2.13.   (a) Applied force; (b) effective force.

The equation of motion is therefore

$$m\ddot{x} + c\dot{x} + kx = 0. \qquad (2.6)$$

This equation of motion pertains to the whole of the cycle: the reader should verify that this is so. (Note: displacements to the left of the equilibrium position are negative, and velocities and accelerations from right to left are also negative.)

Equation (2.6) is a second-order differential equation which can be solved by assuming a solution of the form $x = Xe^{st}$. Substituting this solution into equation (2.6) gives

$$(ms^2 + cs + k)Xe^{st} = 0.$$

Since $Xe^{st} \neq 0$ (otherwise no motion),

$$ms^2 + cs + k = 0.$$

If the roots of the equation are s_1 and s_2, then

$$s_{1,2} = -\frac{c}{2m} \pm \frac{\sqrt{(c^2 - 4mk)}}{2m}$$

Hence

$$x = X_1 e^{s_1 t} + X_2 e^{s_2 t},$$

where X_1 and X_2 are arbitrary constants found from the initial conditions. The system response evidently depends upon whether c is positive or negative, and on whether c^2 is greater than, equal to, or less than $4mk$.

The dynamic behaviour of the system depends upon the numerical value of the radical, so we define critical damping as that value of $c(c_c)$ which makes the radical zero; that is,

$$c_c = 2\sqrt{(km)}.$$

Hence

$$c_c/2m = \sqrt{(k/m)} = \omega, \quad \text{the undamped natural frequency,}$$

and

$$c_c = 2\sqrt{(km)} = 2m\omega.$$

The actual damping in a system can be specified in terms of c_c by introducing the damping ratio ζ.

Thus

$$\zeta = c/c_c,$$

and

$$s_{1,2} = [-\zeta \pm \sqrt{(\zeta^2 - 1)}]\omega. \qquad (2.7)$$

The response evidently depends upon whether c is positive or negative, and upon whether ζ is greater than, equal to, or less than unity. Usually c is positive, so we only need to consider the other possibilities.

Case 1. $\zeta < 1$; that is, damping less than critical

From equation (2.7)

$$s_{1,2} = -\zeta\omega \pm j\sqrt{(1 - \zeta^2)}\omega, \text{ where } j = \sqrt{(-1)},$$

so

$$x = e^{-\zeta\omega t}[X_1 e^{j\sqrt{(1 - \zeta^2)}\omega t} + X_2 e^{-j\sqrt{(1 - \zeta^2)}\omega t}]$$

and

$$x = Xe^{-\zeta\omega t} \sin(\sqrt{(1 - \zeta^2)}\,\omega t + \phi).$$

The motion of the body is therefore an exponentially decaying harmonic oscillation with circular frequency $\omega_v = \omega\sqrt{(1 - \zeta^2)}$, as shown in Fig. 2.14.

The frequency of the viscously damped oscillation ω_v, is given by $\omega_v = \omega\sqrt{(1 - \zeta^2)}$, that is, the frequency of oscillation is reduced by the damping action. However, in many systems this reduction is likely to be small, because very small values of ζ are common; for example, in most engineering structures ζ is rarely greater than 0.02. Even if $\zeta = 0.2$, $\omega_v = 0.98\omega$. This is not true in those cases where ζ is large, for example in motor vehicles where ζ is typically 0.7 for new shock absorbers.
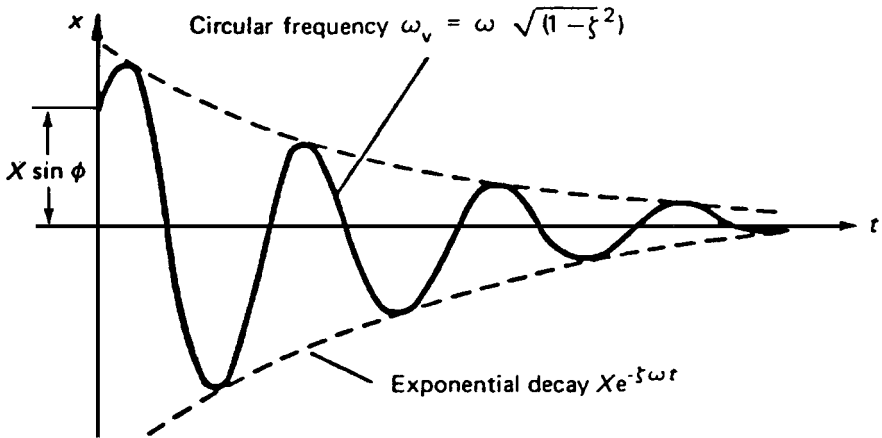
Fig. 2.14. Vibration decay of system with viscous damping, $\zeta < 1$.

Case 2. $\zeta = 1$; that is, critical damping

Both values of s are $-\omega$. However, two constants are required in the solution of equation (2.6); thus $x = (A + Bt)\mathrm{e}^{-\omega t}$ may be assumed.

Critical damping represents the limit of periodic motion; hence the displaced body is restored to equilibrium in the shortest possible time, and without oscillation or overshoot. Many devices, particularly electrical instruments, are critically damped to take advantage of this property.

Case 3. $\zeta > 1$; that is, damping greater than critical

There are two real values of s, so $x = X_1\mathrm{e}^{s_1 t} + X_2\mathrm{e}^{s_2 t}$.

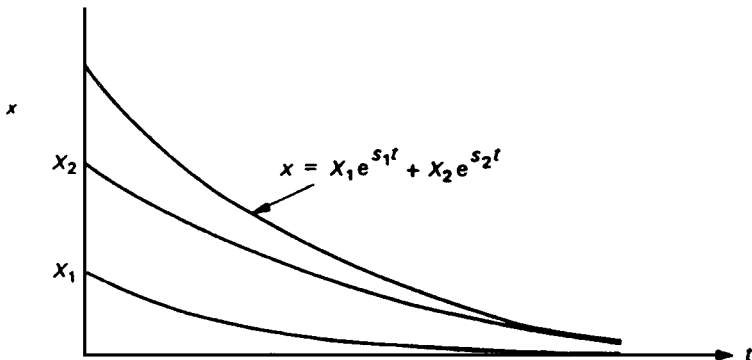Since both values of s are negative the motion is the sum of two exponential decays, as shown in Fig. 2.15.

Fig. 2.15. Disturbance decay of system with viscous damping $\zeta > 1$.

2.2.1.1 Logarithmic decrement Λ

A convenient way of determining the damping in a system is to measure the rate of decay of oscillation. It is usually not satisfactory to measure ω_v and ω because unless $\zeta > 0.2$, $\omega \simeq \omega_v$.

The logarithmic decrement, Λ, is the natural logarithm of the ratio of any two successive amplitudes in the same direction, and so from Fig. 2.16

$$\Lambda = \ln\frac{X_1}{X_{11}}$$

where X_1 and X_{11} are successive amplitudes as shown.

Since

$$x = Xe^{-\zeta\omega t}\sin(\omega_v t + \phi),$$

if

$$X_1 = Xe^{-\zeta\omega t}, \quad \text{then} \quad X_{11} = Xe^{-\zeta\omega(t + \tau_v)},$$

where τ_v is the period of the damped oscillation.
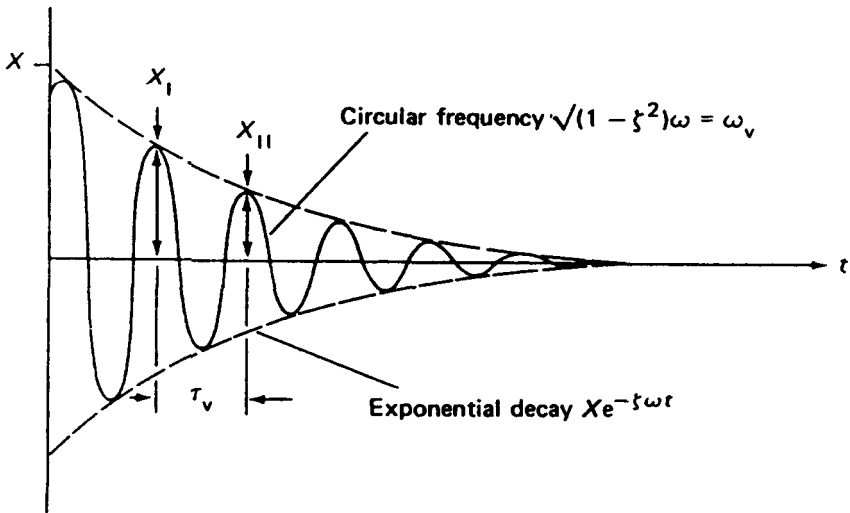

Fig. 2.16.   Vibration decay.

Thus

$$\Lambda = \ln\frac{Xe^{-\zeta\omega t}}{Xe^{-\zeta\omega(t + \tau_v)}} = \zeta\omega\tau_v.$$

Since

$$\tau_v = \frac{2\pi}{\omega_v} = \frac{2\pi}{\omega\sqrt{(1 - \zeta^2)}},$$

then

$$\Lambda = \frac{2\pi\zeta}{\sqrt{(1 - \zeta^2)}} = \ln\left(\frac{X_1}{X_{11}}\right).$$

For small values of $\zeta(\ngtr 0.25)$, $\Lambda \simeq 2\pi\zeta$.

It should be noted that this analysis assumes that the point of maximum displacement in a cycle and the point where the envelope of the decay curve $Xe^{-\zeta\omega t}$ touches the decay curve itself, are coincident. This is usually very nearly so, and the error in making this assumption is usually negligible, except in those cases where the damping is high.

For low damping it is preferable to measure the amplitude of oscillations many cycles apart so that an easily measurable difference exists.

In this case

$$\Lambda = \ln\left(\frac{X_1}{X_{11}}\right) = \frac{1}{N} \ln\left(\frac{X_1}{X_{N+1}}\right)$$

since

$$\frac{X_1}{X_{11}} = \frac{X_{11}}{X_{111}}, \text{ etc.}$$

Example 8

Consider the transverse vibration of a bridge structure. For the fundamental frequency it can be considered as a single degree of freedom system. The bridge is deflected at mid-span (by winching the bridge down) and suddenly released. After the initial disturbance the vibration was found to decay exponentially from an amplitude of 10 mm to 5.8 mm in three cycles with a frequency of 1.62 Hz. The test was repeated with a vehicle of mass 40 000 kg at mid-span, and the frequency of free vibration was measured to be 1.54 Hz.

Find the effective mass, the effective stiffness, and the damping ratio of the structure.

Let m be the effective mass and k the effective stiffness. Then

$$f_1 = 1.62 = \frac{1}{2\pi}\sqrt{\left(\frac{k}{m}\right)} \text{ Hz,}$$

and

$$f_2 = 1.54 = \frac{1}{2\pi}\sqrt{\left(\frac{k}{m + 40 \times 10^3}\right)} \text{ Hz,}$$

if it is assumed that ζ is small enough for $f_v \simeq f$.

Thus $\left(\frac{1.62}{1.54}\right)^2 = \frac{m + 40 \times 10^3}{m},$

so

$$m = 375 \times 10^3 \text{ kg.}$$

Since

$$k = (2\pi f_1)^2 m,$$
$$k = 38\ 850 \text{ kN/m.}$$

Now

$$\Lambda = \ln \frac{X_1}{X_{II}} = \tfrac{1}{3} \ln \frac{X_1}{X_{IV}} = \tfrac{1}{3} \ln \left(\frac{10}{5.8} \right)$$

$$= 0.182.$$

where X_I, X_{II} and X_{IV} are the amplitudes of the first, second and fourth cycles, respectively. Hence

$$\Lambda = \frac{2\pi\zeta}{\sqrt{(1 - \zeta^2)}} = 0.182,$$

and so $\zeta = 0.029$. (This compares with a value of about 0.002 for cast iron material. The additional damping originates mainly in the joints of the structure.) This value of ζ confirms the assumption that $f_v \simeq f$.

Example 9

A light rigid rod of length L is pinned at one end O and has a body of mass m attached at the other end. A spring and viscous damper connected in parallel are fastened to the rod at a distance a from the support. The system is set up in a *horizontal* plane: a plan view is shown.

Assuming that the damper is adjusted to provide critical damping, obtain the motion of the rod as a function of time if it is rotated through a small angle θ_0 and then released. Given that $\theta_0 = 2°$ and the undamped natural frequency of the system is 3 rad/s, calculate the displacement 1 s after release.

Explain the term *logarithmic decrement* as applied to such a system and calculate its value assuming that the damping is reduced to 80% of its critical value.
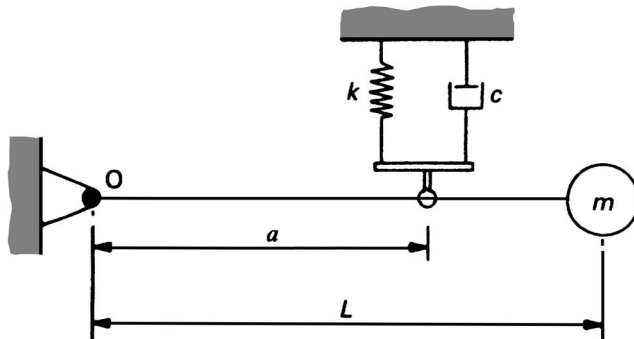
Let the rod turn through an angle θ from the equilibrium position. Note that the system oscillates in the horizontal plane so that the FBDs are:

Spring force = $ka()$
Damper force = $ca()$

Applied forces   Effective forces
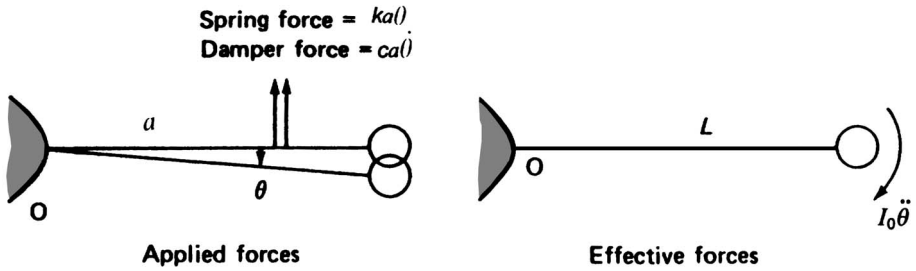
Taking moments about the pivot O gives

$$I_0\ddot{\theta} = -ca^2\dot{\theta} - ka^2\theta,$$

where $I_0 = mL^2$, so the equation of motion is

$$mL^2\ddot{\theta} + ca^2\dot{\theta} + ka^2\theta = 0.$$

Now the system is adjusted for critical damping, so that $\zeta = 1$. The solution to the equation of motion is therefore of the form

$$\theta = (A + Bt)e^{-\omega t}.$$

Now, $\theta = \theta_0$ when $t = 0$, and $d\theta/dt = 0$ when $t = 0$. Hence

$$\theta_0 = A,$$

and

$$0 = Be^{-\omega t} + (A + Bt)(-\omega)e^{-\omega t},$$

so that

$$B = \theta_0\omega.$$

Hence

$$\theta = \theta_0(1 + \omega)e^{-\omega t}.$$

If $\omega = 3$ rad/s, $t = 1$ s and $\theta_0 = 2°$,

$$\theta = 2(1 + 3)\,e^{-3} = 0.4°.$$

The logarithmic decrement

$$\Lambda = \ln\frac{X_1}{X_{11}} = \frac{2\pi\zeta}{\sqrt{(1 - \zeta^2)}},$$

so that if $\zeta = 0.8$,

$$\Lambda = \frac{5.027}{0.6} = 8.38$$

2.2.2 Vibration with Coulomb (dry friction) damping

Steady friction forces occur in many structures when relative motion takes place between adjacent members. These forces are independent of amplitude and frequency; they always oppose the motion and their magnitude may, to a first approximation, be considered constant. Dry friction can, of course, just be one of the damping mechanisms present; however, in some structures it is the main source of damping. In these cases the damping can be modelled as in Fig. 2.17.
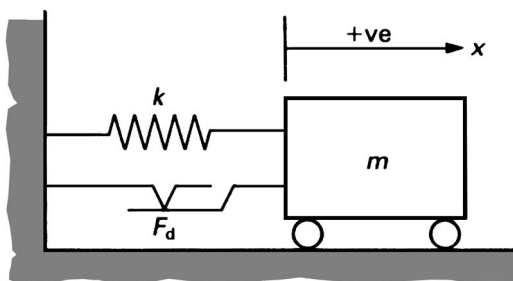

Fig. 2.17.   System with Coulomb damping.

The constant friction force F_d always opposes the motion, so that if the body is displaced a distance x_0 to the right and released from rest we have, for motion from right to left only,

$$m\ddot{x} = F_d - kx$$

or

$$m\ddot{x} + kx = F_d. \tag{2.8}$$

The solution to the complementary function is $x = A \sin \omega t + B \cos \omega t$, and the complete solution is

$$x = A \sin \omega t + B \cos \omega t + \frac{F_d}{k} \tag{2.9}$$

where $\omega = \sqrt{(k/m)}$ rad/s.

Note. The particular integral may be found by using the D-operator. Thus equation (2.8) is

$$(D^2 + \omega^2)x = F_d/m$$

so

$$\begin{aligned} x &= (1/\omega^2)[1 + (D^2/\omega^2)]^{-1}F_d/m \\ &= [1 - (D^2/\omega^2) + \cdots]F_d/m\omega^2 = F_d/k. \end{aligned}$$

The initial conditions were $x = x_0$ at $t = 0$, and $\dot{x} = 0$ at $t = 0$. Substitution into equation (2.9) gives

$$A = 0 \quad \text{and} \quad B = x_0 - \frac{F_d}{k}.$$

Hence

$$x = \left(x_0 - \frac{F_d}{k}\right) \cos \omega t + \frac{F_d}{k}. \tag{2.10}$$

At the end of the half cycle right to left, $\omega t = \pi$ and

$$x_{(t = \pi/\omega)} = -x_0 + \frac{2F_d}{k},$$

that is, there is a reduction in amplitude of $2F_d/k$ per half cycle.

From symmetry, for motion from left to right when the friction force acts in the opposite direction to the above, the initial displacement is $(x_0 - 2F_d/k)$ and the final displacement is therefore $(x_0 - 4F_d/k)$, that is, the reduction in amplitude is $4F_d/k$ per cycle. This oscillation continues until the amplitude of the motion is so small that the maximum spring force is unable to overcome the friction force F_d. This can happen whenever the amplitude is $\leqslant \pm (F_d/k)$. The manner of oscillation decay is shown in Fig. 2.18; the motion is sinusoidal for each half cycle, with successive half cycles centred on points distant $+ (F_d/k)$ and $- (F_d/k)$ from the origin. The oscillation ceases with $|x| \leqslant F_d/k$. The zone $x = \pm F_d/k$ is called the *dead zone*.
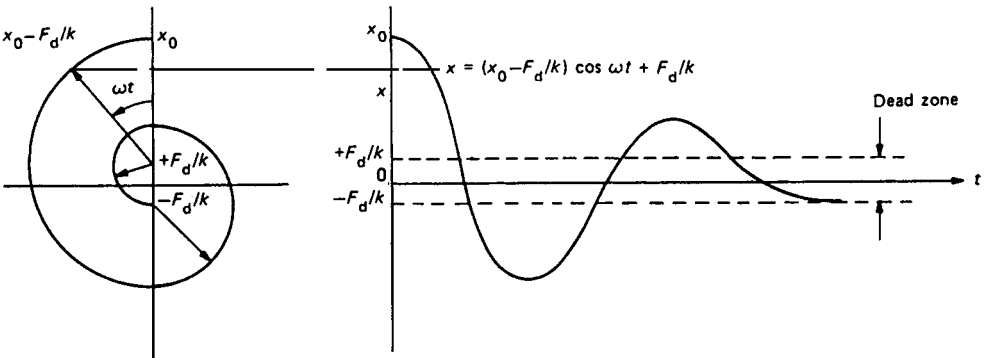

Fig. 2.18. Vibration decay of system with Coulomb damping.

To determine the frequency of oscillation we rewrite the equation of motion (2.8) as

$$m\ddot{x} + k(x - (F_d/k)) = 0.$$

Now if $x' = x - (F_d/k)$, $\ddot{x}' = \ddot{x}$ so that $m\ddot{x}' + kx' = 0$, from which the frequency of oscillation is $(1/2\pi)\sqrt{(k/m)}$ Hz; that is, the frequency of oscillation is not affected by Coulomb friction.

Example 10

Part of a structure can be modelled as a torsional system comprising a bar of stiffness 10 kN m/rad and a beam of moment of inertia about the axis of rotation of 50 kg m². The bottom guide imposes a friction torque of 10 N m (see figure).

Given that the beam is displaced through 0.05 rad from its equilibrium position and released, find the frequency of the oscillation, the number of cycles executed before the beam motion ceases, and the position of the beam when this happens.

Bar

Beam

Guide
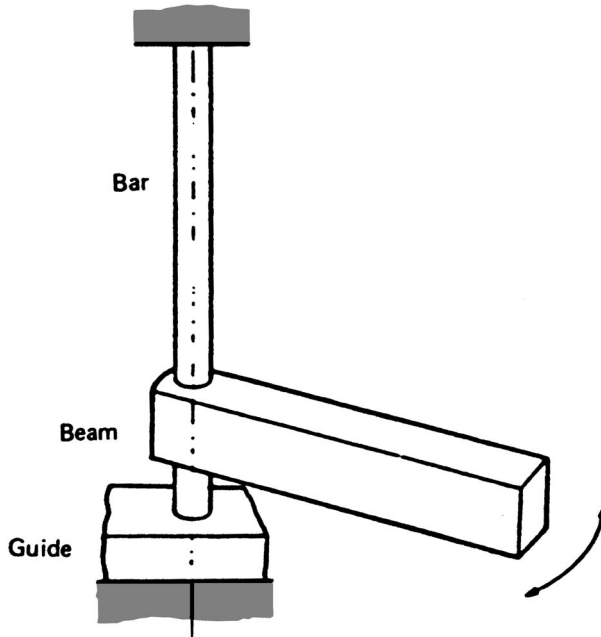
Now

$$\omega = \sqrt{\left(\frac{k_T}{I}\right)} = \sqrt{\left(\frac{10 \times 10^3}{50}\right)} = 14.14 \text{ rad/s.}$$

Hence

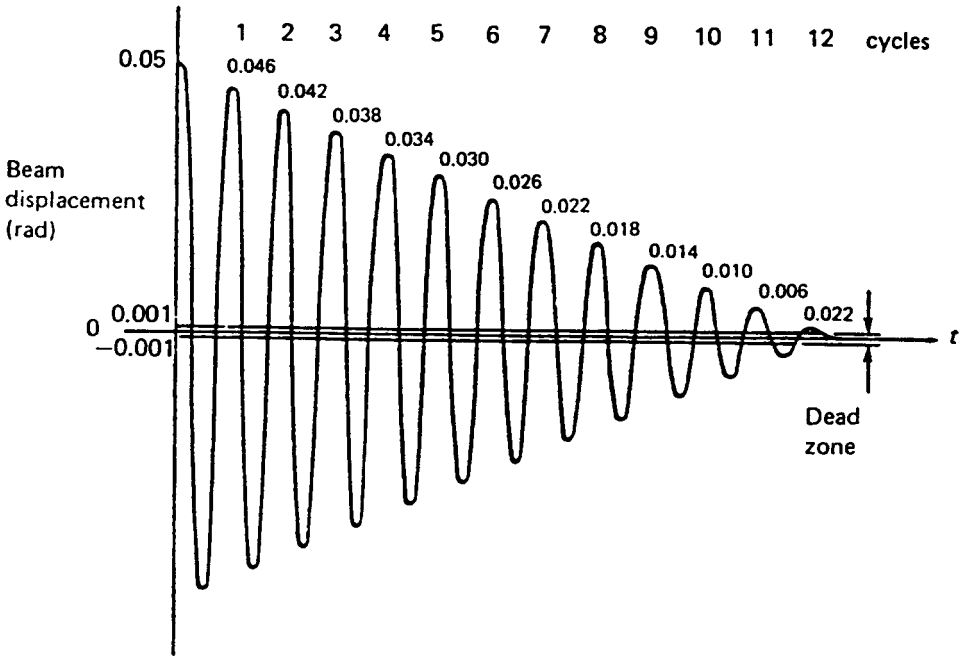$$f = \frac{14.14}{2\pi} = 2.25 \text{ Hz.}$$

$$\text{Loss in amplitude/cycle} = \frac{4F_d}{k} = \frac{4 \times 10}{10^4} \text{ rad}$$

$$= 0.004 \text{ rad.}$$

Number of cycles for motion to cease

$$= \frac{0.05}{0.004} = 12\tfrac{1}{2}.$$

The beam is in the initial (equilibrium) position when motion ceases. The motion is shown below.

2.2.3 Vibration with combined viscous and Coulomb damping

The free vibration of dynamic structures with viscous damping is characterized by an exponential decay of the oscillation, whereas structures with Coulomb damping possess a linear decay of oscillation. Many real structures have both forms of damping, so that their vibration decay is a combination of exponential and linear functions.

The two damping actions are sometimes amplitude-dependent, so that initially the decay is exponential, say, and only towards the end of the oscillation does the Coulomb effect show. In the analyses of these cases the Coulomb effect can easily be separated from the total damping to leave the viscous damping alone. The exponential decay with viscous damping can be checked by plotting the amplitudes on logarithmic–linear axes when the decay should be seen to be linear.

If the Coulomb and viscous effects cannot be separated in this way, a mixture of linear and exponential decay functions have to be found by trial and error in order to conform to the experimental data.

2.2.4 Vibration with hysteretic damping

Experiments on the damping that occurs in solid materials and structures that have been subjected to cyclic stressing have shown the damping force to be independent of frequency. This internal, or material, damping is referred to as hysteretic damping. Since the viscous damping force $c\dot{x}$ is dependent upon the frequency of oscillation, it is not a suitable way of modelling the internal damping of solids and structures. The analysis of systems and structures with this form of damping therefore requires the damping force $c\dot{x}$ to be divided by the frequency of oscillation ω. Thus the equation of motion becomes $m\ddot{x} + (c/\omega)\dot{x} + kx = 0$.

However, it has been observed from experiments carried out on many materials and structures that under harmonic forcing the stress leads the strain by a constant angle, α.

Thus for an harmonic strain, $\varepsilon = \varepsilon_0 \sin vt$, where v is the forcing frequency, the induced stress is $\sigma = \sigma_0 \sin (vt + \alpha)$. Hence

$$\sigma = \sigma_0 \cos \alpha \sin vt + \sigma_0 \sin \alpha \cos vt$$

$$= \sigma_0 \cos \alpha \sin vt + \sigma_0 \sin \alpha \sin \left(vt + \frac{\pi}{2} \right).$$

The first component of stress is in phase with the strain ε, whilst the second component is in quadrature with ε and $\pi/2$ ahead. Putting $j = \sqrt{(-1)}$,

$$\sigma = \sigma_0 \cos \alpha \sin vt + j\sigma_0 \sin \alpha \sin vt.$$

Hence a complex modulus E^* can be formulated, where

$$E^* = \frac{\sigma}{\varepsilon} = \frac{\sigma_0}{\varepsilon_0} \cos \alpha + j \frac{\sigma_0}{\varepsilon_0} \sin \alpha$$

$$= E' + jE'',$$

where E' is the in-phase or storage modulus, and E'' is the quadrature or loss modulus.

The loss factor η, which is a measure of the hysteretic damping in a structure, is equal to E''/E', that is, $\tan \alpha$.

It is not usually possible to separate the stiffness of a structure from its hysteretic damping, so that in a mathematical model these quantities have to be considered together. The complex stiffness k^* is given by $k^* = k(1 + j\eta)$, where k is the static stiffness and η is the hysteretic damping loss factor.

2.2.5 Complex stiffness

In most real structures it is not possible to separate the stiffness and damping effects because they are inherent properties which are often coupled. Realistic mathematical models of structures therefore require these quantities to be considered together in the

form of a complex stiffness. Although this is rather an awkward physical concept it is widely used in analysis.

The complex stiffness k^* is equal to $k(1 + j\eta)$, where k is the static stiffness, $j = \sqrt{-1}$ and η is the hysteretic damping loss factor.

The equation of free motion for a single degree of freedom system with hysteretic damping is therefore $m\ddot{x} + k^*x = 0$. Fig. 2.19 shows a single degree of freedom model with hysteretic damping of coefficient c_H.
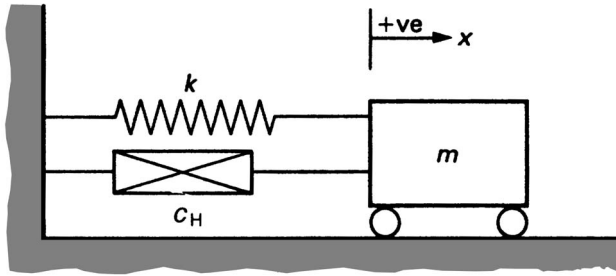

Fig. 2.19.   Single degree of freedom with hysteretic damping.

The equation of motion is

$$m\ddot{x} + (c_H/\omega)\dot{x} + kx = 0.$$

Now if $x = Xe^{j\omega t}$,

$$\dot{x} = j\omega x \quad \text{and} \quad \left(\frac{c_H}{\omega}\right)\dot{x} = jc_Hx.$$

Thus the equation of motion becomes

$$m\ddot{x} + (k + jc_H)x = 0.$$

Since

$$k + jc_H = k\left(1 + \frac{jc_H}{k}\right) = k(1 + j\eta) = k^*,$$

we can write

$$m\ddot{x} + k^*x = 0,$$

that is, the combined effect of the elastic and hysteretic resistance to motion can be represented as a complex stiffness, k^*.

A range of values of η for some common engineering materials is given in the following table. For more detailed information on material damping mechanisms and loss factors, see *Damping of Materials and Members in Structural Mechanics* by B. J. Lazan (Pergamon, 1968), and Chapter 5.

Material	Loss factor
Aluminium–pure	0.00002–0.002
Aluminium alloy–dural	0.0004–0.001
Steel	0.001–0.008
Lead	0.008–0.014
Cast iron	0.003–0.03
Manganese copper alloy	0.05–0.1
Rubber–natural	0.1–0.3
Rubber–hard	1.0
Glass	0.0006–0.002
Concrete	0.01–0.06

2.2.6 Energy dissipated by damping

The energy dissipated per cycle by the viscous damping force in a single degree of freedom vibrating system is approximately

$$4 \int_0^x c\dot{x} \, dx,$$

if $x = X \sin \omega t$ is assumed for the complete cycle. The energy dissipated is therefore

$$4 \int_0^{\pi/2} cX^2 \omega^2 \cos^2 \omega t \, dt = \pi c \omega X^2.$$

The energy dissipated per cycle by Coulomb damping is $4F_d X$ approximately. Thus an equivalent viscous damping coefficient for Coulomb damping c_d can be deduced, where

$$\pi c_d \omega X^2 = 4F_d X,$$

that is,

$$c_d = \frac{4F_d}{\pi \omega X}.$$

The energy dissipated per cycle by a force F acting on a system with hysteretic damping is $\int F \, dx$, where $F = k^*x = k(1 + j\eta)x$, and x is the displacement.

For harmonic motion $x = X \sin \omega t$,

so

$$F = kX \sin \omega t + j\eta kX \sin \omega t$$
$$= kX \sin \omega t + \eta kX \cos \omega t.$$

Now

$$\sin \omega t = \frac{x}{X}, \quad \text{therefore} \quad \cos \omega t = \frac{\sqrt{(X^2 - x^2)}}{X}.$$

Thus

$$F = kx \pm \eta k \sqrt{(X^2 - x^2)}.$$

This is the equation of an ellipse as shown in Fig. 2.20. The energy dissipated is given by the area enclosed by the ellipse.
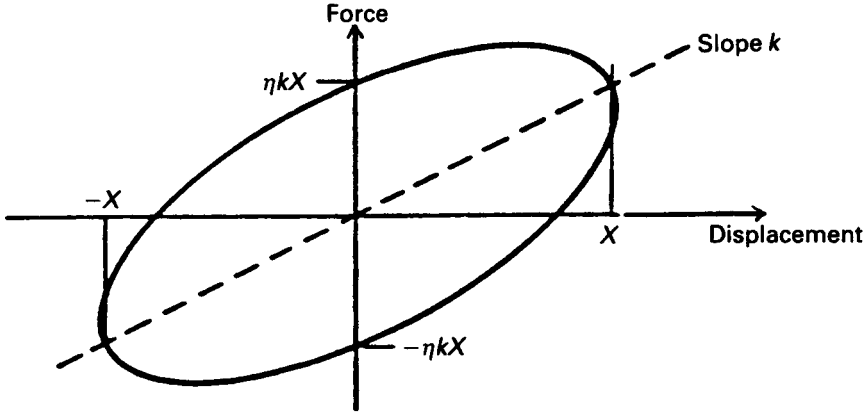

Fig. 2.20.   Elliptical force–displacement relationship for a system with hysteretic damping.

Hence

$$\int F dx = \int_0^x (kx \pm \eta k \sqrt{(X^2 - x^2)}) dx$$
$$= \pi X^2 \eta k.$$

An equivalent viscous damping coefficient c_H is given by

$$\pi c_H \omega X^2 = \pi \eta k X^2,$$

that is

$$c_H = \frac{\eta k}{\omega},$$

Note also that $c = c_H \omega$

Example 11

A single degree of freedom system has viscous damping, with $\zeta = 0.02$. Find the energy dissipated per cycle as a function of the energy in the system at the start of that cycle. Also find the amplitude of the 12th cycle if the amplitude of the 3rd cycle is 1.8 mm.

$$\zeta \ll 1, \text{ so } \ln (X_1/X_2) = 2\pi\zeta = 0.126.$$

Thus

$$X_1/X_2 = e^{0.126} = 1.134.$$

Energy at start of cycle $= \frac{1}{2}kX_1^2$ (stored as strain energy in spring)

Energy at end of cycle $= \frac{1}{2}kX_2^2$

$$\frac{\text{Energy dissipated during cycle}}{\text{Energy at start of cycle}} = \frac{\frac{1}{2}kX_1^2 - \frac{1}{2}kX_2^2}{\frac{1}{2}kX_1^2} = 1 - (X_2/X_1)^2 = 1 - 0.7773 = 0.223,$$

that is, 22.3% of the initial energy is dissipated in one cycle.
   Now

$$X_1/X_2 = 1.134, \quad X_2/X_3 = 1.134, \dots, \quad (X_{n-1})/X_n = 1.134.$$

therefore

$$X_3/X_{12} = (1.134)^9 = 3.107$$

that is

$$X_{12} = \frac{1.8}{3.107} = 0.579 \text{ mm.}$$

2.3   FORCED VIBRATION

Many real structures are subjected to periodic excitation. This may be due to unbalanced rotating or reciprocating components of machinery or equipment, wind or current effects, or a shaking foundation. Usually very low vibration amplitudes are required over a large range of exciting forces and frequencies to keep dynamic stresses, noise, fatigue and other effects to acceptable levels. Some periodic forces are harmonic, but even if they are not, they can be represented as a series of harmonic functions using Fourier analysis techniques. Because of this the response of structures and dynamic systems subjected to harmonic exciting forces and motions must be studied. Non-periodic excitation such as shock, impulse and random, are considered later, in sections 2.3.5 to 2.3.9.

2.3.1   Response of a viscous damped structure to a simple harmonic exciting force with constant amplitude

In the system shown in Fig. 2.21, the body of mass m is connected by a spring and viscous damper to a fixed support, whilst a harmonic force of circular frequency v and amplitude F acts upon it, in the line of motion.
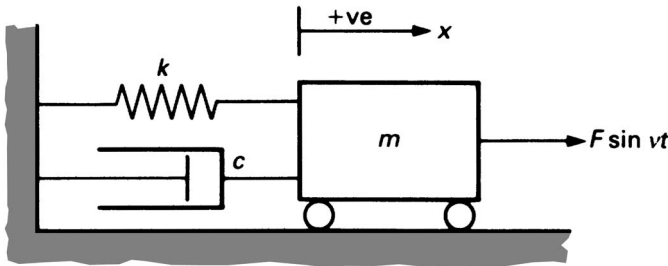

Fig. 2.21.   Single degree of freedom model of a forced system with viscous damping.

The equation of motion is

$$m\ddot{x} + c\dot{x} + kx = F \sin vt. \tag{2.11}$$

The solution to $m\ddot{x} + c\dot{x} + kx = 0$, which has already been studied, is the complementary function; it represents the initial vibration which quickly dies away. The sustained motion is given by the particular solution. A solution $x = X (\sin vt - \phi)$ can be assumed, because this represents simple harmonic motion at the frequency of the exciting force with a displacement vector which lags the force vector by ϕ, that is, the motion occurs after the application of the force.

Assuming $x = X \sin(vt - \phi)$,

$$\dot{x} = Xv \cos (vt - \phi) = Xv \sin (vt - \phi + \tfrac{1}{2}\pi),$$

and

$$\ddot{x} = -Xv^2 \sin (vt - \phi) = Xv^2 \sin (vt - \phi + \pi).$$

The equation of motion (2.11) thus becomes

$$mXv^2 \sin (vt - \phi + \pi) + cXv \sin (vt - \phi + \pi/2) + kX \sin (vt - \phi)$$
$$= F \sin vt.$$

A vector diagram of these forces can now be drawn (Fig. 2.22).

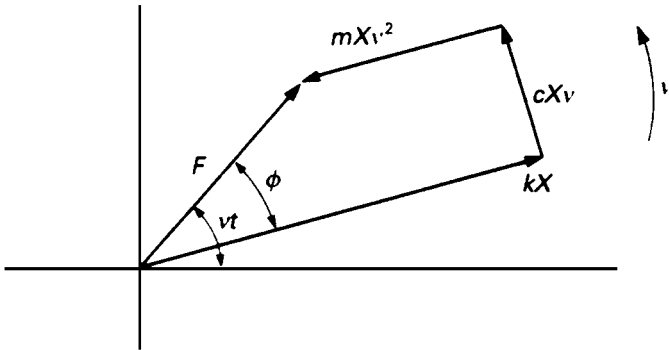Fig. 2.22.   Force vector diagram.

From the diagram,

$$F^2 = (kX - mXv^2)^2 + (cXv)^2,$$

or

$$X = F/\sqrt{((k - mv^2)^2 + (cv)^2)}, \tag{2.12}$$

and

$$\tan \phi = cXv/(kX - mXv^2).$$

Thus the steady-state solution to equation (2.11) is

$$x = \frac{F}{\sqrt{((k - mv^2)^2 + (cv)^2)}} \sin (vt - \phi),$$

where

$$\phi = \tan^{-1}\left(\frac{cv}{k - mv^2}\right).$$

The complete solution includes the transient motion given by the complementary function:

$$x = Ae^{-\zeta\omega t} \sin(\omega\sqrt{(1 - \zeta^2)}t + \alpha).$$

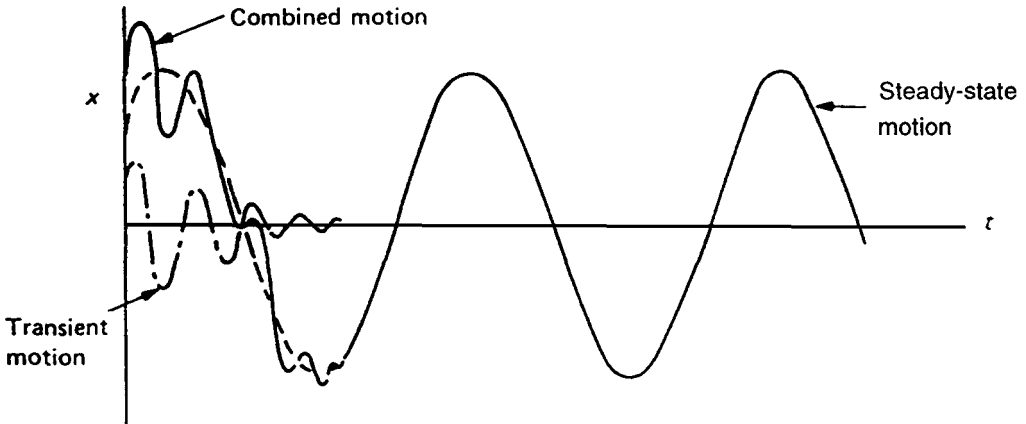Fig. 2.23 shows the combined motion.

Fig. 2.23.   Forced vibration, combined motion.

Equation (2.12) can be written in a more convenient form if we put

$$\omega = \sqrt{\left(\frac{k}{m}\right)} \text{ rad/s} \quad \text{and} \quad X_s = \frac{F}{k}.$$

Then

$$\frac{X}{X_s} = \frac{1}{\sqrt{\left\{\left[1 - \left(\frac{v}{\omega}\right)^2\right]^2 + \left[2\zeta\frac{v}{\omega}\right]^2\right\}}}, \qquad (2.13)$$

and $\phi = \tan^{-1}\left[\dfrac{2\zeta(v/\omega)}{1 - (v/\omega)^2}\right]$

X/X_s is known as the dynamic magnification factor, because X_s is the static deflection of the system under a steady force F, and X is the dynamic amplitude.

By considering different values of the frequency ratio v/ω, we can plot X/X_s and ϕ as functions of frequency for various values of ζ. Figs 2.24 and 2.25 show the results.

The effect of the frequency ratio on the force vector diagram is shown in Fig. 2.26.

The importance of mechanical vibration arises mainly from the large values of X/X_s experienced in practice when v/ω has a value near unity: this means that a small harmonic
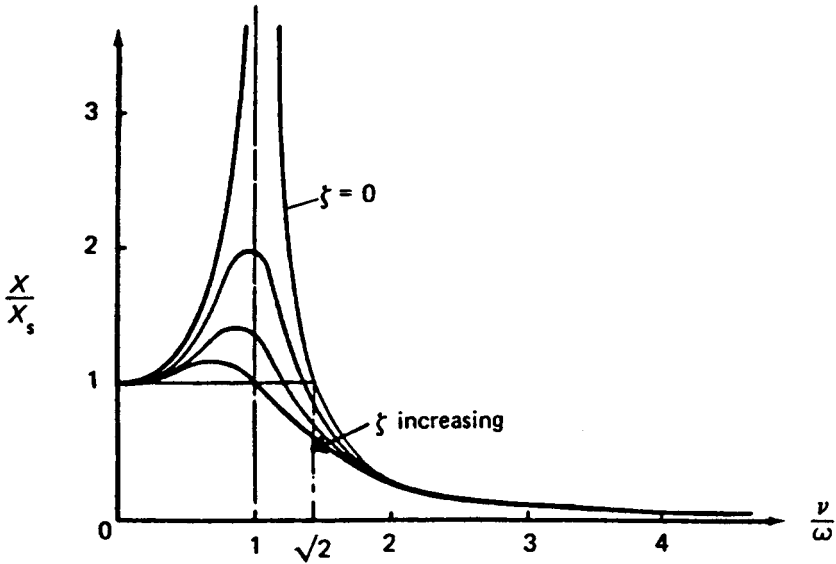
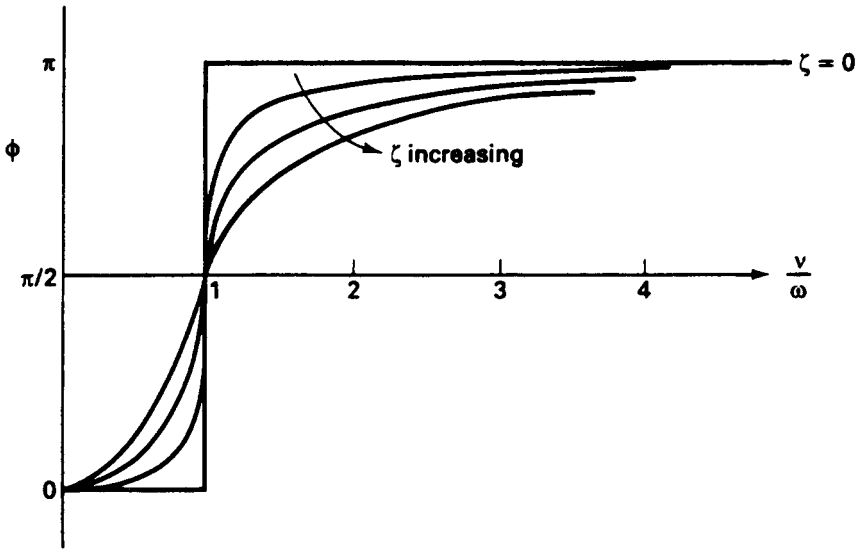Fig. 2.24. Amplitude–frequency response for system of Fig. 2.21.

Fig. 2.25. Phase–frequency response for system of Fig. 2.21.

force can produce a large amplitude of vibration. The phenomenon known as *resonance* occurs when the forcing frequency is equal to the natural frequency, that is when $v/\omega = 1$. The maximum value of X/X_s actually occurs at values of v/ω less than unity: the value can be found by differentiating equation (2.13) with respect to v/ω. Hence

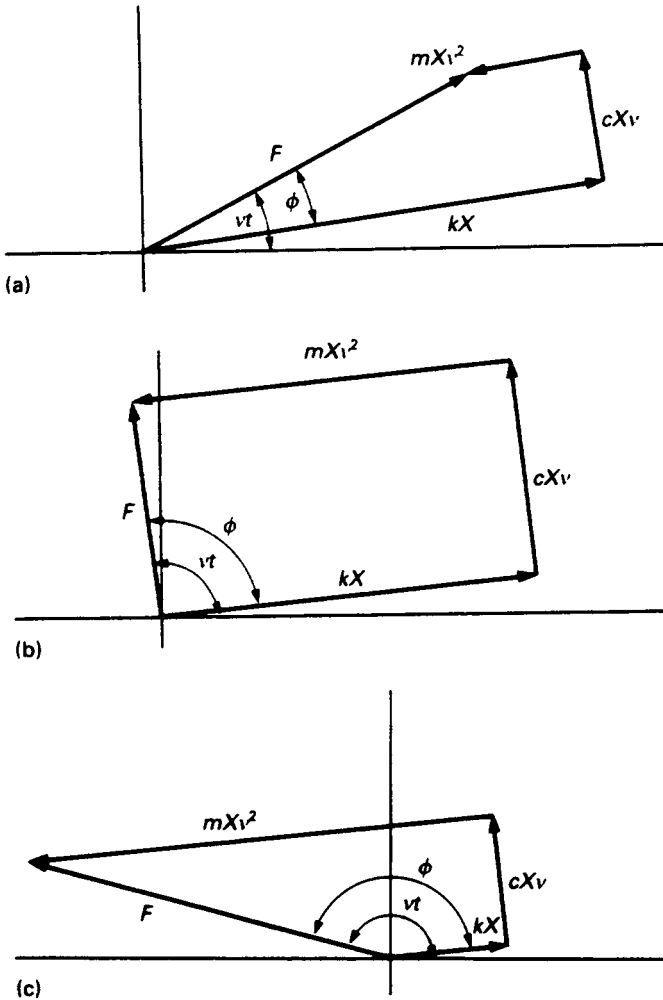$$(v/\omega)_{(X/X_s)\mathrm{max}} = \sqrt{(1 - 2\zeta^2)} \simeq 1 \text{ for } \zeta \text{ small,}$$

Fig. 2.26. Forced vibration vector diagrams: (a) $v/\omega \ll 1$, exciting force approximately equal to spring force; (b) $v/\omega = 1$, exciting force equal to damping force, and inertia force equal to spring force; (c) $v/\omega \gg 1$, exciting force nearly equal to inertia force.

and

$$(X/X_s)_{max} = 1/(2\zeta\sqrt{(1 - \zeta^2)}).$$

For small values of ζ, $(X/X_s)_{max} \simeq 1/2\zeta$ which is the value pertaining to $v/\omega = 1$; $1/2\zeta$ is a measure of the damping in a system and is referred to as the Q factor.

Both *reciprocating* and *rotating unbalance* in a system produce an exciting force of the inertia type and result in the amplitude of the exciting force being proportional to the square of the frequency of excitation.

For an unbalanced body of mass m_r at an effective radius r, rotating at an angular speed v, the exciting force is therefore $m_r rv^2$. If this force is applied to a single degree of freedom

system such as that in Fig. 2.21, the component of the force in the direction of motion is $m_r r v^2 \sin vt$, and the amplitude of vibration is

$$X = \frac{(m_r/m)r(v/\omega)^2}{\sqrt{((1 - (v/\omega)^2)^2 + (2\zeta \, v/\omega)^2)}}. \tag{2.14}$$

(see equation (2.13)).

The value of v/ω for maximum X found by differentiating equation (2.14) is given by

$$(v/\omega)_{Xmax} = 1/\sqrt{(1 - 2\zeta^2)}$$

that is, the peak of the response curve occurs when $v > \omega$. This is shown in Fig. 2.27. Also,

$$X_{max} = (m_r/m)r/2\zeta\sqrt{(1 - \zeta^2)}.$$

It can be seen that away from the resonance condition ($v/\omega = 1$) the system response is not greatly affected by damping unless this happens to be large. Since in most mechanical systems the damping is small ($\zeta < 0.1$) it is often permissible to neglect the damping when evaluating the frequency for maximum amplitude and also the amplitude–frequency response away from the resonance condition.
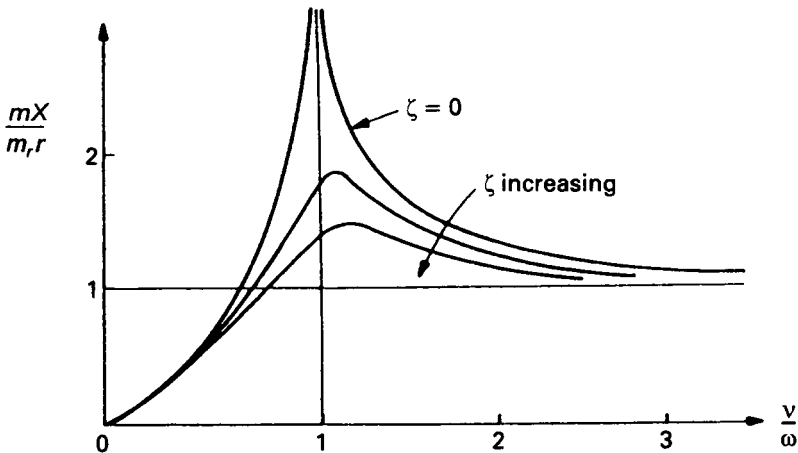

Fig. 2.27.   Amplitude–frequency response, with rotating unbalance excitation.

It can be seen from Figs 2.24, 2.26 and 2.27 that the system response at low frequencies ($\ll \omega$) is stiffness-dependent, and that in the region of resonance the response is damping-dependent, whereas at high frequencies ($\gg \omega$) the response is governed by the system mass. It is most important to realize this when attempting to reduce the vibration of a structure. For example, the application of increased damping will have little effect if the excitation and response frequencies are in a region well away from resonance, such as that controlled by the mass of the structure.

2.3.2 Response of a viscous damped structure supported on a foundation subjected to harmonic vibration

The system considered is shown in Fig. 2.28. The foundation is subjected to harmonic vibration $A \sin vt$ and it is required to determine the response, x, of the body.
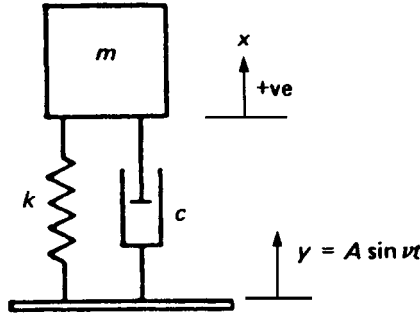

Fig. 2.28.   Single degree of freedom model of a vibrated system with viscous damping.

The equation of motion is

$$m\ddot{x} = c(\dot{y} - \dot{x}) + k(y - x). \tag{2.15}$$

If the displacement of the body relative to the foundation, u, is required, we may write $u = x - y$, and equation (2.15) becomes

$$m\ddot{u} + c\dot{u} + ku = -m\ddot{y} = mv^2 A \sin vt.$$

This equation is similar to (2.11) so that the solution may be written directly as

$$u = \frac{A(v/\omega)^2}{\sqrt{((1 - (v/\omega)^2)^2 + (2\zeta v/\omega)^2)}} \sin\left(vt - \tan^{-1}\frac{2\zeta(v/\omega)}{1 - (v/\omega)^2}\right).$$

If the absolute motion of the body is required we rewrite equation (2.15) as

$$
\begin{aligned}
m\ddot{x} + c\dot{x} + kx &= c\dot{y} + ky \\
&= cAv \cos vt + kA \sin vt \\
&= A\sqrt{(k^2 + (cv)^2)} \sin (vt + \alpha)
\end{aligned}
$$

where

$$\alpha = \tan^{-1}\frac{cv}{k}.$$

Hence, from the previous result,

$$x = \frac{A\sqrt{(k^2 + (cv)^2)}}{\sqrt{((k - mv^2)^2 + (cv)^2)}} \sin (vt - \phi + \alpha).$$

The motion transmissibility is defined as the ratio of the amplitude of the absolute body vibration to the amplitude of the foundation vibration. Thus,

$$\text{motion transmissibility} = \frac{X}{A}$$

$$= \frac{\sqrt{\left[1 + \left(2\zeta\dfrac{v}{\omega}\right)^{2}\right]}}{\sqrt{\left\{\left[1 - \left(\dfrac{v}{\omega}\right)^{2}\right]^{2} + \left[2\zeta\dfrac{v}{\omega}\right]^{2}\right\}}}.$$

2.3.2.1 Vibration isolation

The dynamic forces produced by machinery are often very large. However, the force transmitted to the foundation or supporting structure can be reduced by using flexible mountings with the correct properties; alternatively a machine can be isolated from foundation vibration by using the correct flexible mountings. Fig. 2.29 shows a model of such a system.
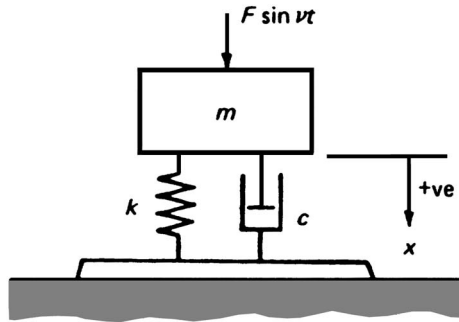

Fig. 2.29.   Single degree of freedom system with foundation.

The force transmitted to the foundation is the sum of the spring force and the damper force. Thus the transmitted force $= kx + c\dot{x}$ and F_T, the amplitude of the transmitted force is given by

$$F_T = \sqrt{[(kX)^2 + (cvX)^2]}.$$

The force transmission ratio or transmissibility, T_R, is given by

$$T_R = \frac{F_T}{F} = \frac{X\sqrt{[k^2 + (cv)^2]}}{F}$$

since

$$X = \frac{F/k}{\sqrt{\left\{\left[1 - \left(\dfrac{v}{\omega}\right)^{2}\right]^{2} + \left[2\zeta\dfrac{v}{\omega}\right]^{2}\right\}}},$$

$$T_{R} = \frac{\sqrt{\left[1 + \left(2\zeta\dfrac{v}{\omega}\right)^2\right]}}{\sqrt{\left\{\left[1 - \left(\dfrac{v}{\omega}\right)^2\right]^2 + \left[2\zeta\dfrac{v}{\omega}\right]^2\right\}}}.$$

Therefore the force and motion transmissibilities are the same.

The effect of v/ω on T_R is shown in Fig. 2.30. It can be seen that for good isolation $v/\omega > \sqrt{2}$, hence a low value of ω is required which implies a low stiffness, that is, a flexible mounting: this may not always be acceptable in practice where a certain minimum stiffness is usually necessary to satisfy operating criteria.
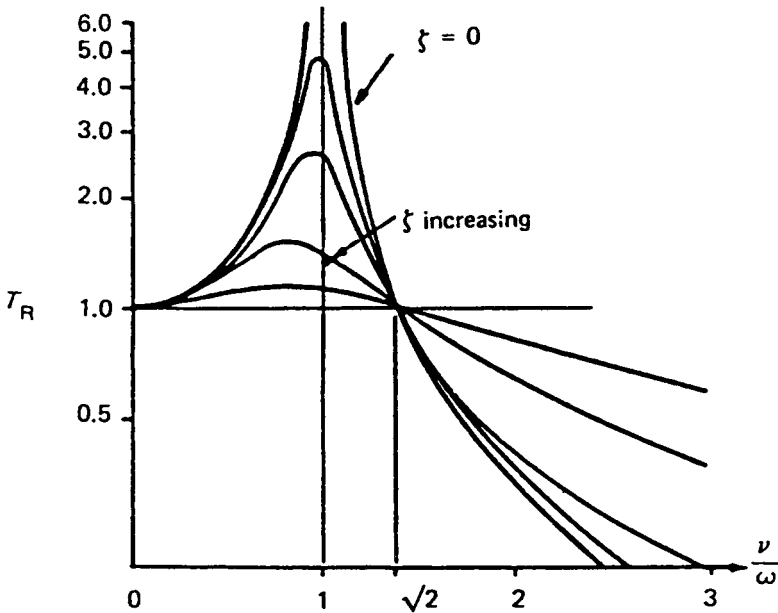

Fig. 2.30.   Transmissibility–frequency ratio response.

It is particularly important to be able to isolate vibration sources because structure-borne vibration can otherwise be easily transmitted to parts that radiate well, and serious noise problems can occur. Theoretically, low stiffness isolators are desirable to give a low natural frequency. However, this often results in isolators that are too soft and stability problems may arise. The system can be attached rigidly to a large block which effectively increases its mass so that stiffer isolators can be used. The centre of mass of the combined system is also lowered, giving improved stability. For the best response a mounting system may be designed with snubbers, which control the large amplitudes while providing little or no damping when the amplitudes are small.

There are four types of spring material commonly used for resilient mountings and vibration isolation: air, metal, rubber and cork. Air springs can be used for very low-frequency suspensions: resonance frequencies as low as 1 Hz can be achieved whereas

metal springs can only be used for resonance frequencies greater than about 1.3 Hz. Metal springs can transmit high frequencies, however, so rubber or felt pads are often used to prohibit metal-to-metal contact between the spring and the structure. Different forms of spring element can be used such as coil, torsion, cantilever and beam. Rubber can be used in shear or compression but rarely in tension. It is important to determine the dynamic stiffness of a rubber isolator because this is generally much greater than the static stiffness. Also rubber possesses some inherent damping although this may be sensitive to amplitude, frequency and temperature. Natural frequencies from 5 Hz upwards can be achieved. Cork is one of the oldest materials used for vibration isolation. It is usually used in compression and natural frequencies of 25 Hz upwards are typical.
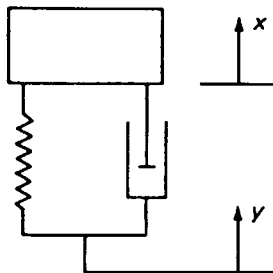
Example 12

A spring-mounted body moves with velocity v along an undulating surface, as shown.

The body has a mass m and is connected to the wheel by a spring of stiffness k, and a viscous damper whose damping coefficient is c. The undulating surface has a wavelength L and an amplitude h.

Derive an expression for the ratio of amplitudes of the absolute vertical displacement of the body to the surface undulations.
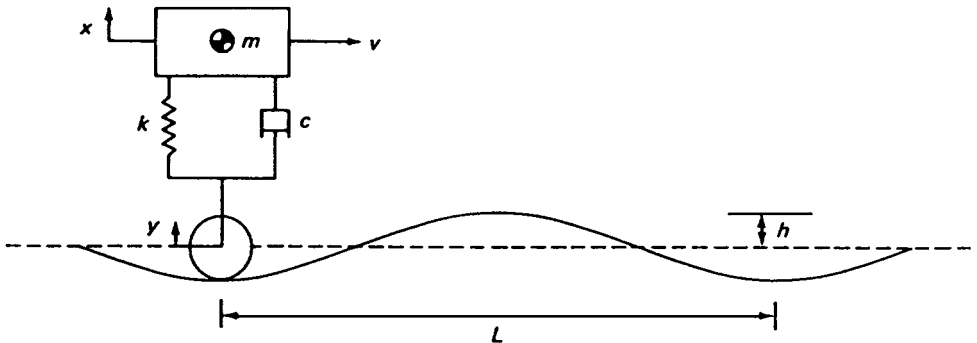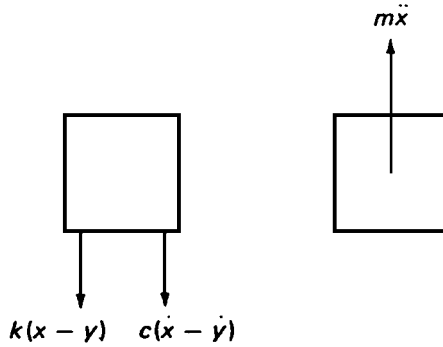

The system can be considered as

where

$$y = h \cos \frac{2\pi z}{L} \text{ and } z = \upsilon t.$$

Hence

$$y = h \cos \frac{2\pi \upsilon}{L} t = h \cos \nu t, \text{ where } \nu = \frac{2\pi \upsilon}{L}.$$

The FBDs are

$$m\ddot{x}$$

$$k(x - y) \quad c(\dot{x} - \dot{y})$$

Hence the equation of motion is

$$m\ddot{x} = -k(x - y) - c(\dot{x} - \dot{y}),$$

or

$$m\ddot{x} + c\dot{x} + kx = c\dot{y} + ky.$$

Now

$$y = h \cos \nu t,$$

so

$$m\ddot{x} + c\dot{x} + kx = \sqrt{[k^2 + (c\nu)^2]}h \sin (\nu t + \phi).$$

Hence, if $x = X_0 \sin (\nu t + \alpha)$, then

$$X_0 = \frac{h\sqrt{[k^2 + (c\nu)^2]}}{\sqrt{[(k - m\nu^2)^2 + (c\nu)^2]}}$$

So,

$$\frac{X_0}{h} = \frac{\sqrt{\left[k^2 + \left(\frac{2\pi\upsilon}{L}c\right)^2\right]}}{\sqrt{\left\{\left[k - \left(\frac{2\pi\upsilon}{L}\right)^2 m\right]^2 + \left(\frac{2\pi\upsilon}{L}c\right)^2\right\}}}$$

Example 13

The vibration on the floor in a building is SHM at a frequency in the range 15–60 Hz. It is desired to install sensitive equipment in the building which must be insulated from floor vibration. The equipment is fastened to a small platform which is supported by three similar springs resting on the floor, each carrying an equal load. Only vertical motion occurs. The combined mass of the equipment and platform is 40 kg, and the equivalent viscous damping ratio of the suspension is 0.2.

Find the maximum value for the spring stiffness, if the amplitude of transmitted vibration is to be less than 10% of the floor vibration over the given frequency range.

$$T_R = \frac{\sqrt{\left[1 + \left(2\zeta\dfrac{v}{\omega}\right)^2\right]}}{\sqrt{\left\{\left[1 - \left(\dfrac{v}{\omega}\right)^2\right]^2 + \left[2\zeta\dfrac{v}{\omega}\right]^2\right\}}}$$

$T_R = 0.1$ with $\zeta = 0.2$ is required, thus

$$\left[1 - \left(\frac{v}{\omega}\right)^2\right]^2 + \left[0.4\left(\frac{v}{\omega}\right)\right]^2 = 100\left[1 + \left(0.4\frac{v}{\omega}\right)^2\right],$$

that is,

$$\left(\frac{v}{\omega}\right)^4 - 17.84\left(\frac{v}{\omega}\right)^2 - 99 = 0.$$

Hence

$$\frac{v}{\omega} = 4.72.$$

When

$$v = 15 \times 2\pi \text{ rad/s}, \ \omega = 19.97 \text{ rad/s}.$$

Since

$$\omega = \sqrt{\left(\frac{k}{m}\right)} \quad \text{and} \quad m = 40 \text{ kg},$$

total $k = 15\ 935$ N/m,

that is, the stiffness of each spring $= 15\ 935/3$ N/m $= 5.3$ kN/m.

The amplitude of the transmitted vibration will be less than 10% at frequencies above 15 Hz.

Example 14

A machine of mass m generates a disturbing force $F \sin vt$; to reduce the force transmitted to the supporting structure, the machine is mounted on a spring of stiffness k with a damper in parallel. Compare the effectiveness of this isolation system for viscous and hysteretic damping.
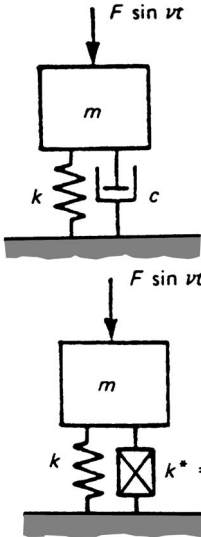
Viscous damping. From section 2.3.2.1,

$$T_R = \frac{F_T}{F} = \frac{\sqrt{\left[1 + \left(2\zeta\frac{v}{\omega}\right)^2\right]}}{\sqrt{\left\{\left[1 - \left(\frac{v}{\omega}\right)^2\right]^2 + \left[2\zeta\frac{v}{\omega}\right]^2\right\}}}$$

Hysteretic damping. From section 2.2.6,

Putting $\eta = \dfrac{cv}{k} = 2\zeta\dfrac{v}{\omega}$,

$$T_R = \frac{F_T}{F} = \frac{\sqrt{(1 + \eta^2)}}{\sqrt{\left\{\left[1 - \left(\frac{v}{\omega}\right)^2\right]^2 + \eta^2\right\}}}$$

The effectiveness of these isolators can be compared using these expressions for T_R. The results are given in the table below.

It can be seen that the isolation effects are similar for the viscous and hysteretically damped isolators, except at high frequency ratios when the hysteretic damping gives much better attentuation of T_R. At these frequencies it is better to decouple the viscous damped isolator by attaching small springs or rubber bushes at each end.

	Viscously damped isolator	Hysteretically damped isolator
Value of T_R when $v = 0^+$	1	1
Frequency ratio v/ω for resonance	1	1
Value of T_R at resonance	$\dfrac{\sqrt{[1 + (2\zeta)^2]}}{2\zeta} \simeq \dfrac{1}{2\zeta}$	$\dfrac{\sqrt{(1 + \eta^2)}}{\eta} \simeq \dfrac{1}{\eta}$
Value of T_R when $v/\omega = \sqrt{2}$	1	1
Frequency ratio v/ω for isolation	$> \sqrt{2}$	$> \sqrt{2}$
High frequency, $v/\omega \gg 1$, attenuation of T_R	$\dfrac{2\zeta}{v/\omega}$	$\dfrac{1}{(v/\omega)^2}$

Example 15

A motor-generator set of mass 100 kg is installed using antivibration (AV) mountings which deflect 1 mm under the static weight of the set. The mountings are effectively undamped and from dynamic test results it is found that the static stiffness and the dynamic stiffness are the same.

When running at 1480 rev/min the amplitude of vibration of the set is measured to be 0.2 mm To reduce this vibration, it is proposed to fasten the motor generator to a concrete block of mass 300 kg which is then to be mounted on the same AV mounts as before. Calculate the new amplitude of vibration.

For undamped mounts,

$$x = \frac{F}{k - mv^2}.$$

Initially,

$$x_1 = \frac{F}{k - m_1 v^2},$$

and when on the block,

$$x_2 = \frac{F}{k - m_2 v^2},$$

that is,

$$\frac{x_1}{x_2} = \frac{k - m_2 v^2}{k - m_1 v^2}.$$

Now

$$k = \frac{100 \times 9.81}{10^{-3}} = 981 \times 10^3 \text{ N/m},$$

$$v = \frac{2\pi \times 1480}{60} = 155 \text{ rad/s},$$

$$m_1 = 100 \text{ kg and } m_2 = 400 \text{ kg}.$$

Thus

$$\frac{x_1}{x_2} = \frac{981 \times 10^3 - 400 (155)^2}{981 \times 10^3 - 100 (155)^2}$$

$$= \frac{981 - 9600}{981 - 2400} = 6.07.$$

Since

$$x_1 = 0.2 \text{ mm}, x_2 = \frac{0.2}{6.07} = 0.033 \text{ mm}.$$

Example 16
A machine of mass 550 kg is flexibly supported on rubber mountings which provide a force proportional to displacement of 210 kN/m, together with a viscous damping force. The machine gives an exciting force of the form $Rv^2 \cos vt$, where R is a constant. At very high speeds of rotation, the measured amplitude of vibration is 0.25 mm, and the maximum amplitude recorded as the speed is slowly increased from zero is 2 mm. Find the value of R and the damping ratio.

Now,

$$X = Rv^2/(\sqrt{(k - mv^2)^2} + c^2v^2).$$

If v is large,

$$X \rightarrow Rv^2/(\sqrt{(m^2v^4)}) = R/m.$$

Hence

$$R = mX = (550 \times 0.25)/1000 = 0.1375 \text{ kg m.}$$

For maximum X, $dX/dv = 0$, hence $v^2 = 2k^2/(2mk - c^2)$, and

$$X_{max} = \frac{R/2m}{\zeta\sqrt{(1 - \zeta^2)}}.$$

So

$$\zeta\sqrt{(1 - \zeta^2)} = R/(2mX_{max}) = 0.1375/(2 \times 550 \times 2 \times 10^{-3}) = 0.0625,$$

that is,

$$\zeta = 0.0625.$$

2.3.3 Response of a Coulomb damped structure to a simple harmonic exciting force with constant amplitude

In the system shown in Fig. 2.31 the damper relies upon dry friction.
    The equation of motion is non-linear because the constant friction force F_d always opposes the motion:
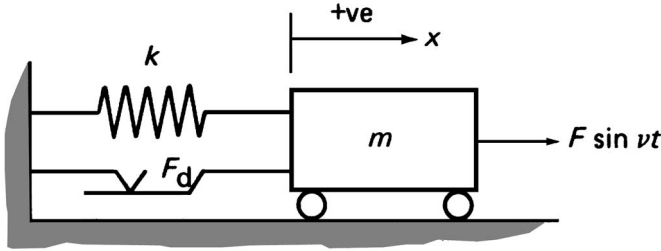
$$m\ddot{x} + kx + F_d = F \sin vt.$$

Fig. 2.31. Single degree of freedom model of a forced system with Coulomb damping.

If F_d is large compared to F, discontinuous motion will occur, but in most structures F_d is usually small so that an approximate continuous solution is valid. The approximate solution is obtained by linearizing the equation of motion; this can be done by expressing F_d in terms of an equivalent viscous damping coefficient, c_d. From section 2.2.6,

$$c_d = \frac{4F_d}{\pi v X}.$$

The solution to the linearized equation of motion gives the amplitude X of the motion as

$$X = \frac{F}{\sqrt{[(k - mv^2)^2 + (c_d v)^2]}}.$$

Hence

$$X = \frac{F}{\sqrt{[(k - mv^2)^2 + (4F_d/\pi X)^2]}},$$

that is,

$$\frac{X}{X_s} = \frac{\sqrt{(1 - (4F_d/\pi F)^2)}}{1 - (v/\omega)^2}.$$

This expression is satisfactory for small damping forces, but breaks down if $4F_d/\pi F < 1$; that is, $F_d > (\pi/4)F$.

At resonance the amplitude is not limited by Coulomb friction.

2.3.4 Response of a hysteretically damped structure to a simple harmonic exciting force with constant amplitude

In the single degree of freedom model shown in Fig. 2.32 the damping is hysteretic.
    The equation of motion is
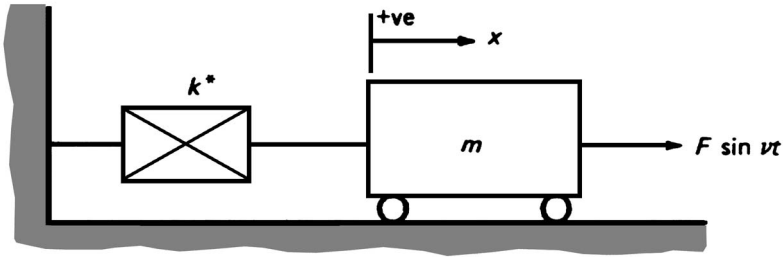
$$m\ddot{x} + k^*x = F \sin vt.$$

Since

Fig. 2.32.   Single degree of freedom model of a forced system with hysteretic damping.

$$k^* = k(1 + j\eta),$$

$$x = \frac{F \sin \nu t}{(k - m\nu^2) + j\eta k}$$

and

$$\frac{X}{X_s} = \frac{1}{\sqrt{([1 - (\nu/\omega)^2]^2 + \eta^2)}}.$$

This result can also be obtained from the analysis of a viscous damped system by substituting $c = \eta k/\nu$.

It should be noted that if $c = \eta k/\nu$, at resonance $c = \eta\sqrt{(km)}$; that is, $\eta = 2\zeta = 1/Q$.

Since $Q = \dfrac{1}{\eta}$,

if a structure is made from a concrete material for which $\eta = 0.02$, a Q factor of 50 may be expected. For a steel, with $\eta = 0.005$ a Q factor of 200 may be expected and for a cast iron with $\eta = 0.01$, $Q = 100$. In practice Q values very much lower than these occur, often by an order of magnitude; that is, Q factors of 10 or less are common. Most of the additional damping found in structures originates in the joints between the connected components of the structure. Joint damping is often the most significant form of damping in a structure and keeps the dynamic response to acceptable levels; it is fully discussed in Chapter 5.

2.3.5 Response of a structure to a suddenly applied force

Consider a single degree of freedom undamped system, such as the system shown in Fig. 2.33, which has been subjected to a suddenly applied force, F. The equation of motion is $m\ddot{x} + kx = F$. The solution to this equation comprises a complementary function $A \sin \omega t + B \cos \omega t$, where $\omega = \sqrt{(k/m)}$ rad/s together with a particular solution. The particular solution may be found by using the D-operator. Thus the equation of motion can
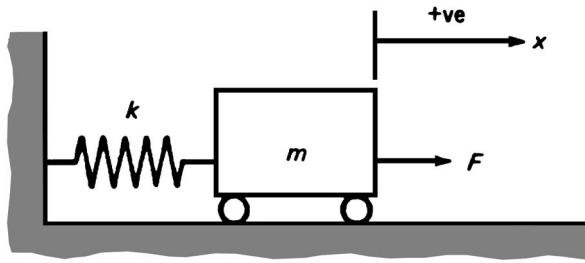
Fig. 2.33.   Single degree of freedom model with constant exciting force.

be written

$$\left(1 + \frac{D^2}{\omega^2}\right)x = \frac{F}{k},$$

and

$$x = \left(1 + \frac{D^2}{\omega^2}\right)^{-1}\frac{F}{k} = \left(1 - \frac{D^2}{\omega^2} - \ldots\right)\frac{F}{k} = \frac{F}{k};$$

that is, the complete solution to the equation of motion is

$$x = A \sin \omega t + B \cos \omega t + \frac{F}{k}.$$

If the initial conditions are such that $x = \dot{x} = 0$ at $t = 0$, then $B = -F/k$ and $A = 0$. Hence

$$x = \frac{F}{k}(1 - \cos \omega t).$$

The motion is shown in Fig. 2.34. It will be seen that the maximum dynamic displacement is twice the static displacement occurring under the same load. This is an important consideration in structures subjected to suddenly applied loads.

If the structure possesses viscous damping of coefficient c, the solution to the equation of motion is $x = Xe^{-\zeta\omega t} \sin (\omega_\nu t + \phi) + F/k$.
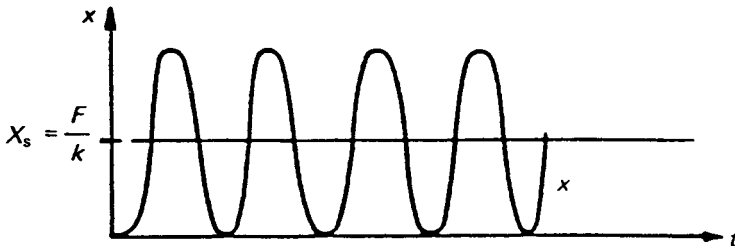

Fig. 2.34.   Displacement–time response for the system shown in Fig. 2.33.

With the same initial conditions as above,

$$x = \frac{F}{k}\left[1 - \frac{e^{-\zeta\omega t}}{\sqrt{(1 - \zeta^2)}} \sin\left(\omega\sqrt{(1 - \zeta^2)}t + \tan^{-1}\frac{\sqrt{(1 - \zeta^2)}}{\zeta}\right)\right].$$

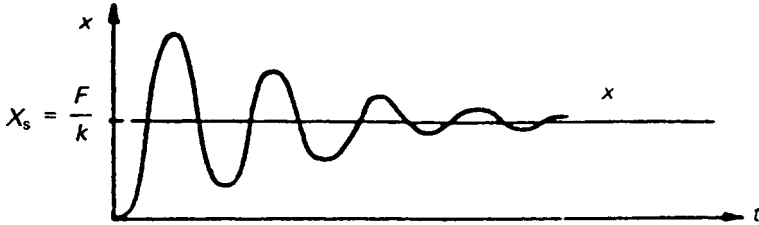This reduces to the undamped case if $\zeta = 0$. The response of the damped system is shown in Fig. 2.35.

Fig. 2.35.   Displacement–time response for a single degree of freedom system with viscous damping.

2.3.6 Shock excitation

Some structures are subjected to shock or impulse loads arising from suddenly applied, non-periodic, short-duration exciting forces.

The impulsive force shown in Fig. 2.36 consists of a force of magnitude F_{max}/ε which has a time duration of ε.
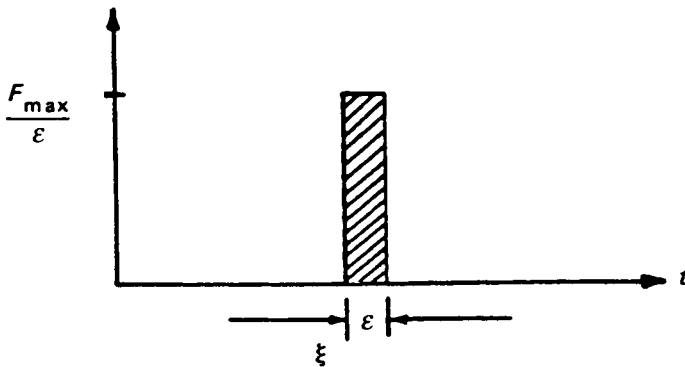

Fig. 2.36.   Impulse.

The impulse is equal to

$$\int_{t}^{t+\varepsilon}\left(\frac{F_{max}}{\varepsilon}\right)dt.$$

When F_{max} is equal to unity, the force in the limiting case $\varepsilon\to 0$ is called either the unit impulse or the delta function, and is identified by the symbol $\delta(t - \xi)$, where

$$\int_{0}^{\infty}\delta(t - \xi)d\xi = 1.$$

Since $F \, dt = m \, dv$, the impulse F_{max} acting on a body will result in a sudden change in its velocity without an appreciable change in its displacement. Thus the motion of a single degree of freedom system excited by an impulse F_{max} corresponds to free vibration with initial conditions $x = 0$ and $\dot{x} = v_0 = F_{max}/m$ at $t = 0$.

Once the response $g(t)$, say, to a unit impulse excitation is known, it is possible to establish the equation for the response of a system to an arbitrary exciting force $F(t)$. For this the arbitrary pulse is considered to comprise a series of impulses as shown in Fig. 2.37.
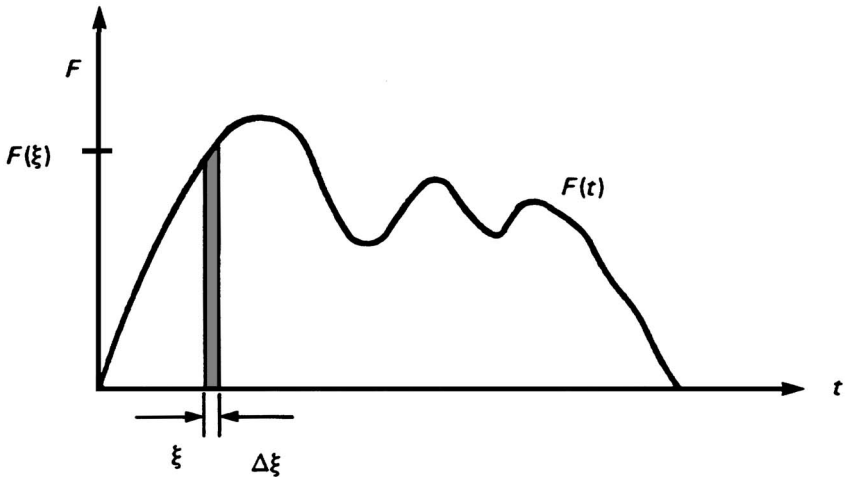

Fig. 2.37.   Force–time pulse.

If one of the impulses is examined which starts at time ξ, its magnitude is $F(\xi)\delta\xi$, and its contribution to the system response at time t is found by replacing the time with the elapsed time $(t - \xi)$ as shown in Fig. 2.38.

If the system can be assumed to be linear, the principle of superposition can be applied, so that

$$x(t) = \int_0^t F(\xi)g(t - \xi)d\xi.$$

This is known as the Duhamel integral.

2.3.7 Wind- or current-excited oscillation

A structure exposed to a fluid stream is subjected to a harmonically varying force in a direction perpendicular to the stream. This is because of eddy, or vortex, shedding on alternate sides of the structure on the leeward side. Tall structures such as masts, bridges and chimneys are susceptible to excitation from steady winds blowing across them. Consider a circular cylinder of diameter D exposed to a fluid which flows past the cylinder with a velocity v. When v is large enough, vortices are formed in the wake which are shed in a regular pattern over a wide range of Reynolds' numbers.

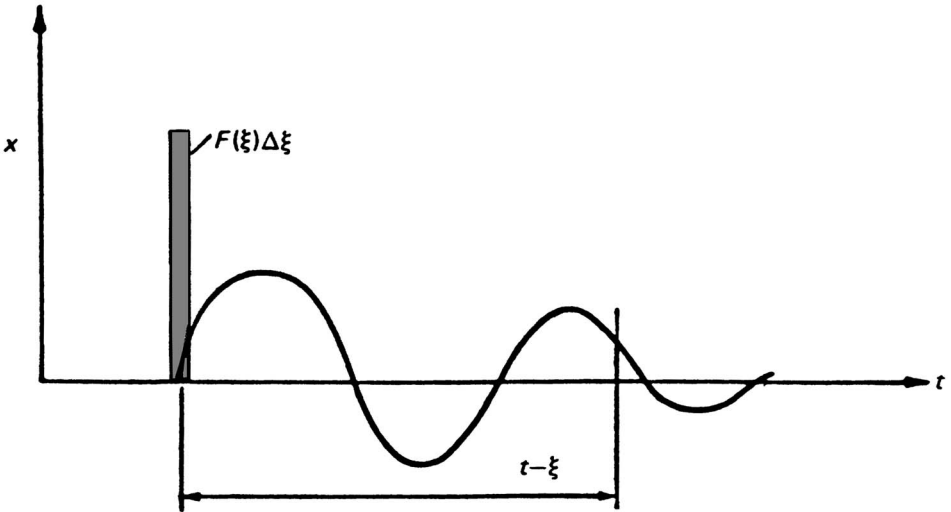$$\text{Reynolds number} = \frac{vD\rho}{\mu},$$

Fig. 2.38.   Displacement–time response to impulse.

where ρ and μ are the mass density and viscosity of the fluid, respectively.

The vortices are shed from opposite sides of the cylinder with a frequency f_s. The Strouhal number relates the excitation frequency, f_s, to the velocity of fluid flow, v (m/s), and the hydraulic mean diameter, D(m), of the structure as follows:

$$\text{Strouhal number } = \frac{f_s D}{v}.$$

This vortex shedding causes an alternating pressure on each side of the cylinder, which acts as a harmonically varying force which is perpendicular to the direction of the undisturbed flow of magnitude

$$\tfrac{1}{2} C_D \rho v^2 A$$

where C_D is the drag coefficient and A is the projected area of the cylinder perpendicular to the direction of flow. If the frequency f_s is close to the natural frequency of the structure, resonance may occur.

For a structure,

$$D = \frac{4 \times \text{area of cross-section}}{\text{circumference}},$$

so that for a chimney of circular cross-section and diameter d,

$$D = \frac{4(\pi/4)\, d^2}{\pi d} = d,$$

and for a building of rectangular cross-section $a \times b$,

$$D = \frac{4ab}{2(a + b)} = \frac{2ab}{(a + b)}.$$

Experimental evidence suggests a value of 0.2–0.24 for the Strouhal number for most flow rates and wind speeds encountered. This value is valid for Reynolds numbers in the range $3 \times 10^5 - 3.5 \times 10^6$.

For a comprehensive discussion of this form of excitation see *Flow Induced Vibration* by R. D. Blevins (Van Nostrand, 1977).
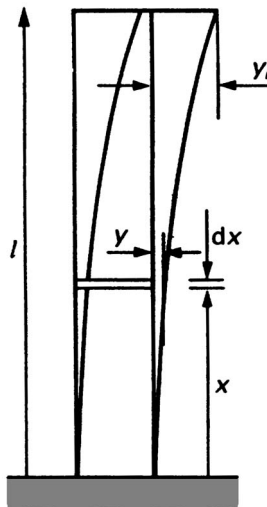
Example 17

For constructing a tanker terminal in a river estuary a number of cylindrical concrete piles were sunk into the river bed and left free-standing. Each pile was 1 m in diameter and protruded 20 m out of the river bed. The density of the concrete was 2400 kg/m^3 and the modulus of elasticity 14×10^6 kN/m^2. Estimate the velocity of the water flowing past a pile which will cause it to vibrate transversely to the direction of the current, assuming a pile to be a cantilever and taking a value for the Strouhal number

$$\frac{f_s D}{v} = 0.22,$$

where f_s is the frequency of flexural vibrations of a pile, D is the diameter and v is the velocity of the current.

Consider the pile to be a cantilever of mass m, diameter D and length l; then the deflection y at a distance x from the root can be taken to be $y = y_l(1 - \cos \pi x/2l)$, where y_l is the deflection at the free end.

Thus

$$\frac{d^2 y}{dx^2} = y_l \left(\frac{\pi}{2l}\right)^2 \cos\left(\frac{\pi x}{2l}\right)$$

$$\int_0^l EI\left(\frac{d^2 y}{dx^2}\right)^2 dx = EI\int_0^l y_l^2 \left(\frac{\pi}{2l}\right)^4 \cos^2\left(\frac{\pi x}{2l}\right) dx$$

$$= EIy_l^2 \left(\frac{\pi}{2l}\right)^4 \frac{l}{2}$$

$$\int y^2 dm = \int_0^l y_l^2 \left(1 - \cos\left(\frac{\pi x}{2l}\right)\right)^2 \left(\frac{m}{l}\right) dx$$

$$= y_l^2 \frac{m}{l}\left(\frac{3}{2} - \frac{4}{\pi}\right) l$$

Hence

$$\omega^2 = \frac{EIy_l^2 \left(\dfrac{\pi}{2l}\right)^4 \dfrac{l}{2}}{y_l^2 \dfrac{m}{l}\left(\dfrac{3}{2} - \dfrac{4}{\pi}\right) l}.$$

Substituting numerical values gives $\omega = 5.53$ rad/s, that is, $f = 0.88$ Hz. When $f_s = 0.88$ Hz resonance occurs; that is, when

$$v = \frac{f_s D}{0.22} = \frac{0.88}{0.22} = 4 \text{ m/s}.$$

2.3.8 Harmonic analysis

A function that is periodic but not harmonic can be represented by the sum of a number of terms, each term representing some multiple of the fundamental frequency. In a *linear* system each of these harmonic terms acts as if it alone were exciting the system, and the system response is the sum of the excitation of all the harmonics.

For example, if the periodic forcing function of a single degree of freedom undamped system is

$$F_1 \sin(vt + \alpha_1) + F_2 \sin(2vt + \alpha_2) + F_3 \sin(3vt + \alpha_3)$$
$$+ \cdots + F_n \sin(nvt + \alpha_n),$$

the steady-state response to $F_1 \sin(vt + \alpha_1)$ is

$$x_1 = \frac{F_1}{k\left(1 - \left(\dfrac{v}{\omega}\right)^2\right)} \sin(vt + \alpha_1),$$

and the response to $F_2 \sin(2vt + \alpha_2)$ is

$$x_2 = \frac{F}{k\left(1 - \left(\dfrac{2v}{\omega}\right)^2\right)} \sin(2vt + \alpha_2),$$

and so on, so that

$$x = \sum_{n=1}^{n} \frac{F_n}{k\left(1 - \left(\dfrac{nv}{\omega}\right)^2\right)} \sin(nvt + \alpha_n).$$

Clearly that harmonic that is closest to the system natural frequency will most influence the response.

A periodic function can be written as the sum of a number of harmonic terms by writing a Fourier series for the function. A Fourier series can be written

$$F(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nvt + b_n \sin nvt),$$

where

$$a_0 = \frac{2}{\tau} \int_0^\tau F(t)dt,$$

$$a_n = \frac{2}{\tau} \int_0^\tau F(t) \cos vtdt,$$

and

$$b_n = \frac{2}{\tau} \int_0^\tau F(t) \sin vtdt.$$

For example, consider the first four terms of the Fourier series representation of the square wave shown in Fig. 2.39 to be required; $\tau = 2\pi$ so $v = 1$ rad/s.

$$F(t) = \frac{a_0}{2} + a_1 \cos vt + a_2 \cos 2vt + \ldots$$

$$+ b_1 \sin vt + b_2 \sin 2vt + \ldots,$$

$F(t)$

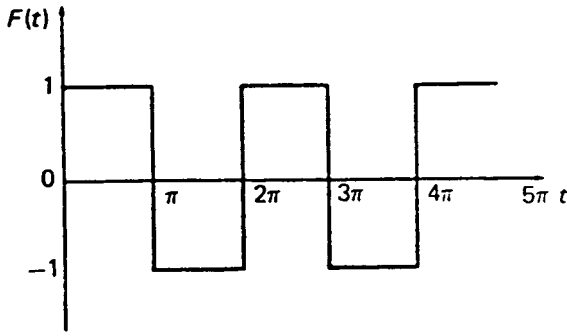Fig. 2.39.   Square wave.

$$a_0 = \frac{2}{\tau}\int_0^\tau F(t)dt = \frac{2}{2\pi}\int_0^\pi 1 \ dt + \frac{2}{2\pi}\int_\pi^{2\pi} -1 \ dt = 0,$$

$$a_1 = \frac{2}{\tau}\int_0^\tau F(t) \cos vt dt$$

$$= \frac{2}{2\pi}\int_0^v \cos vt dt + \frac{2}{2\pi}\int_\pi^{2\pi} - \cos vt dt = 0.$$

Similarly

$$a_2 = a_3 = \dots = 0.$$

$$b_1 = \frac{2}{\tau}\int_0^\tau F(t) \sin vt dt$$

$$= \frac{2}{2\pi}\int_0^\pi \sin vt dt + \frac{2}{2\pi}\int_\pi^{2\pi} - \sin vt dt$$

$$= \frac{1}{\pi v}[-\cos vt]_0^\pi + \frac{1}{\pi v}[\cos vt]_\pi^{2\pi} = \frac{4}{\pi v}.$$

Since $v = 1 \ \text{rad/s}$,

$$b_1 = \frac{4}{\pi}.$$

It is found that $b_2 = 0$, $b_3 = 4/3\pi$ and so on. Thus

$$F(t) = \frac{4}{\pi}\left[\sin t + \frac{1}{3}\sin 3t + \frac{1}{5}\sin 5t + \frac{1}{7}\sin 7t + \dots\right]$$

is the series representation of the square wave shown.

If this stimulus is applied to a simple undamped system with $\omega = 4$ rad/s, say, the steady-state response is given by

$$x = \frac{\dfrac{4}{\pi}\sin t}{1 - \left(\frac{1}{4}\right)^2} + \frac{\dfrac{4}{3\pi}\sin 3t}{1 - \left(\frac{3}{4}\right)^2} + \frac{\dfrac{4}{5\pi}\sin 5t}{1 - \left(\frac{5}{4}\right)^2} + \frac{\dfrac{4}{7\pi}\sin 7t}{1 - \left(\frac{7}{4}\right)^2} \cdots,$$

that is, $x = 1.36 \sin t + 0.97 \sin 3t - 0.45 \sin 5t - 0.09 \sin 7t - \ldots$

Usually three or four terms of the series dominate the predicted response.

It is worth sketching the components of $F(t)$ above to show that they produce a reasonable square wave, whereas the components of x do not. This is an important result.

2.3.9 Random vibration

If the vibration response parameters of a dynamic system are accurately known as functions of time, the vibration is said to be *deterministic*. However, in many systems and processes responses cannot be accurately predicted; these are called *random processes*. Examples of a random process are turbulence, fatigue, the meshing of imperfect gears, surface irregularities, the motion of a car running along a rough road and building vibration excited by an earthquake. Fig. 2.40 shows a random process.
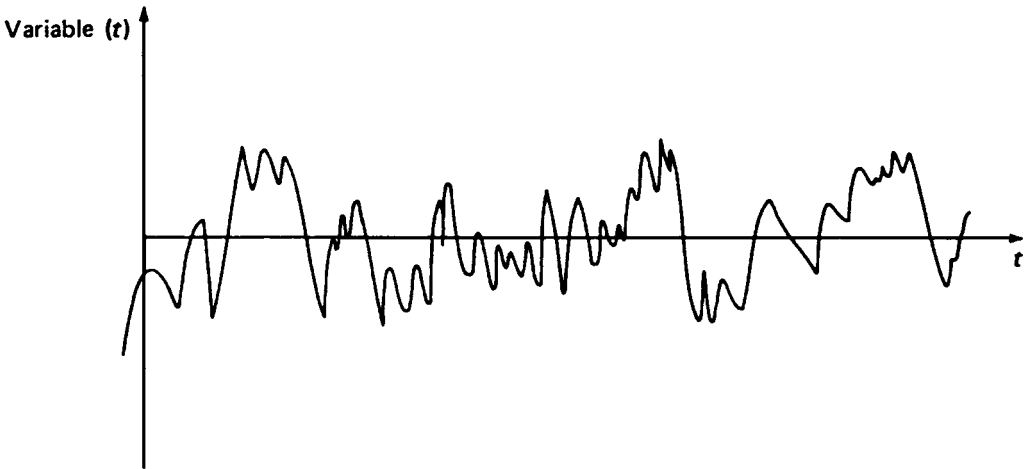

Fig. 2.40.   Example random process variable as a function of t.

A collection of sample functions $x_1(t)$, $x_2(t)$, $x_3(t)$, ..., $x_n(t)$ which make up the random process $x(t)$ is called an *ensemble*, as shown in Fig. 2.41. These functions may comprise, for example, records of noise, pressure fluctuations or vibration levels, taken under the same conditions but at different times.

Any quantity that cannot be precisely predicted is non-deterministic and is known as a *random variable* or a *probabilistic quantity*; that is, if a series of tests is conducted to find

the value of a particular parameter x and that value is found to vary in an unpredictable way that is not a function of any other parameter, then x is a random variable.
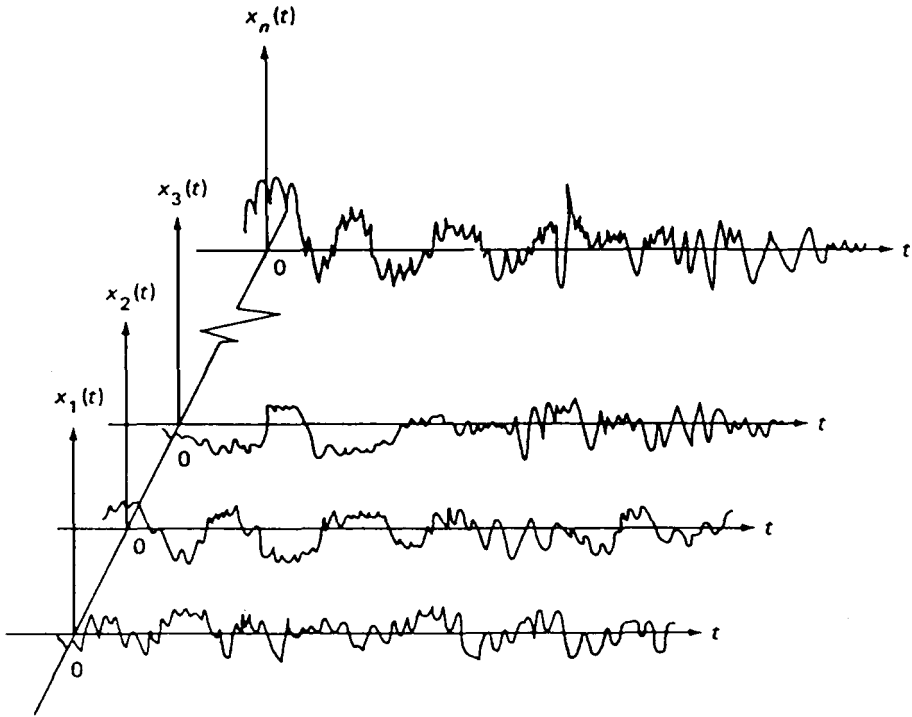

Fig. 2.41.   Ensemble of a random process.

2.3.9.1 Probability distribution

If n experimental values of a variable x are x_1, x_2, x_3, ..., x_n, the probability that the value of x will be less than x' is n'/n, where n' is the number of x values that are less than or equal to x'; that is,

Prob $(x \leqslant x') = n'/n$.

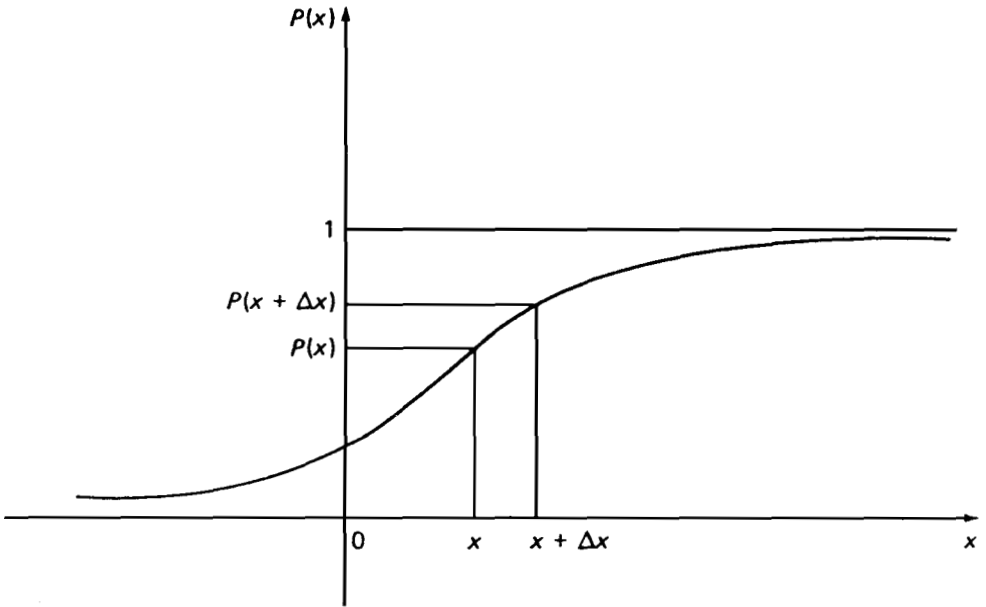When n approaches ∞ this expression is the probability distribution function of x, denoted by $P(x)$, so that

$$P(x) = \underset{n \to \infty}{\text{Lt}} \ (n'/n)$$

The typical variation of $P(x)$ with x is shown in Fig. 2.42. Since $x(t)$ denotes a physical quantity,

Prob $(x < - \infty) = 0$, and Prob $(x < + \infty) = 1$.

The *probability density function* is the derivative of $P(x)$ with respect to x and this is

Fig. 2.42.   Probability distribution function as a function of x.

denoted by $p(x)$; that is,

$$p(x) = \frac{dP(x)}{d(x)}$$

$$= \underset{\Delta x \to 0}{\text{Lt}} \left[\frac{P(x + \Delta x) - P(x)}{\Delta x} \right],$$

where $P(x + \Delta x) - P(x)$ is the probability that the value of $x(t)$ will lie between x and $x + \Delta x$ (Fig. 2.42). Now,

$$p(x) = \frac{dP(x)}{d(x)},$$

so that

$$P(x) = \int_{-\infty}^{x} p(x)dx.$$

Hence

$$P(\infty) = \int_{-\infty}^{\infty} p(x)dx = 1,$$

so that the area under the probability density function curve is unity.

A random process is *stationary* if the joint probability density

$$p(x(t_1), x(t_2), x(t_3), \ ...)$$

depends only upon the time differences $t_2 - t_1$, $t_3 - t_2$ and so on, and not on the actual time instants; that is, the ensemble will look just the same if the time origin is changed. A random process is *ergodic* if every sample function is typical of the entire group.

The expected value of $f(x)$, which is written

$$E[f(x)] \text{ or } \overline{f(x)},$$

is

$$E[f(x)] = \overline{f(x)} = \int_{-\infty}^{\infty} f(x)p(x)dx,$$

so that the expected value of a stationary random process $x(t)$ is

$$E[x(t_1)] = E[x(t_1 + t)]$$

for any value of t.

If $f(x) = x$, the expected value or *mean value* of x,

$$E[x] \text{ or } \bar{x}, \text{ is}$$

$$E[x] = \bar{x} = \int_{-\infty}^{\infty} xp(x)dx.$$

In addition, if $f(x) = x^2$, the *mean square value* $\bar{x}^2$ of x is

$$E[x^2] = \bar{x}^2 = \int_{-\infty}^{\infty} x^2 p(x)dx.$$

The *variance* of x, σ^2, is the mean square value of x about the mean, that is,

$$\sigma^2 = E[(x - \bar{x})^2] = \int_{-\infty}^{\infty} (x - \bar{x})^2 p(x)dx = (\bar{x}^2) - (\bar{x})^2.$$

σ is the *standard deviation* of x, hence

$$\text{variance} = (\text{standard deviation})^2 = \{\text{mean square} - (\text{mean})^2\}$$

If two (or more) random variables x_1 and x_2, represent a random process at two different instants of time, then

$$E[f(x_1,x_2)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x_1,x_2)p(x_1,x_2)dx_1 dx_2,$$

and if t_1 and t_2 are the two instants of time,

$$E[x(t_1),x(t_2)] = R(t_1,t_2),$$

which is the *auto-correlation function* for the random process (Fig. 2.43).

For random processes that are stationary,

$$E[x(t_1), x(t_2)] = R(t_1, t_2) = R(t_2 - t_1) = R(\tau),$$

say, since the average depends only upon time differences. If the process is also ergodic,

then

$$R(\tau) = \underset{T \to \infty}{\text{Lt}} \frac{1}{2\pi} \int_{-T}^{T} x(t)x(t + \tau)\mathrm{d}t.$$

It is worth noting that

$$R(0) = E[x(t)^2] = \underset{T \to \infty}{\text{Lt}} \frac{1}{2T} \int_{-T}^{T} x^2(t)\mathrm{d}t,$$
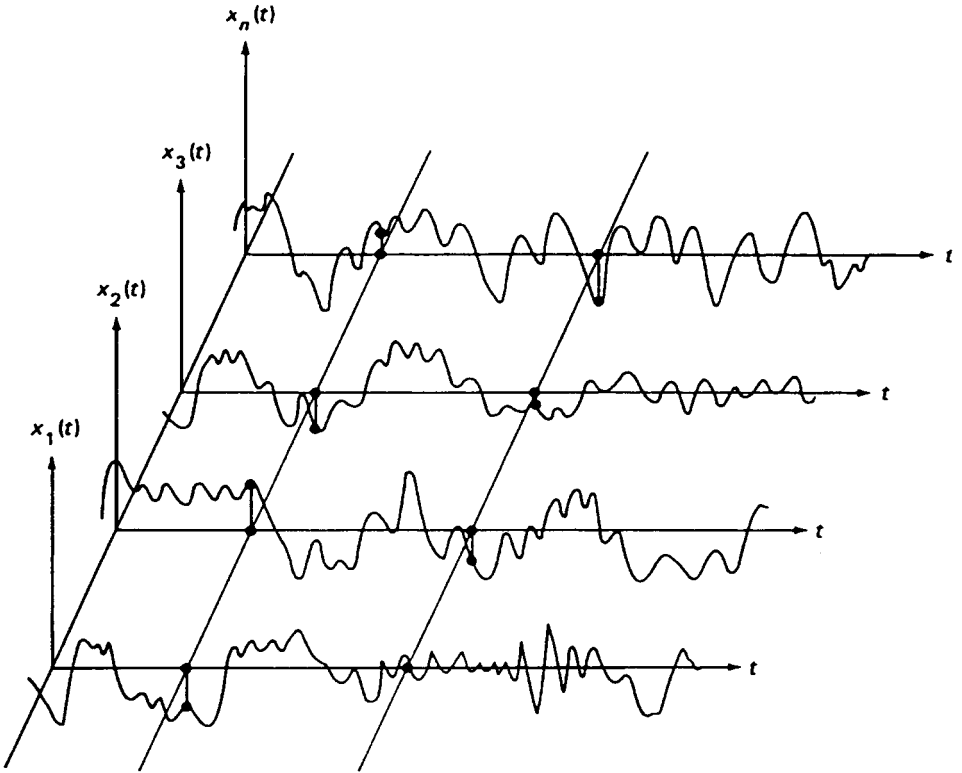
which is the average power in a sample function.

Fig. 2.43.   Random processes.

2.3.9.2 Random processes

The most important random process is the *Gaussian*, or normal random process. This is because a wide range of physically observed random waveforms can be represented as Gaussian processes, and the process has mathematical features which make analysis relatively straightforward.

The probability density function of a Gaussian process $x(t)$ is

$$p(x) = \frac{1}{\sqrt{(2\pi)}\sigma}\exp\left[-\frac{1}{2}\left(\frac{x - \bar{x}}{\sigma}\right)^2\right],$$

where σ is the standard deviation of x, and $\bar{x}$ is the mean value of x.

The values of σ and $\bar{x}$ may vary with time for a non-stationary process but are independent of time if the process is stationary.

One of the most important features of the Gaussian process is that the response of a linear system to this form of excitation is usually another, but still Gaussian, random process. The only changes are that the magnitude and standard deviation of the response may differ from those of the excitation.

A Gaussian probability density function is shown in Fig. 2.44. It can be seen to be symmetric about the mean value $\bar{x}$, and the standard deviation σ controls the spread.
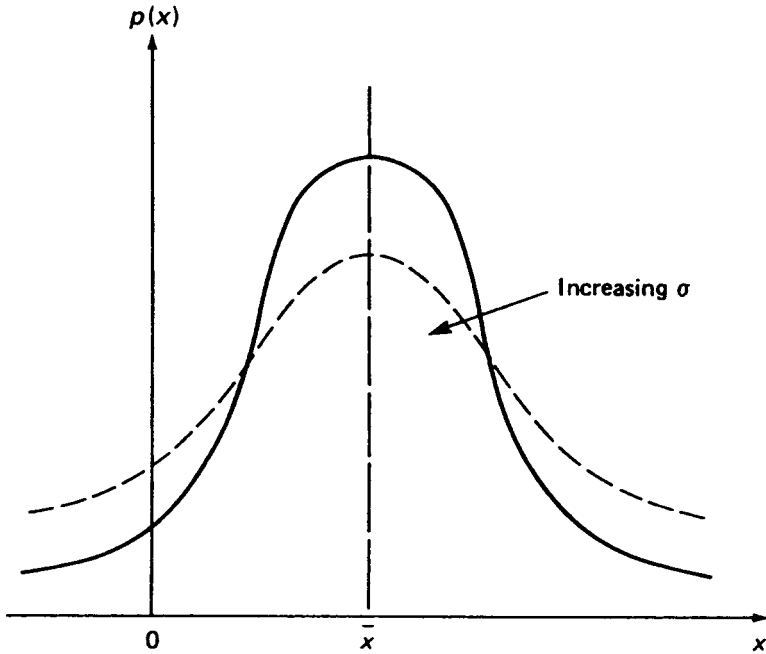

Fig. 2.44.   Gaussian probability density function.

The probability that $x(t)$ lies between $-\lambda\sigma$ and $+\lambda\sigma$, where λ is a positive number, can be found since, if $\bar{x} = 0$,

$$\text{Prob}\left\{-\lambda\sigma \leqslant x(t) \leqslant +\lambda\sigma\right\} = \int_{-\lambda\sigma}^{+\lambda\sigma} \frac{1}{\sqrt{(2\pi)}\sigma}\exp\left(-\frac{1}{2}\frac{x^2}{\sigma^2}\right)\mathrm{d}x.$$

Fig. 2.45 shows the Gaussian probability density function with zero mean.

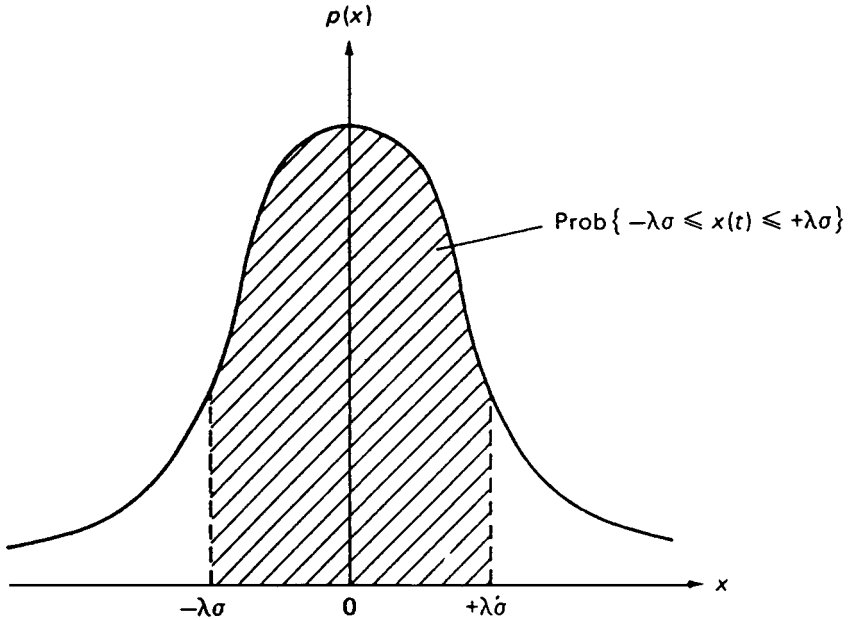Fig. 2.45.   Gaussian probability density function with zero mean.

This integral has been calculated for a range of values of λ; the results are given in the table opposite. The probability that $x(t)$ lies outside the range $-\lambda\sigma$ to $+\lambda\sigma$ is 1 minus the value of the above integral; this probability is also given.

2.3.9.3 Spectral density

The spectral density $S(\omega)$ of a stationary random process is the Fourier transform of the autocorrelation function $R(\tau)$. It is given by

$$S(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} R(\tau)e^{-j\omega\tau}d\tau.$$

The inverse, which also holds true, is

$$R(\tau) = \int_{-\infty}^{\infty} S(\omega)e^{-j\omega\tau}d\omega.$$

If $\tau = 0$

$$R(0) = \int_{-\infty}^{\infty} S(\omega)d\omega = E[x^2],$$

that is, the mean square value of a stationary random process x is the area under the $S(\omega)$ against frequency curve. A typical spectral density function is shown in Fig. 2.46.

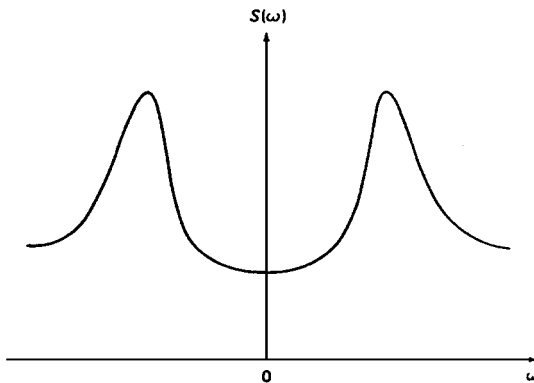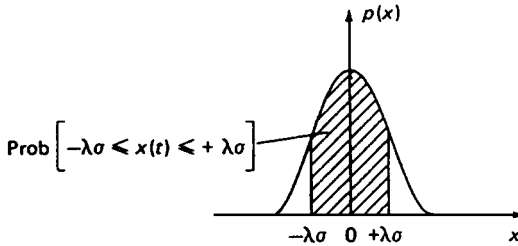| Value of λ	Prob [− λσ ≤ x(t) ≤ λσ]	Prob [|x(t)| > λσ]
0	0	1.0000
0.2	0.1585	0.8415
0.4	0.3108	0.6892
0.6	0.4515	0.5485
0.8	0.5763	0.4237
1.0	0.6827	0.3173
1.2	0.7699	0.2301
1.4	0.8586	0.1414
1.6	0.8904	0.1096
1.8	0.9281	0.0719
2.0	0.9545	0.0455
2.2	0.9722	0.0278
2.4	0.9835	0.0164
2.6	0.9907	0.0093
2.8	0.9949	0.0051
3.0	0.9973	0.0027
3.2	0.9986	0.00137
3.4	0.9993	0.00067
3.6	0.9997	0.00032
3.8	0.9998	0.00014
4.0	0.9999	0.00006

Fig. 2.46. Typical spectral density function.

A random process whose spectral density is constant over a very wide frequency range is called *white noise*. If the spectral density of a process has a significant value over a narrower range of frequencies, but one that is nevertheless still wide compared with the centre frequency of the band, it is termed a *wide-band process* (Fig. 2.47). If the frequency range is narrow compared with the centre frequency it is termed a *narrow-band process* (Fig. 2.48). Narrow-band processes frequently occur in engineering practice because real systems often respond strongly to specific exciting frequencies and thereby effectively act as a filter.
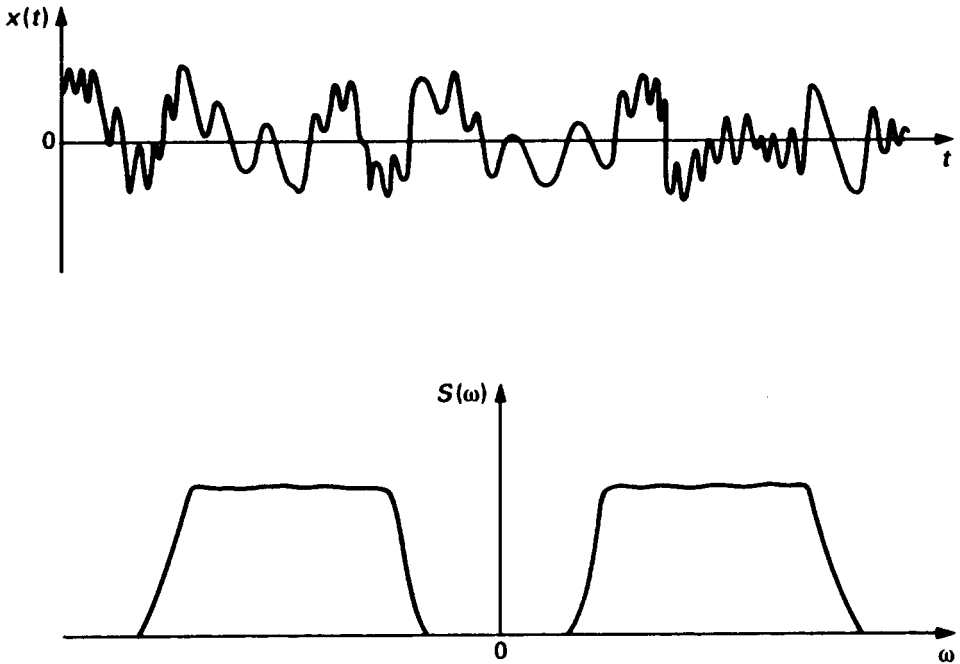

Fig. 2.47.   Wide-band process.

2.3.10 The measurement of vibration

The most commonly used device for vibration measurement is the piezoelectric accelerometer, which gives an electric signal proportional to the vibration acceleration. This signal can readily be amplified, analysed, displayed, recorded, and so on. The principles of this device can be studied by referring to Fig. 2.49 which shows a body of mass m supported by an elastic system of stiffness k and effective viscous damping of coefficient c.

This dynamic system is usually enclosed in a case which is fastened to the surface whose vibration is to be measured. The body has a pointer fixed to it, which moves over a scale fastened to the case, that is, it measures u, the motion of the suspended body relative to that of the vibrating surface so that $u = x - y$.
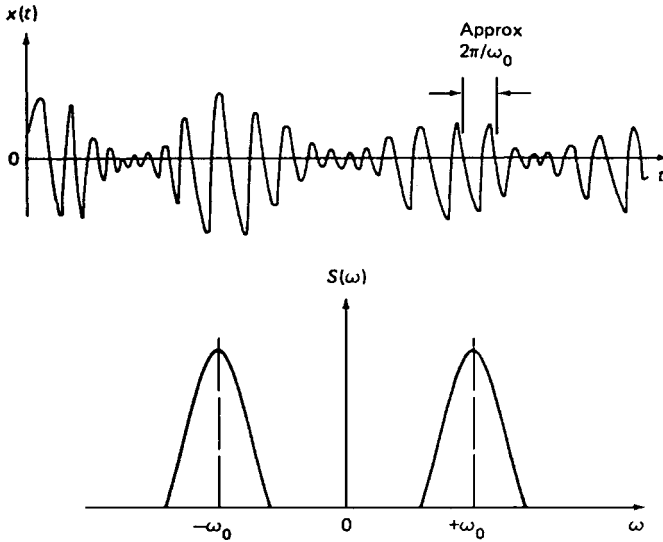
Fig. 2.48.   Narrow-band process.

Now from section 2.3.2, the amplitude of u is

$$U = \frac{A\left(\dfrac{v}{\omega}\right)^2}{\sqrt{\left\{\left[1 - \left(\dfrac{v}{\omega}\right)^2\right]^2 + \left[2\zeta\left(\dfrac{v}{\omega}\right)\right]^2\right\}}}$$

so that if ω is low and $v \gg \omega$,

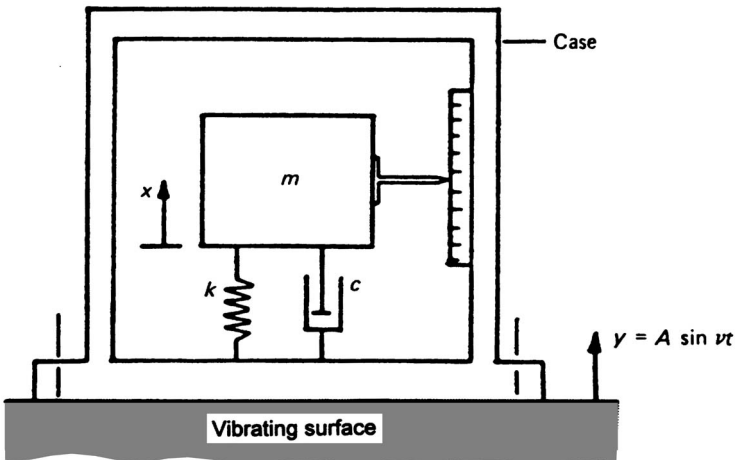Fig. 2.49.   Vibration measuring device.

$$U \simeq \frac{A(v/\omega)^2}{(v/\omega)^2} = A,$$

that is, the device measures the input vibration amplitude; when operating in this mode it is called a *vibrometer*, and if ω is high so that $\omega \gg v$, then

$$U \simeq \frac{A(v/\omega)^2}{1} = \frac{1}{\omega^2}Av^2,$$

that is, the device measures the input vibration acceleration amplitude; when operating in this mode it is called an *accelerometer.*

By adjusting the system parameters correctly it is possible to make

$$\sqrt{\left\{\left[1 - \left(\frac{v}{\omega}\right)^2\right]^2 + \left[2\zeta\left(\frac{v}{\omega}\right)\right]^2\right\}}$$

have a value close to unity for exciting frequencies v up to about 0.3ω. Commercial accelerometers usually have piezoelectric elements instead of a spring and damper, so that the electric signal produced is proportional to the relative motion, u, above.

Piezoelectric accelerometers are widely used for measuring the vibration of structures. The output of these accelerometers is governed by their sensitivity; in general the larger and therefore the heavier the accelerometer, the greater its sensitivity and the greater the output for a given excitation g-level. However, accelerometers have to be attached to the structure and large accelerometers may affect the response of the structure due to their added mass and they may also have a limited frequency range. Smaller accelerometers have stiffer piezoelectric elements which are less sensitive but can operate at higher frequencies. The output of piezoelectric accelerometers is easily amplified, analysed and recorded.

Strain gauges are also often used to measure the dynamic response of a structure. These rely on the change in resistance of a wire caused by a change in its length. Dynamic measurements require using an a.c. bridge circuit, the carrier frequency used determining the range of frequency measurements possible. These gauges are cheap and easy to apply to a structure, and the bridge output is easily recorded and analysed.

Non-contacting capacitance and impedance transducers are also sometimes used.