The Irish Mathematical Olympiads Compendium 1988 – 2021

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Preface

This is a compendium of all Irish Mathematical Olympiads that I began to maintain in 2008. I am very grateful to Mark Flanagan, Marius Ghergu, Bernd Kreussler, and Andrew Smith for providing me with copies of several of the exams.

This annual competition is typically held on a Saturday in late April or early May. The first paper runs from 10am – 1 pm and the second paper from 2pm – 5 pm. There is no (current) intention to provide solutions to these problems, but you might be able to find solutions to these exams by following the discussion boards on `http://www.mathlinks.ro`. Several textbooks have emerged over the years for training in IMO related material and new ones appear every year. Two texts that are particularly relevant to the Irish Mathematical Olympiad are:

- *Irish Mathematical Olympiad Manual* by O'Farrell et al., Logic Press, Maynooth.
- *Irish Mathematical-Olympiad Problems 1988-1998*, edited by Finbarr Holland of UCC, published by the IMO Irish Participation Committee, 1999.

The six highest scoring candidates are invited to attend the IMO. Listings of those people who accepted a place on the team can be found at `www.imo-official.org` (select 'Results' followed by 'IRL'). The latest version of this compendium will always be found at:

`http://www.maths.ucd.ie/~dukes/irmo.html`

If you have found a typo or something that you suspect to be a mistake, then I would be grateful if you could share this with me at the email address below. Further information regarding the Irish Mathematical Olympiad and training can be found at: `http://www.irmo.ie/`

– Mark Dukes (`mark.dukes@ucd.ie`)
May 2021

1st Irish Mathematical Olympiad
30 April 1988, Paper 1

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1. A pyramid with a square base, and all its edges of length 2, is joined to a regular tetrahedron, whose edges are also of length 2, by gluing together two of the triangular faces. Find the sum of the lengths of the edges of the resulting solid.

2. A, B, C, D are the vertices of a square, and P is a point on the arc CD of its circumcircle. Prove that
$$|PA|^2 - |PB|^2 \;=\; |PB|.|PD| - |PA|.|PC|.$$

3. ABC is a triangle inscribed in a circle, and E is the mid-point of the arc subtended by BC on the side remote from A. If through E a diameter ED is drawn, show that the measure of the angle DEA is half the magnitude of the difference of the measures of the angles at B and C.

4. A mathematical moron is given the values b, c, A for a triangle ABC and is required to find a. He does this by using the cosine rule
$$a^2 \;=\; b^2 + c^2 - 2bc\cos A$$
and misapplying the law of the logarithm to this to get
$$\log a^2 \;=\; \log b^2 + \log c^2 - \log(2bc\cos A).$$
He proceeds to evaluate the right-hand side correctly, takes the anti-logarithms and gets the correct answer. What can be said about the triangle ABC?

5. A person has seven friends and invites a different subset of three friends to dinner every night for one week (seven days). In how many ways can this be done so that all friends are invited at least once?

6. Suppose you are given n blocks, each of which weighs an integral number of pounds, but less than n pounds. Suppose also that the total weight of the n blocks is less than $2n$ pounds. Prove that the blocks can be divided into two groups, one of which weighs exactly n pounds.

7. A function f, defined on the set of real numbers $\mathbb{R}$ is said to have a *horizontal chord* of length $a > 0$ if there is a real number x such that $f(a + x) = f(x)$. Show that the cubic
$$f(x) \;=\; x^3 - x \qquad\qquad (x \in \mathbb{R})$$
has a horizontal chord of length a if, and only if, $0 < a \leq 2$.

8. Let $x_1, x_2, x_3, \ldots$ be a sequence of nonzero real numbers satisfying
$$x_n \;=\; \frac{x_{n-2}x_{n-1}}{2x_{n-2} - x_{n-1}}, \qquad n = 3, 4, 5, \ldots$$
Establish necessary and sufficient conditions on x_1, x_2 for x_n to be an integer for infinitely many values of n.

9. The year 1978 was "peculiar" in that the sum of the numbers formed with the first two digits and the last two digits is equal to the number formed with the middle two digits, i.e., $19 + 78 = 97$. What was the last previous peculiar year, and when will the next one occur?

10. Let $0 \leq x \leq 1$. Show that if n is any positive integer, then

$$(1+x)^n \;\geq\; (1-x)^n + 2nx(1-x^2)^{\frac{n-1}{2}}.$$

11. If facilities for division are not available, it is sometimes convenient in determining the decimal expansion of $1/a$, $a > 0$, to use the iteration

$$x_{k+1} \;=\; x_k(2 - ax_k), \qquad k = 0, 1, 2, \ldots,$$

where x_0 is a selected "starting" value. Find the limitations, if any, on the starting values x_0, in order that the above iteration converges to the desired value $1/a$.

12. Prove that if n is a positive integer, then

$$\sum_{k=1}^{n} \cos^4\left(\frac{k\pi}{2n+1}\right) \;=\; \frac{6n-5}{16}.$$

1st Irish Mathematical Olympiad
30 April 1988, Paper 2

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13. The triangles ABG and AEF are in the same plane. Between them the following conditions hold:

    (a) E is the mid-point of AB;

    (b) points A, G and F are on the same line;

    (c) there is a point C at which BG and EF intersect;

    (d) $|CE| = 1$ and $|AC| = |AE| = |FG|$.

    Show that if $|AG| = x$, then $|AB| = x^3$.

14. Let $x_1, \ldots, x_n$ be n integers, and let p be a positive integer, with $p < n$. Put

$$\begin{aligned} S_1 &= x_1 + x_2 + \ldots + x_p, \\ T_1 &= x_{p+1} + x_{p+2} + \ldots + x_n, \\ S_2 &= x_2 + x_3 + \ldots + x_{p+1}, \\ T_2 &= x_{p+2} + x_{p+3} + \ldots + x_n + x_1, \\ &\vdots \\ S_n &= x_n + x_1 + x_2 + \ldots + x_{p-1}, \\ T_n &= x_p + x_{p+1} + \ldots + x_{n-1}. \end{aligned}$$

    For $a = 0, 1, 2, 3$, and $b = 0, 1, 2, 3$, let $m(a, b)$ be the number of numbers i, $1 \leq i \leq n$, such that S_i leaves remainder a on division by 4 and T_i leaves remainder b on division by 4. Show that $m(1, 3)$ and $m(3, 1)$ leave the same remainder when divided by 4 if, and only if, $m(2, 2)$ is even.

15. A city has a system of bus routes laid out in such a way that

    (a) there are exactly 11 bus stops on each route;

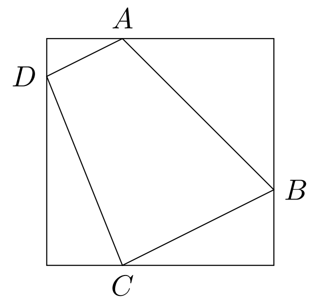
    (b) it is possible to travel between any two bus stops without changing routes;

    (c) any two bus routes have exactly one bus stop in common.

    What is the number of bus routes in the city?

2nd Irish Mathematical Olympiad
29 April 1989, Paper 1

1. A quadrilateral $ABCD$ is inscribed, as shown, in a square of area one unit. Prove that
$$2 \leq |AB|^2 + |BC|^2 + |CD|^2 + |DA|^2 \leq 4.$$



2. A 3×3 magic square, with magic number m, is a 3×3 matrix such that the entries on each row, each column and each diagonal sum to m. Show that if the square has positive integer entries, then m is divisible by 3, and each entry of the square is at most $2n - 1$, where $m = 3n$. [An example of a magic square with $m = 6$ is
$$\begin{pmatrix} 2 & 1 & 3 \\ 3 & 2 & 1 \\ 1 & 3 & 2 \end{pmatrix}.]$$

3. A function f is defined on the natural numbers $\mathbb{N}$ and satisfies the following rules:

   (a) $f(1) = 1$;

   (b) $f(2n) = f(n)$ and $f(2n+1) = f(2n) + 1$ for all $n \in \mathbb{N}$.

   Calculate the maximum value m of the set $\{f(n) : n \in \mathbb{N}, 1 \leq n \leq 1989\}$, and determine the number of natural numbers n, with $1 \leq n \leq 1989$, that satisfy the equation $f(n) = m$.

4. Note that $12^2 = 144$ end in two 4's and $38^2 = 1444$ end in three 4's. Determine the length of the longest string of equal nonzero digits in which the square of an integer can end.

5. Let $x = a_1 a_2 \ldots a_n$ be an n-digit number, where $a_1, a_2, \ldots, a_n$ ($a_1 \neq 0$) are the digits. The n numbers
$$x_1 = x = a_1 a_2 \ldots a_n, \quad x_2 = a_n a_1 \ldots a_{n-1}, \quad x_3 = a_{n-1} a_n a_1 \ldots a_{n-2},$$
$$x_4 = a_{n-2} a_{n-1} a_n a_1 \ldots a_{n-3}, \quad \ldots \quad , \quad x_n = a_2 a_3 \ldots a_n a_1$$
are said to be obtained from x by the cyclic permutation of digits. [For example, if $n = 5$ and $x = 37001$, then the numbers are $x_1 = 37001$, $x_2 = 13700$, $x_3 = 01370 (= 1370)$, $x_4 = 00137 (= 137)$, $x_5 = 70013$.]

   Find, with proof, (i) the smallest natural number n for which there exists an n-digit number x such that the n numbers obtained from x by the cyclic permutation of digits are all divisible by 1989; and (ii) the smallest natural number x with this property.

2nd Irish Mathematical Olympiad
29 April 1989, Paper 2

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6. Suppose L is a fixed line, and A a fixed point not on L. Let k be a fixed nonzero real number. For P a point on L, let Q be a point on the line AP with $|AP|.|AQ| = k^2$. Determine the locus of Q as P varies along the line L.

7. Each of the n members of a club is given a different item of information. They are allowed to share the information, but, for security reasons, only in the following way: A pair may communicate by telephone. During a telephone call only one member may speak. The member who speaks may tell the other member all the information s(he) knows. Determine the minimal number of phone calls that are required to convey all the information to each other.

8. Suppose P is a point in the interior of a triangle ABC, that x, y, z are the distances from P to A, B, C, respectively, and that p, q, r are the perpendicular distances from P to the sides BC, CA, AB, respectively. Prove that
$$xyz \;\geq\; 8pqr,$$
with equality implying that the triangle ABC is equilateral.

9. Let a be a positive real number, and let
$$b \;=\; \sqrt[3]{a+\sqrt{a^2+1}} + \sqrt[3]{a-\sqrt{a^2+1}}.$$
Prove that b is a positive integer if, and only if, a is a positive integer of the form $\frac{1}{2}n(n^2+3)$, for some positive integer n.

10. (i) Prove that if n is a positive integer, then
$$\binom{2n}{n} = \frac{(2n)!}{(n!)^2}$$
is a positive integer that is divisible by all prime numbers p with $n < p \leq 2n$, and that
$$\binom{2n}{n} < 2^{2n}.$$

(ii) For x a positive real number, let $\pi(x)$ denote the number of prime numbers $p \leq x$. [Thus, $\pi(10) = 4$ since there are 4 primes, viz., 2, 3, 5 and 7, not exceeding 10.]
Prove that if $n \geq 3$ is an integer, then

(a) $\pi(2n) \;<\; \pi(n) + \dfrac{2n}{\log_2(n)}$; (b) $\pi(2^n) \;<\; \dfrac{2^{n+1}\log_2(n-1)}{n}$;

(c) Deduce that, for all real numbers $x \geq 8$,
$$\pi(x) \;<\; \frac{4x\log_2(\log_2(x))}{\log_2(x)}.$$

3rd Irish Mathematical Olympiad
5 May 1990, Paper 1

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1. Given a natural number n, calculate the number of rectangles in the plane, the coordinates of whose vertices are integers in the range 0 to n, and whose sides are parallel to the axes.

2. A sequence of primes a_n is defined as follows: $a_1 = 2$, and, for all $n \geq 2$, a_n is the largest prime divisor of $a_1 a_2 \cdots a_{n-1} + 1$. Prove that $a_n \neq 5$ for all n.

3. Determine whether there exists a function $f : \mathbb{N} \to \mathbb{N}$ (where $\mathbb{N}$ is the set of natural numbers) such that
$$f(n) = f(f(n-1)) + f(f(n+1)),$$
for all natural numbers $n \geq 2$.

4. The real number x satisfies all the inequalities
$$2^k < x^k + x^{k+1} < 2^{k+1}$$
for $k = 1, 2, \ldots, n$. What is the greatest possible value of n?

5. Let ABC be a right-angled triangle with right-angle at A. Let X be the foot of the perpendicular from A to BC, and Y the mid-point of XC. Let AB be extended to D so that $|AB| = |BD|$. Prove that DX is perpendicular to AY.

3rd Irish Mathematical Olympiad
5 May 1990, Paper 2

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6. Let n be a natural number, and suppose that the equation
$$x_1x_2 + x_2x_3 + x_3x_4 + x_4x_5 + \cdots + x_{n-1}x_n + x_nx_1 = 0$$
has a solution with all the x_i's equal to ± 1. Prove that n is divisible by 4.

7. Let $n \geq 3$ be a natural number. Prove that
$$\frac{1}{3^3} + \frac{1}{4^3} + \cdots + \frac{1}{n^3} < \frac{1}{12}.$$

8. Suppose that $p_1 < p_2 < \ldots < p_{15}$ are prime numbers in arithmetic progression, with common difference d. Prove that d is divisible by $2, 3, 5, 7, 11$ and 13.

9. Let t be a real number, and let
$$a_n = 2\cos\left(\frac{t}{2^n}\right) - 1, \quad n = 1, 2, 3, \ldots$$
Let b_n be the product $a_1a_2a_3 \cdots a_n$. Find a formula for b_n that does not involve a product of n terms, and deduce that
$$\lim_{n\to\infty} b_n = \frac{2\cos t + 1}{3}.$$

10. Let $n = 2k - 1$, where $k \geq 6$ is an integer. Let T be the set of all n-tuples
$$\mathbf{x} = (x_1, x_2, \ldots, x_n), \text{ where, for } i = 1, 2, \ldots, n,\ x_i \text{ is 0 or 1.}$$
For $\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbf{y} = (y_1, \ldots, y_n)$ in T, let $d(\mathbf{x}, \mathbf{y})$ denote the number of integers j with $1 \leq j \leq n$ such that $x_j \neq y_j$. (In particular, $d(\mathbf{x}, \mathbf{x}) = 0$).

    Suppose that there exists a subset S of T with 2^k elements which has the following property: given any element $\mathbf{x}$ in T, there is a unique $\mathbf{y}$ in S with $d(\mathbf{x}, \mathbf{y}) \leq 3$.

    Prove that $n = 23$.

4th Irish Mathematical Olympiad
4 May 1991, Paper 1

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1. Three points X, Y and Z are given that are, respectively, the circumcentre of a triangle ABC, the mid-point of BC, and the foot of the altitude from B on AC. Show how to reconstruct the triangle ABC.

2. Find all polynomials
$$f(x) = a_0 + a_1 x + \cdots + a_n x^n$$
satisfying the equation
$$f(x^2) = (f(x))^2$$
for all real numbers x.

3. Three operations f, g and h are defined on subsets of the natural numbers $\mathbb{N}$ as follows:
$f(n) = 10n$, if n is a positive integer;
$g(n) = 10n + 4$, if n is a positive integer;
$h(n) = \frac{n}{2}$, if n is an *even* positive integer.
Prove that, starting from 4, every natural number can be constructed by performing a finite number of operations f, g and h in some order.

   [For example: $35 = h(f(h(g(h(h(4))))))$.]

4. Eight politicians stranded on a desert island on January 1st, 1991, decided to establish a parliment.
They decided on the following rules of attendance:

   (a) There should always be at least one person present on each day.

   (b) On no two days should be same subset attend.

   (c) The members present on day N should include for each $K < N$, $(K \geq 1)$ at least one member who was present on day K.

   For how many days can the parliment sit before one of the rules is broken?

5. Find all polynomials
$$f(x) = x^n + a_1 x^{n-1} + \cdots + a_n$$
with the following properties:

   (a) all the coefficients $a_1, a_2, \ldots, a_n$ belong to the set $\{-1, 1\}$

   (b) all the roots of the equation
$$f(x) = 0$$
are real.

4th Irish Mathematical Olympiad
4 May 1991, Paper 2

6. The sum of two consecutive squares can be a square: for instance, $3^2 + 4^2 = 5^2$.

   (a) Prove that the sum of m consecutive squares cannot be a square for the cases $m = 3, 4, 5, 6$.

   (b) Find an example of eleven consecutive squares whose sum is a square.

7. Let
$$a_n = \frac{n^2+1}{\sqrt{n^4+4}}, \quad n = 1, 2, 3, \ldots$$
and let b_n be the product of $a_1, a_2, a_3, \ldots, a_n$. Prove that
$$\frac{b_n}{\sqrt{2}} = \frac{\sqrt{n^2+1}}{\sqrt{n^2+2n+2}},$$
and deduce that
$$\frac{1}{n^3+1} < \frac{b_n}{\sqrt{2}} - \frac{n}{n+1} < \frac{1}{n^3}$$
for all positive integers n.

8. Let ABC be a triangle and L the line through C parallel to the side AB. Let the internal bisector of the angle at A meet the side BC at D and the line L at E, and let the internal bisector of the angle at B meet the side AC at F and the line L at G. If $|GF| = |DE|$, prove that $|AC| = |BC|$.

9. Let $\mathbb{P}$ be the set of positive rational numbers and let $f : \mathbb{P} \to \mathbb{P}$ be such that
$$f(x) + f\left(\frac{1}{x}\right) = 1$$
and
$$f(2x) = 2f(f(x))$$
for all $x \in \mathbb{P}$.
Find, with proof, an explicit expression for $f(x)$ for all $x \in \mathbb{P}$.

10. Let $\mathbb{Q}$ denote the set of rational numbers. A nonempty subset S of $\mathbb{Q}$ has the following properties:

   (a) 0 is not in S;

   (b) for each s_1, s_2 in S, the rational number s_1/s_2 is in S; also

   (c) there exists a nonzero number $q \in \mathbb{Q} \backslash S$ that has the property that every nonzero number in $\mathbb{Q} \backslash S$ is of the form qs, for some s in S.

   Prove that if x belongs to S, then there exist elements y, z in S such that $x = y + z$.

5th Irish Mathematical Olympiad
2 May 1992, Paper 1

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1. Describe in geometric terms the set of points (x, y) in the plane such that x and y satisfy the condition $t^2 + yt + x \geq 0$ for all t with $-1 \leq t \leq 1$.

2. How many ordered triples (x, y, z) of real numbers satisfy the system of equations
$$\begin{aligned} x^2 + y^2 + z^2 &= 9, \\ x^4 + y^4 + z^4 &= 33, \\ xyz &= -4\ ? \end{aligned}$$

3. Let A be a nonempty set with n elements. Find the number of ways of choosing a pair of subsets (B, C) of A such that B is a nonempty subset of C.

4. In a triangle ABC, the points A', B' and C' on the sides opposite A, B and C, respectively, are such that the lines AA', BB' and CC' are concurrent. Prove that the diameter of the circumscribed circle of the triangle ABC equals the product $|AB'|.|BC'|.|CA'|$ divided by the area of the triangle $A'B'C'$.

5. Let ABC be a triangle such that the coordinates of the points A and B are rational numbers. Prove that the coordinates of C are rational if, and only if, $\tan A$, $\tan B$ and $\tan C$, when defined, are all rational numbers.

5th Irish Mathematical Olympiad
2 May 1992, Paper 2

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6. Let $n > 2$ be an integer and let $m = \sum k^3$, where the sum is taken over all integers k with $1 \leq k < n$ that are relatively prime to n. Prove that n divides m. (Note that two integers are *relatively prime* if, and only if, their greatest common divisor equals 1.)

7. If a_1 is a positive integer, form the sequence $a_1, a_2, a_3, \ldots$ by letting a_2 be the product of the digits of a_1, etc.. If a_k consists of a single digit, for some $k \geq 1$, a_k is called a *digital root* of a_1. It is easy to check that every positive integer has a unique digital root. (For example, if $a_1 = 24378$, then $a_2 = 1344$, $a_3 = 48$, $a_4 = 32$, $a_5 = 6$, and thus 6 is the digital root of 24378.) Prove that the digital root of a positive integer n equals 1 if, and only if, all the digits of n equal 1.

8. Let a, b, c and d be real numbers with $a \neq 0$. Prove that if all the roots of the cubic equation
$$az^3 + bz^2 + cz + d = 0$$
lie to the left of the imaginary axis in the complex plane, then
$$ab > 0,\ bc - ad > 0,\ ad > 0.$$

9. A convex pentagon has the property that each of its diagonals cuts off a triangle of unit area. Find the area of the pentagon.

10. If, for $k = 1, 2, \ldots, n$, a_k and b_k are positive real numbers, prove that
$$\sqrt[n]{a_1 a_2 \cdots a_n} + \sqrt[n]{b_1 b_2 \cdots b_n} \leq \sqrt[n]{(a_1 + b_1)(a_2 + b_2) \cdots (a_n + b_n)};$$
and that equality holds if, and only if,
$$\frac{a_1}{b_1} = \frac{a_2}{b_2} = \cdots = \frac{a_n}{b_n}.$$

6th Irish Mathematical Olympiad
8 May 1993, Paper 1

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1. The real numbers α, β satisfy the equations

$$\begin{aligned}\alpha^3 - 3\alpha^2 + 5\alpha - 17 &= 0,\\ \beta^3 - 3\beta^2 + 5\beta + 11 &= 0.\end{aligned}$$

   Find $\alpha + \beta$.

2. A natural number n is called **good** if it can be written in a *unique* way simultaneously as the sum $a_1 + a_2 + \ldots + a_k$ and as the product $a_1 a_2 \ldots a_k$ of some $k \geq 2$ natural numbers $a_1, a_2, \ldots, a_k$. (For example 10 is good because $10 = 5 + 2 + 1 + 1 + 1 = 5.2.1.1.1$ and these expressions are unique.) Determine, in terms of prime numbers, which natural numbers are good.

3. The line l is tangent to the circle S at the point A; B and C are points on l on opposite sides of A and the other tangents from B, C to S intersect at a point P. If B, C vary along l in such a way that the product $|AB|.|AC|$ is constant, find the locus of P.

4. Let $a_0, a_1, \ldots, a_{n-1}$ be real numbers, where $n \geq 1$, and let the polynomial

$$f(x) = x^n + a_{n-1}x^{n-1} + \ldots + a_0$$

   be such that $|f(0)| = f(1)$ and each root α of f is real and satisfies $0 < \alpha < 1$. Prove that the product of the roots does not exceed $1/2^n$.

5. Given a complex number $z = x + iy$ (x, y real), we denote by $P(z)$ the corresponding point (x, y) in the plane. Suppose $z_1, z_2, z_3, z_4, z_5, \alpha$ are nonzero complex numbers such that

   (a) $P(z_1)$, $P(z_2)$, $P(z_3)$, $P(z_4)$, $P(z_5)$ are the vertices of a convex pentagon $\mathbf{Q}$ containing the origin 0 in its interior and

   (b) $P(\alpha z_1)$, $P(\alpha z_2)$, $P(\alpha z_3)$, $P(\alpha z_4)$ and $P(\alpha z_5)$ are all inside $\mathbf{Q}$.

   If $\alpha = p + iq$, where p and q are real, prove that $p^2 + q^2 \leq 1$ and that

$$p + q\tan(\pi/5) \leq 1.$$

6th Irish Mathematical Olympiad
8 May 1993, Paper 2

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6. Given five points P_1, P_2, P_3, P_4, P_5 in the plane having integer coordinates, prove that there is at least one pair (P_i, P_j), with $i \neq j$, such that the line P_iP_j contains a point Q having integer coordinates and lying strictly between P_i and P_j.

7. Let $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n$ be $2n$ real numbers, where $a_1, a_2, \ldots, a_n$ are distinct, and suppose that there exists a real number α such that the product
$$(a_i + b_1)(a_i + b_2) \ldots (a_i + b_n)$$
has the value α for $i = 1, 2, \ldots, n$. Prove that there exists a real number β such that the product
$$(a_1 + b_j)(a_2 + b_j) \ldots (a_n + b_j)$$
has the value β for $j = 1, 2, \ldots, n$.

8. For nonnegative integers n, r, the binomial coefficient $\binom{n}{r}$ denotes the number of combinations of n objects chosen r at a time, with the convention that $\binom{n}{0} = 1$ and $\binom{n}{r} = 0$ if $n < r$. Prove the identity
$$\sum_{d=1}^{\infty} \binom{n-r+1}{d} \binom{r-1}{d-1} = \binom{n}{r}$$
for all integers n and r, with $1 \leq r \leq n$.

9. Let x be a real number with $0 < x < \pi$. Prove that, for all natural numbers n, the sum
$$\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \ldots + \frac{\sin(2n-1)x}{2n-1}$$
is positive.

10. (a) The rectangle $PQRS$ has $|PQ| = \ell$ and $|QR| = m$, where ℓ, m are positive integers. It is divided up into ℓm 1×1 squares by drawing lines parallel to PQ and QR. Prove that the diagonal PR intersects $\ell + m - d$ of these squares, where d is the greatest common divisor, (ℓ, m), of ℓ and m.

    (b) A cuboid (or box) with edges of lengths ℓ, m, n, where ℓ, m, n are positive integers, is divided into ℓmn $1 \times 1 \times 1$ cubes by planes parallel to its faces. Consider a diagonal joining a vertex of the cuboid to the vertex furthest away from it. How many of the cubes does this diagonal intersect?

7th Irish Mathematical Olympiad
7 May 1994, Paper 1

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1. Let x, y be positive integers, with $y > 3$, and
$$x^2 + y^4 = 2[(x-6)^2 + (y+1)^2].$$
Prove that $x^2 + y^4 = 1994$.

2. Let A, B, C be three collinear points, with B between A and C. Equilateral triangles ABD, BCE, CAF are constructed with D, E on one side of the line AC and F on the opposite side. Prove that the centroids of the triangles are the vertices of an equilateral triangle. Prove that the centroid of this triangle lies on the line AC.

3. Determine, with proof, all real polynomials f satisfying the equation
$$f(x^2) = f(x)f(x-1),$$
for all real numbers x.

4. Consider the set of $m \times n$ matrices with every entry either 0 or 1. Determine the number of such matrices with the property that the number of "1"s in each row and in each column is even.

5. Let $f(n)$ be defined on the set of positive integers by the rules: $f(1) = 2$ and
$$f(n+1) = (f(n))^2 - f(n) + 1, \qquad n = 1, 2, 3, \ldots$$
Prove that, for all integers $n > 1$,
$$1 - \frac{1}{2^{2^{n-1}}} < \frac{1}{f(1)} + \frac{1}{f(2)} + \ldots + \frac{1}{f(n)} < 1 - \frac{1}{2^{2^n}}.$$

7th Irish Mathematical Olympiad
7 May 1994, Paper 2

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6. A sequence x_n is defined by the rules: $x_1 = 2$ and
$$nx_n = 2(2n-1)x_{n-1}, \qquad n = 2, 3, \ldots$$
Prove that x_n is an integer for every positive integer n.

7. Let p, q, r be distinct real numbers that satisfy the equations
$$\begin{aligned} q &= p(4-p), \\ r &= q(4-q), \\ p &= r(4-r). \end{aligned}$$
Find all possible values of $p + q + r$.

8. Prove that, for every integer $n > 1$,
$$n\left((n+1)^{2/n} - 1\right) < \sum_{i=1}^{n} \frac{2i+1}{i^2} < n\left(1 - n^{-2/(n-1)}\right) + 4.$$

9. Let w, a, b and c be *distinct* real numbers with the property that there exist real numbers x, y and z for which the following equations hold:
$$\begin{aligned} x \quad + y \quad + z \quad &= 1, \\ xa^2 + yb^2 + zc^2 &= w^2, \\ xa^3 + yb^3 + zc^3 &= w^3, \\ xa^4 + yb^4 + zc^4 &= w^4. \end{aligned}$$
Express w in terms of a, b and c.

10. If a square is partitioned into n convex polygons, determine the maximum number of edges present in the resulting figure.

8th Irish Mathematical Olympiad
6 May 1995, Paper 1

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1. There are n^2 students in a class. Each week all the students participate in a table quiz. Their teacher arranges them into n teams of n players each. For as many weeks as possible, this arrangement is done in such a way that any pair of students who were members of the same team one week are not on the same team in subsequent weeks. Prove that after at most $n+2$ weeks, it is necessary for some pair of students to have been members of the same team on at least two different weeks.

2. Determine, with proof, all those **integers** a for which the equation
$$x^2 + axy + y^2 \;=\; 1$$
has infinitely many distinct **integer** solutions x, y.

3. Let A, X, D be points on a line, with X between A and D. Let B be a point in the plane such that $\angle ABX$ is greater than $120°$, and let C be a point on the line between B and X. Prove the inequality
$$2|AD| \;\geq\; \sqrt{3}(|AB| + |BC| + |CD|).$$

4. Consider the following one-person game played on the x-axis. For each integer k, let X_k be the point with coordinates $(k, 0)$. During the game discs are piled at some of the points X_k. To perform a move in the game, the player chooses a point X_j at which at least two discs are piled and then takes two discs from the pile at X_j and places one of them at X_{j-1} and one at X_{j+1}.
To begin the game, $2n+1$ discs are placed at X_0. The player then proceeds to perform moves in the game for as long as possible. Prove that after $n(n+1)(2n+1)/6$ moves no further moves are possible, and that, at this stage, one disc remains at each of the positions
$$X_{-n}, X_{-n+1}, \ldots, X_{-1}, X_0, X_1, \ldots, X_{n-1}, X_n.$$

5. Determine, with proof, all real-valued functions f satisfying the equation
$$xf(x) - yf(y) \;=\; (x-y)f(x+y),$$
for all real numbers x, y.

8th Irish Mathematical Olympiad
6 May 1995, Paper 2

6. Prove the inequalities

$$n^n \;\le\; (n!)^2 \;\le\; [(n+1)(n+2)/6]^n,$$

for every positive integer n.

7. Suppose that a, b and c are complex numbers, and that all three roots z of the equation

$$x^3 + ax^2 + bx + c \;=\; 0$$

satisfy $|z| = 1$ (where $|\ |$ denotes absolute value). Prove that all three roots w of the equation

$$x^3 + |a|x^2 + |b|x + |c| \;=\; 0$$

also satisfy $|w| = 1$.

8. Let S be the square consisting of all points (x, y) in the plane with $0 \le x, y \le 1$. For each real number t with $0 < t < 1$, let C_t denote the set of all points $(x, y) \in S$ such that (x, y) is on or above the line joining $(t, 0)$ to $(0, 1 - t)$.
Prove that the points common to all C_t are those points in S that are on or above the curve $\sqrt{x} + \sqrt{y} = 1$.

9. We are given three points P, Q, R in the plane. It is known that there is a triangle ABC such that P is the mid-point of the side BC, Q is the point on the side CA with $|CQ|/|QA| = 2$, and R is the point on the side AB with $|AR|/|RB| = 2$. Determine, with proof, how the triangle ABC may be constructed from P, Q, R.

10. For each integer n such that $n = p_1p_2p_3p_4$, where p_1, p_2, p_3, p_4 are distinct primes, let

$$d_1 = 1 < d_2 < d_3 < \cdots < d_{15} < d_{16} = n$$

be the sixteen positive integers that divide n.
Prove that if $n < 1995$, then $d_9 - d_8 \neq 22$.

9th Irish Mathematical Olympiad
4 May 1996, Paper 1

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1. For each positive integer n, let $f(n)$ denote the highest common factor of $n!+1$ and $(n+1)!$ (where ! denotes factorial). Find, with proof, a formula for $f(n)$ for each n. [Note that "highest common factor" is another name for "greatest common divisor".]

2. For each positive integer n, let $S(n)$ denote the sum of the digits of n when n is written in base ten. Prove that, for every positive integer n,
$$S(2n) \leq 2S(n) \leq 10S(2n).$$
Prove also that there exists a positive integer n with
$$S(n) = 1996S(3n).$$

3. Let K be the set of all real numbers x such that $0 \leq x \leq 1$. Let f be a function from K to the set of all real numbers $\mathbb{R}$ with the following properties

   (a) $f(1) = 1$;

   (b) $f(x) \geq 0$ for all $x \in K$;

   (c) if x, y and $x + y$ are all in K, then
   $$f(x+y) \geq f(x) + f(y).$$
   Prove that $f(x) \leq 2x$, for all $x \in K$.

4. Let F be the mid-point of the side BC of a triangle ABC. Isosceles right-angled triangles ABD and ACE are constructed externally on the sides AB and AC with right-angles at D and E respectively. Prove that DEF is an isosceles right-angled triangle.

5. Show, with proof, how to dissect a square into at most five pieces in such a way that the pieces can be re-assembled to form three squares no two of which are the same size.

9th Irish Mathematical Olympiad
4 May 1996, Paper 2

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6. The sequence $F_0, F_1, F_2, \ldots$ is defined as follows: $F_0 = 0$, $F_1 = 1$ and, for all $n \geq 0$,
$$F_{n+2} = F_n + F_{n+1}.$$
(So,
$$F_2 = 1, F_3 = 2, F_4 = 3, F_5 = 5, F_6 = 8 \ldots)$$
Prove that

   (a) The statement "$F_{n+k} - F_n$ is divisible by 10 for all positive integers n" is true if $k = 60$, but not true for any positive integer $k < 60$.

   (b) The statement "$F_{n+t} - F_n$ is divisible by 100 for all positive integers n" is true if $t = 300$, but not true for any positive integer $t < 300$.

7. Prove that the inequality
$$2^{\frac{1}{2}} \cdot 4^{\frac{1}{4}} \cdot 8^{\frac{1}{8}} \cdots (2^n)^{\frac{1}{2^n}} < 4$$
holds for all positive integers n.

8. Let p be a prime number, and a and n positive integers. Prove that if
$$2^p + 3^p = a^n,$$
then $n = 1$.

9. Let ABC be an acute-angled triangle and let D, E, F be the feet of the perpendiculars from A, B, C onto the sides BC, CA, AB, respectively. Let P, Q, R be the feet of the perpendiculars from A, B, C onto the lines EF, FD, DE, respectively. Prove that the lines AP, BQ, CR (extended) are concurrent.

10. We are given a rectangular board divided into 45 squares so that there are five rows of squares, each row containing nine squares. The following game is played:
Initially, a number of discs are randomly placed on some of the squares, no square being allowed to contain more than one disc. A complete move consists of moving every disc from the square containing it to another square, subject to the following rules:

    (a) each disc may be moved one square up or down, or left or right, of the square it occupies to an adjoining square;

    (b) if a particular disc is moved up or down as part of a complete move, then it must be moved left or right in the next complete move;

    (c) if a particular disc is moved left or right as part of a complete move, then it must be moved up or down in the next complete move;

    (d) at the end of each complete move no square can contain two or more discs.

    The game stops if it becomes impossible to perform a complete move. Prove that if initially 33 discs are placed on the board, then the game must eventually stop. Prove also that it is possible to place 32 discs on the board in such a way that the game could go on forever.

10th Irish Mathematical Olympiad
10 May 1997, Paper 1

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1. Find, with proof, all pairs of integers (x, y) satisfying the equation
$$1 + 1996x + 1998y \;=\; xy.$$

2. Let ABC be an equilateral triangle.
For a point M inside ABC, let D, E, F be the feet of the perpendiculars from M onto BC, CA, AB, respectively. Find the locus of all such points M for which $\angle FDE$ is a right-angle.

3. Find all polynomials p satisfying the equation
$$(x-16)p(2x) \;=\; 16(x-1)p(x)$$
for all x.

4. Suppose a, b and c are nonnegative real numbers such that $a+b+c \geq abc$. Prove that $a^2+b^2+c^2 \geq abc$.

5. Let S be the set of all odd integers greater than one. For each $x \in S$, denote by $\delta(x)$ the unique integer satisfying the inequality
$$2^{\delta(x)} \;<\; x \;<\; 2^{\delta(x)+1}.$$
For $a, b \in S$, define
$$a * b \;=\; 2^{\delta(a)-1}(b-3)+a.$$
[For example, to calculate $5 * 7$, note that $2^2 < 5 < 2^3$, so $\delta(5) = 2$, and hence $5 * 7 = 2^{2-1}(7-3)+5 = 13$. Also $2^2 < 7 < 2^3$, so $\delta(7) = 2$ and $7 * 5 = 2^{2-1}(5-3)+7 = 11$].

   Prove that if $a, b, c \in S$, then

   (a) $a * b \in S$ and

   (b) $(a * b) * c = a * (b * c)$.

10th Irish Mathematical Olympiad
10 May 1997, Paper 2

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6. Given a positive integer n, denote by $\sigma(n)$ the sum of all positive integers which divide n. [For example, $\sigma(3) = 1+3 = 4$, $\sigma(6) = 1+2+3+6 = 12$, $\sigma(12) = 1+2+3+4+6+12 = 28$]. We say that n is *abundant* if $\sigma(n) > 2n$. (So, for example, 12 is abundant).
   Let a, b be positive integers and suppose that a is abundant. Prove that ab is abundant.

7. $ABCD$ is a quadrilateral which is circumscribed about a circle Γ (i.e., each side of the quadrilateral is tangent to Γ.) If $\angle A = \angle B = 120°$, $\angle D = 90°$ and BC has length 1, find, with proof, the length of AD.

8. Let A be a subset of $\{0, 1, 2, 3, \ldots, 1997\}$ containing more than 1000 elements. Prove that either A contains a power of 2 (that is, a number of the form 2^k, with k a nonnegative integer) or there exist two distinct elements $a, b \in A$ such that $a + b$ is a power of 2.

9. Let S be the set of all natural numbers n satisfying the following conditions:

   (i) n has 1000 digits;

   (ii) all the digits of n are odd, and

   (iii) the absolute value of the difference between adjacent digits of n is 2.

   Determine the number of distinct elements in S.

10. Let p be a prime number, n a natural number and $T = \{1, 2, 3, \ldots, n\}$. Then n is called *p-partitionable* if there exist p nonempty subsets $T_1, T_2, \ldots, T_p$ of T such that

    (i) $T = T_1 \cup T_2 \cup \cdots \cup T_p$;

    (ii) $T_1, T_2, \ldots, T_p$ are disjoint (that is $T_i \cap T_j$ is the empty set for all i, j with $i \neq j$), and

    (iii) the sum of the elements in T_i is the same for $i = 1, 2, \ldots, p$.

    [For example, 5 is 3-partitionable since, if we take $T_1 = \{1, 4\}$, $T_2 = \{2, 3\}$, $T_3 = \{5\}$, then (i), (ii) and (iii) are satisfied. Also, 6 is 3-partitionable since, if we take $T_1 = \{1, 6\}$, $T_2 = \{2, 5\}$, $T_3 = \{3, 4\}$, then (i), (ii) and (iii) are satisfied.]

    (a) Suppose that n is p-partitionable. Prove that p divides n or $n + 1$.

    (b) Suppose that n is divisible by $2p$. Prove that n is p-partitionable.

11th Irish Mathematical Olympiad
9 May 1998, Paper 1

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1. Show that if x is a nonzero real number, then
$$x^8 - x^5 - \frac{1}{x} + \frac{1}{x^4} \geq 0.$$

2. P is a point inside an equilateral triangle such that the distances from P to the three vertices are 3, 4 and 5, respectively. Find the area of the triangle.

3. Show that no integer of the form $xyxy$ in base 10, where x and y are digits, can be the cube of an integer.
   Find the smallest base $b > 1$ for which there is a perfect cube of the form $xyxy$ in base b.

4. Show that a disc of radius 2 can be covered by seven (possibly overlapping) discs of radius 1.

5. If x is a real number such that $x^2 - x$ is an integer, and, for some $n \geq 3$, $x^n - x$ is also an integer, prove that x is an integer.

11th Irish Mathematical Olympiad
9 May 1998, Paper 2

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6. Find all positive integers n that have exactly 16 positive integral divisors d_1, d_2, …, d_{16} such that
$$1 = d_1 < d_2 < \cdots < d_{16} = n,$$
$d_6 = 18$ and $d_9 - d_8 = 17$.

7. Prove that if a, b, c are positive real numbers, then

(a) $$\frac{9}{a+b+c} \le 2\left(\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a}\right),$$

and

(b) $$\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a} \le \frac{1}{2}\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right).$$

8. Let $\mathbb{N}$ be the set of all natural numbers (i.e., the positive integers).

   (a) Prove that $\mathbb{N}$ can be written as a union of three mutually disjoint sets such that, if $m, n \in \mathbb{N}$ and $|m-n| = 2$ or 5, then m and n are in different sets.

   (b) Prove that $\mathbb{N}$ can be written as a union of four mutually disjoint sets such that, if $m, n \in \mathbb{N}$ and $|m-n| = 2$, 3 or 5, then m and n are in different sets. Show, however, that it is impossible to write $\mathbb{N}$ as a union of three mutually disjoint sets with this property.

9. A sequence of real numbers x_n is defined recursively as follows: x_0, x_1 are arbitrary positive real numbers, and
$$x_{n+2} = \frac{1 + x_{n+1}}{x_n},\ n = 0, 1, 2, \ldots$$
Find x_{1998}.

10. A triangle ABC has positive integer sides, $\angle A = 2\angle B$ and $\angle C > 90^\circ$. Find the minimum length of its perimeter.

12th Irish Mathematical Olympiad
8 May 1999, Paper 1

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1. Find all real values x that satisfy
$$\frac{x^2}{(x+1-\sqrt{x+1})^2} < \frac{x^2+3x+18}{(x+1)^2}.$$

2. Show that there is a positive number in the Fibonacci sequence that is divisible by 1000.
[The Fibonacci sequence F_n is defined by the conditions:
$$F_0 = 0,\ F_1 = 1,\ F_n = F_{n-1} + F_{n-2} \text{ for } n \geq 2$$
So, the sequence begins 0, 1, 1, 2, 3, 5, 8, 13, ...]

3. Let D, E and F, respectively, be points on the sides BC, CA and AB, respectively, of a triangle ABC so that AD is perpendicular to BC, BE is the angle-bisector of $\angle B$ and F is the mid-point of AB. Prove that AD, BE and CF are concurrent if, and only if,
$$a^2(a-c) = (b^2-c^2)(a+c),$$
where a, b and c are the lengths of the sides BC, CA and AB, respectively, of the triangle ABC.

4. A square floor consists of 10000 squares (100 squares $\times$ 100 squares – like a large chessboard) is to be tiled. The only available tiles are rectangular 1×3 tiles, fitting exactly over three squares of the floor.

   (a) If a 2×2 square is removed from the centre of the floor, prove that the remaining part of the floor can be tiles with the available tiles.

   (b) If, instead, a 2×2 square is removed from a corner of the floor, prove that the remaining part of the floor cannot be tiled with the available tiles.

   [There are sufficiently many tiles available. To *tile* the floor – or a portion thereof – means to completely cover it with the tiles, each tile covering three squares, and no pair of tiles overlapping. The tiles may not be broken or cut.]

5. Three real numbers a, b, c with $a < b < c$, are said to be in arithmetic progression if $c - b = b - a$.
Define a sequence u_n, $n = 0, 1, 2, 3, \ldots$ as follows: $u_0 = 0$, $u_1 = 1$ and, for each $n \geq 1$, u_{n+1} is the smallest positive integer such that $u_{n+1} > u_n$ and $\{u_0, u_1, \ldots, u_n, u_{n+1}\}$ contains no three elements that are in arithmetic progression.
Find u_{100}.

12th Irish Mathematical Olympiad
8 May 1999, Paper 2

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6. Solve the system of (simultaneous) equations

$$\begin{aligned} y^2 &= (x+8)(x^2+2), \\ y^2 &= (8+4x)y + 5x^2 - 16x - 16. \end{aligned}$$

7. A function $f : \mathbb{N} \to \mathbb{N}$ (where $\mathbb{N}$ denotes the set of positive integers) satisfies

   (a) $f(ab) = f(a)f(b)$ whenever the greatest common divisor of a and b is 1,

   (b) $f(p+q) = f(p) + f(q)$ for all prime numbers p and q.

   Prove that $f(2) = 2$, $f(3) = 3$ and $f(1999) = 1999$.

8. Let a, b, c and d be positive real numbers whose sum is 1. Prove that

$$\frac{a^2}{a+b} + \frac{b^2}{b+c} + \frac{c^2}{c+d} + \frac{d^2}{d+a} \geq \frac{1}{2},$$

   with equality if, and only if, $a = b = c = d = 1/4$.

9. Find all positive integers m with the property that the fourth power of the number of (positive) divisors of m equals m.

10. $ABCDEF$ is a convex (not necessarily regular) hexagon with $AB = BC$, $CD = DE$, $EF = FA$ and

$$\angle ABC + \angle CDE + \angle EFA = 360^\circ.$$

   Prove that the perpendiculars from A, C and E to FB, BD and DF, respectively, are concurrent.

13th Irish Mathematical Olympiad
6 May 2000, Paper 1

1. Let S be the set of all numbers of the form $a(n) = n^2 + n + 1$, where n is a natural number. Prove that the product $a(n)a(n+1)$ is in S for all natural numbers n. Give, with proof, an example of a pair of elements $s, t \in S$ such that $st \notin S$.

2. Let $ABCDE$ be a regular pentagon with its sides of length one. Let F be the midpoint of AB and let G, H be points on the sides CD and DE, respectively, such that $\langle GFD = \langle HFD = 30°$. Prove that the triangle GFH is equilateral. A square is inscribed in the triangle GFH with one side of the square along GH. Prove that FG has length
$$t = \frac{2\cos 18°(\cos 36°)^2}{\cos 6°},$$
and that the square has sides of length
$$\frac{t\sqrt{3}}{2+\sqrt{3}}.$$

3. Let $f(x) = 5x^{13} + 13x^5 + 9ax$. Find the least positive integer a such that 65 divides $f(x)$ for every integer x.

4. Let
$$a_1 < a_2 < a_3 < \cdots < a_M$$
be real numbers. $\{a_1, a_2, \ldots, a_M\}$ is called a **weak arithmetic progression** of length M if there exist real numbers $x_0, x_1, x_2, \ldots, x_M$ and d such that
$$x_0 \le a_1 < x_1 \le a_2 < x_2 \le a_3 < x_3 \le \cdots \le a_M < x_M$$
and for $i = 0, 1, 2, \ldots, M-1$, $x_{i+1} - x_i = d$ i.e. $\{x_0, x_1, x_2, \ldots, x_M\}$ is an arithmetic progression.

   (a) Prove that if $a_1 < a_2 < a_3$, then $\{a_1, a_2, a_3\}$ is a weak arithmetic progression of length 3.

   (b) Let A be a subset of $\{0, 1, 2, 3, \ldots, 999\}$ with at least 730 members. Prove that A contains a weak arithmetic progression of length 10.

5. Consider all parabolas of the form $y = x^2 + 2px + q$ (p, q real) which intersect the x- and y-axes in three distinct points. For such a pair p, q let $C_{p,q}$ be the circle through the points of intersection of the parabola $y = x^2 + 2px + q$ with the axes. Prove that all the circles $C_{p,q}$ have a point in common.

13th Irish Mathematical Olympiad
6 May 2000, Paper 2

1. Let $x \geq 0$, $y \geq 0$ be real numbers with $x + y = 2$. Prove that
$$x^2y^2(x^2 + y^2) \leq 2.$$

2. Let $ABCD$ be a cyclic quadrilateral and R the radius of the circumcircle. Let a, b, c, d be the lengths of the sides of $ABCD$ and Q its area. Prove that
$$R^2 = \frac{(ab + cd)(ac + bd)(ad + bc)}{16Q^2}.$$
Deduce that
$$R \geq \frac{(abcd)^{3/4}}{Q\sqrt{2}},$$
with equality **if and only if** $ABCD$ is a square.

3. For each positive integer n determine with proof, all positive integers m such that there exist positive integers $x_1 < x_2 < \cdots < x_n$ with
$$\frac{1}{x_1} + \frac{2}{x_2} + \frac{3}{x_3} + \cdots + \frac{n}{x_n} = m.$$

4. Prove that in each set of ten consecutive integers there is one which is coprime with each of the other integers.
For example, taking $114, 115, 116, 117, 118, 119, 120, 121, 122, 123$ the numbers 119 and 121 are each coprime with all the others. [Two integers a, b are coprime if their greatest common divisor is one.]

5. Let $p(x) = a_0 + a_1x + \cdots + a_nx^n$ be a polynomial with non-negative real coefficients. Suppose that $p(4) = 2$ and that $p(16) = 8$. Prove that $p(8) \leq 4$ and find, with proof, all such polynomials with $p(8) = 4$.

14th Irish Mathematical Olympiad
12 May 2001, Paper 1

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1. Find, with proof, all solutions of the equation
$$2^n = a\,! + b\,! + c\,!$$
in positive integers a, b, c and n. (Here, ! means "factorial".)

2. Let ABC be a triangle with sides BC, CA, AB of lengths a, b, c, respectively. Let D, E be the midpoints of the sides AC, AB, respectively. Prove that BD is perpendicular to CE if, and only if,
$$b^2 + c^2 = 5a^2.$$

3. Prove that if an odd prime number p can be expressed in the form $x^5 - y^5$, for some integers x, y, then
$$\sqrt{\frac{4p+1}{5}} = \frac{v^2+1}{2},$$
for some odd integer v.

4. Prove that

   (a) $\displaystyle \frac{2n}{3n+1} \leq \sum_{k=n+1}^{2n} \frac{1}{k},$ and

   (b) $\displaystyle \sum_{k=n+1}^{2n} \frac{1}{k} \leq \frac{3n+1}{4(n+1)},$

   for all positive integers n.

5. Let a, b be real numbers such that $ab > 0$. Prove that
$$\sqrt[3]{\frac{a^2b^2(a+b)^2}{4}} \leq \frac{a^2+10ab+b^2}{12}.$$
Determine when equality occurs.

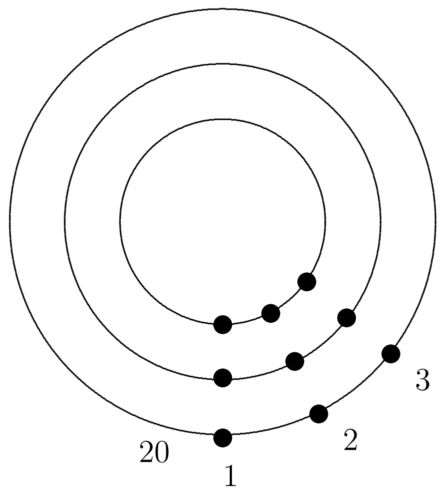
   Hence, or otherwise, prove for *all* real numbers a, b that
$$\sqrt[3]{\frac{a^2b^2(a+b)^2}{4}} \leq \frac{a^2+ab+b^2}{3}.$$
Determine the cases of equality.

14th Irish Mathematical Olympiad
12 May 2001, Paper 2

6. Find the least positive integer a such that 2001 divides $55^n + a32^n$ for some odd integer n.

7. Three hoops are arranged concentrically as in the diagram. Each hoop is threaded with 20 beads, of which 10 are black and 10 are white. On each hoop the positions of the beads are labelled 1 through 20 starting at the bottom and travelling counterclockwise.

   We say there is a *match* at position i if all three beads at position i have the same colour. We are free to slide all of the beads around any hoop (but not to unthread and rethread them).



   Show that it is possible (by sliding) to find a configuration involving at least 5 matches.

8. Let ABC be an acute angled triangle, and let D be the point on the line BC for which AD is perpendicular to BC. Let P be a point on the line segment AD. The lines BP and CP intersect AC and AB at E and F respectively. Prove that the line AD bisects the angle EDF.

9. Determine, with proof, all non-negative real numbers x for which
$$\sqrt[3]{13+\sqrt{x}} + \sqrt[3]{13-\sqrt{x}}$$
is an integer.

10. Determine, with proof, all functions f from the set of positive integers to itself which satisfy
$$f(x + f(y)) = f(x) + y$$
for all positive integers x, y.

15th Irish Mathematical Olympiad
11 May 2002, Paper 1

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1. In a triangle ABC, $AB = 20$, $AC = 21$ and $BC = 29$. The points D and E lie on the line segment BC, with $BD = 8$ and $EC = 9$. Calculate the angle $\angle DAE$.

2. (a) A group of people attends a party. Each person has at most three acquaintances in the group, and if two people do not know each other, then they have a mutual acquaintance in the group. What is the maximum number of people present?

   (b) If, in addition, the group contains three mutual acquaintances (i.e., three people each of whom knows the other two), what is the maximum number of people?

3. Find all triples of positive integers (p, q, n), with p and q primes, satisfying
$$p(p+3) + q(q+3) = n(n+3).$$

4. Let the sequence $a_1, a_2, a_3, a_4, \ldots$ be defined by
$$a_1 = 1,\ a_2 = 1,\ a_3 = 1$$
and
$$a_{n+1}a_{n-2} - a_n a_{n-1} = 2,$$
for all $n \geq 3$. Prove that a_n is a positive *integer* for all $n \geq 1$.

5. Let $0 < a, b, c < 1$. Prove that
$$\frac{a}{1-a} + \frac{b}{1-b} + \frac{c}{1-c} \geq \frac{3\sqrt[3]{abc}}{1-\sqrt[3]{abc}}.$$
Determine the case of equality.

15th Irish Mathematical Olympiad
11 May 2002, Paper 2

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6. A $3 \times n$ grid is filled as follows : the first row consists of the numbers from 1 to n arranged from left to right in ascending order. The second row is a cyclic shift of the top row. Thus the order goes $i, i+1, \ldots, n-1, n, 1, 2, \ldots, i-1$ for some i. The third row has the numbers 1 to n in some order, subject to the rule that in each of the n columns, the sum of the three numbers is the same.

   For which values of n is it possible to fill the grid according to the above rules? For an n for which this is possible, determine the number of different ways of filling the grid.

7. Suppose n is a product of four distinct primes a, b, c, d such that

   (a) $a + c = d$;

   (b) $a(a + b + c + d) = c(d - b)$;

   (c) $1 + bc + d = bd$.

   Determine n.

8. Denote by $\mathbb{Q}$ the set of rational numbers. Determine all functions $f : \mathbb{Q} \longrightarrow \mathbb{Q}$ such that

$$f(x + f(y)) = y + f(x), \text{ for all } x, y \in \mathbb{Q}.$$

9. For each real number x, define $\lfloor x \rfloor$ to be the greatest integer less than or equal to x.

   Let $\alpha = 2 + \sqrt{3}$. Prove that

$$\alpha^n - \lfloor \alpha^n \rfloor = 1 - \alpha^{-n}, \text{ for } n = 0, 1, 2, \ldots$$

10. Let ABC be a triangle whose side lengths are all integers, and let D and E be the points at which the incircle of ABC touches BC and AC respectively. If $||AD|^2 - |BE|^2| \leq 2$, show that $|AC| = |BC|$.

16th Irish Mathematical Olympiad
10 May 2003, Paper 1

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1. Find all solutions in (not necessarily positive) integers of the equation
$$(m^2 + n)(m + n^2) = (m + n)^3.$$

2. P, Q, R and S are (distinct) points on a circle. PS is a diameter and QR is parallel to the diameter PS. PR and QS meet at A. Let O be the centre of the circle and let B be chosen so that the quadrilateral $POAB$ is a parallelogram. Prove that $BQ = BP$.

3. For each positive integer k, let a_k be the greatest integer not exceeding $\sqrt{k}$ and let b_k be the greatest integer not exceeding $\sqrt[3]{k}$. Calculate
$$\sum_{k=1}^{2003} (a_k - b_k).$$

4. Eight players, Ann, Bob, Con, Dot, Eve, Fay, Guy and Hal compete in a chess tournament. No pair plays together more than once and there is no group of five people in which each one plays against all of the other four.

   (a) Write down an arrangement for a tournament of 24 games satisfying these conditions.

   (b) Show that it is impossible to have a tournament of more than 24 games satisfying these conditions.

5. Show that there is no function f defined on the set of positive real numbers such that
$$f(y) > (y - x)(f(x))^2$$
for all x, y with $y > x > 0$.

16th Irish Mathematical Olympiad
10 May 2003, Paper 2

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6. Let T be a triangle of perimeter 2, and let a, b and c be the lengths of the sides of T.

   (a) Show that
   $$abc + \frac{28}{27} \geq ab + bc + ac.$$

   (b) Show that
   $$ab + bc + ac \geq abc + 1.$$

7. $ABCD$ is a quadrilateral. P is at the foot of the perpendicular from D to AB, Q is at the foot of the perpendicular from D to BC, R is at the foot of the perpendicular from B to AD and S is at the foot of the perpendicular from B to CD. Suppose that $\angle PSR = \angle SPQ$. Prove that $PR = SQ$.

8. Find all solutions in integers x, y of the equation
   $$y^2 + 2y = x^4 + 20x^3 + 104x^2 + 40x + 2003.$$

9. Let $a, b > 0$. Determine the largest number c such that
   $$c \leq \max\left(ax + \frac{1}{ax}, bx + \frac{1}{bx}\right)$$
   for all $x > 0$.

10. (a) In how many ways can 1003 distinct integers be chosen from the set $\{1, 2, \ldots, 2003\}$ so that no two of the chosen integers differ by 10?

    (b) Show that there are $(3(5151) + 7(1700))\,101^7$ ways to choose 1002 distinct integers from the set $\{1, 2, \ldots, 2003\}$ so that no two of the chosen integers differ by 10.

17th Irish Mathematical Olympiad
8 May 2004, Paper 1

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1. (a) For which positive integers n, does $2n$ divide the sum of the first n positive integers?

   (b) Determine, with proof, those positive integers n (if any) which have the property that $2n + 1$ divides the sum of the first n positive integers.

2. Each of the players in a tennis tournament played one match against each of the others. If every player won at least one match, show that there is a group A, B, C of three players for which A beat B, B beat C and C beat A.

3. AB is a chord of length 6 of a circle centred at O and of radius 5. Let $PQRS$ denote the square inscribed in the sector OAB such that P is on the radius OA, S is on the radius OB and Q and R are points on the arc of the circle between A and B. Find the area of $PQRS$.

4. Prove that there are only two real numbers x such that
$$(x-1)(x-2)(x-3)(x-4)(x-5)(x-6) \;=\; 720.$$

5. Let $a, b \geq 0$. Prove that
$$\sqrt{2}\left(\sqrt{a(a+b)^3} + b\sqrt{a^2+b^2}\right) \;\leq\; 3(a^2+b^2),$$
with equality if and only if $a = b$.

17th Irish Mathematical Olympiad
8 May 2004, Paper 2

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1. Determine all pairs of prime numbers (p, q), with $2 \leq p, q < 100$, such that $p+6$, $p+10$, $q+4$, $q+10$ and $p+q+1$ are all prime numbers.

2. A and B are distinct points on a circle T. C is a point distinct from B such that $|AB| = |AC|$, and such that BC is tangent to T at B. Suppose that the bisector of $\angle ABC$ meets AC at a point D inside T. Show that $\angle ABC > 72°$.

3. Suppose n is an integer ≥ 2. Determine the first digit after the decimal point in the decimal expansion of the number
$$\sqrt[3]{n^3 + 2n^2 + n}.$$

4. Define the function m of the three real variables x, y, z by
$$m(x, y, z) \ = \ \max(x^2, y^2, z^2), \ \ x, y, z \in \mathbb{R}.$$
Determine, with proof, the minimum value of m if x, y, z vary in $\mathbb{R}$ subject to the following restrictions:
$$x + y + z \ = \ 0, \quad x^2 + y^2 + z^2 \ = \ 1.$$

5. Suppose p, q are distinct primes and S is a subset of $\{1, 2, \ldots, p-1\}$. Let $N(S)$ denote the number of solutions of the equation
$$\sum_{i=1}^{q} x_i \ \equiv \ 0 \quad \mathrm{mod}\ p,$$
where $x_i \in S$, $i = 1, 2, \ldots, q$. Prove that $N(S)$ is a multiple of q.

18th Irish Mathematical Olympiad
7 May 2005, Paper 1

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1. Prove that 2005^{2005} is a sum of two perfect squares, but not the sum of two perfect cubes.

2. Let ABC be a triangle and let D, E and F, respectively, be points on the sides BC, CA and AB, respectively—none of which coincides with a vertex of the triangle—such that AD, BE and CF meet at a point G. Suppose the triangles AGF, CGE and BGD have equal area. Prove that G is the centroid of ABC.

3. Prove that the sum of the lengths of the medians of a triangle is at least three quarters of the sum of the lengths of the sides.

4. Determine the number of different arrangements $a_1, a_2, \ldots, a_{10}$ of the integers $1, 2, \ldots, 10$ such that
$$a_i > a_{2i} \text{ for } 1 \leq i \leq 5,$$
and
$$a_i > a_{2i+1} \text{ for } 1 \leq i \leq 4.$$

5. Suppose a, b and c are non-negative real numbers. Prove that
$$\frac{1}{3}[(a-b)^2 + (b-c)^2 + (c-a)^2] \leq a^2 + b^2 + c^2 - 3\sqrt[3]{a^2b^2c^2} \leq (a-b)^2 + (b-c)^2 + (c-a)^2.$$

18th Irish Mathematical Olympiad
7 May 2005, Paper 2

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6. Let ABC be a triangle, and let X be a point on the side AB that is not A or B. Let P be the incentre of the triangle ACX, Q the incentre of the triangle BCX and M the midpoint of the segment PQ. Show that $|MC| > |MX|$.

7. Using only the digits $1, 2, 3, 4$ and 5, two players A, B compose a 2005-digit number N by selecting one digit at a time as follows: A selects the first digit, B the second, A the third and so on, in that order. The last to play wins if and only if N is divisible by 9. Who will win if both players play as well as possible?

8. Suppose that x is an integer and y, z, w are odd integers. Show that 17 divides $x^{y^{z^w}} - x^{y^z}$.

   [Note: Given a sequence of integers a_n, $n = 1, 2, \ldots$, the terms b_n, $n = 1, 2, \ldots$, of its sequence of "towers" a_1, $a_2^{a_1}$, $a_3^{a_2^{a_1}}$, $a_4^{a_3^{a_2^{a_1}}}$, $\ldots$, are defined recursively as follows: $b_1 = a_1$, $b_{n+1} = a_{n+1}^{b_n}$, $n = 1, 2, \ldots$.]

9. Find the first digit to the left, and the first digit to the right, of the decimal point in the decimal expansion of $(\sqrt{2} + \sqrt{5})^{2000}$.

10. Let m, n be odd integers such that $m^2 - n^2 + 1$ divides $n^2 - 1$. Prove that $m^2 - n^2 + 1$ is a perfect square.

19th Irish Mathematical Olympiad
6 May 2006, Paper 1

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1. Are there integers x, y and z which satisfy the equation
$$z^2 \;=\; (x^2+1)(y^2-1)+n$$
when (a) $n = 2006$ (b) $n = 2007$?

2. P and Q are points on the equal sides AB and AC respectively of an isosceles triangle ABC such that $AP = CQ$. Moreover, neither P nor Q is a vertex of ABC. Prove that the circumcircle of the triangle APQ passes through the circumcentre of the triangle ABC.

3. Prove that a square of side 2.1 units can be completely covered by seven squares of side 1 unit.

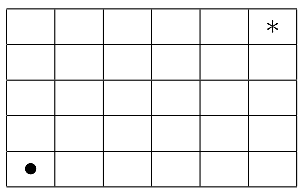
4. Find the greatest value and the least value of $x + y$, where x and y are real numbers, with $x \geq -2$, $y \geq -3$ and
$$x - 2\sqrt{x+2} \;=\; 2\sqrt{y+3} - y.$$

5. Determine, with proof, all functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(1) = 1$, and
$$f(xy + f(x)) \;=\; xf(y) + f(x)$$
for all $x, y \in \mathbb{R}$.

   Notation: $\mathbb{R}$ denotes the set of real numbers.

19th Irish Mathematical Olympiad
6 May 2006, Paper 2

6. The rooms of a building are arranged in a $m \times n$ rectangular grid (as shown below for the 5×6 case). Every room is connected by an open door to each adjacent room, but the only access to or from the building is by a door in the top right room. This door is locked with an elaborate system of mn keys, one of which is located in every room of the building. A person is in the bottom left room and can move from there to any adjacent room. However, as soon as the person leaves a room, all the doors of that room are instantly and automatically locked. Find, with proof, all m and n for which it is possible for the person to collect all the keys and escape the building.



- $\bullet$ starting position
- $*$ room with locked external door

7. ABC is a triangle with points D, E on BC, with D nearer B; F, G on AC, with F nearer C; H, K on AB, with H nearer A. Suppose that $AH = AG = 1$, $BK = BD = 2$, $CE = CF = 4$, $\angle B = 60°$ and that D, E, F, G, H and K all lie on a circle. Find the radius of the incircle of the triangle ABC.

8. Suppose x and y are positive real numbers such that $x + 2y = 1$. Prove that
$$\frac{1}{x} + \frac{2}{y} \geq \frac{25}{1 + 48xy^2}.$$

9. Let n be a positive integer. Find the greatest common divisor of the numbers
$$\binom{2n}{1}, \binom{2n}{3}, \binom{2n}{5}, \ldots, \binom{2n}{2n-1}.$$
Notation: If a and b are nonnegative integers such that $a \geq b$, then
$$\binom{a}{b} = \frac{a!}{(a-b)!b!}.$$

10. Two positive integers n and k are given, with $n \geq 2$. In the plane there are n circles such that any two of them intersect at two points and all these intersection points are distinct. Each intersection point is coloured with one of n given colours in such a way that all n colours are used. Moreover, on each circle there are precisely k different colours present. Find all possible values for n and k for which such a colouring is possible.

20th Irish Mathematical Olympiad
12 May 2007, Paper 1

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1. Find all prime numbers p and q such that p divides $q+6$ and q divides $p+7$.

2. Prove that a triangle ABC is right-angled if and only if
$$\sin^2 A + \sin^2 B + \sin^2 C = 2.$$

3. The point P is a fixed point on a circle and Q is a fixed point on a line. The point R is a variable point on the circle such that P, Q and R are not collinear. The circle through P, Q and R meets the line again at V. Show that the line VR passes through a fixed point.

4. Air Michael and Air Patrick operate direct flights connecting Belfast, Cork, Dublin, Galway, Limerick and Waterford. For each pair of cities exactly one of the airlines operates the route (in both directions) connecting the cities. Prove that there are four cities for which one of the airlines operates a round trip. (Note that a round trip of four cities P, Q, R and S, is a journey that follows the path $P \to Q \to R \to S \to P$.)

5. Let r and n be nonnegative integers such that $r \le n$.

   (a) Prove that
   $$\frac{n+1-2r}{n+1-r}\binom{n}{r}$$
   is an integer.

   (b) Prove that
   $$\sum_{r=0}^{\lfloor n/2 \rfloor} \frac{n+1-2r}{n+1-r}\binom{n}{r} < 2^{n-2}$$
   for all $n \ge 9$.

   (Note that $\binom{n}{r} = \dfrac{n!}{r!(n-r)!}$. Also, if x is a real number then $\lfloor x \rfloor$ is the unique integer such that $\lfloor x \rfloor \le x < \lfloor x \rfloor + 1$.)

20th Irish Mathematical Olympiad
12 May 2007, Paper 2

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6. Let r, s and t be the roots of the cubic polynomial
$$p(x) = x^3 - 2007x + 2002.$$
Determine the value of
$$\frac{r-1}{r+1} + \frac{s-1}{s+1} + \frac{t-1}{t+1}.$$

7. Suppose a, b and c are positive real numbers. Prove that
$$\frac{a+b+c}{3} \le \sqrt{\frac{a^2+b^2+c^2}{3}} \le \frac{\frac{ab}{c} + \frac{bc}{a} + \frac{ca}{b}}{3}.$$
For each of the inequalities, find conditions on a, b and c such that equality holds.

8. Let ABC be a triangle the lengths of whose sides BC, CA, AB, respectively, are denoted by a, b, c, respectively. Let the internal bisectors of the angles $\angle BAC, \angle ABC, \angle BCA$, respectively, meet the sides BC, CA, AB, respectively, at D, E, F, respectively. Denote the lengths of the line segments AD, BE, CF, respectively, by d, e, f, respectively. Prove that
$$def = \frac{4abc(a+b+c)\Delta}{(a+b)(b+c)(c+a)},$$
where Δ stands for the area of the triangle ABC.

9. Find the number of zeros in which the decimal expansion of the integer 2007! ends. Also find its last non-zero digit.

10. Suppose a and b are real numbers such that the quadratic polynomial
$$f(x) = x^2 + ax + b$$
has no nonnegative real roots. Prove that there exist two polynomials g, h, whose coefficients are nonnegative real numbers, such that
$$f(x) = \frac{g(x)}{h(x)},$$
for all real numbers x.

21st Irish Mathematical Olympiad
10 May 2008, Paper 1

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1. Let p_1, p_2, p_3 and p_4 be four different prime numbers satisfying the equations
$$\begin{aligned} 2p_1 + 3p_2 + 5p_3 + 7p_4 &= 162, \\ 11p_1 + 7p_2 + 5p_3 + 4p_4 &= 162. \end{aligned}$$
Find all possible values of the product $p_1p_2p_3p_4$.

2. For positive real numbers a, b, c and d such that $a^2 + b^2 + c^2 + d^2 = 1$ prove that
$$a^2b^2cd + ab^2c^2d + abc^2d^2 + a^2bcd^2 + a^2bc^2d + ab^2cd^2 \le \frac{3}{32},$$
and determine the cases of equality.

3. Determine, with proof, all integers x for which $x(x+1)(x+7)(x+8)$ is a perfect square.

4. How many sequences $a_1, a_2, \ldots, a_{2008}$ are there such that each of the numbers $1, 2, \ldots, 2008$ occurs once in the sequence, and $i \in \{a_1, a_2, \ldots, a_i\}$ for each i such that $2 \le i \le 2008$?

5. A triangle ABC has an obtuse angle at B. The perpendicular at B to AB meets AC at D, and $|CD| = |AB|$. Prove that
$$|AD|^2 = |AB|.|BC| \quad \text{if and only if} \quad \angle CBD = 30^\circ.$$

21st Irish Mathematical Olympiad
10 May 2008, Paper 2

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6. Find, with proof, all triples of integers (a, b, c) such that a, b and c are the lengths of the sides of a right angled triangle whose area is $a + b + c$.

7. Circles S and T intersect at P and Q, with S passing through the centre of T. Distinct points A and B lie on S, inside T, and are equidistant from the centre of T. The line PA meets T again at D. Prove that $|AD| = |PB|$.

8. Find $a_3, a_4, \ldots, a_{2008}$, such that $a_i = \pm 1$ for $i = 3, \ldots, 2008$ and
$$\sum_{i=3}^{2008} a_i 2^i = 2008,$$
and show that the numbers $a_3, a_4, \ldots, a_{2008}$ are uniquely determined by these conditions.

9. Given $k \in \{0, 1, 2, 3\}$ and a positive integer n, let $f_k(n)$ be the number of sequences $x_1, \ldots, x_n$, where $x_i \in \{-1, 0, 1\}$ for $i = 1, \ldots, n$, and
$$x_1 + \cdots + x_n \equiv k \mod 4.$$
   (a) Prove that $f_1(n) = f_3(n)$ for all positive integers n.

   (b) Prove that
$$f_0(n) = \frac{3^n + 2 + (-1)^n}{4}$$
   for all positive integers n.

10. Suppose that x, y and z are positive real numbers such that $xyz \geq 1$.

   (a) Prove that
$$27 \leq (1 + x + y)^2 + (1 + y + z)^2 + (1 + z + x)^2,$$
   with equality if and only if $x = y = z = 1$.

   (b) Prove that
$$(1 + x + y)^2 + (1 + y + z)^2 + (1 + z + x)^2 \leq 3(x + y + z)^2,$$
   with equality if and only if $x = y = z = 1$.

22nd Irish Mathematical Olympiad
9 May 2009, Paper 1

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1. Hamilton Avenue has eight houses. On one side of the street are the houses numbered 1,3,5,7 and directly opposite are houses 2,4,6,8 respectively. An eccentric postman starts deliveries at house 1 and delivers letters to each of the houses, finally returning to house 1 for a cup of tea. Throughout the entire journey he must observe the following rules. The numbers of the houses delivered to must follow an odd-even-odd-even pattern throughout, each house except house 1 is visited exactly once (house 1 is visited twice) and the postman at no time is allowed to cross the road to the house directly opposite. How many different delivery sequences are possible?

2. Let $ABCD$ be a square. The line segment AB is divided internally at H so that $|AB|.|BH| = |AH|^2$. Let E be the mid point of AD and X be the midpoint of AH. Let Y be a point on EB such that XY is perpendicular to BE. Prove that $|XY| = |XH|$.

3. Find all positive integers n for which $n^8 + n + 1$ is a prime number.

4. Given an n-tuple of numbers $(x_1, x_2, \ldots, x_n)$ where each $x_i = +1$ or -1, form a new n-tuple
$$(x_1x_2, x_2x_3, x_3x_4, \ldots, x_nx_1),$$
and continue to repeat this operation. Show that if $n = 2^k$ for some integer $k \geq 1$, then after a certain number of repetitions of the operation, we obtain the n-tuple
$$(1, 1, 1, \ldots, 1)\ .$$

5. Suppose a, b, c are real numbers such that $a + b + c = 0$ and $a^2 + b^2 + c^2 = 1$. Prove that
$$a^2b^2c^2 \ \leq \ \frac{1}{54},$$
and determine the cases of equality.

22nd Irish Mathematical Olympiad
9 May 2009, Paper 2

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6. Let $p(x)$ be a polynomial with rational coefficients. Prove that there exists a positive integer n such that the polynomial $q(x)$ defined by
$$q(x) \;=\; p(x+n) - p(x)$$
has integer coefficients.

7. For any positive integer n define
$$E(n) \;=\; n(n+1)(2n+1)(3n+1)\cdots(10n+1).$$
Find the greatest common divisor of $E(1)$, $E(2)$, $E(3)$, $\ldots$, $E(2009)$.

8. Find all pairs (a, b) of positive integers, such that $(ab)^2 - 4(a+b)$ is the square of an integer.

9. At a strange party, each person knew exactly 22 others.

   For any pair of people X and Y who knew one another, there was no other person at the party that they both knew.

   For any pair of people X and Y who did not know one another, there were exactly 6 other people that they both knew.

   How many people were at the party?

10. In the triangle ABC we have $|AB| < |AC|$. The bisectors of the angles at B and C meet AC and AB at D and E respectively. BD and CE intersect at the incentre I of $\triangle ABC$.

    Prove that $\angle BAC = 60^\circ$ if and only if $|IE| = |ID|$.

23rd Irish Mathematical Olympiad
24 April 2010, Paper 1

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1. Find the least k for which the number 2010 can be expressed as the sum of the squares of k integers.

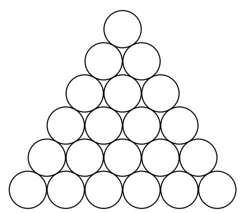
2. Let ABC be a triangle and let P denote the midpoint of the side BC. Suppose that there exist two points M and N interior to the sides AB and AC respectively, such that
$$|AD| = |DM| = 2|DN|,$$
where D is the intersection point of the lines MN and AP. Show that $|AC| = |BC|$.

3. Suppose x, y, z are positive numbers such that $x + y + z = 1$. Prove that

   (a) $xy + yz + zx \geq 9xyz$;

   (b) $xy + yz + zx < \frac{1}{4} + 3xyz$.

4. The country of Harpland has three types of coin: *green*, *white* and *orange*. The unit of currency in Harpland is the shilling. Any coin is worth a positive integer number of shillings, but coins of the same colour may be worth different amounts. A set of coins is stacked in the form of an equilateral triangle of side n coins, as shown below for the case of $n = 6$.



   The stacking has the following properties:

   (a) no coin touches another coin of the same colour;

   (b) the total worth, in shillings, of the coins lying on any line parallel to one of the sides of the triangle is divisible by three.

   Prove that the total worth in shillings of the *green* coins in the triangle is divisible by three.

5. Find all polynomials $f(x) = x^3 + bx^2 + cx + d$, where b, c, d are real numbers, such that $f(x^2 - 2) = -f(-x)f(x)$.

23rd Irish Mathematical Olympiad
24 April 2010, Paper 2

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6. There are 14 boys in a class. Each boy is asked how many other boys in the class have his first name, and how many have his last name. It turns out that each number from 0 to 6 occurs among the answers.

   Prove that there are two boys in the class with the same first name and the same last name.

7. For each odd integer $p \geq 3$ find the number of real roots of the polynomial
$$f_p(x) = (x-1)(x-2)\cdots(x-p+1)+1 \ .$$

8. In the triangle ABC we have $|AB| = 1$ and $\angle ABC = 120^\circ$. The perpendicular line to AB at B meets AC at D such that $|DC| = 1$. Find the length of AD.

9. Let $n \geq 3$ be an integer and $a_1, a_2, \ldots, a_n$ be a finite sequence of positive integers, such that, for $k = 2, 3, \ldots, n$
$$n(a_k+1) - (n-1)a_{k-1} = 1 \ .$$
   Prove that a_n is not divisible by $(n-1)^2$.

10. Suppose a, b, c are the side lengths of a triangle ABC. Show that
$$x = \sqrt{a(b+c-a)}, \ \ y = \sqrt{b(c+a-b)}, \ \ z = \sqrt{c(a+b-c)}$$
   are the side lengths of an acute-angled triangle XYZ, with the same area as ABC, but with a smaller perimeter, unless ABC is equilateral.

24th Irish Mathematical Olympiad
7 May 2011, Paper 1

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1. Suppose $abc \neq 0$. Express in terms of a, b, and c the solutions x, y, z, u, v, w of the equations
$$x + y = a, \quad z + u = b, \quad v + w = c, \quad ay = bz, \quad ub = cv, \quad wc = ax.$$

2. Let ABC be a triangle whose side lengths are, as usual, denoted by $a = |BC|$, $b = |CA|$, $c = |AB|$. Denote by m_a, m_b, m_c, respectively, the lengths of the medians which connect A, B, C, respectively, with the centres of the corresponding opposite sides.

   (a) Prove that $2m_a < b + c$. Deduce that $m_a + m_b + m_c < a + b + c$.

   (b) Give an example of

      (i) a triangle in which $m_a > \sqrt{bc}$;

      (ii) a triangle in which $ma \leq \sqrt{bc}$.

3. The integers $a_0, a_1, a_2, a_3, \ldots$ are defined as follows:
$$a_0 = 1, \quad a_1 = 3, \quad \text{and } a_{n+1} = a_n + a_{n-1} \text{ for all } n \geq 1.$$
Find all integers $n \geq 1$ for which $na_{n+1} + a_n$ and $na_n + a_{n-1}$ share a common factor greater than 1.

4. The incircle $\mathcal{C}_1$ of triangle ABC touches the sides AB and AC at the points D and E, respectively. The incircle $\mathcal{C}_2$ of the triangle ADE touches the sides AB and AC at the points P and Q, and intersects the circle $\mathcal{C}_1$ at the points M and N. Prove that

   (a) the centre of the circle $\mathcal{C}_2$ lies on the circle $\mathcal{C}_1$.

   (b) the four points M, N, P, Q in appropriate order form a rectangle if and only if twice the radius of $\mathcal{C}_1$ is three times the radius of $\mathcal{C}_2$.

5. In the mathematical talent show called "The X^2-factor" contestants are scored by a panel of 8 judges. Each judge awards a score of 0 ('fail'), X ('pass'), or X^2 ('pass with distinction'). Three of the contestants were Ann, Barbara and David. Ann was awarded the same score as Barbara by exactly 4 of the judges. David declares that he obtained different scores to Ann from at least 4 of the judges, and also that he obtained different scores to Barbara from at least 4 judges.

   In how many ways could scores have been allocated to David, assuming he is telling the truth?

24th Irish Mathematical Olympiad
7 May 2011, Paper 2

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6. Prove that
$$\frac{2}{3}+\frac{4}{5}+\cdots+\frac{2010}{2011}$$
is not an integer.

7. In a tournament with N players, $N<10$, each player plays once against each other player scoring 1 point for a win and 0 points for a loss. Draws do not occur. In a particular tournament only one player ended with an odd number of points and was ranked fourth. Determine whether or not this is possible. If so, how many wins did the player have?

8. $ABCD$ is a rectangle. E is a point on AB between A and B, and F is a point on AD between A and D. The area of the triangle EBC is 16, the area of the triangle EAF is 12 and the area of the triangle FDC is 30. Find the area of the triangle EFC.

9. Suppose that x, y and z are positive numbers such that
$$1=2xyz+xy+yz+zx. \tag{1}$$
Prove that

(i)
$$\frac{3}{4}\leq xy+yz+zx<1.$$

(ii)
$$xyz\leq\frac{1}{8}.$$

Using (i) or otherwise, deduce that
$$x+y+z\geq\frac{3}{2}, \tag{2}$$
and derive the case of equality in (2).

10. Find with proof all solutions in nonnegative integers a,b,c,d of the equation
$$11^a5^b-3^c2^d=1.$$

25th Irish Mathematical Olympiad
12 May 2012, Paper 1

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1. Let
$$C = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20\}$$
and let
$$S = \{4, 5, 9, 14, 23, 37\}.$$
Find two sets A and B with the properties

   (a) $A \cap B = \emptyset$.

   (b) $A \cup B = C$.

   (c) The sum of two distinct elements of A is not in S.

   (d) The sum of two distinct elements of B is not in S.

2. A, B, C and D are four points in that order on the circumference of a circle K. AB is perpendicular to BC and BC is perpendicular to CD. X is a point on the circumference of the circle between A and D. AX extended meets CD extended at E and DX extended meets BA extended at F.

   Prove that the circumcircle of triangle AXF is tangent to the circumcircle of triangle DXE and that the common tangent line passes through the centre of the circle K.

3. Find, with proof, all polynomials f such that f has nonnegative integer coefficients, $f(1) = 8$ and $f(2) = 2012$.

4. There exists an infinite set of triangles with the following properties:

   (a) the lengths of the sides are integers with no common factors, and

   (b) one and only one angle is $60°$.

   One such triangle has side lengths 5, 7 and 8. Find two more.

5. (a) Show that if x and y are positive real numbers, then
$$(x + y)^5 \geq 12xy(x^3 + y^3).$$

   (b) Prove that the constant 12 is the best possible. In other words, prove that for any $K > 12$ there exist positive real numbers x and y such that
$$(x + y)^5 < Kxy(x^3 + y^3).$$

25th Irish Mathematical Olympiad
12 May 2012, Paper 2

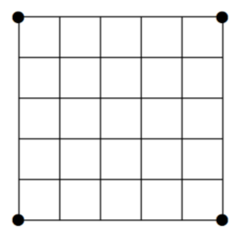
6. Let $S(n)$ be the sum of the decimal digits of n. For example, $S(2012) = 2 + 0 + 1 + 2 = 5$. Prove that there is no integer $n > 0$ for which $n - S(n) = 9990$.

7. Consider a triangle ABC with $|AB| \neq |AC|$. The angle bisector of the angle CAB intersects the circumcircle of $\triangle ABC$ at two points A and D. The circle of centre D and radius $|DC|$ intersects the line AC at two points C and B'. The line BB' intersects the circumcircle of $\triangle ABC$ at B and E. Prove that B' is the orthocentre of $\triangle AED$.

8. Suppose a, b, c are positive numbers. Prove that
$$\left(\frac{a}{b} + \frac{b}{c} + \frac{c}{a} + 1\right)^2 \ \geq \ (2a + b + c)\left(\frac{2}{a} + \frac{1}{b} + \frac{1}{c}\right),$$
with equality if and only if $a = b = c$.

9. Let $x > 1$ be an integer. Prove that $x^5 + x + 1$ is divisible by at least two distinct prime numbers.

10. Let n be a positive integer. A mouse sits at each corner point of an $n \times n$ square board, which is divided into unit squares as shown below for the example $n = 5$.



The mice then move according to a sequence of steps, in the following manner:

(a) In each step, each of the four mice travels a distance of one unit in a horizontal or vertical direction. Each unit distance is called an edge of the board, and we say that each mouse uses an edge of the board.

(b) An edge of the board may not be used twice in the same direction.

(c) At most two mice may occupy the same point on the board at any time.

The mice wish to collectively organise their movements so that each edge of the board will be used twice (not necessarily by the same mouse), and each mouse will finish up at its starting point. Determine, with proof, the values of n for which the mice may achieve this goal.

26th Irish Mathematical Olympiad
11 May 2013, Paper 1

1. Find the smallest positive integer m such that $5m$ is an exact 5^{th} power, $6m$ is an exact 6^{th} power, and $7m$ is an exact 7^{th} power.

2. Prove that
$$1 - \frac{1}{2012}\left(\frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{2013}\right) \geq \frac{1}{\sqrt[2012]{2013}}.$$

3. The altitudes of a triangle ABC are used to form the sides of a second triangle $A_1B_1C_1$. The altitudes of $\triangle A_1B_1C_1$ are then used to form the sides of a third triangle $A_2B_2C_2$. Prove that $\triangle A_2B_2C_2$ is similar to $\triangle ABC$.

4. Each of the 36 squares of a 6×6 table is to be coloured either Red, Yellow or Blue.

   (a) No row or column is contain more than two squares of the same colour.

   (b) In any four squares obtained by intersecting two rows with two columns, no colour is to occur exactly three times.

   In how many different ways can the table be coloured if both of these rules are to be respected?

5. A, B and C are points on the circumference of a circle with centre O. Tangents are drawn to the circumcircles of triangles OAB and OAC at P and Q respectively, where P and Q are diametrically opposite O. The two tangents intersect at K. The line CA meets the circumcircle of $\triangle OAB$ at A and X. Prove that X lies on the line KO.

26th Irish Mathematical Olympiad
11 May 2013, Paper 2

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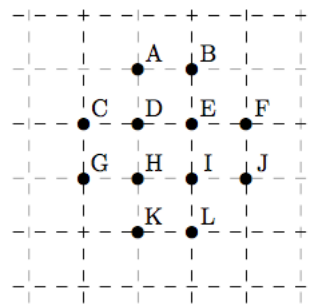
6. The three distinct points B, C, D are collinear with C between B and D. Another point A not on the line BD is such that $|AB| = |AC| = |CD|$. Prove that $\angle BAC = 36°$ if and only if
$$\frac{1}{|CD|} - \frac{1}{|BD|} = \frac{1}{|CD| + |BD|}.$$

7. Consider the collection of different squares which may be formed by sets of four points chosen from the 12 labelled points in the diagram on the right.

   For each possible area such a square may have, determine the number of squares which have this area.

   Make sure to explain why your list is complete.



8. Find the smallest positive integer N for which the equation $(x^2 - 1)(y^2 - 1) = N$ is satisfied by at least two pairs of integers (x, y) with $1 < x \le y$.

9. We say that a doubly infinite sequence
$$\ldots, s_{-2}, s_{-1}, s_0, s_1, s_2, \ldots$$
is *subaveraging* if $s_n = (s_{n-1} + s_{n+1})/4$ for all integers n.

   (a) Find a subaveraging sequence in which all entries are different from each other. Prove that all entries are indeed distinct.

   (b) Show that if (s_n) is a subaveraging sequence such that there exist distinct integers m, n such that $s_m = s_n$, then there are infinitely many pairs of distinct integers i, j with $s_i = s_j$.

10. Let a, b, c be real numbers and let $x = a + b + c$, $y = a^2 + b^2 + c^2$, $z = a^3 + b^3 + c^3$ and $S = 2x^3 - 9xy + 9z$.

    (a) Prove that S is unchanged when a, b, c are replaced by $a + t$, $b + t$, $c + t$, respectively, for any real number t.

    (b) Prove that $(3y - x^2)^3 \ \ge \ 2S^2$.

27th Irish Mathematical Olympiad
10 May 2014, Paper 1

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1. Given an 8×8 chess board, in how many ways can we select 56 squares on the board while satisfying both of the following requirements:

   (a) All black squares are selected.

   (b) Exactly seven squares are selected in each column and in each row.

2. Prove for all integers $N > 1$ that $(N^2)^{2014} - (N^{11})^{106}$ is divisible by $N^6 + N^3 + 1$.

3. In the triangle ABC, D is the foot of the altitude from A to BC, and M is the midpoint of the line segment BC. The three angles $\angle BAD$, $\angle DAM$ and $\angle MAC$ are all equal. Find the angles of the triangle ABC.

4. Three different nonzero real numbers a, b, c satisfy the equations
$$a + \frac{2}{b} = b + \frac{2}{c} = c + \frac{2}{a} = p$$
where p is a real number. Prove that $abc + 2p = 0$.

5. Suppose that $a_1, \ldots, a_n > 0$, where $n > 1$ and $\sum_{i=1}^{n} a_i = 1$. For each $i = 1, 2, \ldots, n$, let $b_i = a_i^2 / \sum_{j=1}^{n} a_j^2$. Prove that
$$\sum_{i=1}^{n} \frac{a_i}{1 - a_i} \leq \sum_{i=1}^{n} \frac{b_i}{1 - b_i}.$$
When does equality occur?

27th Irish Mathematical Olympiad
10 May 2014, Paper 2

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6. Each of the four positive integers $N, N+1, N+2, N+3$ has exactly six positive divisors. There are exactly 20 different positive numbers which are exact divisors of at least one of the numbers. One of these is 27. Find all possible values of N. (Both 1 and m are counted as divisors of the number m.)

7. The square $ABCD$ is inscribed in a circle with centre O. Let E be the midpoint of AD. The line CE meets the circle again at F. The lines FB and AD meet at H. Prove $|HD| = 2|AH|$.

8. (a) Let a_0, a_1, a_2 be real numbers and consider the polynomial $P(x) = a_0 + a_1x + a_2x^2$. Assume that $P(-1)$, $P(0)$ and $P(1)$ are integers. Prove that $P(n)$ is an integer for all integers n.

   (b) Let a_0, a_1, a_2, a_3 be real numbers and consider the polynomial $Q(x) = a_0 + a_1x + a_2x^2 + a_3x^3$. Assume that there exists an integer i such that $Q(i)$, $Q(i+1)$, $Q(i+2)$ and $Q(i+3)$ are integers. Prove that $Q(n)$ is an integer for all integers n.

9. Let n be a positive integer and $a_1, \ldots, a_n$ be positive real numbers. Let $g(x)$ denote the product
$$(x + a_1) \cdots (x + a_n).$$
Let a_0 be a real number and let
$$f(x) \;=\; (x - a_0)g(x) \;=\; x^{n+1} + b_1x^n + b_2x^{n-1} + \ldots + b_nx + b_{n+1}.$$
Prove that all the coefficients $b_1, b_2, \ldots, b_{n+1}$ of the polynomial $f(x)$ are negative if and only if
$$a_0 > a_1 + a_2 + \ldots + a_n.$$

10. Over a period of k consecutive days, a total of 2014 babies were born in a certain city, with at least one baby being born each day. Show that:

   (a) If $1014 < k \leq 2014$, there must be a period of consecutive days during which exactly 100 babies were born.

   (b) By contrast, if $k = 1014$, such a period might not exist.

28th Irish Mathematical Olympiad
25 April 2015, Paper 1

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1. In the triangle ABC, the length of the altitude from A to BC is equal to 1. D is the midpoint of AC. What are the possible lengths of BD?

2. A regular polygon with $n \geq 3$ sides is given. Each vertex is coloured either red, green or blue, and no two adjacent vertices of the polygon are the same colour. There is at least one vertex of each colour.

   Prove that it is possible to draw certain diagonals of the polygon in such a way that they intersect only at the vertices of the polygon and they divide the polygon into triangles so that each such triangle has vertices of three different colours.

3. Find all positive integers n for which both $837+n$ and $837-n$ are cubes of positive integers.

4. Two circles $\mathcal{C}_1$ and $\mathcal{C}_2$, with centres at D and E respectively, touch at B. The circle having DE as diameter intersects the circle $\mathcal{C}_1$ at H and the circle $\mathcal{C}_2$ at K. The points H and K both lie on the same side of the line DE. HK extended in both directions meets the circle $\mathcal{C}_1$ at L and meets the circle $\mathcal{C}_2$ at M. Prove that

   (a) $|LH| = |KM|$;

   (b) the line through B perpendicular to DE bisects HK.

5. Suppose a doubly infinite sequence of real numbers

$$\ldots,\ a_{-2},\ a_{-1},\ a_0,\ a_1,\ a_2,\ \ldots$$

   has the property that

$$a_{n+3} = \frac{a_n + a_{n+1} + a_{n+2}}{3}, \qquad \text{for all integers } n\,.$$

   Show that if this sequence is bounded (i.e., if there exists a number R such that $|a_n| \leq R$ for all n), then a_n has the same value for all n.

28th Irish Mathematical Olympiad
25 April 2015, Paper 2

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6. Suppose x, y are nonnegative real numbers such that $x + y \leq 1$. Prove that
$$8xy \leq 5x(1-x) + 5y(1-y)\,,$$
and determine the cases of equality.

7. Let $n > 1$ be an integer and $\Omega := \{1, 2, \ldots, 2n-1, 2n\}$ the set of all positive integers that are not larger than $2n$.

   A nonempty subset S of Ω is called *sum-free* if, for all elements x, y belonging to S, $x+y$ does not belong to S. We allow $x = y$ in this condition.

   Prove that Ω has more than 2^n distinct sum-free subsets.

8. In triangle $\triangle ABC$, the angle $\angle BAC$ is less than $90°$. The perpendiculars from C on AB and from B on AC intersect the circumcircle of $\triangle ABC$ again at D and E respectively. If $|DE| = |BC|$, find the measure of the angle $\angle BAC$.

9. Let $p(x)$ and $q(x)$ be non-constant polynomial functions with integer coefficients. It is known that the polynomial
$$p(x)q(x) - 2015$$
has at least 33 different integer roots. Prove that neither $p(x)$ nor $q(x)$ can be a polynomial of degree less than three.

10. Prove that, for all pairs of nonnegative integers, j, n,
$$\sum_{k=0}^{n} k^j \binom{n}{k} \geq 2^{n-j} n^j.$$

29th Irish Mathematical Olympiad
26 April 2016, Paper 1

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1. If the three-digit number ABC is divisible by 27, prove that the three-digit numbers BCA and CAB are also divisible by 27.

2. In triangle ABC we have $|AB| \neq |AC|$. The bisectors of $\angle ABC$ and $\angle ACB$ meet AC and AB at E and F, respectively, and intersect at I. If $|EI| = |FI|$ find the measure of $\angle BAC$.

3. Do there exist four polynomials $P_1(x), P_2(x), P_3(x), P_4(x)$ with real coefficients, such that the sum of any three of them always has a real root, but the sum of any two of them has no real root?

4. Let ABC be a triangle with $|AC| \neq |BC|$. Let P and Q be the intersection points of the line AB with the internal and external angle bisectors at C, so that P is between A and B. Prove that if M is any point on the circle with diameter PQ, then $\angle AMP = \angle BMP$.

5. Let $a_1, a_2, \ldots, a_m$ be positive integers, none of which is equal to 10, such that $a_1 + a_2 + \cdots + a_m = 10m$. Prove that
$$(a_1 a_2 a_3 \cdots a_m)^{1/m} \leq 3\sqrt{11}.$$

29th Irish Mathematical Olympiad
26 April 2016, Paper 2

6. Triangle ABC has sides $a = |BC| > b = |AC|$. The points K and H on the segment BC satisfy $|CH| = (a+b)/3$ and $|CK| = (a-b)/3$. If G is the centroid of triangle ABC, prove that $\angle KGH = 90^\circ$.

7. A rectangular array of positive integers has four rows. The sum of the entries in each column is 20. Within each row, all entries are distinct. What is the maximum possible number of columns?

8. Suppose a, b, c are real numbers such that $abc \neq 0$. Determine x, y, z in terms of a, b, c such that
$$bz + cy = a,\ cx + az = b,\ ay + bx = c.$$
Prove also that
$$\frac{1-x^2}{a^2} = \frac{1-y^2}{b^2} = \frac{1-z^2}{c^2}.$$

9. Show that the number
$$\left(\frac{251}{\frac{1}{\sqrt[3]{252} - 5\sqrt[3]{2}} - 10\sqrt[3]{63}} + \frac{1}{\frac{251}{\sqrt[3]{252} + 5\sqrt[3]{2}} + 10\sqrt[3]{63}} \right)^3$$
is an integer and find its value.

10. Let AE be a diameter of the circumcircle of triangle ABC. Join E to the orthocentre, H, of $\triangle ABC$ and extend EH to meet the circle again at D. Prove that the *nine point circle* of $\triangle ABC$ passes through the midpoint of HD.

   [Note. The *nine point circle* of a triangle is a circle that passes through the midpoints of the sides, the feet of the altitudes and the midpoints of the line segments that join the orthocentre to the vertices.]

30th Irish Mathematical Olympiad
6 May 2017, Paper 1

1. Determine, with proof, the smallest positive multiple of 99 all of whose digits are either 1 or 2.

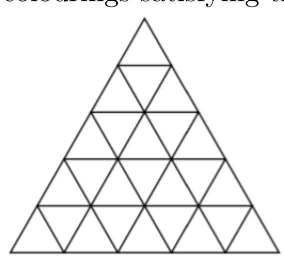
2. Solve the equations
$$a+b+c=0, \quad a^2+b^2+c^2=1, \quad a^3+b^3+c^3=4abc$$
for a, b, and c.

3. Four circles are drawn with the sides of the quadrilateral $ABCD$ as diameters. The two circles passing through A meet again at A', the two circles through B at B', the two circles through C at C' and the two circles through D at D'. Suppose that the points A', B', C' and D' are distinct. Prove that the quadrilateral $A'B'C'D'$ is similar to the quadrilateral $ABCD$.

   (Note: Two quadrilaterals are similar if their corresponding angles are equal to each other and their corresponding side lengths are in proportion to each other.)

4. An equilateral triangle of integer side length $n \geq 1$ is subdivided into small triangles of unit side length, as illustrated in the figure below for the case $n=5$. In this diagram, a subtriangle is a triangle of any size which is formed by connecting vertices of the small triangles along the grid-lines.

   It is desired to colour each vertex of the small triangles either red or blue in such a way that there is no subtriangle with all three of its vertices having the same colour. Let $f(n)$ denote the number of distinct colourings satisfying this condition.



   Determine, with proof, $f(n)$ for every $n \geq 1$.

5. The sequence $a=(a_0,a_1,a_2,\ldots)$ is defined by $a_0=0$, $a_1=2$ and
$$a_{n+2}=2a_{n+1}+41a_n \text{ for all } n \geq 0.$$
Prove that a_{2016} is divisible by 2017.

30th Irish Mathematical Olympiad
6 May 2017, Paper 2

6. Does there exist an even positive integer n for which $n+1$ is divisible by 5 and the two numbers $2^n + n$ and $2^n - 1$ are co-prime?

   (Note: Two integers are said to be *co-prime* if their greatest common divisor is equal to 1.)

7. Five teams play in a soccer competition where each team plays one match against each of the other four teams. A winning team gains 5 points and a losing team 0 points. For a 0-0 draw both teams gain 1 point, and for other draws (1-1, 2-2, etc.) both teams gain 2 points. At the end of the competition, we write down the total points for each team, and we find that they form five consecutive integers. What is the minimum number of goals scored?

8. A line segment B_0B_n is divided into n equal parts at points $B_1, B_2, \ldots, B_{n-1}$ and A is a point such that $\angle B_0AB_n$ is a right angle. Prove that

$$\sum_{k=0}^{n} |AB_k|^2 = \sum_{k=0}^{n} |B_0B_k|^2.$$

9. Show that for all non-negative numbers a, b,

$$1 + a^{2017} + b^{2017} \geq a^{10}b^7 + a^7b^{2000} + a^{2000}b^{10}.$$

   When is equality attained?

10. Given a positive integer m, a sequence of real numbers $a = (a_1, a_2, a_3, \ldots)$ is called m-powerful if it satisfies

$$\left(\sum_{k=1}^{n} a_k\right)^m = \sum_{k=1}^{n} a_k^m \qquad \text{for all positive integers } n.$$

   (a) Show that a sequence is 30-powerful if and only if at most one of its terms is non-zero.

   (b) Find a sequence none of whose terms is zero but which is 2017-powerful.

31st Irish Mathematical Olympiad
12 May 2018, Paper 1

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1. Mary and Pat play the following number game. Mary picks an initial integer greater than 2017. She then multiplies this number by 2017 and adds 2 to the result. Pat will add 2019 to this new number and it will again be Mary's turn. Both players will continue to take alternating turns. Mary will always multiply the current number by 2017 and add 2 to the result when it is her turn. Pat will always add 2019 to the current number when it is his turn. Pat wins if one of the numbers obtained is divisible by 2018. Mary wants to prevent Pat from winning the game. Determine, with proof, the smallest initial integer Mary could choose in order to achieve this.

2. The triangle ABC is right-angled at A. Its incentre is I, and H is the foot of the perpendicular from I on AB. The perpendicular from H on BC meets BC at E, and it meets the bisector of $\angle ABC$ at D. The perpendicular from A on BC meets BC at F. Prove that $\angle EFD = 45°$.

3. Find all functions $f(x) = ax^2 + bx + c$, with $a \neq 0$, such that
$$f(f(1)) = f(f(0)) = f(f(-1)).$$

4. We say that a rectangle with side lengths a and b *fits inside* a rectangle with side lengths c and d if either ($a \leq c$ and $b \leq d$) or ($a \leq d$ and $b \leq c$). For instance, a rectangle with side lengths 1 and 5 fits inside a rectangle with side lengths 6 and 2. Suppose S is a set of 2019 rectangles, all with integer side lengths between 1 and 2018 inclusive. Show that there are three rectangles A, B, and C in S such that A fits inside B, and B fits inside C.

5. Points A, B and P lie on the circumference of a circle Ω_1 such that $\angle APB$ is an obtuse angle. Let Q be the foot of the perpendicular from P on AB. A second circle Ω_2 is drawn with centre P and radius PQ. The tangents from A and B to Ω_2 intersect Ω_1 at F and H respectively. Prove that FH is tangent to Ω_2.

31st Irish Mathematical Olympiad
12 May 2018, Paper 2

6. Find all real-valued functions f satisfying

$$f(2x + f(y)) + f(f(y)) \;=\; 4x + 8y$$

for all real numbers x and y.

7. Let a, b, c be the side lengths of a triangle. Prove that

$$2(a^3 + b^3 + c^3) \;<\; (a + b + c)(a^2 + b^2 + c^2) \;\leq\; 3(a^3 + b^3 + c^3).$$

8. Let M be the midpoint of side BC of an equilateral triangle ABC. The point D is on CA extended such that A is between D and C. The point E is on AB extended such that B is between A and E, and $|MD| = |ME|$. The point F is the intersection of MD and AB. Prove that $\angle BFM = \angle BME$.

9. The sequence of positive integers $a_1, a_2, a_3, \ldots$ satisfies

$$a_{n+1} = a_n^2 + 2018 \qquad \text{for } n \geq 1.$$

Prove that there exists at most one n for which a_n is the cube of an integer.

10. The game of *Greed* starts with an initial configuration of one or more piles of stones. Player 1 and Player 2 take turns to remove stones, beginning with Player 1. At each turn, a player has two choices:
    - take one stone from any one of the piles (a simple move);
    - take one stone from each of the remaining piles (a greedy move).

    The player who takes the last stone wins. Consider the following two initial configurations:

    (a) There are 2018 piles, with either 20 or 18 stones in each pile.

    (b) There are four piles, with 17, 18, 19, and 20 stones, respectively.

    In each case, find an appropriate strategy that guarantees victory to one of the players.

32nd Irish Mathematical Olympiad
11 May 2019, Paper 1

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1. Define the *quasi-primes* as follows.
   - The first quasi-prime is $q_1 = 2$
   - For $n \geq 2$, the n^{th} quasi-prime q_n is the smallest integer greater than q_{n-1} and not of the form $q_i q_j$ for some $1 \leq i \leq j \leq n-1$.

   Determine, with proof, whether or not 1000 is a quasi-prime.

2. Jenny is going to attend a sports camp for 7 days. Each day, she will play exactly one of three sports: hockey, tennis or camogie. The only restriction is that in any period of 4 consecutive days, she must play all three sports.

   Find, with proof, the number of possible sports schedules for Jenny's week.

3. A quadrilateral $ABCD$ is such that the sides AB and DC are parallel, and $|BC| = |AB| + |CD|$. Prove that the angle bisectors of the angles $\angle ABC$ and $\angle BCD$ intersect at right angles on the side AD.

4. Find the set of all quadruplets (x, y, z, w) of non-zero real numbers which satisfy
$$1 + \frac{1}{x} + \frac{2(x+1)}{xy} + \frac{3(x+1)(y+2)}{xyz} + \frac{4(x+1)(y+2)(z+3)}{xyzw} \; = \; 0.$$

5. Let M be a point on the side BC of triangle ABC and let P and Q denote the circumcentres of triangles ABM and ACM respectively. Let L denote the point of intersection of the extended lines BP and CQ and let K denote the reflection of L through the line PQ.

   Prove that M, P, Q and K all lie on the same circle.

   [*We say that K is the reflection of L through the line PQ if PQ is the perpendicular bisector of the segment KL*]

32nd Irish Mathematical Olympiad
11 May 2019, Paper 2

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6. The number 2019 has the following nice properties:

   (a) It is the sum of the fourth powers of five distinct positive integers.

   (b) It is the sum of six consecutive positive integers.

   In fact,
$$2019 = 1^4 + 2^4 + 3^4 + 5^4 + 6^4. \tag{1}$$
$$2019 = 334 + 335 + 336 + 337 + 338 + 339. \tag{2}$$
   Prove that 2019 is the smallest number that satisfies **both** (a) and (b).

   (You may assume that (1) and (2) are correct!)

7. Three non-zero real numbers a, b, c satisfy $a + b + c = 0$ and $a^4 + b^4 + c^4 = 128$. Determine all possible values of $ab + bc + ca$.

8. Consider a point G in the interior of a parallelogram $ABCD$. A circle Γ through A and G intersects the sides AB and AD for the second time at the points E and F respectively. The line FG extended intersects the side BC at H and the line EG extended intersects the side CD at I. The circumcircle of triangle HGI intersects the circle Γ for the second time at $M \neq G$. Prove that M lies on the diagonal AC.

9. Suppose x, y, z are real numbers such that $x^2 + y^2 + z^2 + 2xyz = 1$. Prove that $8xyz \leq 1$, with equality if and only if (x, y, z) is one of the following:
$$\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right) \left(-\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\right) \left(-\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right) \left(\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right).$$

10. *Island Hopping Holidays* offer short holidays to 64 islands, labeled Island i, $1 \leq i \leq 64$. A guest chooses any Island a for the first night of the holiday, moves to Island b for the second night, and finally moves to Island c for the third night. Due to the limited number of boats, we must have $b \in T_a$ and $c \in T_b$, where the sets T_i are chosen so that

    (a) each T_i is non-empty, and $i \notin T_i$,

    (b) $\sum_{i=1}^{64} |T_i| = 128$, where $|T_i|$ is the number of elements of T_i.

    Exhibit a choice of sets T_i giving at least $63 \cdot 64 + 6 = 4038$ possible holidays.

    Note that $c = a$ is allowed, and holiday choices (a, b, c) and (a', b', c') are considered distinct if $a \neq a'$ or $b \neq b'$ or $c \neq c'$.

33rd Irish Mathematical Olympiad
18 July 2020, Paper 1

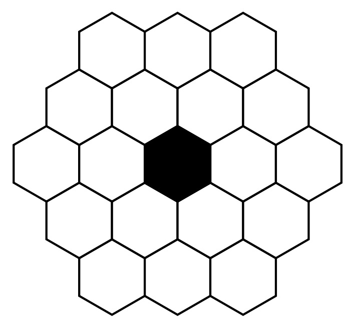
1. We say an integer n is *naoish* if $n \geq 90$ and the second-to-last digit of n (in decimal notation) is equal to 9. For example, 10798, 1999 and 90 are naoish, whereas 9900, 2009 and 9 are not. Nino expresses 2020 as a sum:
$$2020 = n_1 + n_2 + \ldots + n_k$$
where each of the n_j is naoish.

   What is the smallest positive number k for which Nino can do this?

2. A round table has $2N$ chairs around it. Due to social distancing guidelines, no two people are allowed to sit next to each other. How many different ways are there to choose seats around the table on which $N - 1$ guests can be seated?

3. Circles Ω_1, centre Q, and Ω_2, centre R, touch externally at B. A third circle, Ω_3, which contains Ω_1 and Ω_2, touches Ω_1 and Ω_2 at A and C, respectively. Point C is joined to B and the line BC is extended to meet Ω_3 at D. Prove that QR and AD intersect on the circumference of Ω_1.

4. Let n be a positive integer. An *n-level honeycomb* is a plane region covered with regular hexagons of side-length 1 connected along edges, such that the centres of the boundary hexagons are lined up along a regular hexagon of side-length $n\sqrt{3}$.



   The diagram shows a 2-level honeycomb from which the central hexagon has been removed.

   A *trex* is a sequence of 3 hexagons with collinear centres such that the middle hexagon shares an edge with each of its neighbours in the trex.

   An n-level honeycomb from which the central size-1 hexagon has been removed is to be completely covered by trexes without any overlaps.

   Find all values of n for which this is possible.

5. Let $a, b, c > 0$. Prove that
$$\sqrt[7]{\frac{a}{b+c} + \frac{b}{a+c}} + \sqrt[7]{\frac{b}{c+a} + \frac{c}{a+b}} + \sqrt[7]{\frac{c}{a+b} + \frac{a}{b+c}} \geq 3.$$

33rd Irish Mathematical Olympiad
18 July 2020, Paper 2

6. Pat has a pentagon, each of whose vertices is coloured either red or blue. Once an hour, Pat recolours the vertices as follows.

    - Any vertex whose two neighbours were the same colour for the last hour, becomes blue for the next hour.
    - Any vertex whose two neighbours were different colours for the last hour, becomes red for the next hour.

    Show that there is at least one vertex which is blue after the first recolouring and remains blue for ever.

7. Let $\mathbb{N}$ denote the strictly positive integers. A function $f : \mathbb{N} \to \mathbb{N}$ satisfies the following for all $n \in \mathbb{N}$:

$$f(1) = 1$$
$$f(f(n)) = n$$
$$f(2n) = 2f(n) + 1.$$

    Find the value of $f(2020)$.

8. Determine the last (rightmost) three decimal digits of n where:

$$n = 1 \times 3 \times 5 \times 7 \times \ldots \times 2019.$$

9. A trapezium $ABCD$, in which AB is parallel to DC, is inscribed in a circle of radius R and centre O. The non-parallel sides DA and CB are extended to meet at P while diagonals AC and BD intersect at E. Prove that $|OE| \cdot |OP| = R^2$.

10. Show that there exists a hexagon $ABCDEF$ in the plane such that the distance between every pair of vertices is an integer.

34th Irish Mathematical Olympiad
8 May 2021, Paper 1

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1. Let $N = 15! = 15 \cdot 14 \cdot 13 \cdots 3 \cdot 2 \cdot 1$. Prove that N can be written as a product of nine different integers all between 16 and 30 inclusive.

2. An isosceles triangle ABC is inscribed in a circle with $\angle ACB = 90^\circ$ and EF is a chord of the circle such that neither E nor F coincide with C. Lines CE and CF meet AB at D and G respectively.

   Prove $|CE| \cdot |DG| = |EF| \cdot |CG|$.

3. For each integer $n \geq 100$ we define $T(n)$ to be the number obtained from n by moving the two leading digits to the end. For example, $T(12345) = 34512$ and $T(100) = 10$. Find all integers $n \geq 100$ for which:
$$n + T(n) = 10n.$$

4. You have a 3×2021 chessboard from which one corner square has been removed. You also have a set of 3031 identical dominoes, each of which can cover two adjacent chessboard squares. Let m be the number of ways in which the chessboard can be covered with the dominoes, without gaps or overlaps.

   What is the remainder when m is divided by 19?

5. The function $g : [0, \infty) \mapsto [0, \infty)$ satisfies the functional equation:
$$g(g(x)) = \frac{3x}{x+3}, \qquad \text{for all } x \geq 0.$$
   You are also told that for $2 \leq x \leq 3$:
$$g(x) = \frac{x+1}{2}.$$
   (a) Find $g(2021)$.

   (b) Find $g(1/2021)$.

34th Irish Mathematical Olympiad
8 May 2021, Paper 2

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6. A sequence whose first term is positive has the property that any given term is the area of an equilateral triangle whose perimeter is the preceding term. If the first three terms form an arithmetic progression, determine all possible values of the first term.

7. Each square of an $n \times n$ grid is coloured either blue or red, where n is a positive integer. There are k blue cells in the grid. Pat adds the sum of the squares of the numbers of blue cells in each row to the sum of the squares of the numbers of blue cells in each column to form S_B. He then performs the same calculation on the red cells to compute S_R.

   If $S_B - S_R = 50$, determine (with proof) all possible values of k.

8. A point C lies on a line segment AB between A and B and circles are drawn having AC and CB as diameters. A common tangent to both circles touches the circle with AC as diameter at $P \neq= C$ and the circle with CB as diameter at $Q \neq C$.

   Prove that AP, BQ and the common tangent to both circles at C all meet at a single point which lies on the circumference of the circle with AB as diameter.

9. Suppose the real numbers a, A, b, B satisfy the inequalities:
$$|A - 3a| \leq 1 - a, \qquad |B - 3b| \leq 1 - b,$$
   and a, b are positive. Prove that
$$\left|\frac{AB}{3} - 3ab\right| - 3ab \leq 1 - ab.$$

10. Let $P_1, P_2, \ldots, P_{2021}$ be 2021 points in the quarter plane $\{(x, y) : x \geq 0, y \geq 0\}$. The centroid of these 2021 points lies at the point $(1, 1)$.

    Show that there are two distinct points P_i, P_j such that the distance from P_i to P_j is no more than $\sqrt{2}/20$.

Hints to Selected Problems

We gather here some hints to the first problems on selected papers. These hints are intended as a ‘route into’ a problem rather than detailing problem’s solution.

2017 Q1:
As this is a “digits” question, notice that the multiple $k(99)$ can be written as $k(100-1)$. If one expands this expression “digit-ally” and considers subtraction as first taught in school, what can be inferred about the make-up of k?

2017 Q6:
If n is even and $n+1$ is divisible by 5, then what is the precise form for such a number n? On top of this, if a is a number that divides both 2^n+n and 2^n-1, then should a also divide their difference?

2018 Q1:
If the initial number is $m>2017$ then $m=2018+a$ for some non-negative integer a. After Mary has finished her first turn the number will be $2017(2018+a)+2=2018(2017)+(2018-1)a+2=2018(2017+a)+2-a$. The remainder on division by 2018 of this number will be non-zero if and only if $2-a$ is also non-zero on division by 2018. Since everything seems to hinge on an infinite collection of numbers not being divisible by 2018, why not try to keep track of the numbers by always considering them in the form $2018x+y$?

2018 Q6:
It would be nice to first know $f(0)=a$. Can you come up with values for x and y so that $2x+f(y)$ ‘disappears’?

2019 Q1:
Is the product of two quasi-primes also a quasi-prime number, or not a quasi-prime number? What about the product of two non-quasi-prime numbers?

2019 Q6:
Thinking about other possible numbers in terms of sums of powers of four while momentarily forgetting about the second condition, can you find five such distinct positive integeres whose sum of fourth powers is less than $1^4+2^4+3^4+5^4+6^4$? Suppose a is such a number. Can you find an integer solution to the equation $a=n+(n+1)+(n+2)+(n+3)+(n+4)+(n+5)$?

2020 Q1:
Naoish numbers have a very specific form. Write a naoish number as a multiple of 100 plus a remainder. Can you now write down an additive expression involving 2 non-negative unknowns for a general naoish number? Replace this expression into the equation to be solved. Investigate when the sum of the unit digits can take care of the remainder on division by 100.

2020 Q6:
Suppose that there are no blue vertices in the first recolouring. Try to construct an original colouring that will produce such a recolouring. Can you do this? If yes then have you checked that all recolouring rules have been satisfied?

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